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Research article

Three-variable eighth-kind Chebyshev polynomials for two-dimensional multiterm time-fractional diffusion–wave equations

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Abstract: Three-variable eighth-kind Chebyshev polynomials are employed as basis functions to construct a pseudo-operational collocation method. The proposed approach is applied to solve two-dimensional multiterm time-fractional diffusion–wave equations (three variables corresponding to two spatial variables and one temporal variable). The existence and uniqueness of the solution to the considered problem are established using Krasnoselskii's fixed-point theorem. An upper bound for the residual function is derived in a suitable weighted space, showing that the use of a finite number of basis functions provides an accurate approximate solution. Several illustrative examples are presented to demonstrate the applicability, accuracy, and computational efficiency of the proposed scheme for high-dimensional partial differential equations.

Keywords: two-dimensional time-fractional diffusion–wave equation; Chebyshev polynomials of the eighth kind; Caputo derivative; pseudo-operational matrix
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1. Introduction

Fractional partial differential equations (FPDEs) have gained considerable attention as powerful mathematical tools for modeling complex phenomena characterized by memory and nonlocal effects. In particular, time-fractional diffusion and diffusion–wave equations arise in a wide range of applications, including anomalous transport processes, viscoelastic materials, heat conduction, and biological systems [1–4]. Compared with classical integer-order models, fractional formulations provide greater flexibility in capturing intricate physical behaviors. However, the inclusion of fractional derivatives substantially increases the analytical and computational complexity of the governing equations, especially for problems involving multiterm operators and higher-dimensional domains.

Over the past decade, a wide range of numerical techniques has been developed to solve time-fractional diffusion and diffusion–wave equations, including finite difference methods [5–8], finite element methods [9–12], spectral methods, and collocation-based approaches [13–16]. Among these approaches, spectral and pseudospectral methods have received significant attention due to their high-order accuracy and rapid convergence for smooth solutions. Such methods commonly employ orthogonal polynomial bases, such as Jacobi, Legendre, Chebyshev, and Gegenbauer polynomials, to convert the underlying FPDEs into systems of algebraic equations. For example, Sadri et al. employed a combination of classical and shifted Jacobi polynomials to solve two-dimensional time-fractional diffusion–wave equations [15]. Yaghoubi et al. developed a pseudo-operational collocation method based on Gegenbauer polynomials for a class of variable-order time-fractional integro-partial differential equations with weakly singular kernels [17]. In another study, Sadri et al. combined a pseudo-operational collocation approach with the Lagrange multiplier method and sixth-kind Chebyshev polynomials to model a fractional-order glaucoma disease problem [18]. The authors of [19] applied seventh-kind Chebyshev polynomials together with a corrected collocation scheme to variable-order reaction-diffusion and advection–dispersion equations. Moreover, in [20], a matrix collocation method based on shifted fifth-kind Chebyshev polynomials was proposed for solving Volterra integro-partial differential equations with weakly singular kernels. Finally, Ahmed and Hashem introduced a novel spectral tau method using shifted Gegenbauer polynomials for the numerical solution of multidimensional variable-order time-fractional partial differential equations (PDEs) [21]. In a different work, Pleszczyński proposed a modified differential transformation method in [22] to construct a solution using a power series expansion and a delayed term approximation. This method allows one to determine the coefficients in the series solution even in the absence of an explicit analytical solution. Obeidat et al. used the fractional Adomian J-transform method to solve one-dimensional diffusion–wave equations [23].

Chebyshev polynomials are particularly attractive for spectral approximations due to their favorable numerical properties and efficient implementation. Various kinds of Chebyshev polynomials have been introduced and employed to construct operational matrices for solving fractional-order functional equations. For instance, Heydari et al. applied a Hadamard fractional operational approach based on Chebyshev polynomials to solve the fractional-order Ginzburg–Landau equation [24]. In [25], second-kind Chebyshev polynomials were utilized to solve piecewise fractional nonlinear reaction-diffusion equations. The authors of [26] employed third-kind Chebyshev polynomials and corresponding operational matrices to address multiterm variable-order differential equations, and Youssri et al. applied the same class of polynomials to solve nonlinear fractional-order pantograph differential equations [27]. Dincel and Polat developed a fourth-kind Chebyshev wavelet method for multiterm variable-order differential equations [28]. Atta et al. adopted a fifth-kind Chebyshev Galerkin method to solve hyperbolic PDEs [29]. Sadri and Aminikhah proposed an operational collocation method based on sixth-kind Chebyshev polynomials for fractional mobile-immobile advection–dispersion equations [30], and a corrected operational collocation scheme was later employed for variable-order reaction-diffusion and advection–dispersion equations [15]. More recently, Abd–Elhameed et al. introduced a novel spectral collocation approach based on eighth-kind Chebyshev polynomials (EKCPs) to solve the nonlinear time-fractional generalized Kawahara equation [31]. Existing spectral and collocation methods have demonstrated high accuracy for fractional PDEs.

However, most studies employ first- or second-kind Chebyshev polynomials and are primarily restricted to one-dimensional settings. Moreover, relatively little attention has been devoted to higher-kind Chebyshev polynomials in multidimensional space-time approximations. These limitations motivate the development of the present three-variable eighth-kind Chebyshev framework. Despite these advances, most existing studies are restricted to classical Chebyshev polynomials of the first and second kinds and primarily address one-dimensional or single-term fractional problems. The extension of Chebyshev-based pseudo-operational methods to higher-kind polynomials, particularly for multidimensional and multiterm time-fractional diffusion and diffusion–wave equations, remains limited. To the best of the authors' knowledge, the application of EKCPs within a pseudo-operational collocation framework for two-dimensional multiterm time-fractional diffusion and diffusion–wave equations has not yet been reported. Accordingly, in this study, a class of two-dimensional multiterm time-fractional diffusion–wave equations (2DMTTFDWEs) is considered in the following form:

$$\begin{aligned}\sum_{k=1}^{m} \alpha_k\, {}_0^c\mathcal{D}_\tau^{\mu_k}\mathcal{U}(\chi,\zeta,\tau) + \sum_{k=1}^{n} \beta_k\, {}_0^c\mathcal{D}_\tau^{\nu_k}\mathcal{U}(\chi,\zeta,\tau) + \eta_1 \frac{\partial \mathcal{U}(\chi,\zeta,\tau)}{\partial \tau} + \eta_2\, \mathcal{U}(\chi,\zeta,\tau) \\ - \eta_3 \frac{\partial^2 \mathcal{U}(\chi,\zeta,\tau)}{\partial \chi^2} - \eta_4 \frac{\partial^2 \mathcal{U}(\chi,\zeta,\tau)}{\partial \zeta^2} - \gamma_1\, {}_0^c\mathcal{D}_\tau^{\iota} \frac{\partial^2 \mathcal{U}(\chi,\zeta,\tau)}{\partial \chi^2} \\ - \gamma_2\, {}_0^c\mathcal{D}_\tau^{\iota} \frac{\partial^2 \mathcal{U}(\chi,\zeta,\tau)}{\partial \zeta^2} = \mathsf{f}(\chi,\zeta,\tau),\end{aligned} \tag{1.1}$$

where $(\chi,\zeta,\tau) \in \Delta = [0,1]\times[0,1]\times[0,1]$, ${}_0^c\mathcal{D}_\tau^{\mu_k}(.), {}_0^c\mathcal{D}_\tau^{\nu_k}(.)$, and ${}_0^c\mathcal{D}_\tau^{\iota}(.)$ are the Caputo derivative operators of the orders $\mu_k \in (1,2)$, $\nu_k, \iota \in (0,1)$; $\mathsf{f} : \Delta \to \mathbb{R}$ is a continuous function; $\mathcal{U}(\chi,\zeta,\tau) \in H^2(\Delta)$ or $C^2(\Delta)$ is an unknown function such that ${}_0^c\mathcal{D}_\tau^{\theta}\mathcal{U} \in L^2([0,1], L^2(\Delta))$. As can be seen, the governing equation is two-dimensional in space; the proposed approximation is constructed in the space-time domain (χ,ζ,τ). Consequently, three-variable eighth-kind Chebyshev polynomials (TVEKCPs) are employed as basis functions, where the three variables correspond to the two spatial coordinates χ and ζ and the temporal coordinate τ.

Equation (1.1) arises in numerous scientific and engineering applications. In engineering, it is used to model anomalous diffusion, viscoelastic materials, and heat-transfer processes with memory effects [1, 32]. In physics, it describes wave propagation in heterogeneous and complex media [32]. In economics and finance, it has been employed to model systems exhibiting memory and hereditary characteristics [33]. In data science and machine learning, fractional models have recently attracted attention for describing nonlocal dynamics and long-range dependence in complex datasets [34]. Therefore, efficient numerical schemes, such as the proposed method, may serve as useful computational tools for these application areas. Motivated by these observations, the present work develops a novel pseudo-operational collocation method based on TVEKCPs for the numerical solution of Problem (1.1). The proposed approach extends existing Chebyshev-based spectral techniques by employing higher-kind polynomials as basis functions and formulating a multi-variable collocation strategy. By exploiting the approximation properties of EKCPs, the method transforms the governing FPDEs into a system of algebraic equations, leading to an efficient and accurate numerical scheme suitable for high-dimensional problems. pseudo-operational matrices for both integer- and fractional-order integrations are derived using the Kronecker product of the corresponding one-dimensional operational matrices. In addition to the numerical formulation, the theoretical properties of the proposed method are rigorously analyzed. The existence and uniqueness of the solution are established via Krasnoselskii's fixed-point theorem [35]. Moreover, an upper bound for the residual function is obtained in an appropriate weighted space, providing a convergence justification and demonstrating that truncating the basis to a finite number of terms yields an approximation with satisfactory accuracy.

The objectives of the present work are summarized as follows:

- to construct a three-variable spectral approximation that simultaneously treats the spatial variables and time;
- to introduce, to the best of the authors' knowledge, the first pseudo-operational collocation method based on eighth-kind Chebyshev polynomials for solving two-dimensional time-fractional diffusion–wave equations;
- to establish the existence and uniqueness of the solution using Krasnoselskii's fixed-point theorem, a theoretical tool that is rarely incorporated in spectral collocation studies of FPDEs;
- to derive a new upper bound for the residual function, providing theoretical justification for truncation and convergence of the numerical approximation.

The remainder of this paper is organized as follows. Section 2 presents the necessary preliminaries on fractional calculus and EKCPs. Section 3 investigates the existence and uniqueness of the solution of Problem (1.1). Section 4 introduces the shifted EKCPs, their associated matrix relations, and the construction of the TVEKCPs. Section 5 describes the development of the pseudo-operational collocation method. Numerical examples and related discussions are presented in Section 6. Section 7 provides five numerical examples to further demonstrate the applicability and effectiveness of the proposed approach. Finally, Section 8 concludes the paper.

2. Preliminaries and fractional operators

This paper employs the definitions and fundamental properties of fractional calculus, presented in the following section, to formulate and rigorously analyze the proposed numerical method.

Definition 2.1. The Riemann–Liouville fractional integral operator of variable order μ, where $n-1 < \mu < n, n \in \mathbb{N} \cup \{0\}$ for a given function $\mathsf{u}(\chi,\zeta,\tau)$, is defined as follows [36]:

$$ {}_{0}^{RL}\mathcal{I}_{\tau}^{\mu}\mathsf{u}(\chi,\zeta,\tau) = \frac{1}{\Gamma(\mu)}\int_{0}^{\tau}(\tau-t)^{\mu-1}\mathsf{u}(\chi,\zeta,t)dt, \quad (\chi,\zeta,\tau)\in\Delta, \tag{2.1}$$

where $\Gamma(.)$ is the well-known Gamma function, and ${}_{0}^{RL}\mathcal{I}_{\tau}^{0}\mathsf{u}(\chi,\zeta,\tau) = \mathsf{u}(\chi,\zeta,\tau)$.

Definition 2.2. The Caputo fractional derivative of order μ, where $n-1 < \mu < n, n \in \mathbb{N} \cup \{0\}$ for a continuously differentiable function $\mathsf{u}(\chi,\zeta,\tau)$, is defined as follows [36]:

$$ {}_{0}^{c}\mathcal{D}_{\tau}^{\mu}\mathsf{u}(\chi,\zeta,\tau) = \frac{1}{\Gamma(n-\mu)}\int_{0}^{\tau}(\tau-t)^{1-\mu}\frac{\partial^{n}\mathsf{u}(\chi,\zeta,t)}{\partial t^{n}}dt, \quad (\chi,\zeta,\tau)\in\Delta, \tag{2.2}$$

and ${}_{0}^{c}\mathcal{D}_{\tau}^{n}\mathsf{u}(\chi,\zeta,\tau) = \partial^{n}\mathsf{u}(\chi,\zeta,\tau)/\partial\tau^{n}$ for $n \in \mathbb{N} \cup \{0\}$.

