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Research article

Compression fans for mixed-type dark energy supersonic flow past a concave corner and concave bend

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Abstract: This paper mainly considers the problem of the supersonic horizontal upstream flow of mixed-type dark energy fluid turning a concave corner and a concave bend. We obtain explicit expression of the global solutions, which contain compression fans. The results differ from the structure of oblique shock wave solutions for polytropic gas. These are caused by the nonconvexity of mixed-type dark energy fluid.

Keywords: Euler equations; the mixed-type dark energy fluid; concave corner; concave bend; compression fans
Mathematics Subject Classification: 35L53, 58J32, 58J45, 76J20

1. Introduction

Supersonic aircraft, such as fighter jets, missiles, and hypersonic gliders, usually contain corner and bend structures. These structures are present in key components, including air intakes, wing leading edges, and tail cones. Supersonic flows may induce complex phenomena such as shock waves, boundary-layer disturbances, flow separation, and turbulence transition during flow past surfaces. These directly cause bend heat flux to surge severalfold and friction drag to rise significantly, severely compromising aircraft flight stability and the reliability of thermal protection systems. Courant and Friedrichs [1] noted that supersonic flow past a corner and a bend constitutes one of the fundamental flow problems, providing the theoretical foundation for in-depth investigation of a series of complex flow phenomena, including two-dimensional pipe flow, jets, and shock reflection.

In 1936, Prandtl [2] considered the reflection of shock waves past a wedge in steady supersonic flow. For the first time, the shock polar analysis method has demonstrated the presence of either weak shock waves or strong shock waves. For the same upstream state $\vec{q}_0$ and wall turning angle σ, the flow past the shock front may be supersonic or subsonic, the former case corresponding to the weak shock and the latter to the strong shock [1] (see Figure 1). This pioneering research provides crucial foundations for subsequent analyses concerning supersonic flow past corners and bends. In 2007, Chen and Fang [3] considered the stability of transonic shock waves in supersonic steady flow past a wedge using potential flow equations. By perturbing the upstream flow, they demonstrated that the transonic shock wave retained stability. In 2008, Elling and Liu [4] investigated the two-dimensional supersonic flow over a wedge. When a sharp wedge accelerated to supersonic speeds, the structure of a weak shock wave solution at the tip was obtained. In 2009, Yin and Zhou [5] analyzed the globally stable transonic shock for steady Euler equations past a long concave corner. Given an appropriate downstream pressure at infinity, he proved the global existence and stability of the shock solution and obtained the asymptotic behavior of the downstream subsonic solution. In 2017, Chen [6] considered the supersonic flow past a concave corner for two-dimensional steady Euler equations. He analyzed the stability of weak and strong shock waves and elucidated how to transform the wedge problem into an initial boundary value problem and a free-boundary problem. For results regarding flow past a convex corner, refer to [7–9].

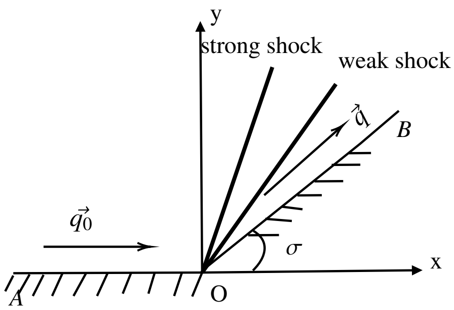


Figure 1. Weak shock and strong shock for the same upstream state $\vec{q}_0$ and wall turning angle σ.

Concave bends may be regarded as smooth transitions at recessed corners, with corresponding changes in flow phenomena. In Chapter 4 of [1], Courant and Friedrichs provided an elucidating example: "Suppose a constant two-dimensional supersonic flow arriving along a straight wall is forced to turn in a curved wedge, an envelope originates inside the flow. This envelope generally has a cusp, and a shock discontinuity forms in the region between the two branches" (see Figure 2). In 2001, Chen [10] studied steady supersonic flow past a curved wedge and proved that a suspended shock with a singularity would form in the flow field. The suspended shock originates from the cusp of the characteristic envelope, and estimates of the solution near the cusp were obtained. In 2006, Yin [11] considered the global existence of shock solutions for supersonic flow past a curved wedge. It is proved that the global existence of an attached shock holds when the small perturbation of a concave corner becomes a concave bend, and the angle of the solid bend is less than a certain critical value. For studies on the flow over convex bends, refer to [1, 12, 13].

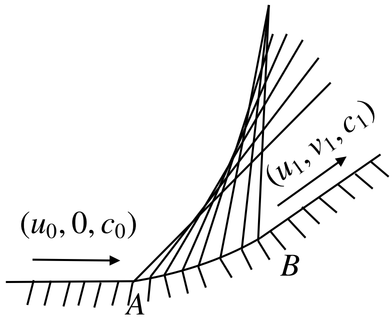


Figure 2. Envelope of a straight characteristic issuing from a bend $\widehat{AB}$.

Polytropic gas can describe the isothermal process of air at normal temperature and the adiabatic compression of air in internal combustion engines. It can also characterize the high-density state of liquids and the evolution of gas behind blast waves. However, numerous nonconvex fluids exist in nature, for instance, Van der Waals gas [14], and the Giacomazzo–González–López Equation of State (GGL EoS) [15]. In 2019, to explain the late-time accelerated expansion of the universe, Oikonomou [16] proposed a nonconvex generalized dark energy fluid EoS

$$P(\rho) = A\rho \ln\left(B\frac{\rho}{H^2}\right) + C\rho^2 + D\frac{1}{\rho} + E\rho, \quad \rho > 0, \tag{1.1}$$

where ρ denotes the density; A, B, C, D, E are constants; and H is the Hubble rate of the universe, which describes the local expansion rate of the universe and evolves with cosmic time. The generalized dark energy fluid (1.1) is used to describe the scenario of dark matter–dark energy interaction [18]. When $A = 0$, $B = H^2$, $C = 0$, $D = -1$, and $E = 0$, EoS (1.1) can be written as

