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Research article

Blow-up dynamics of hyperbolic equations with Hartree-type nonlinearities and internal fractional damping: Analytical results and numerical illustration

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Abstract: In this work, we studied a class of nonlinear wave equations involving a Hartree-type nonlocal source term together with an interior exponentially tempered fractional damping mechanism. The fractional operator was treated through a diffusive representation, which allows the problem to be reformulated as an augmented system suitable for semigroup analysis. Under appropriate assumptions on the exponent and the initial data, we first established local well-posedness by proving the local Lipschitz continuity of the nonlocal Hartree term in the energy space. We then proved the global existence of the solution under suitable assumptions. After that, we investigated the long-time behavior of solutions. In particular, by combining energy estimates with a negative initial energy argument, we derived a finite-time blow-up result. Finally, numerical simulations for a regularized one-dimensional analogue were presented to illustrate qualitatively the rapid-growth mechanism. These computations are not a direct numerical approximation of the multidimensional problem, but they retain its Hartree-type nonlocal interaction, fractional-memory damping, and negative-energy structure.

Keywords: fractional damping; wave equation; Hartree-type nonlinearity; finite-time blow-up; numerical simulations
Mathematics Subject Classification: 26A33, 35L05, 35B44, 65M50

1. Introduction

We consider the following nonlinear wave equation with a Hartree-type nonlinearity and an interior fractional damping term:

$$\begin{cases} y_{tt} - \Delta y + \mu_1 \partial_t^{\alpha,\beta} y = h(y) & \text{in } \Omega \times (0, +\infty), \\ y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x) & \text{in } \Omega, \\ y(x, t) = 0 & \text{on } \partial\Omega \times (0, +\infty), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n > 3)$ is a convex smooth bounded domain and $\mu_1 > 0$ denotes the coefficient of the interior fractional damping. The nonlinear source is given by

$$h(y) = \left(\frac{1}{|x|^{n-2}} * |y|^p \right) |y|^{p-2} y = \int_\Omega \frac{|y(z)|^p}{|x - z|^{n-2}} \, dz \, |y|^{p-2} y, \tag{1.2}$$

where the exponent satisfies

$$1 + \frac{2}{n} < p < \frac{n+2}{n-2}, \qquad n > 3.$$

The admissible exponent range ensures that the Hartree-type nonlinear operator is well-defined in the energy space and possesses the regularity required for the local well-posedness analysis. Moreover, it corresponds to the subcritical regime of the associated nonlinear wave equation, where the nonlocal source can be treated within the functional framework adopted in this paper.

The exponentially tempered fractional derivative of order α is represented by the operator $\partial_t^{\alpha,\beta}$, which is defined as

$$\partial_t^{\alpha,\beta} y(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - r)^{-\alpha} e^{-\beta(t-r)} y'(r) \, dr, \quad 0 < \alpha < 1, \ \beta > 0, \tag{1.3}$$

where Γ denotes the Euler gamma function. This type of operator was introduced by Choi and MacCamy [4]. It differs from the classical Caputo derivative [3] through the presence of the exponential weight in the memory kernel. The corresponding fractional integral is written as

$$[I^{\gamma,\beta} y](t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t - r)^{\gamma-1} e^{-\beta(t-r)} y(r) \, dr, \qquad \gamma > 0, \ \beta > 0. \tag{1.4}$$

Fractional-order operators are widely used to describe systems in which the present state depends on the past history of the process [4, 17]. In the present setting, the exponential kernel in (1.3) represents a fading memory effect with an exponentially decreasing influence of earlier states [4, 17]. Such a mechanism appears naturally in several models arising from viscoelasticity, control theory, and wave propagation in media with internal damping [4,11,17]. Unlike the usual viscous damping, the fractional damping term is nonlocal in time, and this feature has a direct impact on the qualitative properties of the corresponding evolution problem [11, 17].

The term (1.2) introduces another type of nonlocality, namely, a spatially nonlocal interaction. Hartree-type nonlinearities occur in different physical models, including those related to quantum mechanics, plasma physics, and nonlinear optics. They are especially relevant when long-range interactions play a significant role. From the analytical point of view, this convolution-type source is more delicate than a standard local power nonlinearity, since it involves both a singular kernel and a dependence on the solution over the whole spatial domain.

The aim of this paper is to analyze the well-posedness and the qualitative behavior of solutions to problem (1.1). The combination of the interior fractional damping and the Hartree-type source produces several mathematical difficulties. On one hand, the fractional damping term carries memory effects, which require a careful treatment of the associated energy estimates. On the other hand, the convolution structure of the nonlinear source prevents the direct use of many arguments that are commonly applied to local nonlinear wave equations.

To overcome these difficulties, we rewrite the original problem as an augmented system. This reformulation makes it possible to treat the fractional damping term within a semigroup framework. The Hartree-type nonlinearity is handled by using the Hardy–Littlewood–Sobolev inequality together with suitable a priori estimates. After proving local well-posedness, we examine the behavior of solutions through an energy-based argument. In particular, under an appropriate negative-initial-energy condition, we establish a finite-time blow-up result. From a physical viewpoint, Theorem 4.2 ensures a well-defined and unique short-time evolution of the model from admissible initial states. Theorem 5.2 identifies a globally controlled regime in which, under the stated assumptions, the solution remains bounded and exists for all time. In contrast, Theorem 6.5 identifies a negative-initial-energy regime in which the Hartree-type source overcomes the dissipative effect of the fractional-memory mechanism, leading to finite-time blow-up in the energy-space norm. Thus, the main analytical results distinguish between globally controlled evolution and finite-time growth according to the balance between the nonlocal source and fractional-memory dissipation. The present study is theoretical and no direct comparison with physical or engineering experiments is attempted; nevertheless, the competition between nonlocal source effects and memory-type dissipation provides a qualitative mechanism relevant to systems exhibiting long-range interactions and hereditary damping.

Numerical simulations for a regularized one-dimensional analogue are also included to provide a qualitative illustration of the rapid-growth behavior associated with the analytical mechanism. These simulations are not intended as a direct numerical validation of the multidimensional blow-up theorem.

The manuscript has the following structure. In Section 2, literature related to the present work is reviewed and a context is created. In Section 3, we present the preliminary results needed and the description of the functional setting. The proof of local well-posedness is presented in Section 4. Section 5 is devoted to proving the global existence of the solution under suitable assumptions. The finite-time blow-up analysis is provided in Section 6. Finally, Section 7 presents numerical simulations for comparison with the analytical results.

2. Literature review

In the last few decades, the calculus of fractional order has become a vital means of modeling the evolution of problems with memory effects and hereditary behavior, in addition to nonlocal dependence on past states. The basic monographs [15, 17] provide comprehensive treatments of fractional derivatives and fractional integrals, and broader historical surveys and interdisciplinary applications are provided in [20]. In this context, Choi and MacCamy [4] presented a family of exponential-type fractional operators and showed their applications in viscoelasticity and elasticity theory. The most distinguishing characteristic of these operators is that the memory kernel is exponential, which is well-suited for applications in which the fading memory hypothesis is applicable. The Caputo derivative is among the most-used derivatives for solving partial differential equations (PDEs), but the tempered derivative provides more flexibility since it combines the long-range memory property of fractional memory with the damping property of an exponential factor.

Many studies have been directed to the question of the fractional dissipative models of waves, especially those of stability, decay at infinity, and formation in finite time. The nonlinear source term and boundary fractional damping have been studied in [5, 6] and a sufficient condition for blow-up as well as exponential stabilization have been obtained. General decay rates and memory kernels as well as logarithmic growth have then been addressed in [7, 12]. These contributions illustrate the sensitivity of the dynamics of the solutions to the specific form of the dissipative term: Global regularity with decay, relaxation slower than exponential, or even collapse in finite time. From an analytical viewpoint, finite-time blow-up describes the loss of global existence caused by the competition between the nonlinear source and the dissipative mechanism. Identifying sharp conditions that distinguish global existence from finite-time blow-up remains one of the central problems in the analysis of nonlinear evolution equations. Recent developments on analytical and computational methods for nonlinear fractional partial differential equations may also be found in [23, 24].

An important subclass of mathematical models occurs when the nonlocal source terms are of convolution type and in addition to locally defined power-type nonlinearities. Long-range interactions are naturally incorporated in such interactions and are very much involved in quantum mechanics, plasma dynamics, and optical physics, where short-range approximation is not suitable. From the analytical point of view, however, Hartree-type or Riesz-type nonlinearities are different, as the source is globally dependent on the space profile via a singular kernel, requiring more delicate a priori control than the local ones. Additional research on nonlocal memory effects in plate models with fractional damping and infinite history was provided in [1].

The balance between dissipative feedback and source growth has been extensively studied in the classical setting (that is, without fractional terms) with energy techniques and, in particular, using the newest and most powerful tool in stability theory of hyperbolic systems, the multiplier technique introduced by Komornik [9], which remains an indispensable tool in stability analysis. Nevertheless, the majority of existing results concentrate either on local nonlinearities together with standard damping, or else on purely dissipative problems where the fractional mechanism acts via boundary conditions only. The scenario in which a spatially nonlocal Hartree-type nonlinearity coexists with an interior exponentially tempered fractional damping appears to have received comparatively less attention and calls for a dedicated analytic treatment [25, 26].

Although significant progress has been recorded in the study of fractional damped wave equations, Hartree-type nonlocal models, and energy methods, existing works mainly treat these topics separately or under different analytical settings; see, however, the contributions [18, 19, 21] for related developments. See, for example [8],

$$iu_t - (-\Delta)^s u + \left(\frac{1}{|x|^\gamma} * |u|^2\right) u = 0$$

and also [13]

$$\begin{cases}\varphi_{tt} + (-\Delta)^s\varphi + (|x|^{-\gamma} * |\varphi|^2)\varphi = 0 \quad \text{in } \mathbb{R}\times\mathbb{R}^3,\\ \varphi|_{t=0} = \varphi_0, \quad \partial_t\varphi|_{t=0} = \varphi_1.\end{cases}$$

To the best of our knowledge, the combined effect of a Hartree-type convolution nonlinearity and an interior exponentially tempered fractional damping mechanism has not been investigated within a single analytical framework establishing both local well-posedness and finite-time blow-up. In such a coupled model, the spatial nonlocality generated by the Hartree potential and the temporal nonlocality introduced by the fractional operator interact in a way that prevents a direct application of either the classical Hartree analysis or the existing fractional-damping techniques. Consequently, the construction of suitable energy functionals, the treatment of the singular nonlocal interaction, and the derivation of blow-up criteria require additional analytical arguments. Accordingly, the present work addresses this gap by establishing, under explicit assumptions on the initial data, both the local well-posedness of the augmented system and finite-time blow-up under a negative initial energy condition, together with a qualitative numerical illustration of the analytical results.

3. Preliminaries

We begin by recalling the Hardy–Littlewood–Sobolev inequality in the form that will be used throughout the paper.