According to Definitions 2.1 and 2.2, the Riemann–Liouville integral and the Caputo derivative operators satisfy the following properties:

(1) ${}_{0}^{c}\mathcal{D}_{\tau}^{\mu}\mathsf{u}(\chi,\zeta,\tau) = {}_{0}^{RL}\mathcal{I}_{\tau}^{n-\mu}\Big(\frac{\partial^{n}\mathsf{u}(\chi,\zeta,\tau)}{\partial\tau^{n}}\Big)$,

(2) ${}_{0}^{RL}\mathcal{I}_{\tau}^{\mu}({}_{0}^{c}\mathcal{D}_{\tau}^{\mu}\mathsf{u}(\chi,\zeta,\tau)) = \mathsf{u}(\chi,\zeta,\tau) - \sum_{k=0}^{n-1}\frac{\partial^{k}\mathsf{u}(\chi,\zeta,0)}{\partial\tau^{k}}\tau^{k}, \quad n-1<\mu\leq n,\ n\in\mathbb{N}$,

(3) ${}_{0}^{RL}\mathcal{I}_{\tau}^{\mu}\tau^{\nu} = \frac{\Gamma(\nu+1)\,\tau^{\nu+\mu}}{\Gamma(\nu+\mu+1)}, \quad \nu>0$,

(4) ${}_{0}^{c}\mathcal{D}_{\tau}^{\mu}\tau^{\nu} = \frac{\Gamma(\nu+1)\,\tau^{\nu-\mu}}{\Gamma(\nu-\mu+1)}, \quad \nu\geq\lceil\mu\rceil$.

Definition 2.3. Let $\mathbf{F} = [f_{ij}]_{m\times n}$ and $\mathbf{G} = [g_{ij}]_{p\times q}$ be two matrices. The Kronecker product of $\mathbf{F}$ and $\mathbf{G}$, denoted by $\mathbf{F}\otimes\mathbf{G}$, is an $(mp)\times(nq)$ block matrix defined as follows [37]:

$$\mathbf{F}\otimes\mathbf{G} = \begin{bmatrix} f_{11}\,\mathcal{G} & f_{12}\,\mathcal{G} & \dots & f_{1n}\,\mathcal{G} \\ f_{21}\,\mathcal{G} & f_{22}\,\mathcal{G} & \dots & f_{2n}\,\mathcal{G} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1}\,\mathcal{G} & f_{m2}\,\mathcal{G} & \dots & f_{mn}\,\mathcal{G} \end{bmatrix}.$$

3. Existence and uniqueness of solutions

This section investigates the existence and uniqueness of solutions to Problem (1.1) by applying Krasnoselskii's fixed-point theorem [35]. First, rewrite the problem in an equivalent form under the assumption that $\alpha_1 \neq 0, m = n = 2$:

$$\begin{aligned} &\alpha_1\,{}_{0}^{c}\mathcal{D}_{\tau}^{\mu_1}\mathcal{U}(\chi,\zeta,\tau) + \alpha_2\,{}_{0}^{c}\mathcal{D}_{\tau}^{\mu_2}\mathcal{U}(\chi,\zeta,\tau) + \beta_1\,{}_{0}^{c}\mathcal{D}_{\tau}^{\nu_1}\mathcal{U}(\chi,\zeta,\tau) + \beta_2\,{}_{0}^{c}\mathcal{D}_{\tau}^{\nu_2}\mathcal{U}(\chi,\zeta,\tau) + \eta_1\,\frac{\partial\mathcal{U}(\chi,\zeta,\tau)}{\partial\tau} \\ &\quad + \eta_2\,\mathcal{U}(\chi,\zeta,\tau) - \eta_3\,\frac{\partial^2\mathcal{U}(\chi,\zeta,\tau)}{\partial\chi^2} - \eta_4\,\frac{\partial^2\mathcal{U}(\chi,\zeta,\tau)}{\partial\zeta^2} - \gamma_1\,{}_{0}^{c}\mathcal{D}_{\tau}^{\iota}\frac{\partial^2\mathcal{U}(\chi,\zeta,\tau)}{\partial\chi^2} \\ &\quad - \gamma_2\,{}_{0}^{c}\mathcal{D}_{\tau}^{\iota}\frac{\partial^2\mathcal{U}(\chi,\zeta,\tau)}{\partial\zeta^2} = \mathsf{f}(\chi,\zeta,\tau), \end{aligned} \tag{3.1}$$

with the following initial and boundary conditions:

$$\begin{aligned} &\mathcal{U}(\chi,\zeta,0) = \mathsf{g}_1(\chi,\zeta), \quad \frac{\partial\mathcal{U}(\chi,\zeta,0)}{\partial\tau} = \mathsf{g}_2(\chi,\zeta), \quad \mathcal{U}(0,\zeta,\tau) = \mathsf{h}_1(\zeta,\tau), \\ &\mathcal{U}(1,\zeta,\tau) = \mathsf{h}_2(\zeta,\tau), \quad \mathcal{U}(\chi,0,\tau) = w_1(\chi,\tau), \quad \mathcal{U}(\chi,1,\tau) = w_2(\chi,\tau), \end{aligned} \tag{3.2}$$

where $\mathsf{g}_k, \mathsf{h}_k$, and $w_k, k = 1,2$ are continuous known functions. Suppose that $(C^2(\Delta), \|.\|)$ is a Banach space where $\|\mathsf{w}\| = \max_{(\chi,\zeta,\tau)\in\Delta}|\mathsf{w}(\chi,\zeta,\tau)|$. Applying the Riemann–Liouville integral operator of the fractional order $\mu_1 > \mu_2, \mu_k \in (1,2), k = 1,2$ to Eq (3.1) leads to the following equation:

$$\begin{aligned} \mathcal{U}(\chi,\zeta,\tau) = {}& \mathcal{U}(\chi,\zeta,0) + \frac{\partial\mathcal{U}(\chi,\zeta,0)}{\partial\tau}\tau + \frac{1}{\alpha_1\,\Gamma(\mu_1)}\int_0^{\tau}(\tau-t)^{\mu_1-1}\mathsf{f}(\chi,\zeta,t)\,dt \\ & - \frac{\alpha_2}{\alpha_1\,\Gamma(\mu_1-\mu_2)}\int_0^{\tau}(\tau-t)^{\mu_1-\mu_2-1}\mathcal{U}(\chi,\zeta,t)\,dt + \frac{\alpha_2\,\mathcal{U}(\chi,\zeta,0)\,\tau^{\mu_1-\mu_2}}{\alpha_1\,\Gamma(\mu_1-\mu_2+1)} \\ & - \frac{\beta_1}{\alpha_1\,\Gamma(\mu_1-\nu_1)}\int_0^{\tau}(\tau-t)^{\mu_1-\nu_1-1}\,\mathcal{U}(\chi,\zeta,t)\,dt + \frac{\beta_1\,\mathcal{U}(\chi,\zeta,0)\,\tau^{\mu_1-\nu_1}}{\alpha_1\,\Gamma(\mu_1-\nu_1+1)} \\ & - \frac{\beta_2}{\alpha_1\,\Gamma(\mu_1-\nu_2)}\int_0^{\tau}(\tau-t)^{\mu_1-\nu_2-1}\,\mathcal{U}(\chi,\zeta,t)\,dt + \frac{\beta_2\,\mathcal{U}(\chi,\zeta,0)\,\tau^{\mu_1-\nu_2}}{\alpha_1\,\Gamma(\mu_1-\nu_2+1)} \end{aligned}$$

$$
\begin{aligned}
&- \frac{\eta_1}{\alpha_1\,\Gamma(\mu_1-1)} \int_0^\tau (\tau-t)^{\mu_1-2}\,\mathcal{U}(\chi,\zeta,t)\,dt + \frac{\eta_1\,\mathcal{U}(\chi,\zeta,0)\,\tau^{\mu_1-1}}{\alpha_1\,\Gamma(\mu_1)} \\
&- \frac{\eta_2}{\alpha_1\,\Gamma(\mu_1)} \int_0^\tau (\tau-t)^{\mu_1-1}\,\mathcal{U}(\chi,\zeta,t)\,dt + \frac{\eta_3}{\alpha_1\,\Gamma(\mu_1)} \int_0^\tau (\tau-t)^{\mu_1-1} \frac{\partial^2\mathcal{U}(\chi,\zeta,t)}{\partial\chi^2} dt \\
&+ \frac{\eta_4}{\alpha_1\,\Gamma(\mu_1)} \int_0^\tau (\tau-t)^{\mu_1-1} \frac{\partial^2\mathcal{U}(\chi,\zeta,t)}{\partial\zeta^2} dt + \frac{\gamma_1}{\alpha_1\,\Gamma(\mu_1-\iota)} \int_0^\tau (\tau-t)^{\mu_1-\iota-1} \frac{\partial^2\mathcal{U}(\chi,\zeta,t)}{\partial\chi^2} dt \\
&- \frac{\gamma_1\,\tau^{\mu_1-\iota}}{\alpha_1\,\Gamma(\mu_1-\iota+1)} \frac{\partial^2\mathcal{U}(\chi,\zeta,0)}{\partial\chi^2} + \frac{\gamma_2}{\alpha_1\,\Gamma(\mu_1-\iota)} \int_0^\tau (\tau-t)^{\mu_1-\iota-1} \frac{\partial^2\mathcal{U}(\chi,\zeta,t)}{\partial\zeta^2} dt \\
&- \frac{\gamma_2\,\tau^{\mu_1-\iota}}{\alpha_1\,\Gamma(\mu_1-\iota+1)} \frac{\partial^2\mathcal{U}(\chi,\zeta,0)}{\partial\zeta^2}.
\end{aligned} \tag{3.3}
$$

Set the right-hand side of (3.3) as $\mathcal{F}\mathcal{U}(\chi,\zeta,\tau)$. It must be shown that the operator $\mathcal{F}$ is a bounded, continuous, compact, and contraction operator.

Step 1. Define $\mathcal{Y} = \{\mathcal{U} \,|\, \|\mathcal{U}\|, \|\partial^2\mathcal{U}/\partial\chi^2\|, \|\partial^2\mathcal{U}/\partial\zeta^2\| \le r\}$. Then, one has

$$
\begin{aligned}
\|\mathcal{F}\mathcal{U}\| \le &\Big[\|\mathcal{U}(\chi,\zeta,0)\| + \Big\|\frac{\partial\mathcal{U}(\chi,\zeta,0)}{\partial\tau}\Big\| + \frac{\|\mathsf{f}\|}{\alpha_1\,\Gamma(\mu_1+1)}\tau^{\mu_1} + \frac{\alpha_2\,\|\mathcal{U}(\chi,\zeta,0)\|}{\Gamma(\mu_1-\mu_2+1)} + \frac{\beta_1\,\|\mathcal{U}(\chi,\zeta,0)\|}{\alpha_1\,\Gamma(\mu_1-\nu_1+1)} \\
&+ \frac{\beta_2\,\|\mathcal{U}(\chi,\zeta,0)\|}{\alpha_1\,\Gamma(\mu_1-\nu_2+1)} + \frac{\eta_1\,\|\mathcal{U}(\chi,\zeta,0)\|}{\alpha_1\,\Gamma(\mu_1)} + \frac{\gamma_1}{\alpha_1\,\Gamma(\mu_1-\iota+1)}\Big\|\frac{\partial^2\mathcal{U}(\chi,\zeta,0)}{\partial\chi^2}\Big\| \\
&+ \frac{\gamma_2}{\alpha_1\,\Gamma(\mu_1-\iota+1)}\Big\|\frac{\partial^2\mathcal{U}(\chi,\zeta,0)}{\partial\zeta^2}\Big\|\Big] + \Big[\frac{\alpha_2\,\|\mathcal{U}\|\,\tau^{\mu_1-\mu_2}}{\alpha_1\,\Gamma(\mu_1-\mu_2+1)} + \frac{\beta_1\,\|\mathcal{U}\|\,\tau^{\mu_1-\nu_1}}{\alpha_1\,\Gamma(\mu_1-\nu_1+1)} \\
&+ \frac{\beta_2\,\|\mathcal{U}\|\,\tau^{\mu_1-\nu_2}}{\alpha_1\,\Gamma(\mu_1-\nu_2+1)} + \frac{\eta_1\,\|\mathcal{U}\|\,\tau^{\mu_1-1}}{\alpha_1\,\Gamma(\mu_1)} + \frac{\eta_2\,\|\mathcal{U}\|\,\tau^{\mu_1}}{\alpha_1\,\Gamma(\mu_1+1)} + \frac{\eta_3\,\tau^{\mu_1}}{\alpha_1\,\Gamma(\mu_1+1)}\Big\|\frac{\partial^2\mathcal{U}}{\partial\chi^2}\Big\| \\
&+ \frac{\eta_4\,\tau^{\mu_1}}{\alpha_1\,\Gamma(\mu_1+1)}\Big\|\frac{\partial^2\mathcal{U}}{\partial\zeta^2}\Big\| + \frac{\gamma_1\,\tau^{\mu_1-\iota}}{\alpha_1\,\Gamma(\mu_1-\iota+1)}\Big\|\frac{\partial^2\mathcal{U}}{\partial\chi^2}\Big\| + \frac{\gamma_2\,\tau^{\mu_1-\iota}}{\alpha_1\,\Gamma(\mu_1-\iota+1)}\Big\|\frac{\partial^2\mathcal{U}}{\partial\zeta^2}\Big\|\Big]
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
&\le \Big[\frac{\|\mathsf{f}\|}{\alpha_1\,\Gamma(\mu_1+1)} + \|\mathcal{U}(\chi,\zeta,0)\|\Big(1 + \frac{\alpha_2}{\alpha_1\,\Gamma(\mu_1-\mu_2+1)} + \frac{\beta_1}{\alpha_1\,\Gamma(\mu_1-\nu_1+1)} \\
&\quad + \frac{\beta_2}{\alpha_1\,\Gamma(\mu_1-\nu_2+1)} + \frac{\eta_1}{\alpha_1\,\Gamma(\mu_1)}\Big) + \Big\|\frac{\partial\mathcal{U}(\chi,\zeta,0)}{\partial\tau}\Big\| + \frac{\gamma_1}{\alpha_1\,\Gamma(\mu_1-\iota+1)}\Big\|\frac{\partial^2\mathcal{U}(\chi,\zeta,0)}{\partial\chi^2}\Big\| \\
&\quad + \frac{\gamma_2}{\alpha_1\,\Gamma(\mu_1-\iota+1)}\Big\|\frac{\partial^2\mathcal{U}(\chi,\zeta,0)}{\partial\zeta^2}\Big\|\Big] + r\Big[\frac{\alpha_2}{\alpha_1\,\Gamma(\mu_1-\mu_2+1)} + \frac{\beta_1}{\alpha_1\,\Gamma(\mu_1-\nu_1+1)} \\
&\quad + \frac{\beta_2}{\alpha_1\,\Gamma(\mu_1-\nu_2+1)} + \frac{\eta_1}{\alpha_1\,\Gamma(\mu_1)} + \frac{\eta_2+\eta_3+\eta_4}{\alpha_1\,\Gamma(\mu_1+1)} + \frac{\gamma_1+\gamma_2}{\alpha_1\,\Gamma(\mu_1-\iota+1)}\Big] \\
&\le \sigma_1 + r\,\sigma_2 \le r,
\end{aligned}
$$