$$P(\rho) = -\frac{1}{\rho},$$

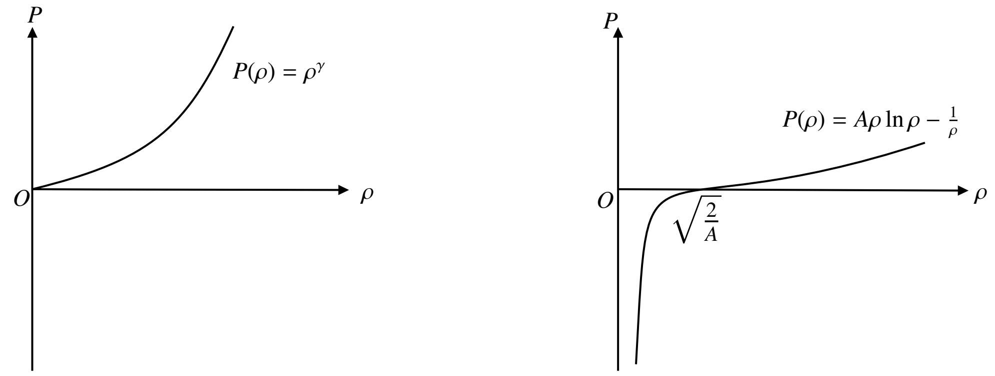
which corresponds to the Chaplygin gas [17]. This model was proposed by Chaplygin in 1904 when he considered transonic flow over two-dimensional airfoils. It was later rediscovered in a cosmological context as a possible explanation for the accelerated expansion of the universe [18]. Furthermore, the generalized Chaplygin gas model was proposed as a unified dark fluid that can describe both dark matter and dark energy within a single framework [19]. When $B = H^2$, $D = -1$, and $C = E = 0$, (1.1) is the mixed-type dark energy fluid EoS

$$P(\rho) = A\rho \ln \rho - \frac{1}{\rho}, \quad A \in (0, 1). \tag{1.2}$$

Thompson [20] showed that, for nonconvex gases, compressive processes do not necessarily evolve into shock waves. Instead, under certain thermodynamic conditions, they may develop into continuous compression waves (compression fans) with smooth pressure gradients. When $\rho^2 \leq 2/A$, Eq (1.2) is nonconvex which differs from classical gas dynamics (see Figure 3). In this paper, we consider the two-dimensional steady isentropic Euler equations with (1.2):

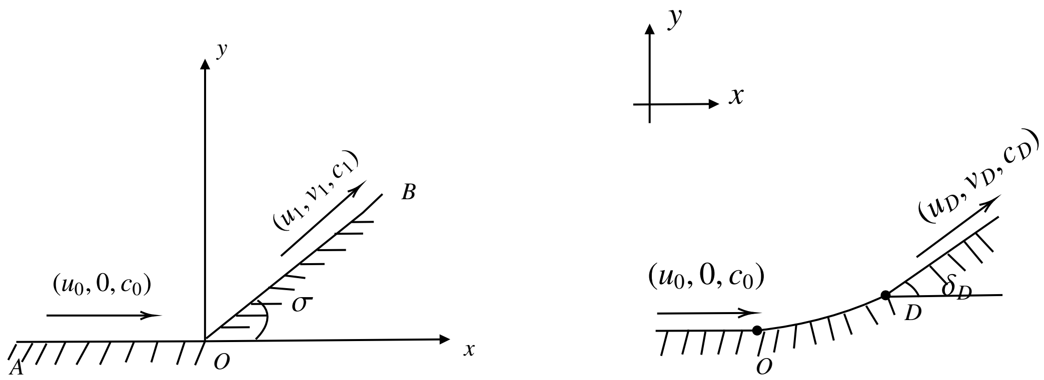
$$\begin{cases} (\rho u)_x + (\rho v)_y = 0, \\ (\rho u^2 + (A\rho\ln\rho - \frac{1}{\rho}))_x + (\rho uv)_y = 0, \\ (\rho uv)_x + (\rho v^2 + (A\rho\ln\rho - \frac{1}{\rho}))_y = 0, \end{cases} \tag{1.3}$$

where (u, v) denotes the flow velocity, and $c = \sqrt{P'(\rho)}$ denotes the speed of sound. We obtain compression fans solutions for the mixed-type dark energy fluid (1.2) under a nonconvexity condition.



(a) Polytropic gas ($\gamma \in (1, 5/3)$ **(b)** Mixed-type dark energy fluid

Figure 3. Polytropic gas and mixed-type dark energy fluid.



(a) Concave corner **(b)** Concave bend

Figure 4. Mixed-type dark energy supersonic flow turning a concave corner and concave bend.

Let $\rho^* = e^{-3/2}$ denote the critical density. We consider the following two problems:

Problem I: Suppose that $\rho \leq \rho^*$. We consider the problem of supersonic horizontal upstream flow of mixed-type dark energy fluid (1.2) past a concave corner, as shown in Figure 4(a). The boundary value conditions for this problem can be written as

$$\begin{cases} (u, v, c)(x, y) = (u_0, 0, c_0), & (x, y) \in \{(x, y)|x < 0, \quad y = 0\}, \\ v_1(x, y) = u_1(x, y)\tan\sigma, & (x, y) \in \{(x, y)|y > 0, \quad y = x\tan\sigma\}, \end{cases} \tag{1.4}$$

where $u_0 > c_0$, u_0, and c_0 represent the upstream flow horizontal velocity and the speed of sound, respectively. σ is the inclination angle of the solid wall OB.

For problem I, when $\rho \leq \rho^*$, we obtain the structure of centered compressive fans for the boundary value problem (1.3) and (1.4) (see Theorem 3.1), as shown in Figure 5.

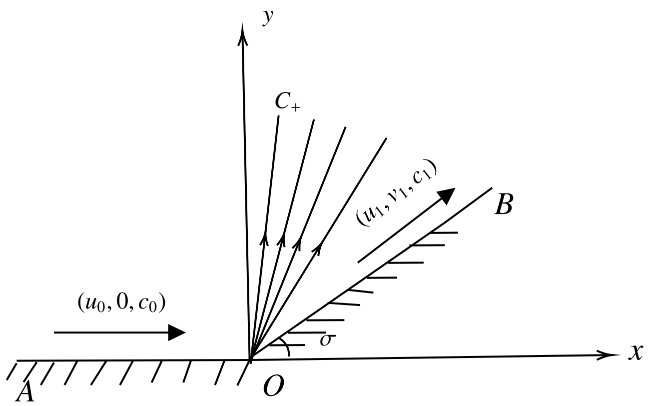


Figure 5. Mixed-type dark energy supersonic flow past the concave corner AOB.

Problem II: Suppose that $\rho \leq \rho^*$. We consider the problem of supersonic horizontal upstream flow of mixed-type dark energy fluid (1.2) past a concave bend, as shown in Figure 4(b).

Let the smooth curve $\widehat{OD}$ be given by

$$y = f(x), \quad x \in (0, x_D), \tag{1.5}$$

where x_D is the x-axis of D, and $f''(x) > 0$. The parametric expression for $\widehat{OD}$ is

$$\begin{cases} x = m(\delta), \\ y = n(\delta), \end{cases} \tag{1.6}$$

where $\delta = \arctan f'(x) \in (0, \delta_D]$, $\delta_D = \arctan f'(x_D)$. The boundary conditions for this problem are

$$\begin{cases} (u, v, c)(x, y) = (u_0, 0, c_0), & (x, y) \in \{(x, y)|x < 0, y = 0\}, \\ v(x, y) = u(x, y) f'(x), & (x, y) \in \{(x, y)|y = f(x), x \in (0, x_D)\}, \\ v_D(x, y) = u_D(x, y) \tan \delta_D, \end{cases} \tag{1.7}$$

where $u_0 > c_0$, and $y = f(x)$ satisfy $f''(x) > 0$, $f'(0) = 0$ and $f(0) = 0$.