Lemma 3.1. *[10, 16] (Hardy–Littlewood–Sobolev inequality) Let $t, r > 1$ and $0 < \mu < n$ satisfy*

$$\frac{1}{t} + \frac{\mu}{n} + \frac{1}{r} = 2, \quad f \in L^t(\mathbb{R}^n), \quad h \in L^r(\mathbb{R}^n).$$

Then there exists a sharp constant $C(t, n, \mu, r) > 0$, independent of f and h, such that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)h(z)}{|x-z|^\mu}\,dx\,dz \le C(t, n, \mu, r)\|f\|_t\|h\|_r. \tag{3.1}$$

Moreover, for every $\mathrm{y} \in H^1(\mathbb{R}^n)$, the integral

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\mathrm{y}(x)|^q|\mathrm{y}(z)|^q}{|x-z|^\mu}\,dx\,dz$$

is well-defined whenever

$$\frac{2n-\mu}{n} \le q \le \frac{2n-\mu}{n-2}.$$

The numbers $\frac{2n-\mu}{n}$ and $\frac{2n-\mu}{n-2}$ are known, respectively, as the lower and upper critical exponents associated with the Hardy–Littlewood–Sobolev inequality.

For functions defined on the bounded domain Ω, we use the standard zero extension. More precisely, if $\mathrm{y} \in H_0^1(\Omega)$, then we set $\mathrm{y}(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$. Hence $\mathrm{y} \in H^1(\mathbb{R}^n)$. As a consequence, for a bounded domain Ω and for $\mathrm{y} \in L^t(\Omega)$, $v \in L^r(\Omega)$, one obtains the following localized version of the Hardy–Littlewood–Sobolev estimate:

$$\int_{\Omega}\int_{\Omega} \frac{\mathrm{y}(x)v(z)}{|x-z|^\mu}\,dx\,dz \le C(t, n, \mu, r, \Omega)\|\mathrm{y}\|_t\|v\|_r. \tag{3.2}$$

In particular, for $y \in H_0^1(\Omega)$*, the double integral*

$$\int_\Omega \int_\Omega \frac{|y(x)|^q |y(z)|^q}{|x-z|^\mu}\, dx\, dz$$

is meaningful provided that

$$\frac{2n-\mu}{n} \le q \le \frac{2n-\mu}{n-2}.$$

Combining (3.2) *with the Sobolev embedding theorem yields, for* $y \in H_0^1(\Omega)$*,*

$$\int_\Omega \int_\Omega \frac{|y(x)|^p |y(z)|^p}{|x-z|^{n-2}}\, dx\, dz \le C_1(n,p,\Omega)\|y\|^{2p}_{\frac{2np}{n+2}} \le C_0 \|\nabla y\|_2^{2p}, \tag{3.3}$$

where

$$C_0 = C_1(n,p,\Omega) C_2^{2p}(n,p,\Omega).$$

Here, $C_1(n,p,\Omega)$ *is the constant appearing in the Hardy–Littlewood–Sobolev inequality, while* $C_2(n,p,\Omega)$ *denotes the Sobolev embedding constant. The above estimate holds under the admissibility condition*

$$1 + \frac{2}{n} < p < \frac{n+2}{n-2}, \quad n > 3. \tag{3.4}$$

We next recall the diffusive representation of the exponentially weighted fractional operator.

Theorem 3.2. *[7] Let*

$$\eta(\zeta) = |\zeta|^{\frac{2\alpha-1}{2}}, \quad -\infty < \zeta < +\infty, \quad 0 < \alpha < 1. \tag{3.5}$$

Consider the auxiliary system describing the relation between the input u and the output O:

$$\begin{cases} \partial_t \varphi(\zeta,t) + (\zeta^2 + \beta)\varphi(\zeta,t) - u(t)\eta(\zeta) = 0, & -\infty < \zeta < +\infty, \quad t > 0, \\ \varphi(\zeta,0) = 0, \\ O(t) = \dfrac{\sin(\alpha\pi)}{\pi} \displaystyle\int_{-\infty}^{+\infty} \eta(\zeta)\varphi(\zeta,t)\, d\zeta. \end{cases} \tag{3.6}$$

Then the output O is given by

$$O(t) = I^{1-\alpha,\beta} u(t), \tag{3.7}$$

where the exponentially weighted fractional integral of order $\gamma > 0$ *is defined by*

$$\left[I^{\gamma,\beta} f \right](t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} e^{-\beta(t-s)} f(s)\, ds.$$

In order to apply Theorem 3.2 to the fractional damping term in the present problem, we take $u(t) = y_t(t)$. Hence,

$$I^{1-\alpha,\beta} y_t(t) = \frac{\sin(\alpha\pi)}{\pi} \int_{-\infty}^{+\infty} \eta(\zeta)\varphi(\zeta,t)\, d\zeta.$$

By the definition of the exponentially tempered fractional derivative,

$$\partial_t^{\alpha,\beta} y(t) = I^{1-\alpha,\beta} y_t(t)$$

and therefore

$$\partial_t^{\alpha,\beta} y(t) = \frac{\sin(\alpha\pi)}{\pi} \int_{-\infty}^{+\infty} \eta(\zeta)\varphi(\zeta,t)\,d\zeta, \tag{3.8}$$

where the corresponding auxiliary variable satisfies

$$\partial_t\varphi(\zeta,t) + (\zeta^2+\beta)\varphi(\zeta,t) = \eta(\zeta)y_t(t), \qquad \varphi(\zeta,0)=0.$$

The diffusive representation acts only on the temporal fractional operator and therefore leaves the spatial differential operator and the prescribed homogeneous Dirichlet boundary conditions unchanged. Consequently, the augmented formulation inherits exactly the same boundary conditions as the original problem.

The auxiliary variable $\varphi(\zeta,t)$ represents a distributed internal memory state associated with the exponentially tempered fractional damping. The parameter ζ describes a continuum of relaxation modes whose superposition reconstructs the nonlocal memory effect represented by the fractional operator.

The following integral identity will also be needed:

Lemma 3.3. *[7] If $\lambda \in D_\beta = \mathbb{C}\backslash]-\infty,-\beta]$, then*

$$\int_{-\infty}^{+\infty} \frac{\eta^2(\zeta)}{\lambda+\beta+\zeta^2}\,d\zeta = \frac{\pi}{\sin\alpha\pi}(\lambda+\beta)^{\alpha-1}.$$

We now pass from the original formulation (1.1) to an augmented system.

By Theorem 3.2 and relation (3.8), we have

$$\mu_1\partial_t^{\alpha,\beta}\mathrm{y}(t) = b\int_{-\infty}^{+\infty} \eta(\zeta)\varphi(\zeta,t)\,d\zeta, \qquad b := \frac{\mu_1\sin(\pi\alpha)}{\pi}.$$

Therefore, model (1.1) is equivalently written in the following augmented form:

$$\begin{cases} \mathrm{y}_{tt} - \Delta\mathrm{y} + b\displaystyle\int_{-\infty}^{+\infty}\eta(\zeta)\varphi(x,\zeta,t)\,d\zeta = \int_\Omega \frac{|\mathrm{y}(z,t)|^p}{|x-z|^{n-2}}\,dz\,|\mathrm{y}(x,t)|^{p-2}\mathrm{y}(x,t), & (x,\zeta,t)\in\Omega\times(-\infty,\infty)\times(0,+\infty),\\ \partial_t\varphi(x,\zeta,t) + \left(\zeta^2+\beta\right)\varphi(x,\zeta,t) - \mathrm{y}_t(x,t)\eta(\zeta) = 0, & (x,\zeta,t)\in\Omega\times(-\infty,\infty)\times(0,+\infty),\\ \mathrm{y}(x,t)=0, & (x,t)\in\partial\Omega\times(0,+\infty),\\ \mathrm{y}(x,0)=\mathrm{y}_0(x), \qquad \mathrm{y}_t(x,0)=\mathrm{y}_1(x), & x\in\Omega,\\ \varphi(x,\zeta,0)=0, & (x,\zeta)\in\Omega\times(-\infty,+\infty). \end{cases} \tag{3.9}$$

4. Well-posedness

We now formulate problem (3.9) as an abstract evolution equation. For this purpose, we set

$$\mathcal{W} = (\mathrm{y}, v, \varphi)^T, \qquad v = \mathrm{y}_t.$$

Then $\mathcal{W}$ satisfies

$$\begin{cases} \mathcal{W}_t - \mathcal{A}\mathcal{W} = J(\mathcal{W}), \quad t > 0, \\ \mathcal{W}(0) = \mathcal{W}_0 = (\mathrm{y}_0, \mathrm{y}_1, 0), \end{cases} \tag{4.1}$$

where

$$J(\mathcal{W}) = (0, h(\mathrm{y}), 0)^T. \tag{4.2}$$

The natural phase space for the augmented problem is

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (-\infty, +\infty)).$$

On this space, the linear operator $\mathcal{A}$ is introduced by

$$\mathcal{A}\begin{pmatrix} \mathrm{y} \\ v \\ \varphi \end{pmatrix} = \begin{pmatrix} v \\ \Delta \mathrm{y} - b\displaystyle\int_{-\infty}^{+\infty} \eta(\zeta)\varphi(x,\zeta,t)\,d\zeta \\ -\left(\zeta^2 + \beta\right)\varphi(x,\zeta,t) + v(x)\eta(\zeta) \end{pmatrix}. \tag{4.3}$$

Its domain is first understood in the following abstract sense:

$$D(\mathcal{A}) = \{\mathcal{W} \in \mathcal{H} : \mathcal{A}\mathcal{W} \in \mathcal{H}\}. \tag{4.4}$$

The space $\mathcal{H}$ is endowed with the inner product

$$\left\langle \mathcal{W}, \overline{\mathcal{W}} \right\rangle_{\mathcal{H}} = \int_\Omega (\nabla \mathrm{y} \cdot \nabla \overline{\mathrm{y}} + v\overline{v})\,dx + b\int_\Omega \int_{-\infty}^{+\infty} \varphi(\zeta, x)\overline{\varphi}(\zeta, x)\,d\zeta\,dx.$$

More precisely, the domain of $\mathcal{A}$ can be characterized as

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (\mathrm{y}, v, \varphi) \in \mathcal{H} : \mathrm{y} \in H^2(\Omega) \cap H_0^1(\Omega), \quad v \in H_0^1(\Omega), \\ -(\zeta^2 + \beta)\varphi + v(x)\eta(\zeta) \in L^2(\Omega \times (-\infty, +\infty)), \\ |\zeta|\varphi \in L^2(\Omega \times (-\infty, +\infty)) \end{array} \right\}. \tag{4.5}$$

We begin by recalling the following result established by Zhang et al. [22].

Lemma 4.1. *The mapping $h(\mathrm{y}) : H_0^1(\Omega) \to L^2(\Omega)$ is locally Lipschitz continuous for*

$$1 + \frac{2}{n} < p < \frac{n+2}{n-2}.$$

We are now ready to state the local well-posedness result.

Theorem 4.2. *Let us assume that* (3.4) *holds true.*

If $\mathcal{W}_0 \in \mathcal{H}$, then there exists a maximal time $T_{\max} > 0$ such that problem (3.9) *admits a unique solution*

$$\mathcal{W} \in C([0, T_{\max}); \mathcal{H}).$$

Moreover, if $\mathcal{W}_0 \in D(\mathcal{A})$, then

$$\mathcal{W} \in C([0, T_{\max}); D(\mathcal{A})) \cap C^1([0, T_{\max}); \mathcal{H}).$$

Besides, we have either

$$T_{\max} = +\infty$$

or

$$\limsup_{t \uparrow T_{\max}} \|\mathcal{W}(t)\|_{\mathcal{H}} = +\infty.$$

Proof. Let $\mathcal{W} = (\mathrm{y}, v, \varphi) \in D(\mathcal{A})$. Applying Green's formula together with the boundary condition in (3.9), we obtain

$$\langle \mathcal{A}\mathcal{W}, \mathcal{W} \rangle_{\mathcal{H}} = -b \int_{\Omega} \int_{-\infty}^{+\infty} \left(\zeta^2 + \beta\right) |\varphi(\zeta, t)|^2 \, d\zeta dx \leq 0.$$

Thus, $\mathcal{A}$ is a dissipative operator on $\mathcal{H}$.