with conditions as follows:

$$
\begin{aligned}
\sigma_1 = &\frac{\|\mathsf{f}\|}{\alpha_1\,\Gamma(\mu_1+1)} + \|\mathcal{U}(\chi,\zeta,0)\|\Big(1 + \frac{\alpha_2}{\alpha_1\,\Gamma(\mu_1-\mu_2+1)} + \frac{\beta_1}{\alpha_1\,\Gamma(\mu_1-\nu_1+1)} \\
&+ \frac{\beta_2}{\alpha_1\,\Gamma(\mu_1-\nu_2+1)} + \frac{\eta_1}{\alpha_1\,\Gamma(\mu_1)}\Big) + \Big\|\frac{\partial\mathcal{U}(\chi,\zeta,0)}{\partial\tau}\Big\| + \frac{\gamma_1}{\alpha_1\,\Gamma(\mu_1-\iota+1)}\Big\|\frac{\partial^2\mathcal{U}(\chi,\zeta,0)}{\partial\chi^2}\Big\| \\
&+ \frac{\gamma_2}{\alpha_1\,\Gamma(\mu_1-\iota+1)}\Big\|\frac{\partial^2\mathcal{U}(\chi,\zeta,0)}{\partial\zeta^2}\Big\| < 1,
\end{aligned}
$$

$$\sigma_2 = \frac{\alpha_2}{\alpha_1\,\Gamma(\mu_1-\mu_2+1)} + \frac{\beta_1}{\alpha_1\,\Gamma(\mu_1-\nu_1+1)} + \frac{\beta_2}{\alpha_1\,\Gamma(\mu_1-\nu_2+1)} + \frac{\eta_1}{\alpha_1\,\Gamma(\mu_1)}$$
$$+ \frac{\eta_2+\eta_3+\eta_4}{\alpha_1\,\Gamma(\mu_1+1)} + \frac{\gamma_1+\gamma_2}{\alpha_1\,\Gamma(\mu_1-\iota+1)} < \frac{1}{2},$$
$$\sigma_1 \le (1-\sigma_2)\frac{r}{2}.$$

This shows that $\mathcal{FU}(\chi,\zeta,\tau)\in\mathcal{Y}$ for all $\mathcal{U}\in\mathcal{Y}$. Inequality (3.4) states that $\mathcal{FU}$ is bounded, as well.
Step 2. For $(\chi,\zeta,\tau_1),(\chi,\zeta,\tau_2)\in\Delta$ such that $\tau_1>\tau_2$, one gets the following:

$$\mathcal{FU}(\chi,\zeta,\tau_1)-\mathcal{FU}(\chi,\zeta,\tau_2) = \frac{1}{\alpha_1\,\Gamma(\mu_1)}\Big(\int_0^{\tau_1}[(\tau_1-t)^{\mu_1-1}-(\tau_2-t)^{\mu_1-1}]\,\mathsf{f}(\chi,\zeta,t)\,dt$$
$$+\int_{\tau_2}^{\tau_1}(\tau_2-t)^{\mu_1-1}\,\mathsf{f}(\chi,\zeta,t)\,dt\Big) - \frac{\alpha_2}{\alpha_1\,\Gamma(\mu_1-\mu_2)}\Big(\int_0^{\tau_1}[(\tau_1-t)^{\mu_1-\mu_2-1}-(\tau_2-t)^{\mu_1-\mu_2-1}]\,\mathcal{U}(\chi,\zeta,t)\,dt$$
$$+\int_{\tau_2}^{\tau_1}(\tau_2-t)^{\mu_1-\mu_2-1}\mathcal{U}(\chi,\zeta,t)\,dt\Big)$$
$$+\frac{\alpha_2\,\mathcal{U}(\chi,\zeta,0)}{\alpha_1\,\Gamma(\mu_1-\mu_2+1)}(\tau_1^{\mu_1-\mu_2}-\tau_2^{\mu_1-\mu_2}) + \frac{\beta_1\,\mathcal{U}(\chi,\zeta,0)}{\alpha_1\,\Gamma(\mu_1-\nu_1+1)}(\tau_1^{\mu_1-\nu_1}-\tau_2^{\mu_1-\nu_1})$$
$$-\frac{\beta_1}{\alpha_1\,\Gamma(\mu_1-\nu_1)}\Big(\int_0^{\tau_1}[(\tau_1-t)^{\mu_1-\nu_1-1}-(\tau_2-t)^{\mu_1-\nu_1-1}]\mathcal{U}(\chi,\zeta,t)\,dt+\int_{\tau_2}^{\tau_1}(\tau_2-t)^{\mu_1-\nu_1-1}\mathcal{U}(\chi,\zeta,t)\,dt\Big)$$
$$-\frac{\beta_2}{\alpha_1\,\Gamma(\mu_1-\nu_2)}\Big(\int_0^{\tau_1}[(\tau_1-t)^{\mu_1-\nu_2-1}-(\tau_2-t)^{\mu_1-\nu_2-1}]\mathcal{U}(\chi,\zeta,t)dt+\int_{\tau_1}^{\tau_2}(\tau_2-t)^{\mu_1-\nu_2-1}\mathcal{U}(\chi,\zeta,t)\,dt\Big)$$
$$+\frac{\beta_2\,\mathcal{U}(\chi,\zeta,0)}{\alpha_1\,\Gamma(\mu_1-\nu_2+1)}(\tau_1^{\mu_1-\nu_2}-\tau_2^{\mu_1-\nu_2}) + \frac{\eta_1\,\mathcal{U}(\chi,\zeta,0)}{\alpha_1\,\Gamma(\mu_1)}(\tau_1^{\mu_1-1}-\tau_2^{\mu_1-1})$$
$$-\frac{\eta_1}{\alpha_1\,\Gamma(\mu_1-1)}\Big(\int_0^{\tau_1}[(\tau_1-t)^{\mu_1-2}-(\tau_2-t)^{\mu_1-2}]\mathcal{U}(\chi,\zeta,t)dt+\int_{\tau_2}^{\tau_1}(\tau_2-t)^{\mu_1-2}\mathcal{U}(\chi,\zeta,t)\,dt\Big)$$
$$-\frac{\gamma_1}{\alpha_1\,\Gamma(\mu_1)}\Big(\int_0^{\tau_1}[(\tau_1-t)^{\mu_1-1}-(\tau_2-t)^{\mu_1-1}]\mathcal{U}(\chi,\zeta,t)\,dt+\int_{\tau_2}^{\tau_1}(\tau_2-t)^{\mu_1-1}\mathcal{U}(\chi,\zeta,t)\,dt\Big)$$
$$+\frac{\eta_3}{\alpha_1\,\Gamma(\mu_1)}\Big(\int_0^{\tau_1}[(\tau_1-t)^{\mu_1-1}-(\tau_2-t)^{\mu_1-1}]\frac{\partial^2\mathcal{U}(\chi,\zeta,t)}{\partial\chi^2}\,dt+\int_{\tau_2}^{\tau_1}(\tau_2-t)^{\mu_1-1}\frac{\partial^2\mathcal{U}(\chi,\zeta,t)}{\partial\chi^2}dt\Big)$$
$$+\frac{\eta_4}{\alpha_1\,\Gamma(\mu_1)}\Big(\int_0^{\tau_1}[(\tau_1-t)^{\mu_1-1}-(\tau_2-t)^{\mu_1-1}]\frac{\partial^2\mathcal{U}(\chi,\zeta,t)}{\partial\zeta^2}dt+\int_{\tau_2}^{\tau_1}(\tau_2-t)^{\mu_1-1}\frac{\partial^2\mathcal{U}(\chi,\zeta,t)}{\partial\zeta^2}dt\Big)$$
$$+\frac{\gamma_1}{\alpha_1\,\Gamma(\mu_1-\iota)}\Big(\int_0^{\tau_1}[(\tau_1-t)^{\mu_1-\iota-1}-(\tau-t)^{\mu_1-\iota-1}]\frac{\partial^2\mathcal{U}(\chi,\zeta,t)}{\partial\chi^2}dt+\int_{\tau_2}^{\tau_1}(\tau_2-t)^{\mu_1-\iota-1}\frac{\partial^2\mathcal{U}(\chi,\zeta,t)}{\partial\chi^2}dt\Big)$$
$$+\frac{\gamma_2}{\alpha_1\,\Gamma(\mu_1-\iota)}\Big(\int_0^{\tau_1}[(\tau_1-t)^{\mu_1-\iota-1}-(\tau-t)^{\mu_1-\iota-1}]\frac{\partial^2\mathcal{U}(\chi,\zeta,t)}{\partial\zeta^2}dt+\int_{\tau_2}^{\tau_1}(\tau_2-t)^{\mu_1-\iota-1}\frac{\partial^2\mathcal{U}(\chi,\zeta,t)}{\partial\zeta^2}dt\Big)$$
$$-\frac{\gamma_2\,(\tau_1^{\mu_1-\iota}-\tau_2^{\mu_1-\iota})}{\alpha_1\,\Gamma(\mu_1-\iota+1)}\frac{\partial^2\mathcal{U}(\chi,\zeta,0)}{\partial\zeta^2} - \frac{\gamma_1\,(\tau_1^{\mu_1-\iota}-\tau_2^{\mu_1-\iota})}{\alpha_1\,\Gamma(\mu_1-\iota+1)}\frac{\partial^2\mathcal{U}(\chi,\zeta,0)}{\partial\zeta^2}.$$

By taking the norm, one has

$$\begin{aligned}
&\|\mathcal{F}\mathcal{U}(\chi,\zeta,\tau_1)-\mathcal{F}\mathcal{U}(\chi,\zeta,\tau_2)\| \le \frac{\|\mathsf{f}\|}{\alpha_1\,\Gamma(\mu_1+1)}[(\tau_1^{\mu_1}-\tau_2^{\mu_1})-(\tau_2-\tau_1)^{\mu_1}]\\
&\quad+\frac{\alpha_2\,\|\mathcal{U}\|}{\alpha_1\,\Gamma(\mu_1-\mu_2+1)}[(\tau_1^{\mu_1-\mu_2}-\tau_2^{\mu_1-\mu_2})-(\tau_2-\tau_1)^{\mu_1-\mu_2}]+\frac{\alpha_2\,\|\mathcal{U}(\chi,\zeta,0)\|}{\alpha_1\,\Gamma(\mu_1-\mu_2+1)}(\tau_1^{\mu_1-\mu_2}-\tau_2^{\mu_1-\mu_2})\\
&\quad+\frac{\beta_1\,\|\mathcal{U}(\chi,\zeta,0)\|}{\alpha_1\,\Gamma(\mu_1-\nu_1+1)}(\tau_1^{\mu_1-\nu_1}-\tau_2^{\mu_1-\nu_1})+\frac{\beta_1\,\|\mathcal{U}\|}{\alpha_1\,\Gamma(\mu_1-\nu_1+1)}[(\tau_1^{\mu_1-\nu_1}-\tau_2^{\mu_1-\nu_1})-(\tau_2-\tau_1)^{\mu_1-\nu_1}]\\
&\quad+\frac{\beta_2\,\|\mathcal{U}\|}{\alpha_1\,\Gamma(\mu_1-\nu_2+1)}[(\tau_1^{\mu_1-\nu_2}-\tau_2^{\mu_1-\nu_2})-(\tau_2-\tau_1)^{\mu_1-\nu_2}]\\
&\quad+\frac{\beta_2\,\|\mathcal{U}(\chi,\zeta,0)\|}{\alpha_1\,\Gamma(\mu_1-\nu_2+1)}(\tau_1^{\mu_1-\nu_2}-\tau_2^{\mu_1-\nu_2})+\frac{\eta_1\,\|\mathcal{U}(\chi,\zeta,0)\|}{\alpha_1\,\Gamma(\mu_1)}(\tau_1^{\mu_1-1}-\tau_2^{\mu_1-1})\\
&\quad+\frac{\eta_1\,\|\mathcal{U}\|}{\alpha_1\,\Gamma(\mu_1)}[(\tau_1^{\mu_1-1}-\tau_2^{\mu_1-1})-(\tau_2-\tau_1)^{\mu_1-1}]+\frac{\gamma_1\,\|\mathcal{U}\|}{\alpha_1\,\Gamma(\mu_1+1)}[(\tau_1^{\mu_1}-\tau_2^{\mu_1})-(\tau_2-\tau_1)^{\mu_1}]\\
&\quad+\frac{\eta_3}{\alpha_1\,\Gamma(\mu_1+1)}\left\|\frac{\partial^2\mathcal{U}}{\partial\chi^2}\right\|[(\tau_1^{\mu_1}-\tau_2^{\mu_1})-(\tau_2-\tau_1)^{\mu_1}]\\
&\quad+\frac{\eta_4}{\alpha_1\,\Gamma(\mu_1+1)}\left\|\frac{\partial^2\mathcal{U}}{\partial\zeta^2}\right\|[(\tau_1^{\mu_1}-\tau_2^{\mu_1})-(\tau_2-\tau_1)^{\mu_1}]\\
&\quad+\frac{\gamma_1}{\alpha_1\,\Gamma(\mu_1-\iota+1)}\left\|\frac{\partial^2\mathcal{U}}{\partial\chi^2}\right\|[(\tau_1^{\mu_1-\iota}-\tau_2^{\mu_1-\iota})-(\tau_2-\tau_1)^{\mu_1-\iota}]\\
&\quad+\frac{\gamma_2}{\alpha_1\,\Gamma(\mu_1-\iota+1)}\left\|\frac{\partial^2\mathcal{U}}{\partial\zeta^2}\right\|[(\tau_1^{\mu_1-\iota}-\tau_2^{\mu_1-\iota})-(\tau_2-\tau_1)^{\mu_1-\iota}]\\
&\quad+\frac{\gamma_2}{\alpha_1\,\Gamma(\mu_1-\iota+1)}\|\frac{\partial^2\mathcal{U}(\chi,\zeta,0)}{\partial\zeta^2}\|(\tau_1^{\mu_1-\iota}-\tau_2^{\mu_1-\iota})+\frac{\gamma_1}{\alpha_1\,\Gamma(\mu_1-\iota+1)}\|\frac{\partial^2\mathcal{U}(\chi,\zeta,0)}{\partial\chi^2}\|(\tau_1^{\mu_1-\iota}-\tau_2^{\mu_1-\iota}).
\end{aligned}$$

