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Research article

Integrability investigation and lump-type wave structures for higher-dimensional generalized Kadomtsev-Petviashvili equations

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Abstract: This paper systematically investigates the integrable properties and lump-type wave structures of higher-dimensional generalized Kadomtsev-Petviashvili (KP) equations, which describe nonlinear wave behaviors in physical systems. For the (3+1)-dimensional KP and Boussinesq equations, a three-soliton condition is established. Based on this condition, a bilinear Bäcklund transformation is constructed, from which the corresponding Lax pair and modified evolutionary equation are derived. The integrability of the family of generalized $(N + 1)$-dimensional KP equations is also verified. By means of the long wave limit approach, lump-type solutions to these equations are constructed, and a (3+2)-dimensional example is presented to demonstrate their dynamical features. Through appropriate coefficient simplifications and linear variable transformations, the studied integrable equations can be reduced to the (2+1)-dimensional KP equation, (1+1)-dimensional Boussinesq equation, and their lower-dimensional reductions. This work enriches existing theoretical findings and provides an effective approach for constructing lump-type wave structures to nonlinear evolution equations.

Keywords: generalized Kadomtsev-Petviashvili equation; Bäcklund transformation; Lax pair; lump-type wave; long wave limit
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1. Introduction

Nonlinear dispersive wave equations, especially nonlinear integrable systems, describe a variety of phenomena in numerous modern scientific and engineering fields, including mechanics, physical chemistry, biology, and atmospheric space science. Furthermore, such mathematical models are central to exploring nonlinear water wave dynamics in ocean science and engineering research. Owing to the fundamental properties of integrable equations, including N-soliton solutions [1–3], Bäcklund transformations [4, 5], Lax representations, and infinitely many conservation laws [6, 7], investigations on nonlinear integrable equations and their integrable properties are of great theoretical value within nonlinear science. Notably, studying the integrable properties of higher-dimensional nonlinear equations holds substantial scientific and practical value. Scientifically, these systems overcome the limitations of low-dimensional models and reveal multi-dimensional physical coupling mechanisms to support understanding and predicting nonlinear wave dynamics.

The Kadomtsev-Petviashvili (KP) equation reads

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0,\ \sigma^2 = \pm 1, \tag{1.1}$$

where $u = u(x, y, t)$ denotes the real-valued wave amplitude, t represents the temporal variable, x and y are spatial variables, and $\sigma^2 = \pm 1$ is the dispersion parameter. The case $\sigma^2 = -1$ corresponds to the KPI equation, while $\sigma^2 = 1$ yields the KPII equation. This equation represents one of the most fundamental integrable frameworks in mathematical physics and is widely recognized as a two-dimensional generalization of the one-dimensional Korteweg-de Vries (KdV) equation. The KP equation acts as a physical model that characterizes two-dimensional water wave motion under weakly nonlinear restoring forces [8], and it finds extensive applications in fluid mechanics as well as theoretical physics. Similar to the KdV equation, the KP equation admits complete integrability. It exhibits remarkable integrable characteristics and abundant mathematical structures [9–11], providing a solid foundation for solving complex water wave problems in the study of nonlinear wave phenomena.

As a great many nonlinear natural phenomena are governed by higher-dimensional nonlinear evolution equations, many higher-dimensional generalizations of the KP Eq (1.1) have attracted scholarly attention from diverse perspectives. Recently, Ma et al. introduced a family of generalized KP and Boussinesq equations [12]

$$(u_{x_1x_1x_1} - 6uu_{x_1})_{x_1} + \sum_{i,j=1}^{M} a_{ij}u_{x_ix_j} = 0, \tag{1.2}$$

where M is a positive integer and the constants a_{ij}'s satisfy $a_{ij} = a_{ji}, 1 \leq i, j \leq M$. It has been verified that the KP Eq (1.1) possesses three soliton solutions, whereas the (3+1)-dimensional KP equations

$$(u_t + 6uu_x + u_{xxx})_x \pm 3u_{yy} \pm 3u_{zz} = 0, \tag{1.3}$$

cannot pass the three-soliton consistency test [12]. The existence of three-soliton solutions is widely regarded as strong evidence for integrability of polynomial bilinear equations. Accordingly, all (3+1)-dimensional KP equations described by formula (1.3) lack integrability [13]. A novel

$(n+1)$-dimensional extended KP equation

$$(u_t + \alpha u u_{x_1} + \beta u_{x_1x_1x_1})_{x_1} + \gamma u_{x_2x_2} + \sum_{i=1}^{n} \sigma_i u_{x_1x_i} = 0, \tag{1.4}$$

where $n \geq 2$ and α, β, γ and $\sigma_i, i = 1, \cdots, n$, denote constant coefficients, has recently been considered [14]. Using the binary Bell polynomial method, the Painlevé property, N-soliton solutions, bilinear Bäcklund transformations, Lax operators, and infinite conservation laws for Eq (1.4) were derived [14], confirming that Eq (1.4) is an integrable generalization of the (2+1)-dimensional KP equation to arbitrary spatial dimensions. Inspired by the above research progress, we aim to explore whether generalized KP equations defined in higher-dimensional spaces can possess integrable properties.

In this paper, we would like to focus on a family of generalized (3+1)-dimensional KP and Boussinesq equations, written as

$$(u_{x_1x_1x_1} + 6\alpha u u_{x_1})_{x_1} + \sum_{i,j=1}^{4} a_{ij} u_{x_ix_j} = 0, \tag{1.5}$$

and a family of $(N+1)$-dimensional generalized KP equations given by

$$(u_{x_1x_1x_1} + 6\alpha u u_{x_1})_{x_1} + a_{22}u_{x_2x_2} + a_{33}u_{x_3x_3} + 2a_{23}u_{x_2x_3} + \sum_{j=1}^{N+1} \sigma_j u_{x_1x_j} = 0, \tag{1.6}$$

where $N > 3$, α is an arbitrary nonzero real constant, the constants a_{ij}'s satisfy $a_{ij} = a_{ji}, 1 \leq i, j \leq 4$, and the coefficients $\sigma_j, 1 \leq j \leq N+1$, are real constants with $\sigma_{N+1} \neq 0$. Our objective is to explore whether integrable evolution equations exist in these two families.

Given the complex mathematical characteristics of nonlinear dispersive wave equations, researchers have developed numerous solution techniques for accurate analytical solutions and numerical simulations. A variety of novel analytical and numerical methods, including the improved tanh-function algorithm [15], generalized G'/G-expansion method [16], modified simplest equation method [17], and differential transform method [18], have been presented to construct soliton, periodic, traveling and rational solutions for nonlinear dispersive wave models. Analyses of stability, modulation instability, and asymptotic properties further clarify the physical essence of wave solutions. Each method has its own advantages and drawbacks, and no universal technique is available for solving all types of nonlinear dispersive wave equations. As an effective supplementary technique, the long wave limit approach generates rational solutions of nonlinear dispersive wave equations by taking the long wave asymptotic limit of multi-soliton solutions derived from direct methods [10].

The layout of the article is presented below. Section 2 employs the three-soliton condition from earlier research to study the existence of three-soliton solutions for the (3+1)-dimensional equation given in (1.5). Furthermore, we construct a standard multi-parameter bilinear Bäcklund transformation for the corresponding equation, from which we further derive the associated Lax pair with a spectral parameter and the modified evolutionary equation. In Section 3, we perform an integrability analysis for the family of generalized $(N+1)$-dimensional KP Eq (1.6), followed by the construction of their lump-type wave structures through the long wave limit approach. Finally, we draw our conclusions.