It remains to verify the range condition. Let $\tilde{\lambda} > 0$. We show that the operator $\tilde{\lambda} I - \mathcal{A}$ is onto. Equivalently, for every triplet $F = (F_1, F_2, F_3)^T \in \mathcal{H}$, we need to find $\mathcal{W} = (\mathrm{y}, v, \varphi)^T \in D(\mathcal{A})$ satisfying

$$(\tilde{\lambda} I - \mathcal{A})\mathcal{W} = F. \tag{4.6}$$

Using the definition of $\mathcal{A}$ in (4.3), Eq (4.6) is equivalent to the system

$$\begin{cases} \tilde{\lambda}\mathrm{y} - v = F_1(x), \\ \tilde{\lambda} v - \Delta \mathrm{y} + b \int_{-\infty}^{+\infty} \eta(\zeta)\varphi(x, \zeta) \, d\zeta = F_2(x), \\ \tilde{\lambda}\varphi + \left(\zeta^2 + \beta\right)\varphi - \eta(\zeta) v = F_3(x, \zeta). \end{cases} \tag{4.7}$$

Assuming first that y is sufficiently regular, the first and third equations of (4.7) give

$$v = \tilde{\lambda}\mathrm{y} - F_1(x) \tag{4.8}$$

and

$$\varphi = \frac{F_3(x, \zeta) + \eta(\zeta) v}{\zeta^2 + \beta + \tilde{\lambda}}. \tag{4.9}$$

Substituting (4.8) and (4.9) into the second equation of (4.7), we obtain

$$\begin{aligned} -\Delta \mathrm{y} \; + \; \tilde{\lambda}^2 \mathrm{y} \; + \; b\tilde{\lambda} \left(\int_{-\infty}^{+\infty} \frac{\eta^2(\zeta)}{\zeta^2 + \beta + \tilde{\lambda}} \, d\zeta \right) \mathrm{y} = F_2 \; + \; \tilde{\lambda} F_1 \; + \; bF_1 \left(\int_{-\infty}^{+\infty} \frac{\eta^2(\zeta)}{\zeta^2 + \beta + \tilde{\lambda}} \, d\zeta \right) \\ - \; b \int_{-\infty}^{+\infty} \frac{\eta(\zeta) F_3(x, \zeta)}{\zeta^2 + \beta + \tilde{\lambda}} \, d\zeta. \end{aligned} \tag{4.10}$$

By Lemma 3.3, set

$$c_{\tilde{\lambda}} := \int_{-\infty}^{+\infty} \frac{\eta^2(\zeta)}{\zeta^2 + \beta + \tilde{\lambda}} \, d\zeta = \frac{\pi}{\sin(\alpha\pi)} (\tilde{\lambda} + \beta)^{\alpha - 1} > 0.$$

Hence the preceding equation becomes

$$-\Delta \mathrm{y} + \left(\tilde{\lambda}^2 + b\tilde{\lambda} c_{\tilde{\lambda}}\right) \mathrm{y} = F_2 + \tilde{\lambda} F_1 + b c_{\tilde{\lambda}} F_1 - b \int_{-\infty}^{+\infty} \frac{\eta(\zeta) F_3(x, \zeta)}{\zeta^2 + \beta + \tilde{\lambda}} \, d\zeta.$$

Multiplying the last equation by $\overline{w} \in H_0^1(\Omega)$, and integrating by parts over Ω, we get

$$\mathcal{B}(\mathrm{y}, w) = \mathcal{L}(w), \ \forall\, w \in H_0^1(\Omega), \tag{4.11}$$

where

$$\mathcal{B}(\mathrm{y}, w) = \int_\Omega \nabla \mathrm{y} \cdot \nabla \overline{w}\, dx + \left(\tilde{\lambda}^2 + b\tilde{\lambda} c_{\tilde{\lambda}}\right) \int_\Omega \mathrm{y}\, \overline{w}\, dx$$

and

$$\mathcal{L}(w) = \int_\Omega \left(F_2 + \tilde{\lambda} F_1 + b c_{\tilde{\lambda}} F_1\right) \overline{w}\, dx - b \int_\Omega \left(\int_{-\infty}^{+\infty} \frac{\eta(\zeta) F_3(x,\zeta)}{\zeta^2 + \beta + \tilde{\lambda}}\, d\zeta\right) \overline{w}\, dx.$$

Applying the Cauchy–Schwarz and Poincaré inequalities yields

$$\begin{aligned} |\mathcal{B}(\mathrm{y}, w)| &\le \|\nabla \mathrm{y}\|_2 \|\nabla w\|_2 + \left(\tilde{\lambda}^2 + b\tilde{\lambda} c_{\tilde{\lambda}}\right) \|\mathrm{y}\|_2 \|w\|_2 \\ &\le C_{\tilde{\lambda}} \|\mathrm{y}\|_{H_0^1(\Omega)} \|w\|_{H_0^1(\Omega)}, \end{aligned}$$

where $C_{\tilde{\lambda}}$ is a positive constant.

Thus, $\mathcal{B}$ is continuous on $H_0^1(\Omega) \times H_0^1(\Omega)$.

Moreover,

$$\begin{aligned} \mathcal{B}(\mathrm{y}, \mathrm{y}) &= \|\nabla \mathrm{y}\|_2^2 + \left(\tilde{\lambda}^2 + b\tilde{\lambda} c_{\tilde{\lambda}}\right) \|\mathrm{y}\|_2^2 \\ &\ge \|\nabla \mathrm{y}\|_2^2. \end{aligned}$$

Hence, $\mathcal{B}$ is coercive on $H_0^1(\Omega) \times H_0^1(\Omega)$.

Furthermore, by the Cauchy–Schwarz and Poincaré inequalities, there exists a constant $\tilde{C}_{\tilde{\lambda}} > 0$ such that

$$|\mathcal{L}(w)| \le \tilde{C}_{\tilde{\lambda}} \|F\|_{\mathcal{H}} \|w\|_{H_0^1(\Omega)}.$$

Therefore, $\mathcal{L}$ is continuous on $H_0^1(\Omega)$. Consequently, by the Lax–Milgram lemma, there exists a unique solution $\mathrm{y} \in H_0^1(\Omega)$ satisfying (4.11).

Using (4.8) and (4.9), we recover v and φ. Standard elliptic regularity, together with the definition of $D(\mathcal{A})$, yields

$$\mathcal{W} = (\mathrm{y}, v, \varphi) \in D(\mathcal{A}).$$

Therefore,

$$\operatorname{Ran}(\tilde{\lambda} I - \mathcal{A}) = \mathcal{H}, \qquad \tilde{\lambda} > 0.$$

Since $\mathcal{A}$ is dissipative and $\tilde{\lambda} I - \mathcal{A}$ is surjective, $\mathcal{A}$ is maximal dissipative. Hence, by the Lumer–Phillips theorem, $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$.

Recall that the nonlinear operator acts only on the second component of the state vector and is defined by

$$J(\mathcal{W}) = (0, h(\mathrm{y}), 0), \qquad \mathcal{W} = (\mathrm{y}, v, \varphi) \in \mathcal{H}.$$

By Lemma 4.1 and the admissibility condition (3.4), $h(\mathrm{y}) \in L^2(\Omega)$ whenever $\mathrm{y} \in H_0^1(\Omega)$. Hence,

$$J : \mathcal{H} \longrightarrow \mathcal{H}$$

is well-defined.

Let

$$U_1 = (\mathrm{y}_1, v_1, \varphi_1), \qquad U_2 = (\mathrm{y}_2, v_2, \varphi_2)$$

belong to a bounded subset of $\mathcal{H}$. By Lemma 4.1, there exists a constant $C_R > 0$, depending on the radius R of this bounded subset, such that

$$\begin{aligned}\|J(U_1) - J(U_2)\|_{\mathcal{H}} &= \|h(\mathrm{y}_1) - h(\mathrm{y}_2)\|_{L^2(\Omega)} \\ &\leq C_R \|\mathrm{y}_1 - \mathrm{y}_2\|_{H_0^1(\Omega)} \\ &\leq C_R \|U_1 - U_2\|_{\mathcal{H}}.\end{aligned}$$

Thus, J is locally Lipschitz continuous from $\mathcal{H}$ into $\mathcal{H}$.

Consequently, the existence and uniqueness of a mild (and strong) solution for problem (3.9) on $[0, T_{\max})$ follows from Theorems 1.4 and 1.6 in Pazy's book [14, Chapter 6]. Moreover either $T_{\max} = +\infty$ or $T_{\max} < +\infty$, and in this case, it is well-known that

$$\limsup_{t \uparrow T_{\max}} \|\mathcal{W}(t)\|_{\mathcal{H}} = +\infty.$$

This completes the proof. □

The energy functional for the regular solution corresponding to problem (3.9) is introduced as follows:

$$E(t) = \frac{1}{2}\left\|\mathrm{y}_t\right\|_2^2 + \frac{1}{2}\|\nabla \mathrm{y}\|_2^2 - \frac{1}{2p}\int_\Omega\int_\Omega \frac{|\mathrm{y}(x)|^p|\mathrm{y}(z)|^p}{|x-z|^{n-2}}\,dx\,dz + \frac{b}{2}\int_\Omega\int_{-\infty}^{+\infty} |\varphi(x,\zeta,t)|^2\,d\zeta\,dx. \tag{4.12}$$

Lemma 4.3. *The energy functional* (4.12) *associated with problem* (3.9) *satisfies*

$$E'(t) = -b\int_\Omega\int_{-\infty}^{+\infty}\left(\zeta^2 + \beta\right)|\varphi(x,\zeta,t)|^2\,d\zeta dx \leq 0. \tag{4.13}$$

Proof. Multiplying the first equation in (3.9) by y_t and integrating over Ω, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathrm{y}_t\|_2^2 + \frac{1}{2}\|\nabla \mathrm{y}\|_2^2\right) + b\int_\Omega \mathrm{y}_t\int_{\mathbb{R}}\eta(\zeta)\varphi(x,\zeta,t)\,d\zeta\,dx = \int_\Omega h(\mathrm{y})\mathrm{y}_t\,dx.$$

Let

$$P(\mathrm{y}) = \int_\Omega\int_\Omega \frac{|\mathrm{y}(x)|^p|\mathrm{y}(z)|^p}{|x-z|^{n-2}}\,dx\,dz.$$

Using the symmetry of the kernel $|x-z|^{2-n}$, we have

$$\frac{d}{dt}P(\mathrm{y}) = 2p\int_\Omega h(\mathrm{y})\mathrm{y}_t\,dx$$

and consequently

$$\frac{d}{dt}\left(\frac{1}{2p}P(\mathrm{y})\right) = \int_\Omega h(\mathrm{y})\mathrm{y}_t\,dx.$$