As τ_2 tends to τ_1, the right-hand side of the above inequality tends to zero, and $\mathcal{F}$ is equicontinuous. Utilizing the Arzela–Ascoli theorem, one can deduce that $\mathcal{F}$ is compact and continuous. Therefore, by Krasnoselskii's fixed-point theorem [35], the given problem has at least one solution on $C^2(\Delta)$.

Step 3. Let $\mathcal{U}_1, \mathcal{U}_2 \in C^2(\Delta)$, and

$$\left\|\frac{\partial^2\mathcal{U}_1}{\partial\chi^2}-\frac{\partial^2\mathcal{U}_2}{\partial\chi^2}\right\| \le s_1\,\|\mathcal{U}_1-\mathcal{U}_2\|, \qquad \left\|\frac{\partial^2\mathcal{U}_1}{\partial\zeta^2}-\frac{\partial^2\mathcal{U}_2}{\partial\zeta^2}\right\| \le s_2\,\|\mathcal{U}_1-\mathcal{U}_2\|. \tag{3.5}$$

One then has

$$\begin{aligned}
&\mathcal{F}\mathcal{U}_1(\chi,\zeta,\tau)-\mathcal{F}\mathcal{U}_2(\chi,\zeta,\tau) = -\frac{\alpha_2}{\alpha_1\,\Gamma(\mu_1-\mu_2)}\int_0^\tau(\tau-t)^{\mu_1-\mu_2-1}(\mathcal{U}_1(\chi,\zeta,t)-\mathcal{U}_2(\chi,\zeta,t))dt\\
&\quad-\frac{\beta_1}{\alpha_1\,\Gamma(\mu_1-\nu_1)}\int_0^\tau(\tau-t)^{\mu_1-\nu_1-1}(\mathcal{U}_1(\chi,\zeta,t)-\mathcal{U}_2(\chi,\zeta,t))dt\\
&\quad-\frac{\beta_2}{\alpha_1\,\Gamma(\mu_1-\nu_2)}\int_0^\tau(\tau-t)^{\mu_1-\nu_2-1}(\mathcal{U}_1(\chi,\zeta,t)-\mathcal{U}_2(\chi,\zeta,t))dt\\
&\quad-\frac{\eta_1}{\alpha_1\,\Gamma(\mu_1-1)}\int_0^\tau(\tau-t)^{\mu_1-2}(\mathcal{U}_1(\chi,\zeta,t)-\mathcal{U}_2(\chi,\zeta,t))dt\\
&\quad-\frac{\eta_2}{\alpha_1\,\Gamma(\mu_1)}\int_0^\tau(\tau-t)^{\mu_1-1}(\mathcal{U}_1(\chi,\zeta,t)-\mathcal{U}_2(\chi,\zeta,t))dt
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\eta_3}{\alpha_1\,\Gamma(\mu_1)}\int_0^{\tau}(\tau-t)^{\mu_1-1}\Big(\frac{\partial^2(\mathcal{U}_1-\mathcal{U}_2)(\chi,\zeta,t)}{\partial\chi^2}\Big)dt\\
&+\frac{\eta_4}{\alpha_1\,\Gamma(\mu_1)}\int_0^{\tau}(\tau-t)^{\mu_1-1}\Big(\frac{\partial^2(\mathcal{U}_1-\mathcal{U}_2)(\chi,\zeta,t)}{\partial\zeta^2}\Big)dt\\
&+\frac{\gamma_1}{\alpha_1\,\Gamma(\mu_1-\iota)}\int_0^{\tau}(\tau-t)^{\mu_1-\iota-1}\Big(\frac{\partial^2(\mathcal{U}_1-\mathcal{U}_2)(\chi,\zeta,t)}{\partial\chi^2}\Big)dt\\
&+\frac{\gamma_2}{\alpha_1\,\Gamma(\mu_1-\iota)}\int_0^{\tau}(\tau-t)^{\mu_1-\iota-1}\Big(\frac{\partial^2(\mathcal{U}_1-\mathcal{U}_2)(\chi,\zeta,t)}{\partial\zeta^2}\Big)dt.
\end{aligned}
$$

Applying the norm and using the triangle inequality and conditions (3.5) yield

$$
\begin{aligned}
\|\mathcal{F}\mathcal{U}_1-\mathcal{F}\mathcal{U}_2\|\le\Big[&\frac{\alpha_2}{\alpha_1\,\Gamma(\mu_1-\mu_2+1)}+\frac{\beta_1}{\alpha_1\,\Gamma(\mu_1-\nu_1+1)}+\frac{\beta_2}{\alpha_1\,\Gamma(\mu_1-\nu_2+1)}\\
&+\frac{\eta_1}{\alpha_1\,\Gamma(\mu_1)}+\frac{\eta_2+s_1\,\eta_3+s_2\,\eta_4}{\alpha_1\,\Gamma(\mu_1+1)}+\frac{s_1\,\gamma_1+s_2\,\gamma_2}{\alpha_1\,\Gamma(\mu_1-\iota+1)}\Big]\|\mathcal{U}_1-\mathcal{U}_2\|.
\end{aligned}
$$

If

$$
\begin{aligned}
&\frac{\alpha_2}{\alpha_1\,\Gamma(\mu_1-\mu_2+1)}+\frac{\beta_1}{\alpha_1\,\Gamma(\mu_1-\nu_1+1)}+\frac{\beta_2}{\alpha_1\,\Gamma(\mu_1-\nu_2+1)}\\
&\quad+\frac{\eta_1}{\alpha_1\,\Gamma(\mu_1)}+\frac{\eta_2+s_1\,\eta_3+s_2\,\eta_4}{\alpha_1\,\Gamma(\mu_1+1)}+\frac{s_1\,\gamma_1+s_2\,\gamma_2}{\alpha_1\,\Gamma(\mu_1-\iota+1)}<1,
\end{aligned}
$$

then the operator $\mathcal{F}$ will be a contraction mapping and has a unique fixed point. Therefore, the given problem has a unique solution.

Remark 3.1. *Convexity and generalized convexity play an important role in modern optimization, variational analysis, control theory, and differential equations. These concepts are frequently employed in establishing existence, uniqueness, stability, and optimality properties of mathematical models. In particular, generalized convexity has found numerous applications in variational inequalities, vector optimization problems, and optimal control systems [38, 39]. Although the present work focuses on the numerical solution of multiterm time-fractional diffusion–wave equations rather than optimization problems, the existence and uniqueness analysis relies on fixed-point arguments in a Banach–space framework where convex subsets naturally arise.*

4. Shifted eighth-kind Chebyshev polynomials and matrix relations

The shifted Chebyshev polynomials of the eighth kind are defined over the interval $\mathbf{I} = [0, 1]$ as follows [31]:

$$
\mathcal{E}_k(\tau)=\sum_{r=0}^{k}\rho_{r,k}\,\tau^r,\quad k=0,1,2,\ldots, \tag{4.1}
$$

where the coefficients $\rho_{r,k}$ are as

$$\rho_{r,k} = \frac{3}{\Gamma(2r+2)} \begin{cases} \displaystyle\sum_{l=\lfloor \frac{r+1}{2} \rfloor}^{\frac{k}{2}} \frac{(-1)^{\frac{k}{2}+l+r}\, 2^{2r-2}\, (\frac{k}{2}-l+1)\, (\frac{k}{2}+l+2)\, \Gamma(\frac{k+3}{2})\, \Gamma(2l+r+2)}{\Gamma(\frac{k+1}{2}+2)\, \Gamma(2l-r+1)} & \text{if } k \text{ even,} \\[2ex] \displaystyle\sum_{l=\lfloor \frac{r}{2} \rfloor}^{\frac{k-1}{2}} \frac{(-1)^{\frac{k+1}{2}+l+r}\, 2^{2r-3}\, (l+1)\, (k-2l+1)\, (\frac{k+5}{2}+l)\, \Gamma(\frac{k+3}{2})\, \Gamma(2l+r+3)}{\Gamma(\frac{k}{2}+3)\, \Gamma(2l-r+2)} & \text{if } k \text{ odd.} \end{cases}$$

The shifted EKCPs are orthogonal with respect to the weight function $\varpi(\tau) = (1-2\tau)^4\, \tau^{1/2}\, (1-\tau)^{1/2}$ on the interval $\mathbf{I}$:

$$\int_0^1 \mathcal{E}_k(\tau)\, \mathcal{E}_l(\tau)\, \varpi(\tau)\, d\tau = \vartheta_k\, \delta_{k,l},$$

where $\delta_{k,l}$ is the Kronecker delta function, and ϑ_k is the normalizing factor below:

$$\vartheta_k = \frac{9\pi}{512} \begin{cases} \frac{(k+2)\,(k+4)}{(k+3)^2}, & k \text{ even,} \\[2ex] \frac{(k+1)\,(k+5)}{(k+2)\,(k+4)}, & k \text{ odd.} \end{cases}$$

A square-integrable function $\mathsf{h}(\tau) \in L^2_{\varpi}(\mathbf{I})$ can be expanded in the shifted EKCPs as a truncated series:

$$\mathsf{h}(\tau) \approx \mathsf{h}_{\mathcal{N}}(\tau) = \sum_{k=0}^{\mathcal{N}} \mathsf{H}_k\, \mathcal{E}_k(\tau) = \mathbf{H}^\top\, \psi(\tau), \tag{4.2}$$

where $\mathbf{H}$ and $\psi(\tau)$ are the vectors of unknown coefficients and basis functions, respectively, as follows:

$$\mathbf{H} = [\mathsf{H}_0, \mathsf{H}_1, \ldots, \mathsf{H}_{\mathcal{N}}]^\top, \qquad \psi(\tau) = [\mathcal{E}_0(\tau), \mathcal{E}_1(\tau), \ldots, \mathcal{E}_{\mathcal{N}}(\tau)]^\top. \tag{4.3}$$

To derive the pseudo-operational matrices of integration of the integer order, integrate the kth component of the vector $\psi(\tau)$ from 0 to τ:

$$\int_0^\tau t^\sigma\, \mathcal{E}_k(t)\, dt = \sum_{r=0}^{k} \rho_{r,k} \int_0^\tau t^{r+\sigma}\, dt = \sum_{r=0}^{k} \rho_{r,k}\, \frac{\tau^{r+\sigma+1}}{r+\sigma+1} = \tau^{\sigma+1} \sum_{r=0}^{k} \rho_{r,k}\, \frac{\tau^r}{r+\sigma+1},$$

where $\sigma \in \mathbb{R}^+ \cup \{0\}$. Now, expand the monomial τ^r in EKCPs:

$$\tau^r = \sum_{l=0}^{\mathcal{N}} c_{l,r}\, \mathcal{E}_l(\tau),$$

where

$$\begin{aligned} c_{l,r} &= \frac{1}{\vartheta_l} \int_0^1 \tau^r\, \mathcal{E}_l(\tau)\, d\tau \\ &= \frac{1}{\vartheta_l} \sum_{m=0}^{l} \rho_{m,l} \int_0^t \tau^{r+m}\, (1-2\tau)^4\, \tau^{\frac{1}{2}}\, (1-\tau)^{\frac{1}{2}}\, d\tau \end{aligned}$$

$$= \frac{1}{\vartheta_l} \sum_{m=0}^{l} \rho_{m,l} \int_0^1 (1-\tau)^{\frac{1}{2}} [16\,\tau^{r+m+\frac{9}{2}} - 32\,\tau^{r+m+\frac{7}{2}} + 24\,\tau^{r+m+\frac{5}{2}} - 8\,\tau^{r+m+\frac{3}{2}} + \tau^{r+m+\frac{1}{2}}]\,d\tau$$

$$= \frac{\sqrt{\pi}}{2\,\vartheta_l} \sum_{m=0}^{l} \rho_{m,l} \Big[16\,\frac{\Gamma(r+m+\frac{11}{2})}{\Gamma(r+m+7)} - 32\,\frac{\Gamma(r+m+\frac{9}{2})}{\Gamma(r+m+6)} + 24\,\frac{\Gamma(r+m+\frac{7}{2})}{\Gamma(r+m+5)} - 8\,\frac{\Gamma(r+m+\frac{5}{2})}{\Gamma(r+m+4)} + \frac{\Gamma(r+m+\frac{3}{2})}{\Gamma(r+m+3)}\Big].$$