2. Integrability for (3+1)-dimensional KP and Boussinesq equations

2.1. Three-soliton condition

Using the results established by Ma [12], we present the three-soliton condition associated with the family of generalized KP and Boussinesq Eq (1.5).

Setting $x_1 = x, x_2 = y, x_3 = z, x_4 = t$, every equation defined by (1.5) may be expressed as

$$(u_{xxx} + 6\alpha uu_x)_x + a_{11}u_{xx} + 2a_{12}u_{xy} + 2a_{13}u_{xz} + 2a_{14}u_{xt} + a_{22}u_{yy}$$
$$+2a_{23}u_{yz} + 2a_{24}u_{yt} + a_{33}u_{zz} + 2a_{34}u_{zt} + a_{44}u_{tt} = 0. \tag{2.1}$$

We introduce the dependent variable transformation

$$u = \frac{2}{\alpha}(\ln f)_{xx}. \tag{2.2}$$

Substituting this transformation into Eq (2.1), all the partial derivative terms of u can be rewritten in terms of f and its derivatives. Using the well-known identities for Hirota's bilinear differential operators

$$2\frac{\partial^2}{\partial x^2}\ln f = \frac{D_x^2 f\cdot f}{f^2},\quad 2\frac{\partial^2}{\partial x\partial t}\ln f = \frac{D_xD_t f\cdot f}{f^2},\quad 2\frac{\partial^4}{\partial x^4}\ln f = \frac{D_x^4 f\cdot f}{f^2} - 3\Big(\frac{D_x^2 f\cdot f}{f^2}\Big)^2,$$

a direct calculation yields the following bilinear form:

$$(D_x^4 + a_{11}D_x^2 + 2a_{12}D_xD_y + 2a_{13}D_xD_z + 2a_{14}D_xD_t + a_{22}D_y^2$$
$$+2a_{23}D_yD_z + 2a_{24}D_yD_t + a_{33}D_z^2 + 2a_{34}D_zD_t + a_{44}D_t^2)f\cdot f = 0. \tag{2.3}$$

Here D_x, D_y, D_z, and D_t are Hirota's bilinear derivatives [19]. Suppose that three wave variables:

$$\xi_m = k_m x + l_{m,2}y + l_{m,3}z + l_{m,4}t + \xi_m^0,\ 1 \le m \le 3, \tag{2.4}$$

where $k_m, l_{m,2}, l_{m,3}, l_{m,4}, 1 \le m \le 3$, are constants to be determined and $\xi_m^0, 1 \le m \le 3$, represent arbitrary constant displacements. We also introduce a group of constants

$$A_{mn} = -\frac{P(k_m - k_n, l_{m,2} - l_{n,2}, l_{m,3} - l_{n,3}, l_{m,4} - l_{n,4})}{P(k_m + k_n, l_{m,2} + l_{n,2}, l_{m,3} + l_{n,3}, l_{m,4} + l_{n,4})}, 1 \le m, n \le 3, \tag{2.5}$$

with the polynomial function P being defined by

$$P(x, y, z, t) = x^4 + a_{11}x^2 + 2a_{12}xy + 2a_{13}xz + 2a_{14}xt + a_{22}y^2$$
$$+2a_{23}yz + 2a_{24}yt + a_{33}z^2 + 2a_{34}zt + a_{44}t^2. \tag{2.6}$$

When the parameters $k_m, l_{m,2}, l_{m,3}, l_{m,4}, 1 \le m \le 3$, satisfy the dispersion relations

$$P(k_m, l_{m,2}, l_{m,3}, l_{m,4}) = 0, 1 \le m \le 3, \tag{2.7}$$

the (3+1)-dimensional generalized nonlinear equation defined by (2.1) possesses a three-soliton solution of the form

$$f = 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + A_{12}e^{\xi_1+\xi_2} + A_{13}e^{\xi_1+\xi_3} + A_{23}e^{\xi_2+\xi_3} + A_{12}A_{13}A_{23}e^{\xi_1+\xi_2+\xi_3}, \tag{2.8}$$

provided that the associated three-soliton condition

$$\sum_{\sigma_1,\sigma_2,\sigma_3=\pm1} P(\sigma_1\bar{p}_1 + \sigma_2\bar{p}_2 + \sigma_3\bar{p}_3)P(\sigma_1\bar{p}_1 - \sigma_2\bar{p}_2)P(\sigma_2\bar{p}_2 - \sigma_3\bar{p}_3)P(\sigma_1\bar{p}_1 - \sigma_3\bar{p}_3) = 0 \tag{2.9}$$

is satisfied, where $\bar{p}_m = (k_m, l_{m,2}, l_{m,3}, l_{m,4})$, $1 \le m \le 3$. In [12], Ma et al. demonstrated that the three-soliton condition (2.9) for (2.1) is equivalent to the following determinant relation:

$$k_1^2k_2^2k_3^2 \sum_{i,j,p,q=2}^{4} a_{ij}a_{pq} \det(K, L_i, L_p) \det(K, L_j, L_q) = 0, \tag{2.10}$$

with

$$K = (k_1, k_2, k_3)^{\mathrm{T}}, \quad L_i = (l_{1,i}, \quad l_{2,i}, l_{3,i})^{\mathrm{T}}, \quad 2 \le i \le 4.$$

For $i, j, p, q \in \{2, 3, 4\}$, the terms on the left-hand-side of Eq (2.10) vanish when $i = p$ or $j = q$, since

$$\det(K, L_i, L_p) = 0 \quad \text{or} \quad \det(K, L_j, L_q) = 0.$$

The six admissible pairs for (i, p) are $\{(2, 3), (2, 4), (3, 2), (3, 4), (4, 2), (4, 3)\}$ and likewise for (j, q), yielding 36 potentially non-zero terms in total. After rearrangement and collecting like-terms, the three-soliton condition (2.10) yields

$$\begin{aligned}
&4(a_{22}a_{34} - a_{23}a_{24}) \det(K, L_2, L_3) \det(K, L_2, L_4) + 2(a_{22}a_{44} - a_{24}^2) \det(K, L_2, L_4)^2 \\
&+4(a_{23}a_{44} - a_{24}a_{34}) \det(K, L_2, L_4) \det(K, L_3, L_4) + 2(a_{33}a_{44} - a_{34}^2) \det(K, L_3, L_4)^2 \\
&+4(a_{23}a_{34} - a_{33}a_{24}) \det(K, L_2, L_3) \det(K, L_3, L_4) + 2(a_{22}a_{33} - a_{23}^2) \det(K, L_2, L_3)^2 = 0.
\end{aligned} \tag{2.11}$$

Define the symmetric coefficient matrix

$$G = \begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{23} & a_{33} & a_{34} \\ a_{24} & a_{34} & a_{44} \end{pmatrix}. \tag{2.12}$$

Clearly, G is either a rank-one matrix with all of its 2×2 minors identically zero or a rank-zero matrix. In both cases, the three-soliton determinant constraint (2.11) is valid for (2.1).