On the other hand, multiplying the auxiliary equation in (3.9) by $b\varphi$ and integrating over $\Omega \times \mathbb{R}$, we obtain

$$\frac{b}{2}\frac{d}{dt}\int_{\Omega}\int_{\mathbb{R}}|\varphi|^2\,d\zeta\,dx + b\int_{\Omega}\int_{\mathbb{R}}(\zeta^2+\beta)|\varphi|^2\,d\zeta\,dx = b\int_{\Omega}\mathrm{y}_t\int_{\mathbb{R}}\eta(\zeta)\varphi\,d\zeta\,dx.$$

Combining the above identities yields

$$E'(t) = -b\int_{\Omega}\int_{\mathbb{R}}(\zeta^2+\beta)|\varphi(x,\zeta,t)|^2\,d\zeta\,dx \le 0,$$

which proves (4.13). □

Thus, the energy is non-increasing. In particular, (4.13) immediately gives

$$E(t) \le E(0). \tag{4.14}$$

5. Global existence

This section is devoted to proving the theorem on the global existence of the solution for problem (3.9). We first define the following functionals:

$$I(t) = \|\nabla \mathrm{y}\|_2^2 - \int_{\Omega}\int_{\Omega}\frac{|\mathrm{y}(x)|^p|\mathrm{y}(z)|^p}{|x-z|^{n-2}}\,dx\,dz + b\int_{\Omega}\int_{-\infty}^{+\infty}|\varphi|^2\,d\xi\,dx \tag{5.1}$$

and

$$J(t) = \frac{1}{2}\|\nabla \mathrm{y}\|_2^2 - \frac{1}{2p}\int_{\Omega}\int_{\Omega}\frac{|\mathrm{y}(x)|^p|\mathrm{y}(z)|^p}{|x-z|^{n-2}}\,dx\,dz + \frac{b}{2}\int_{\Omega}\int_{-\infty}^{+\infty}|\varphi|^2 d\xi dx. \tag{5.2}$$

Therefore, from both functionals' definitions, we get

$$J(t) = \left(\frac{p-1}{2p}\right)\|\nabla \mathrm{y}\|_2^2 + \frac{1}{2p}I(t) + b\left(\frac{p-1}{2p}\right)\int_{\Omega}\int_{-\infty}^{+\infty}|\varphi|^2\,d\xi\,dx \tag{5.3}$$

and

$$E(t) = \frac{1}{2}\|\mathrm{y}_t\|_2^2 + \left(\frac{p-1}{2p}\right)\|\nabla \mathrm{y}\|_2^2 + \frac{1}{2p}I(t) + b\left(\frac{p-1}{2p}\right)\int_{\Omega}\int_{-\infty}^{+\infty}|\varphi|^2\,d\xi\,dx. \tag{5.4}$$

Lemma 5.1. *Assume that*

$$W_0 = (\mathrm{y}_0, \mathrm{y}_1, 0) \in \mathcal{H}$$

satisfies

$$G_1 = C_0\left(\frac{2pE(0)}{p-1}\right)^{p-1} < 1 \quad \textit{and} \quad I(0) > 0. \tag{5.5}$$

Then we get

$$I(t) > 0, \qquad \forall\, t \in [0, T_{\max}).$$

Proof. Since $I(0) > 0$, then, by continuity, there exists $T^* \leq T_{\max}$ such that

$$I(t) > 0, \qquad \forall\, t \in [0, T^*).$$

From (5.3), we have

$$\begin{aligned} J(t) &= \frac{1}{2p} I(t) + \left(\frac{p-1}{2p}\right) \|\nabla \mathrm{y}\|_2^2 + b \left(\frac{p-1}{2p}\right) \int_\Omega \int_{-\infty}^{+\infty} |\varphi|^2 \, d\xi \, dx \\ &\geq \left(\frac{p-1}{2p}\right) \|\nabla y\|_2^2, \qquad \forall\, t \in [0, T^*). \end{aligned} \tag{5.6}$$

Hence,

$$\|\nabla \mathrm{y}\|_2^2 \leq \frac{2p}{p-1} J(t) \leq \frac{2p}{p-1} E(t) \leq \frac{2p}{p-1} E(0), \qquad \forall\, t \in [0, T^*). \tag{5.7}$$

Recall, from (3.3), that

$$\int_\Omega \int_\Omega \frac{|\mathrm{y}(x)|^p |\mathrm{y}(z)|^p}{|x-z|^{n-2}} \, dx \, dz \leq C_0 \|\nabla \mathrm{y}\|_2^{2p}, \tag{5.8}$$

which, together with (5.7), implies that

$$\begin{aligned} \int_\Omega \int_\Omega \frac{|\mathrm{y}(x)|^p |\mathrm{y}(z)|^p}{|x-z|^{n-2}} \, dx \, dz &< C_0 \left(\frac{2pE(0)}{p-1}\right)^{p-1} \|\nabla \mathrm{y}(t)\|_2^2 \\ &< G_1 \|\nabla \mathrm{y}(t)\|_2^2. \end{aligned} \tag{5.9}$$

From (5.1), (5.5), and (5.9), we obtain

$$I(t) > (1 - G_1) \|\nabla \mathrm{y}(t)\|_2^2 > 0, \qquad \forall\, t \in [0, T^*). \tag{5.10}$$

This process can be repeated to extend T^* to $T_{\max}$. □

Theorem 5.2. *Assume that the assumptions of Lemma 5.1 are satisfied. Then the solution of system* (3.9) *is bounded and global in time.*

Proof. To achieve our proof, it is enough to show that

$$\|\mathrm{y}_t\|_2^2 + \|\nabla \mathrm{y}\|_2^2 + b \int_\Omega \int_{-\infty}^{+\infty} |\varphi(x, \zeta, t)|^2 \, d\zeta \, dx$$

is bounded independently of t.

From (5.3), (5.4), and (5.6),

$$\frac{1}{2} \|\mathrm{y}_t\|_2^2 + J(t) = E(t) \leq E(0).$$

Consequently,

$$\frac{1}{2} \|\mathrm{y}_t\|_2^2 + \left(\frac{p-1}{2p}\right) \|\nabla \mathrm{y}\|_2^2 + b \left(\frac{p-1}{2p}\right) \int_\Omega \int_{-\infty}^{+\infty} |\varphi|^2 \, d\xi \, dx \leq E(0).$$

Therefore,

$$\|\mathrm{y}_t\|_2^2 + \|\nabla \mathrm{y}\|_2^2 + \int_\Omega \int_{-\infty}^{+\infty} |\varphi|^2 \, d\xi \, dx \leq \xi_0 E(0),$$

where $\xi_0 > 0$ is a constant that depends on p only. This means that the solution of system (3.9) is bounded and global in time. □

6. Blow up

In this section, we establish the finite-time blow-up of solutions to system (3.9) under the assumption that the initial energy is negative. We shall use the following auxiliary estimates, which are needed to control the Hartree-type nonlocal term and the related norms.

Lemma 6.1. *There exists a constant* $c > 0$ *such that*

$$\left(\int_\Omega\int_\Omega \frac{|y(x)|^p|y(z)|^p}{|x-z|^{n-2}}\,dxdz\right)^{\frac{\varsigma}{2p}} \le c\left(\int_\Omega\int_\Omega \frac{|y(x)|^p|y(z)|^p}{|x-z|^{n-2}}\,dxdz + \|\nabla y\|_2^2\right)$$

for all $y \in H_0^1(\Omega)$ *and* $2 \le \varsigma \le 2p$.

Proof. Let

$$P(y) = \int_\Omega\int_\Omega \frac{|y(x)|^p|y(z)|^p}{|x-z|^{n-2}}\,dxdz.$$

If $P(y) > 1$, then

$$(P(y))^{\frac{\varsigma}{2p}} \le P(y).$$

If $P(y) \le 1$, then $(P(y))^{\frac{\varsigma}{2p}} \le (P(y))^{\frac{1}{p}}$, and by (3.3), we get

$$(P(y))^{\frac{1}{p}} \le C_0^{\frac{1}{p}}\|\nabla y\|_2^2,$$

which implies that

$$(P(y))^{\frac{\varsigma}{2p}} \le C_0^{\frac{1}{p}}\|\nabla y\|_2^2.$$

This completes the proof. □

Lemma 6.2. *Assume that* $p \ge 2$*. Then there exists a constant* $c > 0$ *such that, for every* $y \in H_0^1(\Omega)$*,*

$$\|y\|_2^2 \le c\left[\left(\int_\Omega\int_\Omega \frac{|y(x)|^p|y(z)|^p}{|x-z|^{n-2}}\,dxdz\right)^{\frac{2}{2p}} + \|\nabla y\|_2^{\frac{4}{2p}}\right].$$

Proof. Since Ω is bounded, there exists $D_\Omega > 0$ such that $|x-z| \le D_\Omega$ for all $x, y \in \Omega$. Hence

$$P(y) \ge D_\Omega^{-(n-2)}\left(\int_\Omega |y(x)|^p\,dx\right)^2 = D_\Omega^{-(n-2)}\|y\|_p^{2p}.$$

Therefore,

$$\|y\|_p^2 \le CP(y)^{\frac{1}{p}}.$$

Since $p \ge 2$ and Ω is bounded, $L^p(\Omega) \hookrightarrow L^2(\Omega)$. Thus

$$\|y\|_2^2 \le C\|y\|_p^2 \le CP(y)^{\frac{1}{p}}.$$

The desired estimate follows immediately after enlarging the constant C, since

$$P(y)^{\frac{1}{p}} \le P(y)^{\frac{1}{p}} + \|\nabla y\|_2^{\frac{2}{p}}.$$

□

Lemma 6.3. *Let $p \geq 2$. There exists a constant $c > 0$ such that*

$$\|y\|_p^{\varsigma} \leq c\left(\int_\Omega \int_\Omega \frac{|y(x)|^p |y(z)|^p}{|x-z|^{n-2}}\, dxdz + \|\nabla y\|_2^2\right)$$

for all $y \in H_0^1(\Omega)$ *and* $2 \leq \varsigma \leq 2p$.