Therefore,

$$\int_0^{\tau} t^{\sigma}\,\mathcal{E}_k(t)\,dt = \tau^{\sigma+1} \sum_{l=0}^{\mathcal{N}} \Big\{ \sum_{r=0}^{k} \sum_{m=0}^{l} \frac{\sqrt{\pi}}{2\,\vartheta_l}\,\frac{\rho_{r,k}\,\rho_{m,l}}{r+\sigma+1} \Big[16\,\frac{\Gamma(r+m+\frac{11}{2})}{\Gamma(r+m+7)} - 32\,\frac{\Gamma(r+m+\frac{9}{2})}{\Gamma(r+m+6)} + 24\,\frac{\Gamma(r+m+\frac{7}{2})}{\Gamma(r+m+5)} - 8\,\frac{\Gamma(r+m+\frac{5}{2})}{\Gamma(r+m+4)} + \frac{\Gamma(r+m+\frac{3}{2})}{\Gamma(r+m+3)}\Big]\Big\} \mathcal{E}_l(\tau), \quad k = 1, 2, \ldots, \mathcal{N}.$$

In matrix form, one can have

$$\int_0^{\tau} t^{\sigma}\,\psi(t)\,dt = \tau^{\sigma+1}\,\mathbf{B}^{\sigma}\,\psi(\tau), \tag{4.4}$$

where $\mathbf{B}^{\sigma}$ is the $(\mathcal{N}+1)\times(\mathcal{N}+1)$ pseudo-operational matrix of the integration with the following entries:

$$\mathbf{B}^{\sigma}_{k,l} = \sum_{r=0}^{k} \sum_{m=0}^{l} \frac{\sqrt{\pi}}{2\,\vartheta_l}\,\frac{\rho_{r,k}\,\rho_{m,l}}{r+\sigma+1} \Big[16\,\frac{\Gamma(r+m+\frac{11}{2})}{\Gamma(r+m+7)} - 32\,\frac{\Gamma(r+m+\frac{9}{2})}{\Gamma(r+m+6)} + 24\,\frac{\Gamma(r+m+\frac{7}{2})}{\Gamma(r+m+5)} - 8\,\frac{\Gamma(r+m+\frac{5}{2})}{\Gamma(r+m+4)} + \frac{\Gamma(r+m+\frac{3}{2})}{\Gamma(r+m+3)}\Big], \qquad k, l = 0, 1, \ldots, \mathcal{N}.$$

Let $\mu \in \mathbb{R}^{+}$. To construct the pseudo-operational matrix of the integration of the fractional order, apply the Riemann–Liouville integral operator of the order μ on the kth component of the basis vector $\psi(\tau)$:

$$\begin{aligned} {}_{0}^{RL}\mathcal{I}_{\tau}^{\mu}(\tau^{\sigma}\,\mathcal{E}_k(\tau)) &= \sum_{r=0}^{k} \rho_{r,k}\,\frac{\Gamma(r+\sigma+1)\Gamma(r+\sigma+\mu+1)}{\Gamma(r+\sigma+\mu+1)}\tau^{r+\sigma+\mu} \\ &= \tau^{\sigma+\mu} \sum_{r=0}^{k} \rho_{r,k}\,\frac{\Gamma(r+\sigma+1)\Gamma(r+\sigma+\mu+1)}{\Gamma(r+\sigma+\mu+1)}\tau^{r}. \end{aligned}$$

Similarly, the same expansion is obtained for τ^r in terms of the shifted EKCPs. Therefore, in matrix representation, one has

$${}_{0}^{RL}\mathcal{I}_{\tau}^{\mu}(\tau^{\sigma}\,\psi(\tau)) = \tau^{\sigma+\mu}\,\mathbf{B}^{\sigma,\mu}\,\psi(\tau), \tag{4.5}$$

where $\mathbf{B}^{\sigma,\mu}$ is the $(\mathcal{N}+1)\times(\mathcal{N}+1)$ pseudo-operational matrix of the integration of the fractional order with the following entries:

$$\mathbf{B}^{\sigma,\mu}_{k,l} = \sum_{r=0}^{k}\sum_{m=0}^{l} \frac{\sqrt{\pi}}{2\,\vartheta_l}\,\frac{\rho_{r,k}\,\rho_{m,l}\,\Gamma(r+\sigma+1)}{\Gamma(r+\sigma+\mu+1)}\Big[16\,\frac{\Gamma(r+m+\frac{11}{2})}{\Gamma(r+m+7)} - 32\,\frac{\Gamma(r+m+\frac{9}{2})}{\Gamma(r+m+6)} + 24\,\frac{\Gamma(r+m+\frac{7}{2})}{\Gamma(r+m+5)}$$
$$- 8\,\frac{\Gamma(r+m+\frac{5}{2})}{\Gamma(r+m+4)} + \frac{\Gamma(r+m+\frac{3}{2})}{\Gamma(r+m+3)}\Big], \qquad k,l = 0,1,\ldots,\mathcal{N}.$$

Because the given problem is a two-dimensional time-fractional PDE, a set of basis functions depending on three variables is required. Accordingly, the three-variable Chebyshev polynomials of the eighth kind are constructed by employing the Kronecker product of the one-variable shifted EKCPs over the domain $\Delta = [0,1]\times[0,1]\times[0,1]$ as follows:

$$\mathbb{E}_{k,l,m}(\chi,\zeta,\tau) = \mathcal{E}_k(\chi)\,\mathcal{E}_l(\zeta)\,\mathcal{E}_m(\tau), \quad k,l,m = 0,1,2,\ldots,\mathcal{N}.$$

These polynomials have the orthogonality characteristic with respect to the weight function $\Omega(\chi,\zeta,\tau) = \varpi(\chi)\,\varpi(\zeta)\,\varpi(\tau)$:

$$\int_0^1\int_0^1\int_0^1 \mathbb{E}_{k,l,m}(\chi,\zeta,\tau)\,\mathbb{E}_{k',l',m'}(\chi,\zeta,\tau)\,\Omega(\chi,\zeta,\tau)\,d\chi\,d\zeta\,d\tau = \vartheta_k\,\vartheta_l\,\vartheta_m\,\delta_{k,k'}\,\delta_{l,l'}\,\delta_{m,m'},$$
$$k,l,m,k',l',m' = 0,1,2,\ldots,\mathcal{N}.$$

Thus, a square-integrable function $\mathsf{h}(\chi,\zeta,\tau) \in L^2_\Omega(\Delta)$ in the three variables can be expanded below:

$$\mathsf{h}(\chi,\zeta,\tau) \approx \mathsf{h}_{\mathcal{N}}(\chi,\zeta,\tau) = \sum_{k=0}^{\mathcal{N}}\sum_{l=0}^{\mathcal{N}}\sum_{m=0}^{\mathcal{N}} \mathsf{H}_{k,l,m}\,\mathcal{E}_k(\chi)\,\mathcal{E}_l(\zeta)\,\mathcal{E}_m(\tau) = \mathbb{H}^\top\,\mathbf{\Psi}(\chi,\zeta,\tau),$$

where $\mathbb{H}$ and $\mathbf{\Psi}(\chi,\zeta,\tau)$ are the $(\mathcal{N}+1)^3\times 1$ vectors of the unknown coefficients and tri-variable basis functions as follows:

$$\begin{aligned}
\mathbb{H} &= [\mathsf{H}_{0,0,0},\mathsf{H}_{0,0,1},\ldots,\mathsf{H}_{0,0,\mathcal{N}},\mathsf{H}_{0,1,0},\mathsf{H}_{0,1,1},\ldots,\mathsf{H}_{0,1,\mathcal{N}},\ldots,\mathsf{H}_{\mathcal{N},\mathcal{N},0},\mathsf{H}_{\mathcal{N},\mathcal{N},1},\ldots,\mathsf{H}_{\mathcal{N},\mathcal{N},\mathcal{N}}]^\top,\\
\mathbf{\Psi}(\chi,\zeta,\tau) &= [\mathbb{E}_{0,0,0}(\chi,\zeta,\tau),\mathbb{E}_{0,0,1}(\chi,\zeta,\tau),\ldots,\mathbb{E}_{0,0,\mathcal{N}}(\chi,\zeta,\tau),\mathbb{E}_{0,1,0}(\chi,\zeta,\tau),\mathbb{E}_{0,1,1}(\chi,\zeta,\tau)\\
&\qquad,\ldots,\mathbb{E}_{0,1,\mathcal{N}}(\chi,\zeta,\tau),\ldots,\mathbb{E}_{\mathcal{N},\mathcal{N},0}(\chi,\zeta,\tau),\mathbb{E}_{\mathcal{N},\mathcal{N},1}(\chi,\zeta,\tau),\ldots,\mathbb{E}_{\mathcal{N},\mathcal{N},\mathcal{N}}(\chi,\zeta,\tau)]^\top\\
&= \psi(\chi)\otimes\psi(\zeta)\otimes\psi(\tau),
\end{aligned} \tag{4.6}$$

where the symbol $\otimes$ denotes the Kronecker product.

Assume that $\sigma \in \mathbb{R}^+\cup\{0\}, \mu \in \mathbb{R}^+$. pseudo-operational matrices of the integration of the integer and fractional orders, corresponding to the variables $\chi,\zeta,$ and τ can be derived as follows:

Pseudo-operational matrix of the integration corresponding to χ:

$$\begin{aligned}
\int_0^\chi x^\sigma\,\mathbf{\Psi}(x,\zeta,\tau)\,dx &= \int_0^\chi x^\sigma\,\psi(x)\,dx\otimes\psi(\zeta)\otimes\psi(\tau)\\
&= (\chi^{\sigma+1}\,\mathbf{B}^\sigma\,\psi(\chi))\otimes\psi(\zeta)\otimes\psi(\tau)\\
&= \chi^{\sigma+1}\,(\mathbf{B}^\sigma\otimes I\otimes I)(\psi(\chi)\otimes\psi(\zeta)\otimes\psi(\tau))\\
&= \chi^{\sigma+1}\,\mathbb{B}^\sigma_\chi\,\mathbf{\Psi}(\chi,\zeta,\tau),
\end{aligned} \tag{4.7}$$

where $\mathbb{B}_{\chi}^{\sigma}$ is the $(\mathcal{N}+1)^3 \times (\mathcal{N}+1)^3$ pseudo-operational matrix of the integration of the integer order.
Pseudo-operational matrix of the integration corresponding to ζ:

$$\begin{aligned}\int_0^{\zeta} y^{\sigma}\,\mathbf{\Psi}(\chi,y,\tau)\,dy &= \psi(\chi)\otimes\int_0^{\zeta} y^{\sigma}\,\psi(y)\,dy\otimes\psi(\tau)\\ &= \psi(\chi)\otimes(\zeta^{\sigma+1}\,\mathbf{B}^{\sigma}\,\psi(\zeta))\otimes\psi(\tau)\\ &= \zeta^{\sigma+1}\,(I\otimes\mathbf{B}^{\sigma}\otimes I)(\psi(\chi)\otimes\psi(\zeta)\otimes\psi(\tau))\\ &= \zeta^{\sigma+1}\,\mathbb{B}_{\zeta}^{\sigma}\,\mathbf{\Psi}(\chi,\zeta,\tau),\end{aligned} \tag{4.8}$$

where $\mathbb{B}_{\zeta}^{\sigma}$ is the $(\mathcal{N}+1)^3 \times (\mathcal{N}+1)^3$ pseudo-operational matrix of the integration of the integer order.
Pseudo-operational matrix of the integration corresponding to τ:

$$\begin{aligned}\int_0^{\tau} t^{\sigma}\,\mathbf{\Psi}(\chi,\zeta,t)\,dt &= \psi(\chi)\otimes\psi(\zeta)\otimes\int_0^{\tau} t^{\sigma}\,\psi(t)\,dt\\ &= \psi(\chi)\otimes\psi(\zeta)(\tau^{\sigma+1}\,\mathbf{B}^{\sigma}\,\psi(\tau))\\ &= \tau^{\sigma+1}\,(I\otimes I\otimes\mathbf{B}^{\sigma})(\psi(\chi)\otimes\psi(\zeta)\otimes\psi(\tau))\\ &= \tau^{\sigma+1}\,\mathbb{B}_{\tau}^{\sigma}\,\mathbf{\Psi}(\chi,\zeta,\tau),\end{aligned} \tag{4.9}$$

where $\mathbb{B}_{\tau}^{\sigma}$ is the $(\mathcal{N}+1)^3 \times (\mathcal{N}+1)^3$ pseudo-operational matrix of the integration of the integer order.
Pseudo-operational matrix of the integration of the fractional order corresponding to τ:

$$\begin{aligned}{}_{0}^{RL}\mathcal{I}_{\tau}^{\mu}(\tau^{\sigma}\,\mathbf{\Psi}(\chi,\zeta,\tau)) &= \psi(\chi)\otimes\psi(\zeta)\otimes{}_{0}^{RL}\mathcal{I}_{\tau}^{\mu}(\tau^{\sigma}\,\psi(\tau))\\ &= \psi(\chi)\otimes\psi(\zeta)(\tau^{\sigma+\mu}\,\mathbf{B}^{\sigma,\mu}\,\psi(\tau))\\ &= \tau^{\sigma+\mu}\,(I\otimes I\otimes\mathbf{B}^{\sigma,\mu})(\psi(\chi)\otimes\psi(\zeta)\otimes\psi(\tau))\\ &= \tau^{\sigma+\mu}\,\mathbb{B}_{\tau}^{\sigma,\mu}\,\mathbf{\Psi}(\chi,\zeta,\tau),\end{aligned} \tag{4.10}$$

where $\mathbb{B}_{\tau}^{\sigma,\mu}$ is the $(\mathcal{N}+1)^3 \times (\mathcal{N}+1)^3$ pseudo-operational matrix of the integration of the fractional order.