Now, we take the choice

$$a_{22} = \beta^2, \quad a_{33} = \gamma^2, \quad a_{44} = \delta^2, \quad a_{23} = \beta\gamma, \quad a_{24} = \beta\delta, \quad a_{34} = \gamma\delta, \tag{2.13}$$

where β, γ, and δ are arbitrary constants not all zero. As a consequence of the rank-one property of the coefficient matrix G, the three-soliton determinant constraint (2.11) holds for (2.1). Under the choice of (2.13), the original equation defined by (2.1) takes the bilinear form

$$[D_x^4 + a_{11}D_x^2 + 2a_{12}D_xD_y + 2a_{13}D_xD_z + 2a_{14}D_xD_t + (\beta D_y + \gamma D_z + \delta D_t)^2]f \cdot f = 0, \tag{2.14}$$

which passes the three-soliton solution test and has the three-soliton solution f given by (2.8). Under the transformation (2.2), Eq (2.14) is mapped into

$$\begin{aligned}&(u_{xxx}+6\alpha uu_x)_x+a_{11}u_{xx}+2a_{12}u_{xy}+2a_{13}u_{xz}+2a_{14}u_{xt}+\beta^2u_{yy}\\&+2\beta\gamma u_{yz}+2\beta\delta u_{yt}+\gamma^2u_{zz}+2\gamma\delta u_{zt}+\delta^2u_{tt}=0.\end{aligned}\tag{2.15}$$

When $\delta=0$ and $a_{14}\beta\gamma\neq 0$, Eq (2.14) reduces to the generalized KP equation [20]

$$\begin{aligned}&(D_x^4+a_{11}D_x^2+2a_{12}D_xD_y+2a_{13}D_xD_z+2a_{14}D_xD_t\\&+\beta^2D_y^2+2\beta\gamma D_yD_z+\gamma^2D_z^2)f\cdot f=0,\end{aligned}\tag{2.16}$$

which differs from Eq (1.4) for $n=3$ due to the nonzero constants β and γ. The N-soliton solution of the above bilinear Eq (2.16) reads

$$f_N=\sum_{\mu_1,\mu_2,\ldots,\mu_N\in\{0,1\}}\exp\left(\sum_{m=1}^{N}\mu_m\xi_m+\sum_{1\le m<n\le N}\mu_m\mu_n\ln A_{mn}\right),\tag{2.17}$$

where the summation runs over all possible combinations of $\mu_m\in\{0,1\}$ for $m=1,2,\ldots,N$, and

$$\begin{aligned}\xi_m&=k_mx+l_{m,2}y+l_{m,3}z+l_{m,4}t+\xi_m^0,\\ l_{m,4}&=-\frac{\beta^2l_{m,2}^2+2\beta\gamma l_{m,2}l_{m,3}+\gamma^2l_{m,3}^2+k_m^4+a_{11}k_m^2+2a_{12}k_ml_{m,2}+2a_{13}k_ml_{m,3}}{2a_{14}k_m},\quad 1\le m\le N,\\ A_{mn}&=\frac{3k_m^2k_n^2(k_m-k_n)^2-[\beta(k_nl_{m,2}-k_ml_{n,2})+\gamma(k_nl_{m,3}-k_ml_{n,3})]^2}{3k_m^2k_n^2(k_m+k_n)^2-[\beta(k_nl_{m,2}-k_ml_{n,2})+\gamma(k_nl_{m,3}-k_ml_{n,3})]^2},\quad 1\le m,n\le N.\end{aligned}$$

Taking $N=3$, expression (2.17) yields the three-soliton solution (2.8).

When $\gamma=0$ and $a_{14}\beta\delta\neq 0$, Eq (2.14) yields the (3+1)-dimensional generalized Boussinesq equation [21, 22]

$$(D_x^4+a_{11}D_x^2+2a_{12}D_xD_y+2a_{13}D_xD_z+2a_{14}D_xD_t+\beta^2D_y^2+2\beta\delta D_yD_t+\delta^2D_t^2)f\cdot f=0.\tag{2.18}$$

In a sense, the obtained three-soliton solutions provide strong evidence for integrability characteristics of the given equation. It is worth noting that the (3+1)-dimensional Eq (2.15) is not a novel integrable system. When the coefficient matrix G defined by (2.12) is of rank one, this system can be reduced to the classical (2+1)-dimensional KP Eq (1.1), the Boussinesq equation $(u_{xxx}-6uu_x)_x+u_{xx}\pm u_{tt}=0$, and their dimensional reductions through suitable simplifications of coefficients and invertible linear transformations of independent variables [12].

2.2. *Bäcklund transformation and Lax representation*

The bilinear Bäcklund transformation describes the relationship between different solutions of a bilinear differential equation and serves as an effective approach for constructing exact solutions to nonlinear equations [23–25]. In what follows, we present a bilinear Bäcklund transformation associated with the bilinear form (2.14).

Replacing $a_{11}D_x + 2a_{12}D_y + 2a_{13}D_z + 2a_{14}D_t$ and $\beta D_y + \gamma D_z + \delta D_t$ in (2.14) by D_t and $\sqrt{3}\sigma D_y$, respectively, we find that Eq (2.14) reduces to the bilinear KP equation in (2+1) dimensions

$$(D_x^4 + D_xD_t + 3\sigma^2 D_y^2)f \cdot f = 0, \tag{2.19}$$

which is equivalent to Eq (1.1) under the transformation $u = 2(\ln f)_{xx}$. It is widely known that the KP Eq (1.1) possesses the following bilinear Bäcklund transformation [19]:

$$(D_x^2 + \sigma D_y + \lambda)f \cdot f' = 0, \tag{2.20a}$$

$$(D_x^3 + D_t - 3\sigma D_xD_y - 3\lambda D_x)f \cdot f' = 0, \tag{2.20b}$$

where f and f' denote two distinct solutions for Eq (2.19) and λ denotes a free parameter. Consequently, by replacing D_t and σD_y in the system (2.20) by $a_{11}D_x + 2a_{12}D_y + 2a_{13}D_z + 2a_{14}D_t$ and $\frac{1}{\sqrt{3}}(\beta D_y + \gamma D_z + \delta D_t)$, respectively, we can present a bilinear Bäcklund transformation of Eq (2.14) as follows:

$$\Big[D_x^2 + \frac{1}{\sqrt{3}}(\beta D_y + \gamma D_z + \delta D_t) + \lambda\Big]f \cdot f' = 0, \tag{2.21a}$$

$$\Big[D_x^3 + a_{11}D_x + 2a_{12}D_y + 2a_{13}D_z + 2a_{14}D_t - \sqrt{3}D_x(\beta D_y + \gamma D_z + \delta D_t) - 3\lambda D_x\Big]f \cdot f' = 0, \tag{2.21b}$$

where f and f' denote two distinct solutions of Eq (2.14) and λ is an arbitrary parameter.