Proof. We have

$$P(y) \geq D_\Omega^{-(n-2)} \left(\int_\Omega |y(x)|^p\, dx\right)^2 = D_\Omega^{-(n-2)} \|y\|_p^{2p}.$$

Therefore,

$$\|y\|_p^{\varsigma} \leq CP(y)^{\frac{\varsigma}{2p}}.$$

Let

$$\vartheta = \frac{\varsigma}{2p}.$$

Since $2 \leq \varsigma \leq 2p$, we have

$$\frac{1}{p} \leq \vartheta \leq 1.$$

If $P(y) \geq 1$, then

$$P(y)^{\vartheta} \leq P(y).$$

If $P(y) < 1$, then the estimate (3.3) gives

$$P(y) \leq C_0 \|\nabla y\|_2^{2p}.$$

Hence

$$P(y)^{\vartheta} \leq C\|\nabla y\|_2^{2p\vartheta} = C\|\nabla y\|_2^{\varsigma}.$$

Since $\varsigma \geq 2$, this term is controlled by $C\|\nabla y\|_2^2$ when $\|\nabla y\|_2 \leq 1$, while it is controlled by $C\|\nabla y\|_2^2$ trivially when $\|\nabla y\|_2 > 1$, because $P(y)^{\vartheta} < 1$. Combining the above estimates gives the desired result. □

Remark 6.4. *The constants in Lemmas 6.2 and 6.3 depend only on the bounded domain Ω, the exponent p, and the functional inequalities used in their proofs; they do not depend on the particular spatial profile of* y*. Consequently, these estimates remain valid for every* $y \in H_0^1(\Omega)$*, even when the solution develops spatial concentration, as long as it remains in the energy space on its maximal existence interval. Such concentration is manifested through the growth of the relevant norms, rather than through a failure of the estimates.*

Theorem 6.5. *Assume that $n \in \{4, 5\}, 2 < p < \frac{n+2}{n-2}$, and $E(0) < 0$. Then the solution of system* (3.9) *blows up in finite time.*

Proof. We introduce the following functional:

$$\begin{aligned} H(t) = -E(t) = -\frac{1}{2}\|\nabla y\|_2^2 + \frac{1}{2p}\int_\Omega \int_\Omega \frac{|y(x)|^p |y(z)|^p}{|x-z|^{n-2}}\, dxdz \\ -\frac{1}{2}\|y_t\|_2^2 - \frac{b}{2}\int_\Omega \int_{-\infty}^{+\infty} |\varphi(\zeta, t)|^2\, d\zeta\, dx. \end{aligned} \tag{6.1}$$

Since the energy is nonincreasing, the functional H satisfies

$$H'(t) = -E'(t) \geq b \int_\Omega \int_{-\infty}^{+\infty} \left(\zeta^2 + \beta\right) |\varphi(\zeta, t)|^2 \, d\zeta dx \geq 0. \tag{6.2}$$

Moreover, from (6.1), we obtain

$$0 < H(0) \leq H(t) \leq \frac{1}{2p} \int_\Omega \int_\Omega \frac{|\mathrm{y}(x)|^p |\mathrm{y}(z)|^p}{|x - z|^{n-2}} \, dx \, dz. \tag{6.3}$$

We define the auxiliary functional

$$R(t) = H^{1-\sigma}(t) + \varepsilon \int_\Omega \mathrm{y}\mathrm{y}_t \, dx, \tag{6.4}$$

where $\varepsilon > 0$ is a small parameter to be chosen later and

$$\frac{p-1}{p^2} < \sigma < \frac{p-1}{2p} < \frac{p-1}{p} < 1. \tag{6.5}$$

Since $p > 2$, then $p^2 > 2p > p$ and so the interval in (6.5) is nonempty. The lower and upper bounds on σ ensure that the exponents used below in Lemmas 6.1 and 6.3 belong to the required admissible range $[2, 2p]$.

Differentiating (6.4), we obtain

$$\begin{aligned} R'(t) &= (1 - \sigma) H^{-\sigma}(t) H'(t) + \varepsilon \frac{d}{dt} \int_\Omega y y_t \, dx \\ &= (1 - \sigma) H^{-\sigma}(t) H'(t) + \varepsilon \|y_t\|_2^2 + \varepsilon \int_\Omega y y_{tt} \, dx. \end{aligned}$$

Using the first equation of the augmented system (3.9), namely,

$$y_{tt} = \Delta y - b \int_{-\infty}^{+\infty} \eta(\zeta) \phi(x, \zeta, t) \, d\zeta + h(y),$$

we have

$$\int_\Omega y y_{tt} \, dx = \int_\Omega y \Delta y \, dx - b \int_\Omega y \int_{-\infty}^{+\infty} \eta(\zeta) \phi(x, \zeta, t) \, d\zeta \, dx + \int_\Omega h(y) y \, dx.$$

Since $y = 0$ on $\partial\Omega$, integration by parts yields

$$\int_\Omega y \Delta y \, dx = -\int_\Omega |\nabla y|^2 \, dx = -\|\nabla y\|_2^2.$$

Consequently,

$$\begin{aligned} R'(t) = (1 - \sigma) H^{-\sigma}(t) H'(t) + \varepsilon \|\mathrm{y}_t\|_2^2 + \varepsilon \int_\Omega h(\mathrm{y}) \mathrm{y} \, dx - \varepsilon \|\nabla \mathrm{y}\|_2^2 \\ - \varepsilon b \int_\Omega \mathrm{y} \int_{-\infty}^{+\infty} \eta(\zeta) \varphi(x, \zeta, t) d\zeta \, dx. \end{aligned} \tag{6.6}$$

The last term in (6.6) can be estimated by Young's inequality as follows:

$$\begin{aligned} b\int_\Omega \mathrm{y}\int_{-\infty}^{+\infty}\eta(\zeta)\,\varphi(x,\zeta,t)\,d\zeta\,dx &\le \delta C_1\|\mathrm{y}\|_2^2+\frac{b}{4\delta}\int_\Omega\int_{-\infty}^{+\infty}(\zeta^2+\beta)\,|\varphi(x,\zeta,t)|^2\,d\zeta\,dx \\ &\le \delta C_1\|\mathrm{y}\|_2^2+\frac{1}{4\delta}H'(t), \end{aligned} \tag{6.7}$$

where

$$C_1=b\int_{-\infty}^{+\infty}\frac{\eta^2(\zeta)}{\zeta^2+\beta}\,d\zeta$$

and $\delta>0$.

Inserting (6.7) into (6.6) gives

$$\begin{aligned} R'(t)\ge \Big((1-\sigma)H^{-\sigma}(t)-\frac{\varepsilon}{4\delta}\Big)H'(t)+\varepsilon\|\mathrm{y}_t\|_2^2-\varepsilon\|\nabla \mathrm{y}\|_2^2 \\ -\varepsilon\delta C_1\|\mathrm{y}\|_2^2+\varepsilon\int_\Omega\int_\Omega\frac{|\mathrm{y}(x)|^p\,|\mathrm{y}(z)|^p}{|x-z|^{n-2}}\,dx\,dz. \end{aligned} \tag{6.8}$$

We choose δ such that

$$\frac{1}{4\delta}=\theta\,H^{-\sigma}(t), \tag{6.9}$$

where $\theta>0$ will be fixed later. By substituting (6.9) into (6.8), we obtain

$$\begin{aligned} R'(t)\ge\ (1-\sigma-\varepsilon\theta)H^{-\sigma}(t)\,H'(t)+\varepsilon\|\mathrm{y}_t\|_2^2-\varepsilon\|\nabla \mathrm{y}\|_2^2 \\ -\frac{\varepsilon C_1}{4\theta}\,H^{\sigma}(t)\,\|\mathrm{y}\|_2^2+\varepsilon\int_\Omega\int_\Omega\frac{|\mathrm{y}(x)|^p\,|\mathrm{y}(z)|^p}{|x-z|^{n-2}}\,dx\,dz. \end{aligned} \tag{6.10}$$

We next split the Hartree potential term in the following way:

$$\begin{aligned} \varepsilon\int_\Omega\int_\Omega\frac{|\mathrm{y}(x)|^p\,|\mathrm{y}(z)|^p}{|x-z|^{n-2}}\,dx\,dz \quad &= \quad \varepsilon a\int_\Omega\int_\Omega\frac{|\mathrm{y}(x)|^p\,|\mathrm{y}(z)|^p}{|x-z|^{n-2}}\,dx\,dz \\ &\quad +\varepsilon p(1-a)\|\mathrm{y}_t\|_2^2+2\varepsilon p(1-a)H(t)+\varepsilon p(1-a)\|\nabla \mathrm{y}\|_2^2 \\ &\quad +\varepsilon p(1-a)b\int_\Omega\int_{-\infty}^{+\infty}|\varphi(x,\zeta,t)|^2\,d\zeta\,dx, \end{aligned} \tag{6.11}$$

where $0<a<1$ will be specified later. Substituting (6.11) into (6.10), we derive

$$\begin{aligned} R'(t)\ge{}&[1-\sigma-\varepsilon\theta]H^{-\sigma}(t)\,H'(t)+\varepsilon[p(1-a)+1]\|\mathrm{y}_t\|_2^2 \\ &+\varepsilon[p(1-a)-1]\|\nabla \mathrm{y}\|_2^2+2\varepsilon p(1-a)H(t) \\ &+\varepsilon p(1-a)b\int_\Omega\int_{-\infty}^{+\infty}|\varphi(x,\zeta,t)|^2\,d\zeta\,dx \\ &+a\varepsilon\int_\Omega\int_\Omega\frac{|\mathrm{y}(x)|^p|\mathrm{y}(z)|^p}{|x-z|^{n-2}}\,dx\,dz-\frac{\varepsilon C_1}{4\theta}\,H^{\sigma}(t)\,\|\mathrm{y}\|_2^2. \end{aligned} \tag{6.12}$$

We now estimate the last term in (6.12). By (6.3), Lemma 6.2, and Young's inequality, we have

$$H^{\sigma}(t)\|\mathrm{y}\|_2^2 \quad\le\quad \left(\int_\Omega\int_\Omega\frac{|\mathrm{y}(x)|^p|\mathrm{y}(z)|^p}{|x-z|^{n-2}}\,dxdz\right)^{\sigma}\|\mathrm{y}\|_2^2$$

$$\leq \ c\Big[\Big(\int_\Omega\int_\Omega \frac{|\mathrm{y}(x)|^p|\mathrm{y}(z)|^p}{|x-z|^{n-2}}\,dxdz\Big)^{\sigma+\frac{1}{p}} + \Big(\int_\Omega\int_\Omega \frac{|\mathrm{y}(x)|^p|\mathrm{y}(z)|^p}{|x-z|^{n-2}}\,dxdz\Big)^{\sigma}\|\nabla \mathrm{y}\|_2^{\frac{2}{p}}\Big]$$
$$\leq \ c\Big[\Big(\int_\Omega\int_\Omega \frac{|\mathrm{y}(x)|^p|\mathrm{y}(z)|^p}{|x-z|^{n-2}}\,dxdz\Big)^{\frac{p\sigma+1}{p}} + \Big(\int_\Omega\int_\Omega \frac{|\mathrm{y}(x)|^p|\mathrm{y}(z)|^p}{|x-z|^{n-2}}\,dxdz\Big)^{\frac{\sigma p}{p-1}} + \|\nabla \mathrm{y}\|_2^2\Big].$$

Moreover, (6.5) implies that

$$2 < 2(p\sigma+1) \leq 2p \quad \text{and} \quad 2 < \frac{2\sigma p^2}{p-1} \leq 2p.$$

Therefore, applying Lemma 6.1, we obtain

$$H^\sigma(t)\|\mathrm{y}\|_2^2 \leq c\Big(\int_\Omega\int_\Omega \frac{|\mathrm{y}(x)|^p|\mathrm{y}(z)|^p}{|x-z|^{n-2}}\,dxdz + \|\nabla \mathrm{y}\|_2^2\Big). \tag{6.13}$$

Combining (6.12) with (6.13) yields

$$\begin{aligned} R'(t) \geq{} & [1-\sigma-\varepsilon\theta]H^{-\sigma}(t)\,H'(t) + \varepsilon[p(1-a)+1]\|\mathrm{y}_t\|_2^2 \\ & + \varepsilon[p(1-a)-1-\frac{C_1c}{4\theta}]\|\nabla \mathrm{y}\|_2^2 + 2\varepsilon p(1-a)H(t) \\ & + \varepsilon p(1-a)b\int_\Omega\int_{-\infty}^{+\infty}|\varphi(x,\zeta,t)|^2\,d\zeta\,dx \\ & + \varepsilon\Big(a-\frac{C_1c}{4\theta}\Big)\int_\Omega\int_\Omega \frac{|\mathrm{y}(x)|^p|\mathrm{y}(z)|^p}{|x-z|^{n-2}}\,dxdz. \end{aligned} \tag{6.14}$$