5. Methodology

From Eq (3.1), the highest orders of derivatives with respect to the space variables χ and ζ are two. The fractional-order derivative $\mu_k \in (1,2)$, so $\lceil \mu_k \rceil = 2$, and the highest order of derivative with respect to the time variable is two, as well. Thus, to obtain an approximate solution to Problems (3.1)–(3.3), an initial approximation is constructed based on the highest-order derivatives with respect to the independent variables χ, ζ, and τ as follows. Subsequent approximations can then be derived through successively backward integrations:

$$\frac{\partial^6 \mathcal{U}(\chi,\zeta,\tau)}{\partial\chi^2\,\partial\zeta^2\,\partial\tau^2} \approx \mathbb{U}^{\top}\,\mathbf{\Psi}(\chi,\zeta,\tau), \tag{5.1}$$

where $\mathbb{U}$ is the vector of unknown coefficients. By integrating (5.1) with respect to τ and using the initial condition $\partial\mathcal{U}(\chi,\zeta,0)/\partial\tau = \mathsf{g}_2(\chi,\zeta)$, one has

$$\frac{\partial^5 \mathcal{U}(\chi,\zeta,\tau)}{\partial\chi^2\,\partial\zeta^2\,\partial\tau} \approx \tau\,\mathbb{U}^{\top}\,\mathbb{B}_{\tau}^{0}\,\mathbf{\Psi}(\chi,\zeta,\tau) + \frac{\partial^5 \mathcal{U}(\chi,\zeta,0)}{\partial\chi^2\,\partial\zeta^2\,\partial\tau} = \tau\,\mathbb{U}^{\top}\,\mathbb{B}_{\tau}^{0}\,\mathbf{\Psi}(\chi,\zeta,\tau) + \frac{\partial^4 \mathsf{g}_2(\chi,\zeta)}{\partial\chi^2\,\partial\zeta^2}. \tag{5.2}$$

Integrating (5.2) with respect to τ and using the initial condition $\mathcal{U}(\chi,\zeta,0)=\mathsf{g}_1(\chi,\zeta)$ yields

$$\begin{aligned}\frac{\partial^4\mathcal{U}(\chi,\zeta,\tau)}{\partial\chi^2\,\partial\zeta^2} &\approx \tau^2\,\mathbb{U}^\top\,\mathbb{B}_\tau^0\,\mathbb{B}_\tau^1\,\mathbf{\Psi}(\chi,\zeta,\tau)+\frac{\partial^4\mathsf{g}_2(\chi,\zeta)}{\partial\chi^2\,\partial\zeta^2}\,\tau+\frac{\partial^4\mathcal{U}(\chi,\zeta,0)}{\partial\chi^2\,\partial\zeta^2}\\ &= \tau^2\,\mathbb{U}^\top\,\mathbb{B}_\tau^0\,\mathbb{B}_\tau^1\,\mathbf{\Psi}(\chi,\zeta,\tau)+\frac{\partial^4\mathsf{g}_2(\chi,\zeta)}{\partial\chi^2\,\partial\zeta^2}\,\tau+\frac{\partial^4\mathsf{g}_1(\chi,\zeta)}{\partial\chi^2\,\partial\zeta^2}.\end{aligned}\tag{5.3}$$

Integrating (5.3) to ζ leads to the following approximation:

$$\begin{aligned}\frac{\partial^3\mathcal{U}(\chi,\zeta,\tau)}{\partial\chi^2\,\partial\zeta} \approx{}& \zeta\,\tau^2\,\mathbb{U}^\top\,\mathbb{B}_\tau^0\,\mathbb{B}_\tau^1\,\mathbb{B}_\zeta^0\,\mathbf{\Psi}(\chi,\zeta,\tau)+\frac{\partial^3\mathsf{g}_2(\chi,\zeta)}{\partial\chi^2\,\partial\zeta}\,\tau-\frac{\partial^3\mathsf{g}_2(\chi,0)}{\partial\chi^2\,\partial\zeta}\,\tau+\frac{\partial^3\mathsf{g}_1(\chi,\zeta)}{\partial\chi^2\,\partial\zeta}\\ &-\frac{\partial^3\mathsf{g}_1(\chi,0)}{\partial\chi^2\,\partial\zeta}+\frac{\partial^3\mathcal{U}(\chi,0,\tau)}{\partial\chi^2\,\partial\zeta},\end{aligned}\tag{5.4}$$

where $\partial^3\mathcal{U}(\chi,0,\tau)/\partial\chi^2\,\partial\zeta$ is approximated as $Z_1(\chi,\tau)$, which can be given by Part I in Appendix A. Integrating (5.4) to ζ and using the boundary condition $\mathcal{U}(\chi,0,\tau)=w_1(\chi,\tau)$ yields

$$\begin{aligned}\frac{\partial^2\mathcal{U}(\chi,\zeta,\tau)}{\partial\chi^2} \approx{}& \zeta^2\,\tau^2\,\mathbb{U}^\top\,\mathbb{B}_\tau^0\,\mathbb{B}_\tau^1\,\mathbb{B}_\zeta^0\,\mathbb{B}_\zeta^1\,\mathbf{\Psi}(\chi,\zeta,\tau)+\frac{\partial^2\mathsf{g}_2(\chi,\zeta)}{\partial\chi^2}\,\tau-\frac{\partial^2\mathsf{g}_2(\chi,0)}{\partial\chi^2}\,\tau-\frac{\partial^3\mathsf{g}_2(\chi,0)}{\partial\chi^2\,\partial\zeta}\,\zeta\,\tau\\ &+\frac{\partial^2\mathsf{g}_1(\chi,\zeta)}{\partial\chi^2}-\frac{\partial^2\mathsf{g}_1(\chi,0)}{\partial\chi^2}-\frac{\partial^3\mathsf{g}_1(\chi,0)}{\partial\chi^2\,\partial\zeta}\,\zeta+Z_1(\chi,\tau)\,\zeta+\frac{\partial^2 w_1(\chi,\tau)}{\partial\chi^2}.\end{aligned}\tag{5.5}$$

Integrating (5.5) with respect to χ leads to the following approximation:

$$\begin{aligned}\frac{\partial\mathcal{U}(\chi,\zeta,\tau)}{\partial\chi} \approx{}& \chi\,\zeta^2\,\tau^2\,\mathbb{U}^\top\,\mathbb{B}_\tau^0\,\mathbb{B}_\tau^1\,\mathbb{B}_\zeta^0\,\mathbb{B}_\zeta^1\,\mathbb{B}_\chi^0\,\mathbf{\Psi}(\chi,\zeta,\tau)+\frac{\partial\mathsf{g}_2(\chi,\zeta)}{\partial\chi}\,\tau-\frac{\partial\mathsf{g}_2(0,\zeta)}{\partial\chi}\,\tau-\frac{\partial\mathsf{g}_2(\chi,0)}{\partial\chi}\,\tau\\ &+\frac{\partial\mathsf{g}_2(0,0)}{\partial\chi}\,\tau-\frac{\partial^2\mathsf{g}_2(\chi,0)}{\partial\chi\,\partial\zeta}\,\zeta\,\tau+\frac{\partial^2\mathsf{g}_2(0,0)}{\partial\chi\,\partial\zeta}\,\zeta\,\tau+\frac{\partial\mathsf{g}_1(\chi,\zeta)}{\partial\chi}-\frac{\partial\mathsf{g}_1(0,\zeta)}{\partial\chi}\\ &-\frac{\partial\mathsf{g}_1(\chi,0)}{\partial\chi}+\frac{\partial\mathsf{g}_1(0,0)}{\partial\chi}-\frac{\partial^2\mathsf{g}_1(\chi,0)}{\partial\chi\,\partial\zeta}\,\zeta+\frac{\partial^2\mathsf{g}_1(0,0)}{\partial\chi\,\partial\zeta}\,\zeta+\int_0^\chi Z_1(x,\zeta)\,dx\\ &+\frac{\partial w_1(\chi,\tau)}{\partial\chi}-\frac{\partial w_1(0,\tau)}{\partial\chi}+\frac{\partial\mathcal{U}(0,\zeta,\tau)}{\partial\chi},\end{aligned}\tag{5.6}$$

where $\partial\mathcal{U}(0,\zeta,\tau)/\partial\chi$ is approximated as $Z_2(\zeta,\tau)$ in Part II of Appendix A. By integrating (5.6) with respect to χ and using the boundary condition $\mathcal{U}(0,\zeta,\tau)=\mathsf{h}_1(\zeta,\tau)$, one sums up

$$\begin{aligned}\mathcal{U}(\chi,\zeta,\tau) \approx{}& \chi^2\,\zeta^2\,\tau^2\,\mathbb{U}^\top\,\mathbb{B}_\tau^0\,\mathbb{B}_\tau^1\,\mathbb{B}_\zeta^0\,\mathbb{B}_\zeta^1 0\,\mathbb{B}_\zeta^1\,\mathbb{B}_\chi^0\,\mathbb{B}_\chi^1\,\mathbf{\Psi}(\chi,\zeta,\tau)+\mathsf{g}_2(\chi,\zeta)\,\tau-\mathsf{g}_2(0,\zeta)\,\tau\\ &-\frac{\partial\mathsf{g}_2(0,\zeta)}{\partial\chi}\,\chi\,\tau-\mathsf{g}_2(\chi,0)\,\tau+\mathsf{g}_2(0,0)\,\tau+\frac{\partial\mathsf{g}_2(0,0)}{\partial\chi}\,\chi\,\tau-\frac{\partial\mathsf{g}_2(\chi,0)}{\partial\zeta}\,\zeta\,\tau\\ &+\frac{\partial\mathsf{g}_2(0,0)}{\partial\zeta}\,\zeta\,\tau+\frac{\partial^2\mathsf{g}_2(0,0)}{\partial\chi\,\partial\zeta}\,\chi\,\zeta\,\tau+\mathsf{g}_1(\chi,\zeta)-\mathsf{g}_1(0,\zeta)-\frac{\partial\mathsf{g}_1(0,\zeta)}{\partial\chi}\,\chi\\ &-\mathsf{g}_1(\chi,0)+\mathsf{g}_1(0,0)+\frac{\partial\mathsf{g}_1(0,0)}{\partial\chi}\,\chi-\frac{\partial\mathsf{g}_1(\chi,0)}{\partial\zeta}\,\zeta+\frac{\partial\mathsf{g}_1(0,0)}{\partial\zeta}\,\zeta\\ &+\frac{\partial^2\mathsf{g}_1(0,0)}{\partial\chi\,\partial\zeta}\,\chi\,\zeta+\int_0^\chi\int_0^x Z_1(s,\zeta)\,ds\,dx+w_1(\chi,\tau)-w_1(0,\tau)\\ &-\frac{\partial w_1(0,\tau)}{\partial\chi}\,\chi+Z_2(\zeta,\tau)\,\chi+\mathsf{h}_1(\zeta,\tau).\end{aligned}\tag{5.7}$$

Integrating (5.3) with respect to χ leads to

$$\begin{aligned}\frac{\partial^3 \mathcal{U}(\chi,\zeta,\tau)}{\partial\chi\,\partial\zeta^2} \approx{}& \chi\,\tau^2\,\mathbb{U}^\top\,\mathbb{B}^0_\tau\,\mathbb{B}^1_\tau\,\mathbb{B}^0_\chi\,\boldsymbol{\Psi}(\chi,\zeta,\tau) + \frac{\partial^3 \mathsf{g}_2(\chi,\zeta)}{\partial\chi\,\partial\zeta^2}\tau - \frac{\partial^3 \mathsf{g}_2(0,\zeta)}{\partial\chi\,\partial\zeta^2}\tau \\ &+ \frac{\partial^3 \mathsf{g}_1(\chi,\zeta)}{\partial\chi\,\partial\zeta^2} - \frac{\partial^3 \mathsf{g}_1(0,\zeta)}{\partial\zeta^2} + \frac{\partial^2 Z_2(\zeta,\tau)}{\partial\zeta^2}.\end{aligned} \tag{5.8}$$