We next set

$$\psi = \frac{f}{f'},\ u = \frac{2}{\alpha}(\ln f')_{xx}, \tag{2.22}$$

and employ the identities for rational transformations [19]

$$\left\{\begin{array}{l} (D_xf \cdot f')/f'^2 = \psi_x, \\ (D_x^2 f \cdot f')/f'^2 = \psi_{xx} + \alpha u\psi, \\ (D_x^3 f \cdot f')/f'^2 = \psi_{xxx} + 3\alpha u\psi_x, \\ (D_xD_y f \cdot f')/f'^2 = \psi_{xy} + \alpha\partial_x^{-1}u_y\psi, \\ \cdots\cdots, \end{array}\right.$$

where

$$\partial_x^{-1}u(x,y,z,t) = \int_{-\infty}^{x} u(\xi,y,z,t)\,d\xi.$$

All potentials are assumed to satisfy the vanishing at infinity condition $u \to 0$ as $x \to -\infty$. The bilinear Bäcklund transformation (2.21) can be converted into

$$\psi_{xx} + \frac{1}{\sqrt{3}}\Big(\beta\psi_y + \gamma\psi_z + \delta\psi_t\Big) + \alpha u\psi + \lambda\psi = 0, \tag{2.23a}$$

$$4\psi_{xxx} + 6\alpha u\psi_x + a_{11}\psi_x + 2a_{12}\psi_y + 2a_{13}\psi_z + 2a_{14}\psi_t + 3\alpha u_x\psi - \sqrt{3}\alpha\partial_x^{-1}(\beta u_y + \gamma u_z + \delta u_t)\psi = 0. \tag{2.23b}$$

Let us take the operators L and A as

$$L = \partial_x^2 + \frac{1}{\sqrt{3}}(\beta\partial_y + \gamma\partial_z + \delta\partial_t) + \alpha u + \lambda, \tag{2.24a}$$

$$A = 4\partial_x^3 + (6\alpha u + a_{11})\partial_x + 2a_{12}\partial_y + 2a_{13}\partial_z + 2a_{14}\partial_t + 3\alpha u_x - \sqrt{3}\alpha\partial_x^{-1}(\beta u_y + \gamma u_z + \delta u_t). \qquad (2.24b)$$

Then, the compatibility condition $[L, A] = 0$ reduces to (2.15), which indicates the system (2.24) serves as a Lax pair associated with Eq (2.15). We can perform an explicit substitution check to confirm the compatibility relation.

In addition, upon applying the dependent variable transformation [19]

$$\phi = \frac{1}{\alpha}\ln\frac{f}{f'},\ \rho = \ln(ff'), \qquad (2.25)$$

and setting $\lambda = 0$ for convenience, the bilinear Bäcklund transformation (2.21) is written as

$$\rho_{xx} + \alpha^2\phi_x^2 + \frac{1}{\sqrt{3}}\alpha\Big(\beta\phi_y + \gamma\phi_z + \delta\phi_t\Big) = 0, \qquad (2.26a)$$

$$\begin{aligned}&\alpha\phi_{xxx} + 3\alpha\phi_x\rho_{xx} + \alpha^3\phi_x^3 + \alpha(a_{11}\phi_x + 2a_{12}\phi_y + 2a_{13}\phi_z + 2a_{14}\phi_t)\\ &- \sqrt{3}(\beta\rho_{xy} + \beta\alpha^2\phi_x\phi_y + \gamma\rho_{xz} + \gamma\alpha^2\phi_x\phi_z + \delta\rho_{xt} + \delta\alpha^2\phi_x\phi_t) = 0. \qquad (2.26b)\end{aligned}$$

By eliminating ρ in the coupled system (2.26), we have

$$\begin{aligned}&\phi_{xxxx} - 6\alpha^2\phi_x^2\phi_{xx} + a_{11}\phi_{xx} + 2a_{12}\phi_{xy} + 2a_{13}\phi_{xz}\\ &+2a_{14}\phi_{xt} - 2\sqrt{3}\alpha\phi_{xx}(\beta\phi_y + \gamma\phi_z + \delta\phi_t) + \beta^2\phi_{yy}\\ &+2\beta\gamma\phi_{yz} + 2\beta\delta\phi_{yt} + \gamma^2\phi_{zz} + 2\gamma\delta\phi_{zt} + \delta^2\phi_{tt} = 0, \qquad (2.27)\end{aligned}$$

which may be considered as a modified form of (2.15). If introducing a new variable v by $v = \phi_x$, it is easy to see that (2.27) can be expressed as

$$\begin{aligned}&v_{xxx} - 6\alpha^2 v^2 v_x + a_{11}v_x + 2a_{12}v_y + 2a_{13}v_z + 2a_{14}v_t\\ &-2\sqrt{3}\alpha v_x\partial_x^{-1}(\beta v_y + \gamma v_z + \delta v_t) + \partial_x^{-1}(\beta^2 v_{yy} + 2\beta\gamma v_{yz}\\ &+2\beta\delta v_{yt} + \gamma^2 v_{zz} + 2\gamma\delta v_{zt} + \delta^2 v_{tt}) = 0, \qquad (2.28)\end{aligned}$$

where

$$\partial_x^{-1}u(x, y, z, t) = \int_{-\infty}^{x} u(\xi, y, z, t)\, d\xi$$

and $u \to 0$ as $x \to -\infty$. Accordingly, from Eqs (2.16) and (2.18), the (3+1)-dimensional generalized modified KP and modified Boussinesq equations read

$$\begin{aligned}&v_{xxx} - 6\alpha^2 v^2 v_x + a_{11}v_x + 2a_{12}v_y + 2a_{13}v_z + 2a_{14}v_t\\ &-2\sqrt{3}\alpha v_x\partial_x^{-1}(\beta v_y + \gamma v_z) + \partial_x^{-1}(\beta^2 v_{yy} + 2\beta\gamma v_{yz} + \gamma^2 v_{zz}) = 0 \qquad (2.29)\end{aligned}$$

and

$$\begin{aligned}&v_{xxx} - 6\alpha^2 v^2 v_x + a_{11}v_x + 2a_{12}v_y + 2a_{13}v_z + 2a_{14}v_t\\ &-2\sqrt{3}\alpha v_x\partial_x^{-1}(\beta v_y + \delta v_t) + \partial_x^{-1}(\beta^2 v_{yy} + 2\beta\delta v_{yt} + \delta^2 v_{tt}) = 0, \qquad (2.30)\end{aligned}$$

respectively.

Further, setting $\lambda = 0$ and applying the double logarithmic transformation formula [19], an alternative representation for the left-hand side of (2.21a) is given by

$$\begin{aligned}
&\Big[D_x^2 + \frac{1}{\sqrt{3}}(\beta D_y + \gamma D_z + \delta D_t)\Big] f \cdot f'/(ff') \\
&= [\ln(ff')]_{xx} + [\ln(f/f')]_x^2 + \frac{1}{\sqrt{3}}\Big(\beta[\ln(f/f')]_y + \gamma[\ln(f/f')]_z + \delta[\ln(f/f')]_t\Big) \\
&= 2[\ln(f)]_{xx} - \big[\ln(f/f')\big]_{xx} + [\ln(f/f')]_x^2 \\
&+ \frac{1}{\sqrt{3}}\Big(\beta[\ln(f/f')]_y + \gamma[\ln(f/f')]_z + \delta[\ln(f/f')]_t\Big),
\end{aligned} \tag{2.31}$$

which implies that the bilinear form (2.21a) is identical to

$$u = v_x - \alpha v^2 - \frac{1}{\sqrt{3}}\partial_x^{-1}(\beta v_y + \gamma v_z + \delta v_t), \tag{2.32}$$

under the following associated variable transformation

$$u = \frac{2}{\alpha}[\ln(f)]_{xx},\ v = \frac{1}{\alpha}(\ln\frac{f}{f'})_x. \tag{2.33}$$

Equation (2.32) is a Miura transformation relating the solution u of Eq (2.15) to the solution v of Eq (2.28) [19]. Accordingly, the (3+1)-dimensional generalized modified KP and modified Boussinesq equations determined by Eqs (2.29) and (2.30) are integrable and admit their corresponding Lax pairs.