Choose $a > 0$ sufficiently small such that

$$m = p(1-a)-1 > 0.$$

Then choose θ sufficiently large so that

$$a - \frac{C_1c}{4\theta} > 0 \quad \text{and} \quad m - \frac{C_1c}{4\theta} > 0.$$

After a and θ have been fixed, take $\varepsilon > 0$ sufficiently small such that

$$(1-\sigma)-\varepsilon\theta > 0$$

and

$$R(0) > 0. \tag{6.15}$$

With these choices, estimate (6.14) gives, for some $m_1 > 0$,

$$\begin{aligned} R'(t) \geq m_1\Big[& H(t) + \|\mathrm{y}_t\|_2^2 + \|\nabla \mathrm{y}\|_2^2 \\ & + \int_\Omega\int_\Omega \frac{|\mathrm{y}(x)|^p|\mathrm{y}(z)|^p}{|x-z|^{n-2}}\,dxdz \\ & + b\int_\Omega\int_{-\infty}^{+\infty}|\varphi(x,\zeta,t)|^2\,d\zeta\,dx\Big]. \end{aligned} \tag{6.16}$$

We derive, from (6.3), (6.15), and (6.16), that

$$R(t) \geq R(0) > 0.$$

It remains to relate the functional R to the quantities appearing on the right-hand side of (6.16). Since $p \geq 2$ and Ω is bounded, $L^p(\Omega) \hookrightarrow L^2(\Omega)$. Hence, by Young's and Hölder's inequalities, we have

$$\left|\int_\Omega \mathrm{y}\,\mathrm{y}_t\,dx\right|^{\frac{1}{1-\sigma}} \leq c\left[\|\mathrm{y}\|_p^{\frac{2}{1-2\sigma}} + \|\mathrm{y}_t\|_2^2\right]. \tag{6.17}$$

By (6.5), we have

$$2 \leq \frac{2}{1-2\sigma} \leq 2p.$$

Therefore, applying Lemma 6.3 with

$$\varsigma = \frac{2}{1-2\sigma},$$

we obtain

$$\begin{aligned}\left|\int_\Omega \mathrm{y}\,\mathrm{y}_t\,dx\right|^{\frac{1}{1-\sigma}} &\leq c\left[\|\mathrm{y}\|_p^{\frac{2}{1-2\sigma}} + \|\mathrm{y}_t\|_2^2\right]\\ &\leq c\left[\int_\Omega\int_\Omega \frac{|\mathrm{y}(x)|^p|\mathrm{y}(z)|^p}{|x-z|^{n-2}}\,dxdz + \|\mathrm{y}_t\|_2^2 + \|\nabla \mathrm{y}\|_2^2\right].\end{aligned} \tag{6.18}$$

Consequently,

$$\begin{aligned}R^{\frac{1}{1-\sigma}}(t) &= \left(H^{1-\sigma}(t) + \varepsilon\int_\Omega \mathrm{y}\,\mathrm{y}_t\,dx\right)^{\frac{1}{1-\sigma}}\\ &\leq c\left\{H(t) + \left|\int_\Omega \mathrm{y}\,\mathrm{y}_t\,dx\right|^{\frac{1}{1-\sigma}}\right\}\\ &\leq c\left\{H(t) + \|\mathrm{y}_t\|_2^2 + \|\nabla \mathrm{y}\|_2^2\right.\\ &\qquad \left. + \int_\Omega\int_\Omega \frac{|\mathrm{y}(x)|^p|\mathrm{y}(z)|^p}{|x-z|^{n-2}}\,dxdz\right\}\\ &\leq c\left\{H(t) + \|\mathrm{y}_t\|_2^2 + \|\nabla \mathrm{y}\|_2^2\right.\\ &\qquad + \int_\Omega\int_\Omega \frac{|\mathrm{y}(x)|^p|\mathrm{y}(z)|^p}{|x-z|^{n-2}}\,dxdz\\ &\qquad \left. + b\int_\Omega\int_{-\infty}^{+\infty} |\varphi(x,\zeta,t)|^2\,d\zeta\,dx\right\}.\end{aligned} \tag{6.19}$$

From (6.16) and (6.19), we deduce that

$$R'(t) \geq \mathcal{B}R^{\frac{1}{1-\sigma}}(t), \tag{6.20}$$

where $\mathcal{B} > 0$. Integrating (6.20) over $(0,t)$, we arrive at

$$R^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{R^{-\frac{\sigma}{1-\sigma}}(0) - \mathcal{B}\frac{\sigma}{1-\sigma}t}.$$

Thus, $R(t)$ becomes unbounded in finite time. In particular, the blow-up time satisfies

$$T \le T^* = \frac{1-\sigma}{\mathcal{B}\sigma R^{\frac{\sigma}{1-\sigma}}(0)}.$$

Therefore,

$$T_{\max} \le T^* < +\infty.$$

By the continuation alternative established in Theorem 4.2,

$$\limsup_{t\uparrow T_{\max}} \|\mathcal{W}(t)\|_{\mathcal{H}} = +\infty.$$

Hence, the solution blows up in finite time in the energy-space norm. This completes the proof. □

7. Numerical illustration of the blow-up mechanism

This section presents a qualitative numerical illustration of the rapid-growth behavior associated with a reduced one-dimensional analogue of the Hartree-type wave equation with fractional damping. The analytical problem considered in the preceding sections is posed on a bounded domain $\Omega \subset \mathbb{R}^n$ with $n > 3$ and involves the singular multidimensional Hartree kernel. A direct numerical approximation of that problem would require a substantially more demanding multidimensional computation, together with a careful treatment of the singular nonlocal interaction.

For this reason, the computations below are carried out on a one-dimensional interval using a regularized Hartree-type kernel. The reduced model retains the principal structural ingredients of the analytical problem, namely, the wave operator, the Hartree-type nonlocal source, the augmented diffusive representation of the fractional damping term, the associated energy functional, and the negative-initial-energy rapid-growth mechanism.

However, the one-dimensional regularized problem is not a direct discretization of the multidimensional model. Consequently, the numerical results should not be interpreted as a quantitative validation of the finite-time blow-up theorem. Their purpose is instead to provide a qualitative illustration of the competition between nonlinear amplification and fractional-memory dissipation, and to examine the behavior of the discrete energy, the blow-up functional, and the maximum norm during the computed rapid-growth phase. The analytical identities are derived directly rather than through symbolic computation; no computer-algebra verification is used in the analysis.

For the qualitative numerical illustration, we consider the interval

$$\Omega = (0,1),$$

with homogeneous Dirichlet boundary conditions

$$\mathrm{y}(0,t) = \mathrm{y}(1,t) = 0.$$

The one-dimensional augmented model used in the computation is

$$\begin{cases} \mathrm{y}_{tt} - \mathrm{y}_{xx} + b\displaystyle\int_{-\infty}^{+\infty} \eta(\zeta)\varphi(x,\zeta,t)\,d\zeta = \left(\int_0^1 K_\varepsilon(x,y)|\mathrm{y}(y,t)|^p\,dy\right)|\mathrm{y}(x,t)|^{p-2}\mathrm{y}(x,t), \\ \varphi_t(x,\zeta,t) + (\zeta^2+\beta)\varphi(x,\zeta,t) = \eta(\zeta)\mathrm{y}_t(x,t), \end{cases}$$

where

$$\eta(\zeta) = |\zeta|^{\frac{2\alpha-1}{2}}, \qquad b = \frac{\mu_1 \sin(\pi\alpha)}{\pi}.$$

The Hartree kernel is approximated by the regularized kernel

$$K_{\varepsilon_K}(x,\xi) = \frac{1}{\sqrt{(x-\xi)^2 + \varepsilon_K^2}},$$

where the regularization parameter is fixed as

$$\varepsilon_K = \frac{1}{120}$$

for the baseline computation. Consequently, all spatial meshes approximate the same regularized problem, while the influence of the kernel regularization is examined separately in the numerical sensitivity study.

This regularization removes the point singularity at $x = \xi$ and still preserves the nonlocal character of the Hartree-type interaction.

The initial data are chosen as

$$\mathrm{y}(x,0) = A_0 \sin(\pi x), \qquad \mathrm{y}_t(x,0) = 0,$$

with

$$\varphi(x,\zeta,0) = 0.$$

The numerical parameters used in the experiment are reported in Table 1.

Table 1. Numerical parameters used in the simulation.

Parameter	Value	Description
L	1	Length of the computational interval
N_x	121	Number of spatial grid points
Δx	$1/(N_x - 1)$	Spatial mesh size
p	2.3	Hartree nonlinearity exponent
n	4	Dimension used for admissibility check
α	0.55	Fractional order
β	0.8	Tempering/damping parameter
μ_1	1	Fractional damping coefficient
A_0	3.0	Initial amplitude
N_ζ	35	Number of quadrature points for ζ
ε_K	1/120	Hartree-kernel regularization parameter
$\zeta_{\max}$	8	Truncation bound for ζ-integration
$\|\|\mathrm{y}\|\|_\infty$ threshold	25	Baseline stopping threshold

For consistency with the exponent regime used in the analytical problem, we choose the representative theoretical dimension $n = 4$. In this case, the admissibility condition becomes

$$1+\frac{2}{4}<p<\frac{4+2}{4-2}, \qquad \text{that is,} \qquad \frac{3}{2}<p<3.$$

Hence, the numerical value $p = 2.3$ lies within the admissible exponent range associated with the four-dimensional analytical model. This choice only ensures consistency of the nonlinear exponent with the theoretical assumptions; it does not identify the one-dimensional numerical analogue with the original problem posed in dimension $n > 3$.

7.1. Numerical method

The spatial discretization is carried out by a finite difference scheme, while the resulting system is advanced in time by the method of lines. Let

$$x_i = i\Delta x, \qquad i = 0, 1, \dots, N_x - 1$$

be a uniform partition of $[0, 1]$. The second derivative is approximated by the standard second-order central difference formula

$$\mathrm{y}_{xx}(x_i, t) \approx \frac{\mathrm{y}_{i-1}(t) - 2\mathrm{y}_i(t) + \mathrm{y}_{i+1}(t)}{(\Delta x)^2}.$$

The homogeneous Dirichlet boundary conditions are imposed by setting

$$\mathrm{y}_0(t) = \mathrm{y}_{N_x-1}(t) = 0.$$

The Hartree-type nonlocal potential is approximated by the quadrature formula

$$\mathcal{H}_i(t) = \sum_{j=1}^{N_x-2} K_{\varepsilon_K}(x_i, x_j)|\mathrm{y}_j(t)|^p \,\Delta x.$$

Therefore, the discrete Hartree source term is

$$\mathcal{N}_i(t) = \mathcal{H}_i(t)|\mathrm{y}_i(t)|^{p-2}\mathrm{y}_i(t).$$

The fractional damping contribution is evaluated through the diffusive representation. Such fractional damping mechanisms are commonly used in the study of stabilization and energy decay for wave equations; see, for instance, [11]. The infinite integral with respect to ζ is truncated to $[0, \zeta_{\max}]$. Since the integrand is treated symmetrically with respect to ζ, the integral over $(-\infty, +\infty)$ is approximated by

$$\int_{-\infty}^{+\infty} \eta(\zeta)\varphi(x,\zeta,t)\, d\zeta \approx 2\sum_{k=1}^{N_\zeta} \eta(\zeta_k)\varphi(x,\zeta_k,t)w_k,$$

where w_k are trapezoidal quadrature weights. Hence, the discrete memory term is

$$\mathcal{M}_i(t) = 2b\sum_{k=1}^{N_\zeta} \eta(\zeta_k)\varphi_{i,k}(t)w_k.$$

The resulting semi-discrete system is

$$\begin{cases} \dot{y}_i = v_i, \\ \dot{v}_i = \dfrac{y_{i-1} - 2y_i + y_{i+1}}{(\Delta x)^2} - \mathcal{M}_i + \mathcal{N}_i, \\ \dot{\varphi}_{i,k} = -(\zeta_k^2 + \beta)\varphi_{i,k} + \eta(\zeta_k) v_i. \end{cases}$$

The resulting semi-discrete problem forms a stiff system of ordinary differential equations because the discrete Laplacian contributes rapidly varying spatial modes, while the diffusive variables introduce additional decay rates $\zeta_k^2 + \beta$. Consequently, the method of lines is coupled with the implicit MATLAB solver `ode15s`, whose adaptive time-stepping strategy is well-suited for stiff evolution problems.