Again, integrating (5.8) with respect to χ leads to an approximation of $\partial^2\mathcal{U}/\partial\zeta^2$.

$$\begin{aligned}\frac{\partial^2 \mathcal{U}(\chi,\zeta,\tau)}{\partial\zeta^2} \approx{}& \chi^2\,\tau^2\,\mathbb{U}^\top\,\mathbb{B}^0_\tau\,\mathbb{B}^1_\tau\,\mathbb{B}^0_\chi\,\mathbb{B}^1_\chi\,\boldsymbol{\Psi}(\chi,\zeta,\tau) + \frac{\partial^2 \mathsf{g}_2(\chi,\zeta)}{\partial\zeta^2}\tau - \frac{\partial^2 \mathsf{g}_2(0,\zeta)}{\partial\zeta^2}\frac{\partial^2 Z_2(\zeta,\tau)}{\partial\zeta^2}\chi \\ &- \frac{\partial^3 \mathsf{g}_2(0,\zeta)}{\partial\chi\,\partial\zeta^2}\chi\,\tau + \frac{\partial^2 \mathsf{g}_1(\chi,\zeta)}{\partial\zeta^2} - \frac{\partial^2 \mathsf{g}_1(0,\zeta)}{\partial\zeta^2} - \frac{\partial^3 \mathsf{g}_1(0,\zeta)}{\partial\,\partial\zeta^2}\chi + \frac{\partial^2 Z_2(\zeta,\tau)}{\partial\zeta^2}\chi \\ &+ \frac{\partial^2 \mathsf{h}_1(\zeta,\tau)}{\partial\zeta^2}.\end{aligned} \tag{5.9}$$

To find an approximation of $\partial^2\mathcal{U}/\partial\tau^2$, integrate (5.1) with respect to χ along with $\partial^3\mathcal{U}(0,\zeta,\tau)/\partial\chi\,\partial\zeta^2 = \partial^2 Z_2(\zeta,\tau)/\partial\zeta^2$. Therefore, one has

$$\frac{\partial^5 \mathcal{U}(\chi,\zeta,\tau)}{\partial\chi\,\partial\zeta^2\,\partial\tau^2} \approx \chi\,\mathbb{U}^\top\,\mathbb{B}^0_\chi\,\boldsymbol{\Psi}(\chi,\zeta,\tau) + \frac{\partial^4 Z_2(\zeta,\tau)}{\partial\zeta^2\,\partial\tau^2}. \tag{5.10}$$

Integrating (5.10) with respect to χ leads to the following expression:

$$\frac{\partial^4 \mathcal{U}(\chi,\zeta,\tau)}{\partial\zeta^2\,\partial\tau^2} \approx \chi^2\,\mathbb{U}^\top\,\mathbb{B}^0_\chi\,\mathbb{B}^1_\chi\,\boldsymbol{\Psi}(\chi,\zeta,\tau) + \frac{\partial^4 Z_2(\zeta,\tau)}{\partial\zeta^2\,\partial\tau^2}\chi + \frac{\partial^4 \mathsf{h}_1(\zeta,\tau)}{\partial\zeta^2\,\partial\tau^2}. \tag{5.11}$$

Again, integrating (5.11) with respect to ζ yields the following approximation:

$$\begin{aligned}\frac{\partial^3 \mathcal{U}(\chi,\zeta,\tau)}{\partial\zeta\,\partial\tau^2} \approx{}& \chi^2\,\zeta\,\mathbb{U}^\top\,\mathbb{B}^0_\chi\,\mathbb{B}^1_\chi\,\mathbb{B}^0_\zeta\,\boldsymbol{\Psi}(\chi,\zeta,\tau) + \frac{\partial^3 Z_2(\zeta,\tau)}{\partial\zeta\,\partial\tau^2}\chi - \frac{\partial^3 Z_2(0,\tau)}{\partial\zeta\,\partial\tau^2} + \frac{\partial^3 \mathsf{h}_1(\zeta,\tau)}{\partial\zeta\,\partial\tau^2} \\ &- \frac{\partial^3 \mathsf{h}_1(0,\tau)}{\partial\zeta\,\partial\tau^2} + \frac{\partial^3 \mathcal{U}(\chi,0,\tau)}{\partial\zeta\,\partial\tau},\end{aligned} \tag{5.12}$$

where $\partial^3\mathcal{U}(\chi,0,\tau)/\partial\zeta\,\partial\tau$ is approximated as $Z_3(\chi,\tau)$ in Part III of Appendix A. Integrating (5.12) with respect to ζ and using the boundary condition $\mathcal{U}(\chi,0,\tau) = w_1(\chi,\tau)$ leads to

$$\begin{aligned}\frac{\partial^2 \mathcal{U}(\chi,\zeta,\tau)}{\partial\tau^2} \approx{}& \chi^2\,\zeta^2\,\mathbb{U}^\top\,\mathbb{B}^0_\chi\,\mathbb{B}^1_\chi\,\mathbb{B}^0_\zeta\,\mathbb{B}^1_\zeta\,\boldsymbol{\Psi}(\chi,\zeta,\tau) + \frac{\partial^2 Z_2(\zeta,\tau)}{\partial\tau^2}\chi - \frac{\partial^2 Z_2(0,\tau)}{\partial\tau^2}\chi \\ &- \frac{\partial^3 Z_2(0,\tau)}{\partial\zeta\,\partial\tau^2}\zeta + \frac{\partial^2 \mathsf{h}_1(\zeta,\tau)}{\partial\tau^2} - \frac{\partial^2 \mathsf{h}_1(0,\tau)}{\partial\tau^2} - \frac{\partial^3 \mathsf{h}_1(0,\tau)}{\partial\zeta\,\partial\tau^2}\zeta \\ &+ Z_3(\chi,\tau)\,\zeta + \frac{\partial^2 w_1(\chi,\tau)}{\partial\tau^2}.\end{aligned} \tag{5.13}$$

Integrating (5.13) with respect to τ along with the initial condition leads to an approximation of $\partial\mathcal{U}/\partial\tau$:

$$\begin{aligned}\frac{\partial\mathcal{U}(\chi,\zeta,\tau)}{\partial\tau} &\approx \chi^2\,\zeta^2\,\tau\,\mathbb{U}^\top\,\mathbb{B}_\chi^0\,\mathbb{B}_\chi^1\,\mathbb{B}_\zeta^0\,\mathbb{B}_\zeta^1\,\mathbb{B}_\tau^0\,\mathbf{\Psi}(\chi,\zeta,\tau)+\frac{\partial Z_2(\zeta,\tau)}{\partial\tau}\chi-\frac{\partial Z_2(\zeta,0)}{\partial\tau}\chi-\frac{\partial Z_2(0,\tau)}{\partial\tau}\chi\\&+\frac{\partial Z_2(0,0)}{\partial\tau}\chi-\frac{\partial^2 Z_2(0,\tau)}{\partial\zeta\,\partial\tau}\zeta+\frac{\partial^2 Z_2(0,0)}{\partial\zeta\,\partial\tau}\zeta+\frac{\partial\mathsf{h}_1(\zeta,\tau)}{\partial\tau}-\frac{\partial\mathsf{h}_1(\zeta,0)}{\partial\tau}\\&-\frac{\partial\mathsf{h}_1(0,\tau)}{\partial\tau}+\frac{\partial\mathsf{h}_1(0,0)}{\partial\tau}-\frac{\partial^2\mathsf{h}_1(0,\tau)}{\partial\zeta\,\partial\tau}\zeta+\frac{\partial^2\mathsf{h}_1(0,0)}{\partial\zeta\,\partial\tau}\zeta+\zeta\int_0^\tau Z_3(\chi,t)\,dt\\&+\frac{\partial w_1(\chi,\tau)}{\partial\tau}-\frac{\partial w_1(\chi,0)}{\partial\tau}+\mathsf{g}_2(\chi,\zeta).\end{aligned}\tag{5.14}$$

Now, approximations to terms with the fractional-order derivatives are found using approximations (5.13) and (5.14):

$$\begin{aligned}{}_0^c\mathcal{D}_\tau^{\mu_i}\mathcal{U}(\chi,\zeta,\tau)&={}_0^{RL}\mathcal{I}_\tau^{2-\mu_i}\Big(\frac{\partial^2\mathcal{U}(\chi,\zeta,\tau)}{\partial\tau^2}\Big)\approx\chi^2\,\zeta^2\,\tau^{2-\mu_i}\,\mathbb{U}^\top\,\mathbb{B}_\chi^0\,\mathbb{B}_\chi^1\,\mathbb{B}_\zeta^0\,\mathbb{B}_\zeta^1\,\mathbb{B}_\tau^{0,2-\mu_i}\,\mathbf{\Psi}(\chi,\zeta,\tau)\\&+\chi\,{}_0^c\mathcal{D}_\tau^{\mu_i}Z_2(\zeta,\tau)-{}_0^c\mathcal{D}_\tau^{\mu_i}Z_2(0,\tau)-\zeta\,{}_0^c\mathcal{D}_\tau^{\mu_i}\frac{\partial Z_2(0,\tau)}{\partial\zeta}\\&+{}_0^c\mathcal{D}_\tau^{\mu_i}\mathsf{h}_1(\zeta,\tau)-{}_0^c\mathcal{D}_\tau^{\mu_i}\mathsf{h}_1(0,\tau)-\zeta\,{}_0^c\mathcal{D}_\tau^{\mu_i}\frac{\partial\mathsf{h}_1(0,\tau)}{\partial\zeta}+\zeta\,{}_0^{RL}\mathcal{I}_\tau^{2-\mu_i}Z_3(\chi,\tau)\\&+{}_0^c\mathcal{D}_\tau^{\mu_i}w_1(\chi,\tau),\quad i=1,2,\end{aligned}\tag{5.15}$$

$$\begin{aligned}{}_0^c\mathcal{D}_\tau^{\nu_i}\mathcal{U}(\chi,\zeta,\tau)&={}_0^{RL}\mathcal{I}_\tau^{1-\nu_i}\Big(\frac{\partial\mathcal{U}(\chi,\zeta,\tau)}{\partial\tau}\Big)\approx\chi^2\,\zeta^2\,\tau^{2-\nu_i}\,\mathbb{U}^\top\,\mathbb{B}_\chi^0\,\mathbb{B}_\chi^1\,\mathbb{B}_\zeta^0\,\mathbb{B}_\zeta^1\,\mathbb{B}_\tau^0\,\mathbb{B}_\tau^{1,1-\nu_i}\,\mathbf{\Psi}(\chi,\zeta,\tau)\\&+\chi\,{}_0^c\mathcal{D}_\tau^{\nu_i}Z_2(\zeta,\tau)-\chi\,\frac{\partial Z_2(\zeta,0)}{\partial\tau}\,\frac{\tau^{1-\nu_i}}{\Gamma(2-\nu_i)}-{}_0^c\mathcal{D}_\tau^{\nu_i}Z_2(0,\tau)+\frac{\partial Z_2(0,0)}{\partial\tau}\,\frac{\tau^{1-\nu_i}}{\Gamma(2-\nu_i)}\\&-\zeta\,\frac{\partial^2 Z_2(0,\tau)}{\partial\zeta\,\partial\tau}\,\frac{\tau^{1-\nu_i}}{\Gamma(2-\nu_i)}+\zeta\,\frac{\partial^2 Z_2(0,0)}{\partial\zeta\,\partial\tau}\,\frac{\tau^{1-\nu_i}}{\Gamma(2-\nu_i)}+{}_0^c\mathcal{D}_\tau^{\nu_i}\mathsf{h}_1(\zeta,\tau)\\&-\frac{\partial\mathsf{h}_1(\zeta,0)}{\partial\tau}\,\frac{\tau^{1-\nu_i}}{\Gamma(2-\nu_i)}-{}_0^c\mathcal{D}_\tau^{\nu_i}\mathsf{h}_1(0,\tau)+\frac{\partial\mathsf{h}_1(0,0)}{\partial\tau}\,\frac{\tau^{1-\nu_i}}{\Gamma(2-\nu_i)}-\zeta\,{}_0^c\mathcal{D}_\tau^{\nu_i}\frac{\partial\mathsf{h}_1(0,\tau)}{\partial\zeta}\\&+\zeta\,\frac{\partial^2\mathsf{h}_1(0,0)}{\partial\zeta\,\partial\tau}\,\frac{\tau^{1-\nu_i}}{\Gamma(2-\nu_i)}+\zeta\,{}_0^{RL}\mathcal{I}_\tau^{2-\nu_i}Z_3(\chi,\tau)+{}_0^c\mathcal{D}_\tau^{\nu_i}w_1(\chi,\tau)\\&-\frac{\partial w_1(\chi,0)}{\partial\tau}\,\frac{\tau^{1-\nu_i}}{\Gamma(2-\nu_i)}+\mathsf{g}_2(\chi,\zeta)\,\frac{\tau^{1-\nu_i}}{\Gamma(2-\nu_i)},\quad i=1,2.\end{aligned}\tag{5.16}$$

Integrating (5.1) with respect to ζ and using $\partial^3\mathcal{U}(\chi,0,\tau)/\partial\chi^2\,\partial\tau=Z_1(\chi,\tau)$ leads to:

$$\frac{\partial^5\mathcal{U}(\chi,\zeta,\tau)}{\partial\chi^2\,\partial\zeta\,\partial\tau}\approx\zeta\,\mathbb{U}^\top\,\mathbb{B}_\zeta^0\,\mathbf{\Psi}(\chi,\zeta,\tau)+\frac{\partial^2 Z_1(\chi,\tau)}{\partial\tau^2}.\tag{5.17}$$