3. Integrability and Lump-type solutions for generalized $(N+1)$-dimensional KP equations

The family of generalized (3+1)-dimensional KP and Boussinesq Eq (1.5) possesses full second-order quadratic derivative terms, while its higher-dimensional extension (1.6) is constructed with a second-order mixed-derivative structure. By virtue of the dependent variable transformation

$$u = \frac{2}{\alpha}(\ln f)_{x_1x_1}, \tag{3.1}$$

every equation defined by (1.6) takes the form of a bilinear structure

$$(D_{x_1}^4 + a_{22}D_{x_2}^2 + a_{33}D_{x_3}^2 + 2a_{23}D_{x_2}D_{x_3} + \sum_{j=1}^{N+1}\sigma_j D_{x_1}D_{x_j})f \cdot f = 0. \tag{3.2}$$

Setting $N = 3$, the family of generalized higher-dimensional equations defined by (1.6) reduces to a special case of (1.5). Lump-type solutions of (1.5) have been extensively investigated [21, 22, 26]. In this section, we aim to explore the integrability of (1.6) and derive their lump-type wave structures.

3.1. Integrability analysis

From the three-soliton constraint condition (2.10) [12], the bilinear Eq (3.2) possesses a three-soliton solution when the coefficient matrix

$$H = \begin{pmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{pmatrix}, \tag{3.3}$$

has rank 0 or 1. In particular, such a solution exists under the coefficient choice

$$a_{22} = \beta^2, a_{33} = \gamma^2, a_{23} = \beta\gamma, \tag{3.4}$$

and σ_j, $1 \leq j \leq N+1$, are arbitrary constants satisfying $\sigma_{N+1} \neq 0$. With this coefficient setting (3.4) and based on the results derived in Section 2, we directly give the bilinear form of Bäcklund transformation for (3.2):

$$\left[D_{x_1}^2 + \frac{1}{\sqrt{3}}(\beta D_{x_2} + \gamma D_{x_3}) + \lambda\right] f \cdot f' = 0, \tag{3.5a}$$

$$\left[D_{x_1}^3 + \sum_{j=1}^{N+1} \sigma_j D_{x_j} - \sqrt{3} D_{x_1}(\beta D_{x_2} + \gamma D_{x_3}) - 3\lambda D_{x_1}\right] f \cdot f' = 0, \tag{3.5b}$$

in which f and f' are two distinct solutions to Eq (3.2) and λ is an arbitrary parameter. Accordingly, under the coefficient choice (3.4), the corresponding Lax pair of (1.6) reads

$$\psi_{x_1 x_1} + \frac{1}{\sqrt{3}}\left(\beta\psi_{x_2} + \gamma\psi_{x_3}\right) + \alpha u\psi + \lambda\psi = 0, \tag{3.6a}$$

$$4\psi_{x_1 x_1 x_1} + 6\alpha u\psi_{x_1} + \sum_{j=1}^{N+1} \sigma_j \psi_{x_j} + 3\alpha u_{x_1}\psi - \sqrt{3}\alpha\partial_{x_1}^{-1}(\beta u_{x_2} + \gamma u_{x_3})\psi = 0. \tag{3.6b}$$

Similarly, if the matrix H defined by (3.3) has rank 0 or 1, the generalized $(N + 1)$-dimensional KP equations defined by (1.6) can be reduced to the KP Eq (1.1), the (1+1)-dimensional Boussinesq equation, as well as their dimensionally-reduced counterparts, via invertible linear mappings for the independent variables and proper coefficient adjustments [12].

3.2. Lump-type wave structures

The investigation of lump and lump-type wave structures has attracted considerable attention in recent research. In ocean engineering, lump waves are extreme nonlinear waves whose energy is highly localized in space. Characterized by steep crests, short duration evolution, and intense impact, they pose a significant potential threat to the safety and stability of marine structures. From a mathematical perspective, lump waves are defined as analytical rational function solutions that decay to zero along all spatial directions as the spatial coordinates tend to infinity [26–28]. On the other hand, lump-type waves refer to rationally constructed solutions that may exhibit only partial spatial localization in higher-dimensional settings. Unlike the (2+1)-dimensional case, such rational solutions do not necessarily decay in every spatial direction [26]. There may exist certain non-decaying directions belonging to the common null space of the associated affine-linear forms. A large family of exact lump wave solutions has been constructed for the (2+1)-dimensional KPI

Eq (1.1) with $\sigma^2 = -1$ [11]. In contrast, for the family of generalized $(N+1)$-dimensional KP Eq (1.6) with $N > 3$, genuine fully localized lump solutions cannot be obtained from the quadratic ansatz functions under the transformation (3.1) [26, 29]. In this section, we aim to construct lump-type rational solutions of (1.6) for $N > 3$ by means of the long wave limit approach.

Herein, the wave variables are defined as

$$\xi_i = k_i(\sum_{j=1}^{N+1} p_{ij}x_j + \delta_i) + \xi_i^0, i = 1, 2, \tag{3.7}$$

where k_i and p_{ij} with $1 \le i \le 2, 1 \le j \le N+1$ are undetermined parameters that jointly form the wave vector components $k_i p_{ij}$ along each independent variable direction x_j, δ_i denotes introduced auxiliary constants, and ξ_i^0 represents arbitrary constant shifts. All these parameters satisfy the dispersion relations

$$p_{i(N+1)} = -\frac{k_i^2 p_{i1}^4 + a_{22}p_{i2}^2 + a_{33}p_{i3}^2 + 2a_{23}p_{i2}p_{i3} + \sum_{j=1}^{N} \sigma_j p_{i1}p_{ij}}{\sigma_{N+1}p_{i1}}, \quad i = 1, 2. \tag{3.8}$$

For convenience, we introduce the auxiliary quantities

$$P_j = p_{11}p_{2j} - p_{1j}p_{21}, \quad j = 2, 3. \tag{3.9}$$

This notation will be adopted throughout the rest of the paper. By applying Hirota's bilinear method, the bilinear Eq (3.2) admits the explicit two-soliton solution listed below:

$$f = 1 + e^{\xi_1} + e^{\xi_2} + A_{12}e^{\xi_1+\xi_2}, \tag{3.10}$$

where $\xi_i, i = 1, 2$, are defined by (3.7) with (3.8) and

$$A_{12} = \frac{R_{12}}{Q_{12}},$$

with R_{12} and Q_{12} being given by

$$\begin{aligned} R_{12} &= 3p_{11}^2p_{21}^2(k_1p_{11} - k_2p_{21})^2 - a_{22}P_2^2 - a_{33}P_3^2 - 2a_{23}P_2P_3, \\ Q_{12} &= 3p_{11}^2p_{21}^2(k_1p_{11} + k_2p_{21})^2 - a_{22}P_2^2 - a_{33}P_3^2 - 2a_{23}P_2P_3. \end{aligned}$$