The approximation of the memory term involves two numerical approximations. First, the infinite interval $(0, \infty)$ is truncated to the finite interval $(0, \zeta_{\max})$. Second, the resulting integral is evaluated by the trapezoidal rule using N_ζ quadrature nodes. Therefore, the computed fractional damping contains both truncation and quadrature errors. The influence of these numerical parameters is examined later through sensitivity studies.

To prevent numerical overflow during the rapid-growth phase, the computation is terminated when the solution reaches a prescribed maximum-norm threshold,

$$\|y(\cdot, t)\|_\infty = M_{\text{stop}}.$$

Throughout the baseline simulation, we take $M_{\text{stop}} = 25$. The attainment of this threshold is used solely as a numerical stopping criterion and should not be interpreted as an independent proof of finite-time blow-up. Instead, it provides a practical indicator of the onset of rapid solution growth. The dependence of the computed stopping time on the choice of M_{stop} is investigated separately in the sensitivity analysis.

7.2. Discrete energy and dissipation

The continuous energy associated with the augmented formulation is

$$E(t) = \frac{1}{2}\|y_t\|_2^2 + \frac{1}{2}\|\nabla y\|_2^2 - \frac{1}{2p}\int_\Omega\int_\Omega \frac{|y(x)|^p |y(z)|^p}{|x - z|^{n-2}}\, dx\, dz + \frac{b}{2}\int_\Omega\int_{-\infty}^{+\infty} |\varphi(x, \zeta, t)|^2\, d\zeta\, dx.$$

For the one-dimensional numerical analogue, this quantity is approximated by

$$E_h(t) = \frac{1}{2}\sum_i v_i^2 \Delta x + \frac{1}{2}\sum_i \left(\frac{y_{i+1} - y_i}{\Delta x}\right)^2 \Delta x - \frac{1}{2p}\sum_{i,j} K_{\varepsilon_K}(x_i, x_j)|y_i|^p |y_j|^p (\Delta x)^2 + b\sum_{i,k} |\varphi_{i,k}|^2 w_k \Delta x.$$

The final term contains the coefficient b, rather than $b/2$, because the symmetric integration over $(-\infty, \infty)$ is represented by twice the integral over $[0, \zeta_{\max}]$.

The corresponding discrete dissipation functional is defined by

$$D_h(t) = 2b\sum_{i,k} (\zeta_k^2 + \beta)|\varphi_{i,k}(t)|^2 w_k\, \Delta x.$$

The discrete functionals $E_h(t)$ and $D_h(t)$ provide the numerical counterparts of the continuous energy and dissipation associated with the reduced one-dimensional model.

At the continuous level, the analytical energy satisfies

$$E'(t) = -b \int_{\Omega} \int_{-\infty}^{+\infty} \left(\zeta^2 + \beta\right) |\varphi(x, \zeta, t)|^2 \, d\zeta dx = -D(t) \leq 0.$$

The semi-discrete approximation is designed to preserve the same qualitative energy structure. Nevertheless, the computed energy is also affected by the finite-difference approximation, the quadrature of the Hartree interaction, the truncation of the diffusive integral, and the adaptive time integration. Consequently, the monotonicity of $E_h(t)$ is verified numerically through the computed solution.

We define the corresponding discrete blow-up functional by

$$H_h(t) = -E_h(t).$$

Since $H_h(t)$ is defined directly as the negative of the computed discrete energy, the two quantities are exact reflections of one another throughout the computation.

For the chosen initial data, the computed initial energy satisfies

$$E_h(0) < 0.$$

Hence, the reduced numerical model starts from the same negative-energy regime considered in the analytical blow-up analysis.

For the baseline computation, the maximum observed positive increment of the discrete energy is approximately

$$9.53 \times 10^{-11},$$

which is at the level of numerical roundoff relative to the magnitude of the computed energy. Moreover, the relative energy-balance residual is approximately

$$1.31 \times 10^{-2},$$

indicating good agreement between the computed energy evolution and the analytical dissipation mechanism despite the effects of spatial discretization, quadrature approximation, truncation of the diffusive representation, and adaptive time integration.

7.3. Results and discussion

Figure 1 presents the time evolution of the discrete energy functional $E_h(t)$. The computed discrete energy remains negative and decreases throughout the simulation, in agreement with the dissipative behavior $E'(t) \leq 0$ established in the theoretical analysis.

Figure 2 displays the evolution of the discrete blow-up functional $H_h(t) = -E_h(t)$. Since $E_h(0) < 0$, one has $H_h(0) > 0$ and the observed growth of $H_h(t)$ is consistent with the energy-based functional argument used in the analytical proof of blow-up. Related analytical and numerical studies for fractionally damped wave equations with strong damping and infinite memory can be found in [2].

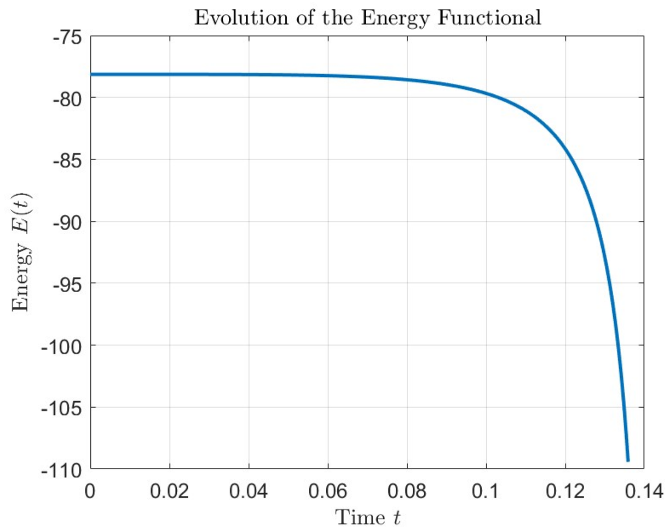

Figure 1. Time evolution of the computed energy functional $E(t)$. The monotone decay agrees with the analytical dissipation property $E'(t) \leq 0$. Parameters: $p = 2.3$, $\alpha = 0.55$, $\beta = 0.8$, and $N_x = 121$.

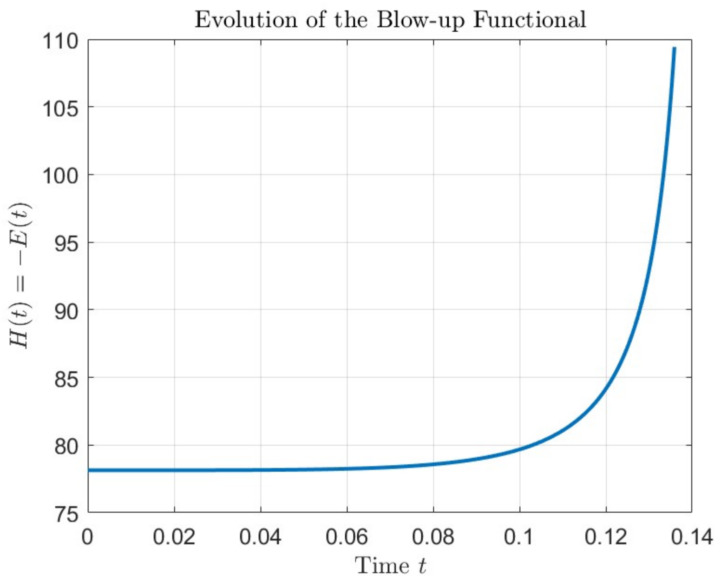

Figure 2. Time evolution of the blow-up functional $H(t) = -E(t)$. Since $E(0) < 0$, the functional remains positive and increases throughout the computation, consistent with the analytical blow-up functional. Parameters: $p = 2.3$, $\alpha = 0.55$, $\beta = 0.8$, and $N_x = 121$.

Figure 3 shows the time evolution of the maximum norm $\|y(\cdot, t)\|_\infty$. The norm grows increasingly rapidly as the computation approaches the stopping time. This behavior provides numerical evidence consistent with a finite-time blow-up scenario; however, reaching the prescribed stopping threshold should be interpreted as an indication of rapid numerical growth rather than as an independent proof of divergence.

A three-dimensional representation of the numerical solution $y(x, t)$ is presented in Figure 4. The solution remains concentrated near the central part of the spatial interval while its amplitude increases rapidly with time. The homogeneous Dirichlet boundary conditions keep the numerical solution equal to zero at the endpoints.

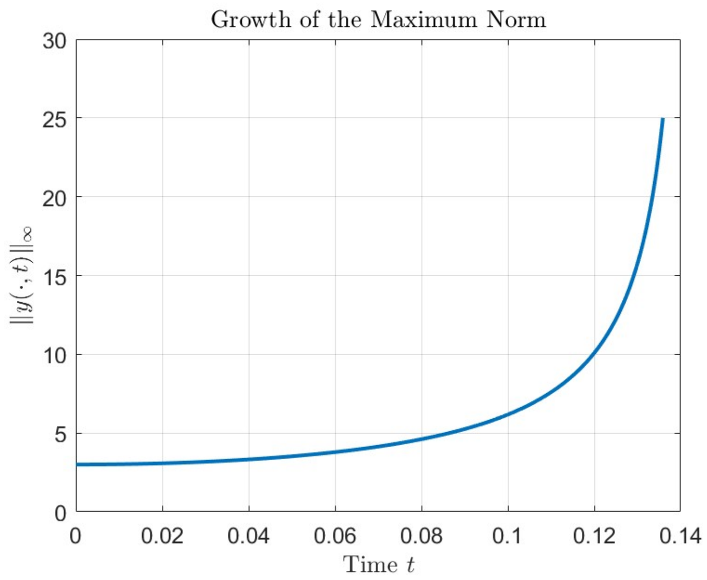

Figure 3. Growth of the maximum norm $||y(\cdot, t)||_\infty$. The rapid increase of the maximum norm is consistent with the analytical rapid-growth mechanism and the negative-energy blow-up scenario. Parameters: $p = 2.3$, $\alpha = 0.55$, $\beta = 0.8$, and $N_x = 121$.