For Problem II, when $\rho \leq \rho^*$, we obtain the structure of compressive fans for the boundary value problem (1.3) and (1.7) (see Theorem 4.1), as shown in Figure 6.

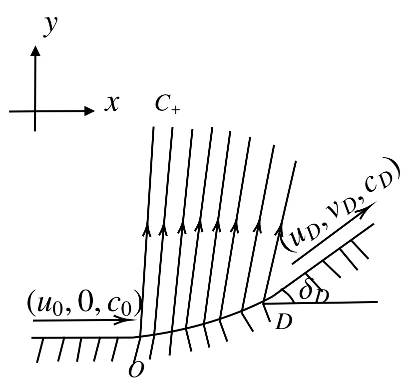


Figure 6. Mixed-type dark energy supersonic flow past the concave bend $\widehat{OD}$.

Remark 1. *For the problem of supersonic flow turning a concave corner and a concave bend, we obtain the compression fans for mixed-type dark energy fluid* (1.2)*, which differ from the results of the shock waves solution for polytropic gas [1].*

This paper is structured as follows. In Section 2, we present the characteristic analysis of System (1.3). In Section 3, we construct centered compression fans structure for the boundary value Problems (1.3) and (1.4). In Section 4, we construct compression fans structure for the boundary value Problems (1.3) and (1.7).

2. Characteristic analysis

For smooth flow, System (1.3) can be written as

$$\begin{cases} (\rho u)_x + (\rho v)_y = 0, \\ u u_x + v u_y + \dfrac{1}{\rho}\left(A\rho \ln\rho - \frac{1}{\rho}\right)_x = 0, \\ u v_x + v v_y + \dfrac{1}{\rho}\left(A\rho \ln\rho - \frac{1}{\rho}\right)_y = 0. \end{cases} \tag{2.1}$$

If the flow is isentropic, and the upstream state is irrotational, the flow remains irrotational throughout the flow field [9]. Under these assumptions, System (2.1) can be rewritten as

$$\begin{bmatrix} c^2-u^2 & -uv \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_x + \begin{bmatrix} -uv & c^2-v^2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.2}$$

Define

$$B = \frac{1}{2}(u^2+v^2) + \int_{\rho_0}^{\rho} \frac{P'(s)}{s}\,ds.$$

Using the last two equations of (2.1), we obtain

$$B_x = -v\varpi, \qquad B_y = u\varpi,$$

where $\varpi := u_y - v_x$ denotes the vorticity. Because $\varpi = 0$, we have $\nabla B = 0$. Hence, B is constant, throughout the connected smooth irrotational flow region. For the uniform upstream state $(u_0, 0, c_0)$, we gain

$$B = \frac{1}{2}u_0^2.$$

Thus, Bernoulli's law is

$$\frac{1}{2}q^2 + \int_{\rho_0}^{\rho} \frac{P'(s)}{s}ds = \frac{1}{2}u_0^2, \tag{2.3}$$

where $q = \sqrt{u^2+v^2}$, so the characteristic equation of System (2.2) is given by

$$(v-\lambda u)^2 - c^2(1+\lambda^2) = 0. \tag{2.4}$$

When $q > c$, from (2.4), we obtain

$$\lambda_{\pm} = \frac{uv \pm c\sqrt{q^2 - c^2}}{u^2 - c^2} \tag{2.5}$$

and

$$c = \frac{|(u, v) \cdot (-\lambda, 1)|}{\sqrt{1 + \lambda^2}}. \tag{2.6}$$

When the upstream flow is supersonic, let $dy/dx = \lambda_{\pm}$, and let α, β, θ, μ represent the C_+ characteristic angle, C_- characteristic angle, flow angle, and Mach angle, respectively. Then,

$$\tan\alpha = \lambda_+, \quad \tan\beta = \lambda_-, \quad \theta = \alpha - \mu = \frac{\alpha + \beta}{2}, \quad \mu = \frac{\alpha - \beta}{2}, \quad u = c\frac{\cos\theta}{\sin\mu}, \quad v = c\frac{\sin\theta}{\sin\mu}. \tag{2.7}$$

The characteristic form of System (2.2) is

$$\begin{cases} \cos\beta\,\bar{\partial}_+ u + \sin\beta\,\bar{\partial}_+ v = 0, \\ \cos\alpha\,\bar{\partial}_- u + \sin\alpha\,\bar{\partial}_- v = 0, \end{cases} \tag{2.8}$$

where $\bar{\partial}_+ = \cos\alpha\,\partial_x + \sin\alpha\,\partial_y$, $\bar{\partial}_- = \cos\beta\,\partial_x + \sin\beta\,\partial_y$ denote the directional derivatives along the C_+ and C_- characteristics (see Figure 7), respectively.

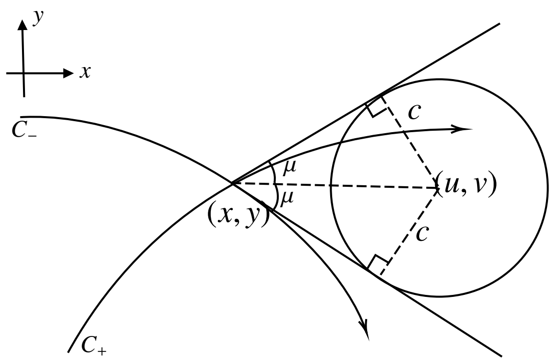


Figure 7. Characteristic directions.