In the two-soliton solution (3.10), setting $e^{\xi_i^0} = -1$ and $k_i \to 0$ with $k_1/k_2 = O(1), p_{i1} = O(1)$ and $p_{i2} = O(1)$ for $i = 1, 2$, we may derive

$$A_{12} = 1 + \frac{12k_1k_2p_{11}^3p_{21}^3}{a_{22}P_2^2 + a_{33}P_3^2 + 2a_{23}P_2P_3} + O(\mathbf{k}^3) \tag{3.11}$$

and

$$f = k_1k_2\Big(\theta_1\theta_2 + \frac{12p_{11}^3p_{21}^3}{a_{22}P_2^2 + a_{33}P_3^2 + 2a_{23}P_2P_3}\Big) + O(\mathbf{k^3}), \tag{3.12}$$

where $\mathbf{k} = \max(k_1, k_2)$ and

$$\theta_i = \sum_{j=1}^{N} p_{ij}x_j - \frac{a_{22}p_{i2}^2 + a_{33}p_{i3}^2 + 2a_{23}p_{i2}p_{i3} + \sum_{j=1}^{N} \sigma_j p_{i1}p_{ij}}{\sigma_{N+1}p_{i1}} x_{N+1} + \delta_i, \quad i = 1, 2. \tag{3.13}$$

Here, we impose non-degeneracy assumptions on the involved parameters to ensure that the denominators of A_{12} and θ_i are nonzero. All parameter combinations yielding zero denominators are excluded. Owing to the logarithmic derivative transformation (3.1), f is equivalent to $\theta_1\theta_2 + B_{12}$, where

$$B_{12} = \frac{12p_{11}^3 p_{21}^3}{a_{22}P_2^2 + a_{33}P_3^2 + 2a_{23}P_2P_3}. \tag{3.14}$$

By means of the dependent variable transformation (3.1), the rational solution corresponding to (1.6) may be expressed as

$$u = \frac{2}{\alpha}(\ln f)_{x_1x_1}, f = \theta_1\theta_2 + B_{12}, \tag{3.15}$$

where θ_1, θ_2, and B_{12} are defined by (3.13) and (3.14), respectively. The function f may vanish for some parameters and then produce singular rational solutions. The zero-set of f can form a high-dimensional algebraic manifold rather than isolated singular points depending on the rank of coefficient vectors and active spatial variables. In the present work, we focus on the nonsingular branch of rational solutions.

To obtain lump-type solutions, a class of nonsingular rational solutions, we select the following parameters:

$$p_{1j} = b_j + c_j\mathrm{I},\ \delta_1 = l + m\mathrm{I},\ p_{2j} = p_{1j}^*,\ \delta_2 = \delta_1^*,\ b_j, c_j, l, m \in \mathbb{R},\ \mathrm{I} = \sqrt{-1},\ 1 \le j \le N, \tag{3.16}$$

which gives rise to $\theta_2 = \theta_1^*$. The asterisk here stands for the complex conjugate. Substituting (3.16) into (3.14) and (3.15) yields

$$f = \Big(\sum_{j=1}^{N} b_jx_j + b_{N+1}x_{N+1} + l\Big)^2 + \Big(\sum_{j=1}^{N} c_jx_j + c_{N+1}x_{N+1} + m\Big)^2 + d, \tag{3.17}$$

where

$$b_{N+1} = -\frac{\sum_{j=2}^{3} a_{jj}[b_1(b_j^2 - c_j^2) + 2c_1b_jc_j]}{\sigma_{N+1}(b_1^2 + c_1^2)} - \frac{2a_{23}[b_1(b_2b_3 - c_2c_3) + c_1(b_2c_3 + b_3c_2)]}{\sigma_{N+1}(b_1^2 + c_1^2)} - \frac{\sum_{j=1}^{N}\sigma_jb_j}{\sigma_{N+1}}, \tag{3.18}$$

$$c_{N+1} = -\frac{\sum_{j=2}^{3} a_{jj}[c_1(c_j^2 - b_j^2) + 2b_1b_jc_j]}{\sigma_{N+1}(b_1^2 + c_1^2)} - \frac{2a_{23}[b_1(b_2c_3 + b_3c_2) + c_1(c_2c_3 - b_2b_3)]}{\sigma_{N+1}(b_1^2 + c_1^2)} - \frac{\sum_{j=1}^{N}\sigma_jc_j}{\sigma_{N+1}}, \tag{3.19}$$

$$d = \frac{-3(b_1^2 + c_1^2)^3}{\sum_{j=2}^{3} a_{jj}(b_jc_1 - b_1c_j)^2 + 2a_{23}(b_2c_1 - b_1c_2)(b_3c_1 - b_1c_3)}. \tag{3.20}$$

Note that the involved real parameters $b_j, c_j, l, m, 1 \le j \le N$, are arbitrary, with the restriction that

$$(b_1^2 + c_1^2)\Big[\sum_{j=2}^{3} a_{jj}(b_jc_1 - b_1c_j)^2 + 2a_{23}(b_2c_1 - b_1c_2)(b_3c_1 - b_1c_3)\Big] \ne 0. \tag{3.21}$$

The solutions given in (3.17) are positive quadratic function solutions precisely when $d > 0$. Accordingly, we impose two fundamental positivity requirements to ensure $d > 0$, namely,

$$(b_1^2 + c_1^2) \ne 0, \tag{3.22}$$

$$\Big[\sum_{j=2}^{3} a_{jj}(b_jc_1 - b_1c_j)^2 + 2a_{23}(b_2c_1 - b_1c_2)(b_3c_1 - b_1c_3)\Big] < 0. \tag{3.23}$$

These two conditions ensure that f defined by (3.17) is positive, whereby the lump-type solutions of (1.6) can be obtained via the transformation (3.1). It is easy to verify that the constraints (3.22) and (3.23) depend on a_{22}, a_{33}, and a_{23} and are independent of $\sigma_j, 1 \leq j \leq N+1$, the coefficients of $D_{x_1}D_{x_j}$.

Observe that the determinant condition

$$(b_2c_1 - b_1c_2)^2 + (b_3c_1 - b_1c_3)^2 = \left(\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\right)^2 + \left(\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}\right)^2 \neq 0, \tag{3.24}$$

implies $(b_1^2 + c_1^2) \neq 0$. Under the condition (3.24), we analyze the associated symmetric 2×2 matrix H given in (3.3). If

$$a_{22} < 0, a_{33} < 0, a_{23}^2 < a_{22}a_{33}, \tag{3.25}$$

then H is negative definite. This ensures the validity of (3.23) and further yields $d > 0$. The requirement guaranteeing lump-type wave solutions under the setting (3.25) simplifies to (3.24). For example, taking

$$\begin{aligned} &N = 4, \alpha = 1, x_1 = x, x_2 = y, x_3 = z, x_4 = \rho, x_5 = t, \\ &a_{22} = a_{33} = -1, a_{23} = 0.5, \sigma_1 = \sigma_2 = \sigma_3 = 0, \sigma_4 = 2, \sigma_5 = 1, \end{aligned} \tag{3.26}$$

in (1.6), we obtain a (3+2)-dimensional generalized KP equation written as

$$(u_{xxx} + 6uu_x)_x - u_{yy} - u_{zz} + u_{yz} + 2u_{x\rho} + u_{xt} = 0, \tag{3.27}$$