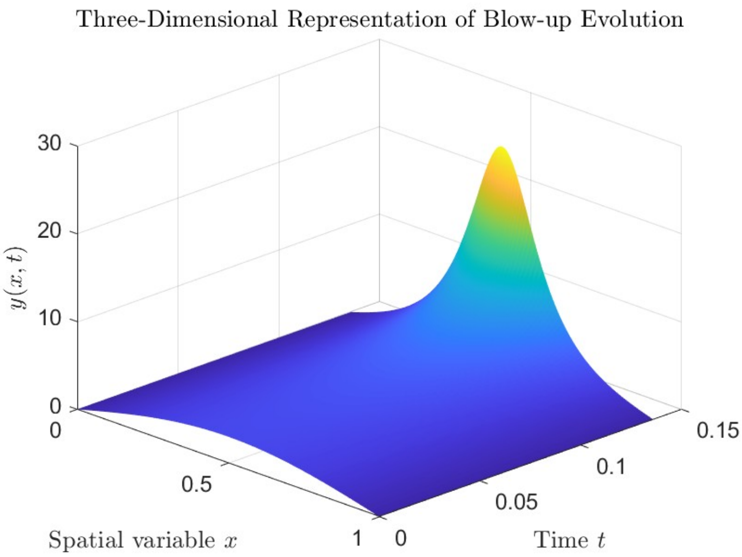

Figure 4. Three-dimensional representation of $y(x, t)$, showing the spatial and temporal evolution of the computed solution and its rapid growth near the final computation time. Parameters: $p = 2.3$, $\alpha = 0.55$, $\beta = 0.8$, and $N_x = 121$.

Figure 5 illustrates the influence of the initial amplitude A_0 on the growth process. Larger values of A_0 produce faster growth of $\|y(\cdot, t)\|_\infty$ and earlier attainment of the prescribed stopping threshold. The corresponding estimated stopping times T_{stop} for each value of A_0 are reported in the figure legend. Thus, the computed stopping time is sensitive to the size of the initial data.

Solution profiles at selected time levels are displayed in Figure 6. These profiles preserve the homogeneous boundary conditions and show a progressive increase in the interior amplitude, thereby illustrating the spatial development of the rapid-growth behavior.

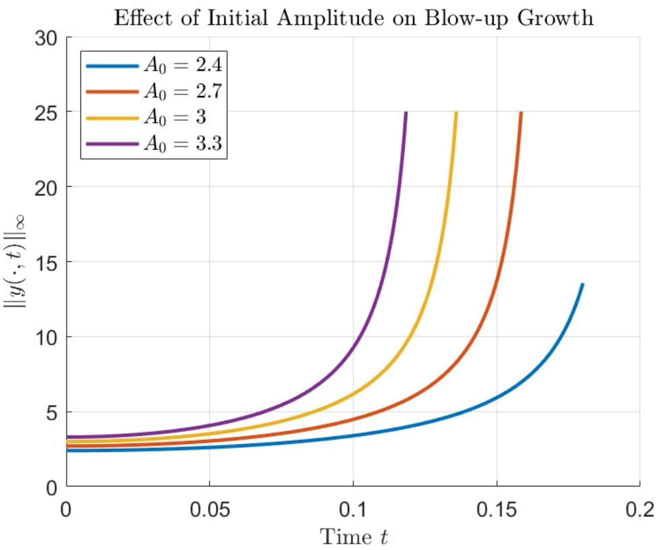

Figure 5. Influence of the initial amplitude A_0 on the solution growth. Larger initial amplitudes produce faster growth and earlier attainment of the prescribed stopping threshold. Parameters: $p = 2.3$, $\alpha = 0.55$, $\beta = 0.8$, $N_x = 121$, and $A_0 \in \{2.4, 2.7, 3.0, 3.3\}$.

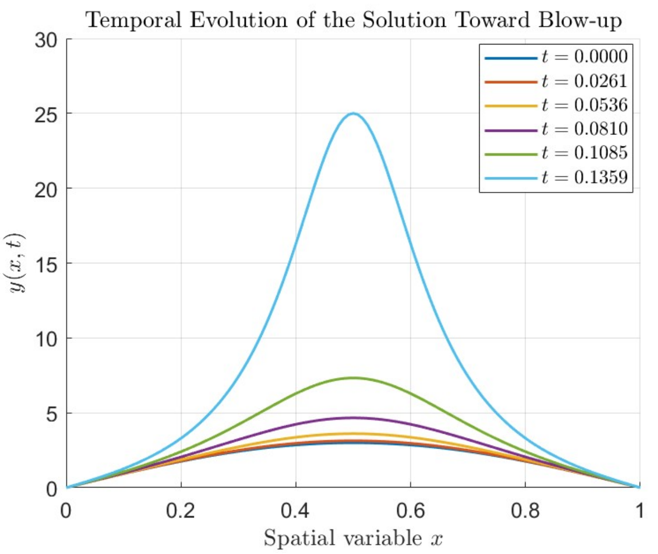

Figure 6. Solution profiles at selected time levels, illustrating the temporal evolution of the computed solution and its progressive growth. Parameters: $p = 2.3$, $\alpha = 0.55$, $\beta = 0.8$, and $N_x = 121$.

To assess the sensitivity of the computation, additional simulations were performed by varying the spatial mesh, the tolerance parameters of MATLAB's `ode15s`, and the prescribed stopping threshold. These tests are not intended to constitute a complete convergence analysis of the original multidimensional Hartree problem. Rather, they examine whether the qualitative rapid-growth behavior observed in the reduced one-dimensional model depends strongly on a particular mesh, solver tolerance, or stopping criterion. The results are summarized in Tables 2–5.

Table 2. Spatial refinement study with fixed kernel regularization $\varepsilon_K = 1/120$.

$\mathbf{N_x}$	$\mathbf{\Delta x}$	$\boldsymbol{\varepsilon_K}$	$\mathbf{T}_{\text{stop}}$	$\mathbf{E_h(0)}$
61	0.016667	0.0083333	0.13477	−79.560
121	0.0083333	0.0083333	0.13589	−78.137
241	0.0041667	0.0083333	0.13593	−78.093
481	0.0020833	0.0083333	0.13593	−78.092

Table 2 presents the spatial-refinement study while keeping the kernel regularization parameter fixed at $\varepsilon_K = 1/120$. As the mesh is refined, the computed stopping time rapidly stabilizes from 0.13477 to 0.13593, whereas the initial discrete energy remains nearly unchanged. These results indicate that the observed rapid-growth behavior is not an artifact of a coarse spatial mesh. Instead, the reduced numerical solution exhibits a consistent stopping time under successive mesh refinements, providing additional confidence in the qualitative numerical illustration.

Table 3 shows that tightening the relative and absolute tolerances of MATLAB's `ode15s` solver does not change the computed stopping time. The identical values of T_{stop} obtained for all three tolerance pairs indicate that the numerical results are insensitive to the selected solver tolerances within the tested range, providing additional confidence in the temporal integration of the reduced model.

The influence of the diffusive quadrature was also examined by varying the number of quadrature points and the truncation bound used in the ζ-integration. For $(N_\zeta, \zeta_{\max}) = (35, 8), (50, 8), (70, 10)$, and $(90, 12)$, the computed stopping times were 0.13589, 0.13589, 0.13591, and 0.13592, respectively. The observed variations are negligible, indicating that the computed rapid-growth behavior is insensitive to the tested quadrature and truncation parameters.

Table 3. Sensitivity of the computed stopping time with respect to the `ode15s` solver tolerances.

RelTol	AbsTol	T_{stop}
10^{-6}	10^{-8}	0.13589
10^{-8}	10^{-10}	0.13589
10^{-10}	10^{-12}	0.13589

Table 4 shows that the computed stopping time depends noticeably on the regularization parameter. As ε_K decreases, the regularized kernel becomes more strongly concentrated near $x = \xi$, the initial discrete energy becomes more negative, and the prescribed threshold is reached earlier. This experiment is therefore interpreted as a kernel-regularization sensitivity assessment and not as a convergence study toward the singular multidimensional Hartree problem.

Table 4. Sensitivity of the computed stopping time with respect to the Hartree-kernel regularization parameter, with $N_x = 241$.

ε_K	T_{stop}	$E_h(0)$
1/60	0.15135	−61.519
1/120	0.13593	−78.093
1/240	0.12428	−94.752
1/480	0.11449	−112.800

Table 5 shows that increasing the prescribed stopping threshold results in only a gradual increase in the computed stopping time. As the threshold becomes larger, the changes in T_{stop} become progressively smaller, indicating that the numerical solution has already entered a rapid-growth phase before the stopping criterion is reached. The stopping threshold is introduced solely to prevent numerical overflow and does not generate the observed rapid-growth behavior.

Table 5. Sensitivity of the computed stopping time with respect to the stopping threshold.

Stopping threshold	T_{stop}	$\|y(\cdot, T_{\text{stop}})\|_\infty$
15	0.12916	15
20	0.13346	20
25	0.13589	25
30	0.13743	30
35	0.13849	35
40	0.13926	40
50	0.14030	50

Overall, the sensitivity studies indicate that the qualitative numerical behavior remains consistent under variations of the spatial mesh, the `ode15s` solver tolerances, and the stopping threshold. Moreover, the discrete energy decreases monotonically, the discrete blow-up functional $H_h(t) = -E_h(t)$ increases accordingly, and the maximum norm $\|y(\cdot, t)\|_\infty$ exhibits rapid growth. These numerical observations are qualitatively consistent with the analytical results obtained for the reduced one-dimensional model.

8. Conclusions

In this paper, we investigated a nonlinear wave equation involving a Hartree-type nonlocal source and an interior exponentially tempered fractional damping term. By employing the diffusive representation of the fractional operator, the original problem was reformulated as an augmented evolution system. This reformulation made it possible to apply semigroup arguments, together with the local Lipschitz continuity of the Hartree-type nonlinear operator, to establish the local well-posedness of solutions.

We also analyzed the finite-time blow-up behavior of solutions under a negative initial energy condition. By constructing a suitable auxiliary functional and combining it with estimates adapted to the Hartree-type nonlocal source, we proved that the corresponding solution cannot exist globally in time. This result shows that, for suitable initial data, the nonlocal source is sufficiently strong to dominate the dissipative effect induced by the fractional damping mechanism.

Finally, a one-dimensional numerical analogue was presented to illustrate the analytical blow-up mechanism. The simulations show that the energy remains negative and decreases in time, whereas the blow-up functional and the maximum norm grow rapidly near the terminal time. Additional mesh-refinement, solver-tolerance, diffusive-quadrature, kernel-regularization, and stopping-threshold tests show that the qualitative rapid-growth behavior remains consistent under the principal numerical variations examined in the reduced one-dimensional simulation. These numerical findings are qualitatively consistent with the analytical blow-up mechanism.

One limitation of the present study is that the numerical experiments are performed for a regularized one-dimensional analogue of the original multidimensional problem. Consequently, the computations provide a qualitative illustration of the analytical blow-up mechanism rather than a direct numerical approximation or validation of the multidimensional model.

The present work may be extended in several directions. Possible topics for future research include multidimensional numerical approximations of the Hartree-type model, the development of structure-preserving numerical methods for tempered fractional damping, and the investigation of related coupled models with more general nonlinearities and damping mechanisms.

Author contributions

Muhammad Fahim Aslam, Jianghao Hao, Iqra Kanwal, Imran Shabir Chuhan, Mohamed Balegh, and Zayd Hajjej: Conceptualization, methodology, investigation, writing original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

Availability of data and materials

The data supporting the findings of this study are contained within the manuscript. The MATLAB code used to generate the numerical results and figures is available from the corresponding author upon reasonable request.

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