From (2.7), we have

$$\begin{cases} \bar{\partial}_{\pm} u = \frac{\cos\theta}{\sin\mu}\bar{\partial}_{\pm} c + c\dfrac{\cos\bar{\alpha}\bar{\partial}_{\pm}\beta - \cos\beta\bar{\partial}_{\pm}\alpha}{2\sin^2\mu}, \\ \bar{\partial}_{\pm} v = \frac{\sin\theta}{\sin\mu}\bar{\partial}_{\pm} c + c\dfrac{\sin\alpha\bar{\partial}_{\pm}\beta - \sin\beta\bar{\partial}_{\pm}\alpha}{2\sin^2\mu}. \end{cases} \tag{2.9}$$

Combining (2.8) and (2.9), we obtain

$$\begin{cases} \bar{\partial}_- c = c\dfrac{\cos 2\mu\bar{\partial}_-\alpha - \bar{\partial}_-\beta}{\sin 2\mu}, \\ \bar{\partial}_+ c = c\dfrac{\bar{\partial}_+\alpha - \cos 2\mu\bar{\partial}_+\beta}{\sin 2\mu}. \end{cases} \tag{2.10}$$

According to (2.3),

$$\begin{cases} u\bar{\partial}_+ u + v\bar{\partial}_+ v + \frac{c^2}{\rho}\bar{\partial}_+\rho = 0, \\ u\bar{\partial}_- u + v\bar{\partial}_- v + \frac{c^2}{\rho}\bar{\partial}_-\rho = 0. \end{cases} \tag{2.11}$$

From the definition of the speed of sound $c^2 = dP/d\rho$, we obtain

$$2c\bar{\partial}_\pm c = P''(\rho)\bar{\partial}_\pm\rho. \tag{2.12}$$

Now, we insert (2.12) into (2.11):

$$\left(\frac{1}{\sin^2\mu} + k(\rho)\right)\bar{\partial}_\pm c = c\frac{\cos\mu\left(\bar{\partial}_\pm\alpha - \bar{\partial}_\pm\beta\right)}{2\sin^3\mu}, \tag{2.13}$$

where $k(\rho) = 2P'(\rho)/[\rho P''(\rho)]$. Substituting (2.10) into (2.13), we have

$$\begin{cases} \bar{\partial}_-\beta = \frac{\cos 2\mu k(\rho)-1}{1+k(\rho)}\bar{\partial}_-\alpha, \\ \bar{\partial}_+\alpha = \frac{\cos 2\mu k(\rho)-1}{1+k(\rho)}\bar{\partial}_+\beta, \end{cases} \tag{2.14}$$

which yields

$$\begin{cases} \bar{\partial}_- c = c\dfrac{\cot\mu}{1+k(\rho)}\bar{\partial}_-\alpha, \\ \bar{\partial}_+ c = -c\dfrac{\cot\mu}{1+k(\rho)}\bar{\partial}_+\beta. \end{cases} \tag{2.15}$$

The Riemann invariants r_+ and r_- are defined as

$$\begin{cases} r_+ = \theta + \int_{u_0}^{q}\dfrac{\sqrt{s^2-c^2}}{sc}ds, \\ r_- = \theta - \int_{u_0}^{q}\dfrac{\sqrt{s^2-c^2}}{sc}ds. \end{cases} \tag{2.16}$$

Using (2.8) and (2.16), we simply derive

$$\begin{cases} \bar{\partial}_- r_+ = 0, \\ \bar{\partial}_+ r_- = 0. \end{cases} \tag{2.17}$$

3. The supersonic flow around a concave corner

In this section, we obtain a centered compressive fans solution R_0^+ for the boundary value problem (1.3) and (1.4), as shown in Figure 5.

Theorem 3.1. *Let*

$$\sigma_0 := \int_{\rho_0}^{\rho^*}\frac{\sqrt{P'(\rho)}\sqrt{u_0^2 - 2\displaystyle\int_{\rho_0}^{\rho}\frac{P'(s)}{s}\,ds - P'(\rho)}}{\rho\left(u_0^2 - 2\displaystyle\int_{\rho_0}^{\rho}\frac{P'(s)}{s}\,ds\right)}\,d\rho. \tag{3.1}$$

(i) If $0 < \sigma < \sigma_0$, then $\rho_0 \le \rho < \rho^*$, and there exists a centered compressive fan R_0^+ for the boundary value problem (1.3) *and* (1.4)*, given by*

$$R_0^+:\begin{cases} u(x,y) = \hat{u}(\alpha) = \sqrt{u_0^2 - P'(\rho_0) - \displaystyle\int_{\rho_0}^{\rho(\alpha)} \Big(P''(\varphi) + \frac{2}{\varphi} P'(\varphi) \Big)\, d\varphi} \cos\alpha + c(\alpha)\sin\alpha, \\ v(x,y) = \hat{v}(\alpha) = \sqrt{u_0^2 - P'(\rho_0) - \displaystyle\int_{\rho_0}^{\rho(\alpha)} \Big(P''(\varphi) + \frac{2}{\varphi} P'(\varphi) \Big)\, d\varphi} \sin\alpha - c(\alpha)\cos\alpha, \\ c(x,y) = \hat{c}(\alpha) = c_0 + \displaystyle\int_{\alpha_0}^{\alpha} \frac{\sqrt{u_0^2 - P'(\rho_0) - \displaystyle\int_{\rho_0}^{\rho(h)} \Big(P''(\varphi) + \frac{2}{\varphi} P'(\varphi) \Big)\, d\varphi}}{1 + k(\rho(h))} dh, \end{cases} \tag{3.2}$$

where $\alpha = \arctan y/x$ is the C_+ characteristic angle, and $\alpha_0 = \arcsin c_0/u_0$.

(ii) If $\sigma = \sigma_0$, then $\rho_0 \le \rho \le \rho^*$. For the boundary value problem (1.3) *and* (1.4)*, the solution is a centered compressive fan. For $\rho < \rho^*$, the solution is given by R_0^+. At $\rho = \rho^*$, the downstream state is defined by continuous extension as*

$$\lim_{\rho\to\rho^*} \big(\hat{u}(\alpha(\rho)), \hat{v}(\alpha(\rho)), \hat{c}(\alpha(\rho)) \big) = (u^*, v^*, c^*),$$

where

$$u^* = \sqrt{u_0^2 - 2\int_{\rho_0}^{\rho^*} \frac{P'(s)}{s}\, ds} \cos\sigma_0, \qquad v^* = \sqrt{u_0^2 - 2\int_{\rho_0}^{\rho^*} \frac{P'(s)}{s}\, ds} \sin\sigma_0, \qquad c^* = \sqrt{P'(\rho^*)}.$$

Proof. (i) Assume $0 < \sigma < \sigma_0$. Combining (2.12) and (2.15), we obtain

$$\bar{\partial}_-\alpha = \frac{2c^2 + \rho P''(\rho)}{2\rho c\sqrt{q^2 - c^2}} \bar{\partial}_-\rho. \tag{3.3}$$

Integrating along the C_- characteristic,

$$\alpha(x,y) := \tilde{\alpha}(\rho) = \alpha_0 + \int_{\rho_0}^{\rho} \frac{2c^2 + sP''(s)}{2cs\sqrt{q^2 - c^2}} ds. \tag{3.4}$$

By (3.3), we derive

$$\frac{d\alpha}{d\rho} = \frac{2c^2 + \rho P''(\rho)}{2c\rho\sqrt{q^2 - c^2}} < 0. \tag{3.5}$$