which is equivalent to

$$(D_x^4 - D_y^2 - D_z^2 + D_yD_z + 2D_xD_\rho + D_xD_t)f \cdot f = 0, \tag{3.28}$$

under the dependent variable conversion

$$u = 2(\ln f)_{xx}. \tag{3.29}$$

A second time-like variable ρ is involved in this equation. This parameter acts as an auxiliary mathematical evolution parameter rather than a genuine physical time or spatial coordinate. It is introduced to describe the dependence of the underlying dynamical process on the intrinsic properties of the system [30]. Utilizing the derived results (3.17)–(3.20), we have

$$f = (b_1x + b_2y + b_3z + b_4\rho + b_5t + l)^2 + (c_1x + c_2y + c_3z + c_4\rho + c_5t + m)^2 + d, \tag{3.30}$$

where

$$\begin{aligned} b_5 = &\frac{[b_1(b_2^2 - c_2^2) + 2c_1b_2c_2] + [b_1(b_3^2 - c_3^2) + 2c_1b_3c_3]}{(b_1^2 + c_1^2)} \\ &- \frac{b_1(b_2b_3 - c_2c_3) + c_1(b_2c_3 + b_3c_2) + 2b_4(b_1^2 + c_1^2)}{(b_1^2 + c_1^2)}, \end{aligned} \tag{3.31}$$

$$c_5 = \frac{[c_1(c_2^2 - b_2^2) + 2b_1b_2c_2] + [c_1(c_3^2 - b_3^2) + 2b_1b_3c_3]}{(b_1^2 + c_1^2)}$$
$$-\frac{b_1(b_2c_3 + b_3c_2) + c_1(c_2c_3 - b_2b_3) + 2c_4(b_1^2 + c_1^2)}{(b_1^2 + c_1^2)}, \tag{3.32}$$

$$d = \frac{3(b_1^2 + c_1^2)^3}{(b_2c_1 - b_1c_2)^2 + (b_3c_1 - b_1c_3)^2 - (b_2c_1 - b_1c_2)(b_3c_1 - b_1c_3)}. \tag{3.33}$$

The determinant condition (3.24) guarantees the existence of lump-type solutions to Eq (3.27).

Moreover, taking the following specific values for the free parameters:

$$b_1 = 1,\ b_2 = 1,\ b_3 = -1,\ b_4 = 2,\ c_1 = 1,\ c_2 = -3,\ c_3 = 1,\ c_4 = 2,\ l = 4,\ m = 3, \tag{3.34}$$

the corresponding function f given by (3.17) is expressed as

$$f = (x + y - z + 2\rho - 15t + 4)^2 + (x - 3y + z + 2\rho - 5t + 3)^2 + \frac{6}{7}, \tag{3.35}$$

which yields a lump-type solution of the reduced case (3.27)

$$u = \frac{2ff_{xx} - 2(f_x)^2}{f^2} = \frac{8f - 2(4x - 4y + 8\rho - 40t + 14)^2}{f^2}, \tag{3.36}$$

by means of the logarithmic transformation (3.29). Figures 1–3 show the three-dimensional graphics, density plots, and curves of the lump-type solution (3.36) corresponding to (3.35).

Figures 1a, 1b, 2a, and 2b depict the temporal propagation of the lump-type wave (3.36) in the (x, y)-plane with fixed parameters

$$z = 0 \quad \text{and} \quad \rho = 0.$$

The waveform features a sharp and narrow positive peak surrounded by symmetric negative troughs, decaying algebraically to zero in nearly all spatial directions. Analytical calculations verify that the solution achieves a local maximum amplitude

$$u_{\max} = \frac{28}{3}$$

at the peak position

$$(x, y) = (\frac{50t - 15}{4}, \frac{10t - 1}{4}).$$

For fixed z, ρ, and t, the solution decays to zero as $(x, y) \to \infty$, presenting spatial localization characteristics. Nevertheless, the solution derived from Eq (3.36) cannot decay to zero on the degenerate surface defined by

$$z = 2y - 5t + \frac{1}{2}.$$

Figures 1c and 2c are visualized on this specific degenerate surface, intuitively demonstrating that the solution maintains a non-vanishing value along the infinite ridge-shaped direction. Consequently, the constructed rational solution cannot be regarded as a strict lump solution in the high-dimensional system. Figure 3a shows x-direction profiles for varying y. Curves exhibit a positive peak and two symmetric negative troughs with declining amplitude. Figure 3b presents y-direction profiles for varying x. The peak shifts rightward and weakens, confirming lump localization in the (x, y)-plane. Figure 3c plots profiles at

$$t = 0, \quad z = 2y + \frac{1}{2}$$

for varying ρ. Sharp peaks shift toward positive y, with amplitude rising near the degenerate ridge.

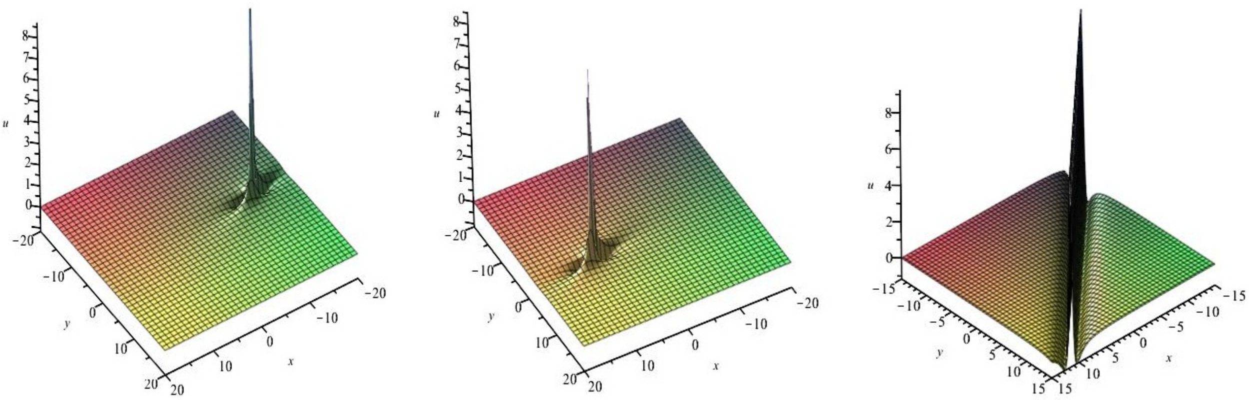

Figure 1. Three-dimensional plots of the solution u governed by (3.36) with fixed $\rho = 0$: (a) $z = 0, t = -0.7$, (b) $z = 0, t = 1$, (c) $z = 2y + \frac{1}{2}, t = 0$.