It follows from (3.3) and (3.4) that ρ is the function of α; that is,

$$\rho(x,y) = \hat{\rho}(\alpha). \tag{3.6}$$

Combining (3.6) and the definition of the sound speed $c^2 = dP/d\rho$,

$$c(x,y) = \hat{c}(\alpha). \tag{3.7}$$

Using Bernoulli's law (2.3) and (3.7), q is the function of α:

$$q(x,y) := \hat{q}(\alpha) = \sqrt{u_0^2 - 2\int_{\rho_0}^{\rho(\alpha)} \frac{P'(s)}{s} ds}. \tag{3.8}$$

From (2.7), (3.7), and (3.8), we deduce

$$\theta(x,y) := \hat{\theta}(\alpha) = \alpha - \mu(\alpha) = \alpha - \arcsin \frac{\hat{c}(\alpha)}{\hat{q}(\alpha)}. \tag{3.9}$$

Differentiating (2.3) and using $c^2 = P'(\rho)$, we obtain

$$\frac{dq}{d\rho} = -\frac{c^2}{\rho q}, \qquad \frac{dc}{d\rho} = \frac{P''(\rho)}{2c}. \tag{3.10}$$

Because (2.7) and $\sin\mu = c/q$, combining these relations with (3.5) gives

$$\frac{d\theta}{d\rho} = \frac{d\alpha}{d\rho} - \frac{d\mu}{d\rho} = \frac{c\sqrt{q^2 - c^2}}{\rho q^2} > 0. \tag{3.11}$$

By (2.3) and (3.5), we derive

$$\frac{d\hat{\theta}}{d\alpha} = \frac{d\hat{\theta}}{d\rho}\frac{d\rho}{d\alpha} = \frac{2\hat{c}^2(\hat{q}^2 - \hat{c}^2)}{\hat{q}^2\big(2\hat{c}^2 + \rho P''(\rho)\big)} < 0. \tag{3.12}$$

Thus, $\theta(\alpha)$ is strictly monotone:

$$\begin{cases} u(x,y) = \hat{q}\cos\theta, \\ v(x,y) = \hat{q}\sin\theta. \end{cases} \tag{3.13}$$

Combining (3.8) and (3.9), we find that

$$(u,v)(x,y) = (\hat{u},\hat{v})(\alpha). \tag{3.14}$$

From (2.6), we decompose (u,v) along the direction parallel to the C_+ characteristic and the direction perpendicular to the C_+ characteristic; thus, we have the relation

$$\begin{cases} \hat{g} = \hat{u}\cos\alpha + \hat{v}\sin\alpha, \\ \hat{c} = \hat{u}\sin\alpha - \hat{v}\cos\alpha. \end{cases} \tag{3.15}$$

By (3.15), we obtain

$$\begin{cases} \hat{u} = \hat{g}\cos\alpha + \hat{c}\sin\alpha, \\ \hat{v} = \hat{g}\sin\alpha - \hat{c}\cos\alpha, \end{cases} \tag{3.16}$$

and

$$\hat{g}^2 + \hat{c}^2 = \hat{u}^2 + \hat{v}^2 = \hat{q}^2. \tag{3.17}$$

According to (2.8) and (3.14),

$$\cos\alpha\frac{d\hat{u}}{d\alpha} + \sin\alpha\frac{d\hat{v}}{d\alpha} = 0. \tag{3.18}$$

Differentiating both sides of (3.16), we have

$$\begin{cases} \frac{d\hat{u}}{d\alpha} = \frac{d\hat{g}}{d\alpha}\cos\alpha + \frac{d\hat{c}}{d\alpha}\sin\alpha - \hat{g}\sin\alpha + \hat{c}\cos\alpha, \\ \frac{d\hat{v}}{d\alpha} = \frac{d\hat{g}}{d\alpha}\sin\alpha - \frac{d\hat{c}}{d\alpha}\cos\alpha + \hat{g}\cos\alpha + \hat{c}\sin\alpha. \end{cases} \tag{3.19}$$

Inserting (3.19) into (3.18),

$$\frac{d\hat{g}}{d\alpha} = -\hat{c}. \tag{3.20}$$

Differentiate the relation of (3.17) and (2.3) to get

$$2\hat{g}\frac{d\hat{g}}{d\alpha} + 2\hat{c}\frac{d\hat{c}}{d\alpha} = 2\hat{q}\frac{d\hat{q}}{d\alpha}, \tag{3.21}$$

$$\hat{q}\frac{d\hat{q}}{d\alpha} + \frac{P'(\rho)}{\rho}\frac{d\rho}{d\alpha} = 0. \tag{3.22}$$

With (3.21), (3.22), and (2.12), it follows that

$$\hat{g}(\alpha) = (1 + k(\rho))\,\frac{d\hat{c}}{d\alpha}. \tag{3.23}$$

By using (3.23) and (2.8), we derive

$$(1 + k(\rho))\,\frac{d^2\hat{c}}{d\alpha^2} + \frac{2k'(\rho)\sqrt{P'(\rho)}}{P''(\rho)}\left(\frac{d\hat{c}}{d\alpha}\right)^2 + \hat{c}(\alpha) = 0. \tag{3.24}$$

Then, $c(x, y)$ is given by

$$\begin{cases} c(x, y) = \hat{c}(\alpha) = c_0 + \displaystyle\int_{\alpha_0}^{\alpha} \frac{\sqrt{u_0^2 - P'(\rho_0) - \int_{\rho_0}^{\rho(h)} \left[P''(\varphi) + \frac{2}{\varphi}P'(\varphi)\right] d\varphi}}{1 + k(\rho(h))}\, dh, \\ c(\alpha_0) = c_0. \end{cases} \tag{3.25}$$

From (3.23) and (3.25), we obtain

$$g(x, y) = \hat{g}(\alpha) = \sqrt{u_0^2 - P'(\rho_0) - \int_{\rho_0}^{\rho(\alpha)} P''(\varphi) + \frac{2}{\varphi}P'(\varphi)d\varphi}. \tag{3.26}$$

Using (3.16), (3.25), and (3.26), one computes

$$\begin{cases} u(x, y) = \hat{u}(\alpha) = \hat{g}(\alpha)\cos\alpha + \hat{c}(\alpha)\sin\alpha, \\ v(x, y) = \hat{v}(\alpha) = \hat{g}(\alpha)\sin\alpha - \hat{c}(\alpha)\cos\alpha. \end{cases} \tag{3.27}$$

When $\sigma \in (0, \sigma_0)$ for $\gamma \in [\sigma + \arcsin c_1/q_1, \arcsin c_0/u_0]$, define the ray

$$l : \begin{cases} x = r\cos\gamma, \\ y = r\sin\gamma, \end{cases} \tag{3.28}$$

where $r \geq 0$, and $q_1 = \sqrt{u_1^2 + v_1^2}$. Let

$$\begin{cases} u(x,y) := \bar{u}(\gamma) = \sqrt{u_0^2 - P'(\rho_0) - \int_{\rho_0}^{\rho(\gamma)} \left(P''(\varphi) + \frac{2}{\varphi} P'(\varphi) \right) d\varphi} \cos\gamma + \bar{c}(\gamma)\sin\gamma, \\ v(x,y) := \bar{v}(\gamma) = \sqrt{u_0^2 - P'(\rho_0) - \int_{\rho_0}^{\rho(\gamma)} \left(P''(\varphi) + \frac{2}{\varphi} P'(\varphi) \right) d\varphi} \sin\gamma - \bar{c}(\gamma)\cos\gamma, \\ c(x,y) := \bar{c}(\gamma) = c_0 + \int_{\alpha_0}^{\gamma} \frac{\sqrt{u_0^2 - P'(\rho_0) - \int_{\rho_0}^{\rho(h)} \left(P''(\varphi) + \frac{2}{\varphi} P'(\varphi) \right) d\varphi}}{1 + k(\rho(h))} dh. \end{cases} \tag{3.29}$$

The proof is divided into the following three steps.

Step I. We prove that l is a C_+ characteristic.

For any point (x, y) on l, it satisfies

$$u(x,y)\sin\gamma - v(x,y)\cos\gamma = c_0 + \int_{\alpha_0}^{\gamma} \frac{\sqrt{u_0^2 - P'(\rho_0) - \int_{\rho_0}^{\rho(h)} \left(P''(\varphi) + \frac{2}{\varphi} P'(\varphi) \right) d\varphi}}{1 + k(\rho(h))} dh = c(x,y), \tag{3.30}$$

and thus, $\gamma = \alpha$, and the ray l is a C_+ characteristic.

Step II. We prove that (3.29) satisfies (2.8) and (2.3).

Along the C_+ characteristic

$$\cos\beta\,\bar{\partial}_+ u + \sin\beta\,\bar{\partial}_+ v = 0,$$

from (3.18) and (3.29),

$$\cos\alpha\,\bar{\partial}_- u + \sin\alpha\,\bar{\partial}_- v = \left(\cos\alpha \frac{du}{d\alpha} + \sin\alpha \frac{dv}{d\alpha} \right) \bar{\partial}_- \alpha = 0. \tag{3.31}$$

Furthermore, (3.29) satisfies

$$\frac{1}{2}(\bar{u}(\gamma)^2 + \bar{v}(\gamma)^2) + \int_{\rho_0}^{\rho} \frac{P'(s)}{s} ds = \frac{1}{2} u_0^2. \tag{3.32}$$

Step III. We prove that R_0^+ is a centered compressive fan and that the downstream flow remains supersonic.

When $u_0 > c_0$, and $\rho < \rho^*$, we have

$$P''(\rho) < -\frac{2c^2}{\rho}, \tag{3.33}$$

$$q^2 - c^2 = u_0^2 - c_0^2 - \int_{\rho_0}^{\rho} \left(P''(s) + \frac{2}{s} P'(s) \right) ds > 0. \tag{3.34}$$

By (3.34), we have $q^2 > c^2$, so the downstream flow remains supersonic. By (3.5), we derive

$$\frac{d\lambda_+}{d\rho} = \sec^2\alpha \frac{d\alpha}{d\rho} < 0. \tag{3.35}$$

Using (3.11) and (3.35), the C_+ characteristic direction decreases monotonically as the density increases. Hence, the constructed simple wave is compressive.

Accordingly, if $\sigma \in (0, \sigma_0)$, because $\alpha_1 = \sigma + \mu_1$, the terminal flow angle satisfies $\theta_1 = \alpha_1 - \mu_1 = \sigma$. Therefore, $v_1/u_1 = \tan\theta_1 = \tan\sigma$, and the solution to the boundary value problem (1.3) and (1.4) is given by

$$(u, v, c)(x, y) = \begin{cases} (u_0, 0, c_0), & (x, y) \in \{(x, y) : x < 0,\ y > 0\} \cup \{(x, y) : x \geq 0,\ y > x\tan\alpha_0\}, \\ \big(\hat{u}(\alpha), \hat{v}(\alpha), \hat{c}(\alpha)\big), & (x, y) \in \{(x, y) : x \geq 0,\ x\tan\alpha_1 \leq y \leq x\tan\alpha_0\}, \\ (u_1, v_1, c_1), & (x, y) \in \{(x, y) : x \geq 0,\ x\tan\sigma \leq y < x\tan\alpha_1\}. \end{cases}$$

(ii) If $\sigma = \sigma_0$, for $\rho < \rho^*$, the centered compressive fan R_0^+ is given by case (i). At $\rho = \rho^*$, with (2.3) and the upstream state $(u_0, 0, c_0)$, we get

$$\frac{1}{2}(q^*)^2 + \int_{\rho_0}^{\rho^*} \frac{P'(s)}{s} ds = \frac{1}{2}u_0^2. \tag{3.36}$$

Subsequently derive,

$$q^* = \sqrt{u_0^2 - 2\int_{\rho_0}^{\rho^*} \frac{P'(s)}{s}\, ds}, \tag{3.37}$$

where $\rho^* = e^{-3/2}$. By (3.11) and $\theta_0 = 0, \theta = \sigma$, the maximum inclination angle σ_0 satisfies

$$\sigma_0 := \int_{\rho_0}^{\rho^*} \frac{\sqrt{P'(\rho)}\sqrt{u_0^2 - 2\int_{\rho_0}^{\rho} \frac{P'(s)}{s}\, ds - P'(\rho)}}{\rho\left(u_0^2 - 2\int_{\rho_0}^{\rho} \frac{P'(s)}{s}\, ds\right)}\, d\rho.$$

Using (3.5) and making the change of variable $h = \alpha(s)$ in (3.25), we obtain

$$\hat{c}(\alpha(\rho)) = c_0 + \int_{\rho_0}^{\rho} \frac{\sqrt{q^2 - c^2}}{1 + k(s)} \frac{d\alpha}{ds}\, ds = c_0 + \int_{\rho_0}^{\rho} \frac{P''(s)}{2c(s)}\, ds.$$

Therefore,

$$\lim_{\rho \to \rho^*} \hat{c}(\alpha(\rho)) = \sqrt{P'(\rho^*)} = c^*.$$

(2.3) and (3.37) give

$$\lim_{\rho \to \rho^*} q(\rho) = \sqrt{u_0^2 - 2\int_{\rho_0}^{\rho^*} \frac{P'(s)}{s}\, ds} = q^*.$$