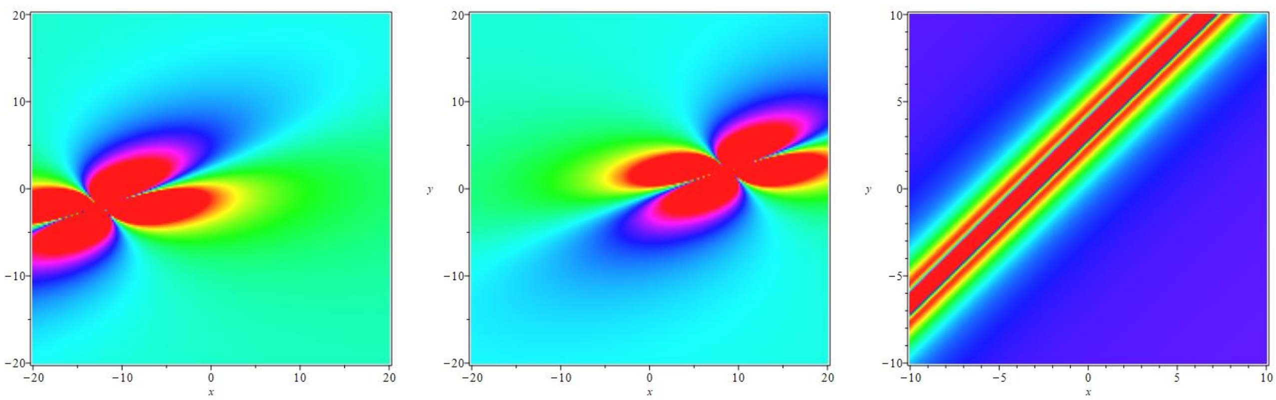

Figure 2. Density plots of the solution u governed by (3.36) with fixed $\rho = 0$: (a) $z = 0, t = -0.7$, (b) $z = 0, t = 1$, (c) $z = 2y + \frac{1}{2}, t = 0$.

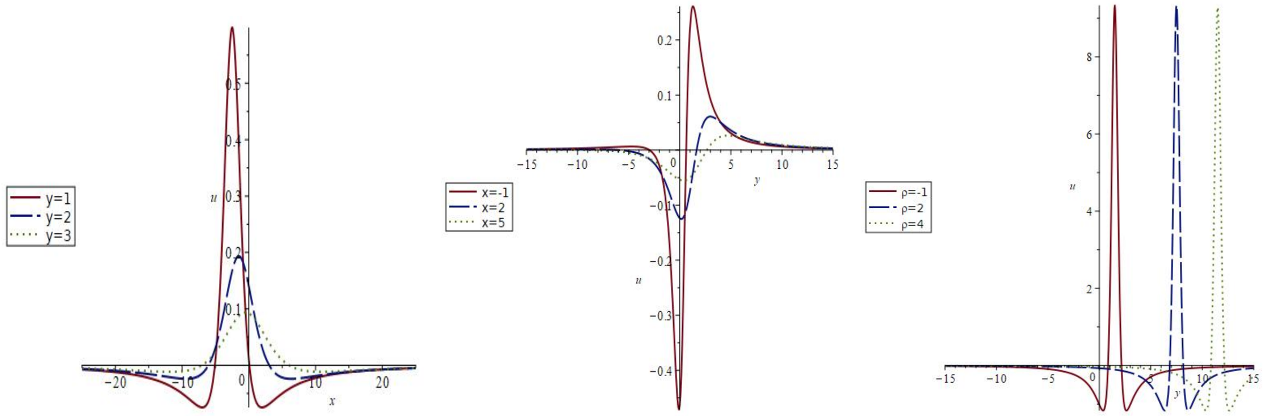

Figure 3. Curves of the solution u governed by (3.36): (a) x curves with $z = \rho = t = 0$, (b) y curves with $z = \rho = t = 0$, (c) y curves with $x = t = 0, z = 2y + \frac{1}{2}$.

If

$$a_{22} > 0, a_{33} > 0, a_{23}^2 < a_{22}a_{33}, \tag{3.37}$$

the symmetric matrix H defined by (3.3) becomes positive definite. In this case, not every equation in (1.6) admits lump-type solutions but instead possesses singular rational solutions through the transformation (3.1). The singularity exhibited by the resulting solutions is attributed to the negative value of the constant term d.

If $a_{22} = a_{33} = 0, a_{23} \neq 0$, the condition (3.23) becomes

$$a_{23}(b_2c_1 - b_1c_2)(b_3c_1 - b_1c_3) = a_{23}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} < 0, \tag{3.38}$$

ensuring that f in (3.17) is positive. The above case can be viewed as an extended version of the (4+1)-dimensional bilinear Fokas equation [31, 32].

If

$$a_{22} < 0, a_{33} < 0, a_{23}^2 = a_{22}a_{33}, \tag{3.39}$$

the symmetric matrix H is rank-deficient, which corresponds to a degenerate quadratic form case. Under this degeneracy condition, the parameter d in (3.20) reduces to

$$d = \frac{3(b_1^2 + c_1^2)^3}{[\sqrt{-a_{22}}(b_2c_1 - b_1c_2) \pm \sqrt{-a_{33}}(b_3c_1 - b_1c_3)]^2}. \tag{3.40}$$

We introduce a fundamental constraint

$$\sqrt{-a_{22}}(b_2c_1 - b_1c_2) \pm \sqrt{-a_{33}}(b_3c_1 - b_1c_3) \neq 0, \tag{3.41}$$

from which we derive a family of positive quadratic function solutions satisfying the bilinear relation (3.2).

4. Conclusions

In the present study, we conducted a systematic investigation into the integrable properties and exact solutions of two types of higher-dimensional generalized KP equations. For the family of generalized KP and Boussinesq Eqs (1.5), we first derived the three-soliton condition based on existing studies and confirmed that the equations satisfy this condition when their symmetric coefficient matrices have rank 0 or 1. We then constructed a standard multi-parameter bilinear Bäcklund transformation for these equations, from which their spectral-parameter Lax pairs and modified equations have been obtained. Noticeably, such (3+1)-dimensional integrable systems are not new, since they can be degenerated into the classic KP Eq (1.1), (1+1)-dimensional Boussinesq equation, and their dimensionally-reduced counterparts via appropriate simplifications of coefficients and invertible linear transformations of independent variables [12].

For the family of generalized KP Eqs (1.6) in $(N + 1)$-dimensions, we first investigated the bilinear Bäcklund transformation and Lax pair through expanding the (3+1)-dimensional analytical results. Then, we applied the long wave limit approach on two-soliton solutions and successfully derived lump-type rational waves. The existence of these lump-type solutions depends on the positivity of the quadratic function f, which is determined by specific constraints on the coefficients and parameters involved in the solutions. For example, when the coefficients satisfy $a_{22} < 0, a_{33} < 0$, and $a_{23}^2 < a_{22}a_{33}$, the determinant condition (3.24) guarantees the existence of lump-type waves.

In summary, the present work complements and improves the investigations of higher-dimensional integrable systems and provides a feasible technique to derive analytical solutions for such higher-dimensional nonlinear evolution equations. The obtained lump-type solutions supplement the theoretical results of higher-dimensional nonlinear equations and may provide a basis for subsequent physical investigations.

For future work, multiple lump-type wave structures will be generated from multi-soliton solutions via the long wave limit and their asymptotic behaviors will be analyzed. We will extend the exact solution construction method proposed in this paper to more complex higher-dimensional generalized KP systems, including those with variable coefficients [33] or nonlocal terms. Furthermore, extended KP-like equations will be formulated using generalized bilinear derivatives based on the prime number three [34]. As another promising research direction, our investigations on soliton solutions may be further extended to stochastic wave models considering Brownian motion and white noise effects, as reported in [35].

Author contributions

Li Cheng: writing-original draft, writing-review & editing, visualization; Wen-Xiu Ma: supervision, methodology; Xiao-Fei Hu: funding acquisition, software. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare they have no conflicts of interest.

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