By (3.11) and σ_0 , we have

$$\lim_{\rho \to \rho^*} \theta(\rho) = \sigma_0.$$

Therefore,

$$\lim_{\rho \to \rho^*} \hat{u}(\alpha(\rho)) = \sqrt{u_0^2 - 2\int_{\rho_0}^{\rho^*} \frac{P'(s)}{s}\, ds}\cos\sigma_0 = u^*,$$

and

$$\lim_{\rho\to\rho^*}\hat{v}(\alpha(\rho)) = \sqrt{u_0^2 - 2\int_{\rho_0}^{\rho^*}\frac{P'(s)}{s}\,ds}\,\sin\sigma_0 = v^*.$$

Consequently,

$$\lim_{\rho\to\rho^*}\big(\hat{u}(\alpha(\rho)), \hat{v}(\alpha(\rho)), \hat{c}(\alpha(\rho))\big) = (u^*, v^*, c^*),$$

and the downstream flow is (u^*, v^*, c^*). □

4. The supersonic flow around a concave bend

In this section, we discuss the boundary value problem (1.3) and (1.7), deriving the compression fans R_1^+ for (1.3) and (1.7), as shown in Figure 6.

Theorem 4.1. *Let*

$$\delta_0 := \int_{\rho_0}^{\rho^*}\frac{\sqrt{P'(\rho)}\sqrt{u_0^2 - 2\int_{\rho_0}^{\rho}\frac{P'(s)}{s}\,ds - P'(\rho)}}{\rho\left(u_0^2 - 2\int_{\rho_0}^{\rho}\frac{P'(s)}{s}\,ds\right)}\,d\rho \tag{4.1}$$

and

$$\delta = \arctan f'(x) \in (0, \delta_D].$$

(i) *If $\delta_D < \delta_0$, then the density satisfies $\rho_0 \le \rho \le \rho_D < \rho^*$, and there exists a compressive fan R_1^+ for the boundary value problem* (1.3) *and* (1.7)*, which is given by*

$$R_1^+:\ \begin{cases} \tilde{u}(x,y) = \tilde{u}(\delta) = \sqrt{u_0^2 - P'(\rho_0) - \displaystyle\int_{\rho_0}^{\rho(\delta)}\left(P''(\varphi) + \frac{2}{\varphi}P'(\varphi)\right)d\varphi}\,\cos\alpha(\delta) + \tilde{c}(\delta)\sin\alpha(\delta),\\ \tilde{v}(x,y) = \tilde{v}(\delta) = \sqrt{u_0^2 - P'(\rho_0) - \displaystyle\int_{\rho_0}^{\rho(\delta)}\left(P''(\varphi) + \frac{2}{\varphi}P'(\varphi)\right)d\varphi}\,\sin\alpha(\delta) - \tilde{c}(\delta)\cos\alpha(\delta),\\ \tilde{c}(x,y) = \tilde{c}(\delta) = c_0 + \displaystyle\int_{\alpha_0}^{\alpha(\delta)}\frac{\sqrt{u_0^2 - P'(\rho_0) - \int_{\rho_0}^{\rho(h)}\left(P''(\varphi) + \frac{2}{\varphi}P'(\varphi)\right)d\varphi}}{1 + k(\rho(h))}\,dh, \end{cases} \tag{4.2}$$

where $\alpha_0 = \arcsin c_0/u_0$.

(ii) *If $\delta_D = \delta_0$, then $\rho_0 \le \rho \le \rho_D = \rho^*$. For the boundary value problem* (1.3) *and* (1.7)*, the solution is a compressive fan. For $\rho < \rho^*$, the solution is given by R_1^+. At $\rho = \rho^*$, the downstream state is defined by continuous extension as*

$$\lim_{\rho\to\rho^*}\big(\tilde{u}(\delta(\rho)), \tilde{v}(\delta(\rho)), \tilde{c}(\delta(\rho))\big) = (u_D^*, v_D^*, c_D^*),$$

where

$$u_D^* = \sqrt{u_0^2 - 2\int_{\rho_0}^{\rho^*}\frac{P'(s)}{s}\,ds}\,\cos\delta_0, \qquad v_D^* = \sqrt{u_0^2 - 2\int_{\rho_0}^{\rho^*}\frac{P'(s)}{s}\,ds}\,\sin\delta_0, \qquad c_D^* = \sqrt{P'(\rho^*)}.$$

Proof. Using (3.11), we obtain

$$\delta(\rho) = \int_{\rho_0}^{\rho} \frac{c(s)\sqrt{q^2(s) - c^2(s)}}{sq^2(s)}\,ds.$$

Letting $\rho = \rho^*$, the maximum wall turning angle is

$$\delta_0 = \int_{\rho_0}^{\rho^*} \frac{\sqrt{P'(\rho)}\sqrt{u_0^2 - 2\int_{\rho_0}^{\rho} \frac{P'(s)}{s}\,ds - P'(\rho)}}{\rho\left(u_0^2 - 2\int_{\rho_0}^{\rho} \frac{P'(s)}{s}\,ds\right)}\,d\rho.$$

(i) If $\delta_D < \delta_0$ for $\delta \in (0, \delta_D]$, define

$$l' : \begin{cases} x = m(\delta) + r\cos\psi(\delta), \\ y = n(\delta) + r\sin\psi(\delta), \end{cases} \tag{4.3}$$

where $r \geq 0$. For any point (x, y) on the ray l', we define

$$\begin{cases} u(x,y) = \tilde{u}(\delta) = \sqrt{u_0^2 - P'(\rho_0) - \int_{\rho_0}^{\rho(\delta)} \left(P''(\varphi) + \frac{2}{\varphi}P'(\varphi)\right) d\varphi}\,\cos\psi(\delta) + \tilde{c}(\delta)\sin\psi(\delta), \\ v(x,y) = \tilde{v}(\delta) = \sqrt{u_0^2 - P'(\rho_0) - \int_{\rho_0}^{\rho(\delta)} \left(P''(\varphi) + \frac{2}{\varphi}P'(\varphi)\right) d\varphi}\,\sin\psi(\delta) - \tilde{c}(\delta)\cos\psi(\delta), \\ c(x,y) = \tilde{c}(\delta) = c_0 + \int_{\alpha_0}^{\psi(\delta)} \frac{\sqrt{u_0^2 - P'(\rho_0) - \int_{\rho_0}^{\rho(h)} \left(P''(\varphi) + \frac{2}{\varphi}P'(\varphi)\right) d\varphi}}{1 + k(\rho(h))} dh. \end{cases} \tag{4.4}$$

The proof proceeds in three steps.

Step I. We prove that l' is a C_+ characteristic.

For any point (x, y) on l', (4.4) satisfies

$$u(x,y)\sin\psi(\delta) - v(x,y)\cos\psi(\delta) = c(x,y). \tag{4.5}$$

Thus $\psi(\delta) = \alpha(\delta)$, and the ray l' is a C_+ characteristic.

Step II. We prove that (4.4) satisfies the characteristic (2.8) and Bernoulli's law (2.3).

In view of $\bar{\partial}_+ u = \bar{\partial}_+ v = 0$

$$\cos\beta\bar{\partial}_+ u + \sin\beta\bar{\partial}_+ v = 0. \tag{4.6}$$

Owing to (3.13) and (3.17),

$$\cos\alpha\bar{\partial}_- u + \sin\alpha\bar{\partial}_- v = \left(\cos\alpha\frac{du}{d\alpha} + \sin\alpha\frac{dv}{d\alpha}\right)\bar{\partial}_-\alpha = 0. \tag{4.7}$$

From the definition of (4.4), we obtain

$$\bar{\partial}_+\left(\frac{u^2 + v^2}{2} + \int_{\rho_0}^{\rho} \frac{P'(s)}{s}ds\right) = 0 \tag{4.8}$$

and

$$\bar{\partial}_{-}\left(\frac{u^2+v^2}{2}+\int_{\rho_0}^{\rho}\frac{P'(s)}{s}\,ds\right)=\left(u\frac{du}{d\alpha}+v\frac{dv}{d\alpha}+\frac{P'(\rho)}{\rho}\frac{d\rho}{d\alpha}\right)\bar{\partial}_{-}\alpha=0. \tag{4.9}$$

Step III. We prove that R_1^+ is a compressive fan and that the downstream flow remains supersonic.

By (3.11), we have

$$\frac{d\rho}{d\delta}=\frac{\rho q^2}{c\sqrt{q^2-c^2}}>0. \tag{4.10}$$

Because $\lambda_+=\tan\alpha$. Using (3.5) and the chain rule yields

$$\frac{d\lambda_+}{d\delta}=\frac{d\lambda_+}{d\rho}\frac{d\rho}{d\delta}=\sec^2\alpha\frac{q^2\big(2c^2+\rho P''(\rho)\big)}{2c^2(q^2-c^2)}<0. \tag{4.11}$$

The density increases monotonically with the wall turning angle, and hence, the constructed simple wave is compressive. Moreover, by (4.11), the C_+ characteristic direction decreases monotonically along the concave wall. In accordance with the proof of Theorem 3.1, we have $q_D^2>c_D^2$, so the downstream flow remains supersonic. Accordingly, if $\delta_D\in(0,\delta_0)$, the terminal flow angle is δ_D, and $v_D/u_D=\tan\delta_D$, so the solution to the boundary value problem (1.3) and (1.7) is given by

$$(u,v,c)(x,y)=\begin{cases}(u_0,0,c_0), & (x,y)\in\{(x,y):x<0,\ y>0\}\cup\{(x,y):x\ge 0,\ y\ge x\tan\alpha_0\},\\[2mm] \big(\tilde{u}(\delta),\tilde{v}(\delta),\tilde{c}(\delta)\big), & (x,y)\in\left\{(x,y):\begin{array}{c}x=m(\delta)+r\cos\alpha(\delta),\\ y=n(\delta)+r\sin\alpha(\delta),\\ 0<\delta\le\delta_D,\ r\ge 0\end{array}\right\},\\[2mm] \big(u_D,v_D,c_D\big), & (x,y)\in\left\{(x,y):\begin{array}{c}x\ge m(\delta_D),\\ n(\delta_D)+(x-m(\delta_D))\tan\delta_D\le y\\ <n(\delta_D)+(x-m(\delta_D))\tan\alpha(\delta_D)\end{array}\right\}.\end{cases}$$

(ii) If $\delta_D=\delta_0$, for $\rho<\rho^*$, the compressive fan R_1^+ is given by case (i). For $\rho=\rho^*$, Bernoulli's law (2.3) gives

$$q_D^*=\lim_{\rho\to\rho^*}q(\rho)=\sqrt{u_0^2-2\int_{\rho_0}^{\rho^*}\frac{P'(s)}{s}\,ds},\qquad c_D^*=\lim_{\rho\to\rho^*}c(\rho)=\sqrt{P'(\rho^*)}. \tag{4.12}$$

Hence, by (3.13), (4.12),

$$u_D^*=q_D^*\cos\delta_0,\qquad v_D^*=q_D^*\sin\delta_0;$$

therefore, the downstream state is (u_D^*,v_D^*,c_D^*).

□

Remark 4.2. *Because* $\tan\delta=f'(m(\delta))$*, and* $f''>0$*, we have*

$$m'(\delta)=\frac{\sec^2\delta}{f''(m(\delta))}>0.$$

For (4.3)*, the Jacobian with respect to* (δ, r) *is*

$$J(\delta, r) = \frac{m'(\delta)}{\cos\delta}\sin\mu(\delta) - r\alpha'(\delta) > 0, \qquad r \geq 0. \tag{4.13}$$

Thus, the parametrization is locally invertible, and no characteristic envelope is generated by a vanishing Jacobian. We define

$$L(x, \delta) = n(\delta) + (x - m(\delta))\lambda_+(\delta),$$

where $\lambda_+(\delta) = \tan\alpha(\delta)$*. Differentiating both sides with respect to δ, we obtain*

$$\frac{\partial L}{\partial\delta} = m'(\delta)\big(\tan\delta - \lambda_+(\delta)\big) + \big(x - m(\delta)\big)\frac{d\lambda_+}{d\delta} < 0, \qquad x \geq m(\delta). \tag{4.14}$$

If $0 < \delta_1 < \delta_2$ *,* (4.14) *gives*

$$L(x, \delta_1) > L(x, \delta_2).$$

Therefore, the characteristic parametrization is one-to-one and locally invertible throughout the compression fan region.

5. Conclusions

In this paper, we consider the two-dimensional steady isentropic supersonic flow of the mixed-type dark energy fluid past a concave corner and a concave bend. Unlike classical polytropic gases, where compressive waves are typically associated with shock formation due to the convexity of the equation of state, the mixed-type dark energy fluid possesses a nonconvex equation of state. This nonconvexity leads to a distinct wave solution: Compressive processes may develop into compression fans characterized by continuous variations of the flow variables rather than shock discontinuities.

By employing characteristic analysis and Bernoulli's law, we construct explicit compression fan solutions for the corresponding boundary value for concave corner and concave bend. For the concave corner problem, we obtain a centered compressive fan and determine the critical turning angle. For the concave bend problem, we get compressive simple wave solution and determine the corresponding critical wall turning angle.

Author contributions

Hanxuan Wang: Conceptualization, formal analysis, investigation, writing–original draft; Lihui Guo: Supervision, writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest in relation to this article.

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