Integrating (5.17) with respect to ζ and using the boundary condition $\mathcal{U}(\chi,0,\tau)=w_1(\chi,\tau)$ yields:

$$\frac{\partial^4\mathcal{U}(\chi,\zeta,\tau)}{\partial\chi^2\,\partial\tau}\approx\zeta^2\,\mathbb{U}^\top\,\mathbb{B}_\zeta^0\,\mathbb{B}_\zeta^1\,\mathbf{\Psi}(\chi,\zeta,\tau)+\frac{\partial^2 Z_1(\chi,\tau)}{\partial\tau^2}\zeta+\frac{\partial^4 w_1(\chi,\tau)}{\partial\chi^2\,\partial\tau^2}.\tag{5.18}$$

Integrating (5.18) with respect to τ and using the initial condition results in

$$\begin{aligned}\frac{\partial^3\mathcal{U}(\chi,\zeta,\tau)}{\partial\chi^2\,\partial\tau} \approx \zeta^2\,\tau\,\mathbb{U}^\top\,\mathbb{B}^0_\zeta\,\mathbb{B}^1_\zeta\,\mathbb{B}^0_\tau\,\mathbf{\Psi}(\chi,\zeta,\tau) + \frac{\partial \mathrm{Z}_1(\chi,\tau)}{\partial\tau}\zeta - \frac{\partial \mathrm{Z}_1(\chi,0)}{\partial\tau}\zeta + \frac{\partial^3 w_1(\chi,\tau)}{\partial\chi^2\,\partial\tau}\\ - \frac{\partial^3 w_1(\chi,0)}{\partial\chi^2\,\partial\tau} + \frac{\partial^2 \mathsf{g}_2(\chi,\zeta)}{\partial\chi^2}.\end{aligned} \tag{5.19}$$

Integrating (5.11) with respect to τ and using the initial condition yields

$$\begin{aligned}\frac{\partial^3\mathcal{U}(\zeta,\tau)}{\partial\zeta^2\,\partial\tau} \approx \chi^2\,\tau\,\mathbb{U}^\top\,\mathbb{B}^0_\chi\,\mathbb{B}^1_\chi\,\mathbb{B}^0_\tau\,\mathbf{\Psi}(\chi,\zeta,\tau) + \frac{\partial^3 \mathrm{Z}_2(\zeta,\tau)}{\partial\zeta^2\,\partial\tau}\chi - \frac{\partial^3 \mathrm{Z}_2(\zeta,0)}{\partial\zeta^2\,\partial\tau}\chi + \frac{\partial^3 \mathsf{h}_1(\zeta,\tau)}{\partial\zeta^2\,\partial\tau}\\ - \frac{\partial^3 \mathsf{h}_1(\zeta,0)}{\partial\zeta^2\,\partial\tau} + \frac{\partial^2 \mathsf{g}_2(\chi,\zeta)}{\partial\zeta^2}.\end{aligned} \tag{5.20}$$

From (5.19), one has

$$\begin{aligned}{}^{c}_{0}\mathcal{D}^{\iota}_{\tau}\frac{\partial^2\mathcal{U}(\chi,\zeta,\tau)}{\partial\chi^2} = {}^{RL}_{0}\mathcal{I}^{1-\iota}_{\tau}\Big(\frac{\partial^3\mathcal{U}(\chi,\zeta,\tau)}{\partial\chi^2\,\partial\tau}\Big) \approx \zeta^2\,\tau^{2-\iota}\,\mathbb{U}^\top\,\mathbb{B}^0_\zeta\,\mathbb{B}^1_\zeta\,\mathbb{B}^0_\tau\,\mathbb{B}^{1,1-\iota}_\tau\,\mathbf{\Psi}(\chi,\zeta,\tau) + \zeta\,{}^{c}_{0}\mathcal{D}^{\iota}_{\tau}\mathrm{Z}_1(\chi,\tau)\\ - \zeta\,\frac{\partial \mathrm{Z}_1(\chi,0)}{\partial\tau}\,\frac{\tau^{1-\iota}}{\Gamma(2-\iota)} + {}^{c}_{0}\mathcal{D}^{\iota}_{\tau}\frac{\partial^2 w_1(\chi,\tau)}{\partial\chi^2} - \frac{\partial^3 w_1(\chi,0)}{\partial\chi^2\,\partial\tau}\,\frac{\tau^{1-\iota}}{\Gamma(2-\iota)}\\ + \frac{\partial^2 \mathsf{g}_2(\chi,\zeta)}{\partial\chi^2}\,\frac{\tau^{1-\iota}}{\Gamma(2-\iota)}.\end{aligned} \tag{5.21}$$

It follows that from (5.20),

$$\begin{aligned}{}^{c}_{0}\mathcal{D}^{\iota}_{\tau}\frac{\partial^2\mathcal{U}(\chi,\zeta,\tau)}{\partial\zeta^2} = {}^{RL}_{0}\mathcal{I}^{1-\iota}_{\tau}\Big(\frac{\partial^3\mathcal{U}(\chi,\zeta,\tau)}{\partial\zeta^2\,\partial\tau}\Big) \approx \chi^2\,\tau^{2-\iota}\,\mathbb{U}^\top\,\mathbb{B}^0_\chi\,\mathbb{B}^1_\chi\,\mathbb{B}^0_\tau\,\mathbb{B}^{1,1-\iota}_\tau\,\mathbf{\Psi}(\chi,\zeta,\tau)\\ + {}^{c}_{0}\mathcal{D}^{\iota}_{\tau}\frac{\partial^2 \mathrm{Z}_2(\zeta,\tau)}{\partial\zeta^2} + {}^{c}_{0}\mathcal{D}^{\iota}_{\tau}\frac{\partial^2 \mathsf{h}_1(\zeta,\tau)}{\partial\zeta^2} - \chi\,\frac{\partial^3 \mathrm{Z}_2(\zeta,0)}{\partial\zeta^2\,\partial\tau}\,\frac{\tau^{1-\iota}}{\Gamma(2-\iota)}\\ - \frac{\partial^3 \mathsf{h}_1(\zeta,0)}{\partial\zeta^2\,\partial\tau}\,\frac{\tau^{1-\iota}}{\Gamma(2-\iota)} + \frac{\partial^2 \mathsf{g}_2(\chi,\zeta)}{\partial\zeta^2}\,\frac{\tau^{1-\iota}}{\Gamma(2-\iota)}.\end{aligned} \tag{5.22}$$

Substituting approximations (5.5), (5.7), (5.9), (5.14)–(5.16), (5.21), and (5.22) into (3.1) leads to a residual function as $\mathcal{R}_{\mathcal{N}}(\chi,\zeta,\tau)$. Collocating $\mathcal{R}(\chi,\zeta,\tau)$ at tensor points $(\chi_i,\zeta_j,\tau_k), i, j, k = 0, 1, \ldots, \mathcal{N}$, in which χ_i,ζ_j,τ_k are roots of $\mathcal{E}_{\mathcal{N}+1}(\chi)$, $\mathcal{E}_{\mathcal{N}+1}(\zeta)$, and $\mathcal{E}_{\mathcal{N}+1}(\tau)$, respectively, provides a system of $(\mathcal{N}+1)^3$ algebraic equations that by solving it, the unknown vector $\mathbb{U}$ is achieved. Therefore, an approximate solution $\mathcal{U}(\chi,\zeta,\tau)$ is acquired from (5.7).

To understand the mechanism of the proposed scheme, refer to Algorithm 1 below.

Algorithm 1 Solution procedure.

Input: $\mathcal{N}, \mathsf{f}(\chi, \zeta, \tau), \mu_k, \nu_k, k = 1, 2, \iota, \mathsf{g}_k, \mathsf{h}_k, w_k, k = 1, 2.$

Step 1. Construct the pseudo-operational matrices of integration using (4.4) and (4.5) for one-variable basis vector.

Step 2. Derive the pseudo-operational matrices corresponding to the three-variable basis vector using (4.7)–(4.10).

Step 3. Find approximations to terms in (3.1) using (5.5), (5.7), (5.9), (5.14)–(5.16), (5.21), and (5.22).

Step 4. Collocate the residual function $\mathcal{R}_{\mathcal{N}}(\chi, \zeta, \tau)$ at tensor points $(\chi_i, \zeta_j, \tau_k), i, j, k = 0, 1, \ldots, \mathcal{N}$.

Step 5. Solve the resulting linear system to obtain the coefficient vector $\mathbb{U}$ and then compute the approximate solution using Eq (5.7).

Output: $\mathcal{U}_{\mathcal{N}}(\chi, \zeta, \tau)$.

6. Error bounds

This section focuses on establishing the convergence of the proposed method. A rigorous theoretical analysis is carried out to show that the approximate solution converges to the exact solution as the number of basis functions increases, thereby guaranteeing the reliability and accuracy of the numerical scheme:

$$\mathcal{A}^m_{\varpi}(\mathbf{I}) = \left\{ u(\chi) \,\middle|\, \frac{d^k u(\chi)}{d\chi^k} \in L^2_{\varpi}(\mathbf{I}), \quad 0 \le k \le m \right\},$$

$$\mathcal{B}^n_{\varpi}(\mathbf{I}) = \left\{ w(\zeta) \,\middle|\, \frac{d^k w(\zeta)}{d\zeta^k} \in L^2_{\varpi}(\mathbf{I}), \quad 0 \le k \le n \right\},$$

$$C^r_{\varpi}(\mathbf{I}) = \left\{ z(\tau) \,\middle|\, \frac{d^{k+\eta} z(\tau)}{d\tau^{k+\eta}} \in L^2_{\varpi}(\mathbf{I}), \quad 0 \le k \le r, \quad \eta \in \mathbb{R}^+ \cup \{0\} \right\},$$

where $\mathbf{I} = [0, 1]$. The above spaces are equipped with the following inner product, norm, and semi-norm:

$$\langle u_1(\chi), u_2(\chi) \rangle_{\mathcal{A}^m_{\varpi}(\mathbf{I})} = \sum_{k=0}^{m} \left\langle \frac{d^k u_1(\chi)}{d\chi^k}, \frac{d^k u_2(\chi)}{d\chi^k} \right\rangle_{L^2_{\varpi}(\mathbf{I})}, \quad \|u\|^2_{\mathcal{A}^m_{\varpi}(\mathbf{I})} = \langle u(\chi), u(\chi) \rangle_{\mathcal{A}^m_{\varpi}(\mathbf{I})},$$

$$|u(\chi)|_{\mathcal{A}^m_{\varpi}(\mathbf{I})} = \left\| \frac{d^m u}{d\chi^m} \right\|_{L^2_{\varpi}(\mathbf{I})},$$

$$\langle w_1(\zeta), w_2(\zeta) \rangle_{\mathcal{B}^n_{\varpi}(\mathbf{I})} = \sum_{k=0}^{n} \left\langle \frac{d^k w_1(\zeta)}{d\zeta^k}, \frac{d^k w_2(\zeta)}{d\zeta^k} \right\rangle_{L^2_{\varpi}(\mathbf{I})}, \quad \|w\|^2_{\mathcal{B}^n_{\varpi}(\mathbf{I})} = \langle w(\zeta), w(\zeta) \rangle_{\mathcal{B}^n_{\varpi}(\mathbf{I})},$$

$$|w(\zeta)|_{\mathcal{B}^n_{\varpi}(\mathbf{I})} = \left\| \frac{d^n w}{d\zeta^n} \right\|_{L^2_{\varpi}(\mathbf{I})},$$

$$\langle z_1(\tau), z_2(\tau) \rangle_{C^r_{\varpi}(\mathbf{I})} = \sum_{k=0}^{r} \left\langle \frac{d^{k+\eta} z_1(\tau)}{d\tau^{k+\eta}}, \frac{d^{k+\eta} z_2(\tau)}{d\tau^{k+\eta}} \right\rangle_{L^2_{\varpi}(\mathbf{I})}, \quad \|z\|^2_{C^r_{\varpi}(\mathbf{I})} = \langle z(\tau), z(\tau) \rangle_{C^r_{\varpi}(\mathbf{I})},$$

$$|z(\tau)|_{C^r_{\varpi}(\mathbf{I})} = \left\| \frac{d^{r+\eta} z}{d\tau^{r+\eta}} \right\|_{L^2_{\varpi}(\mathbf{I})},$$

where $m,n,r \in \mathbb{N}\cup\{0\}$. A three-dimensional weighted space is defined as follows:

$$\mathbb{B}^{m,n,r}_{\Omega}(\Delta)=\left\{Z(\chi,\zeta,\tau)\,\middle|\,\frac{\partial^{i+j+k+\eta}Z(\chi,\zeta,\tau)}{\partial\chi^{i}\,\partial\zeta^{j}\,\partial\tau^{k+\eta}}\in L^{2}_{\Omega}(\Delta),\quad 0\le i\le m, 0\le j\le n, 0\le k\le r\right\},$$

where $\eta\in(0,1)$ or $(1,2)$. The above space is equipped with the following norm and semi-norm:

$$\|Z\|_{\mathbb{B}^{r,s}_{\Omega}(\Delta)}=\Big(\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{r}\left\|\frac{\partial^{i+j+k+\eta}Z}{\partial\chi^{i}\,\partial\zeta^{j}\,\partial\tau^{k+\eta}}\right\|^{2}_{L^{2}_{\Omega}(\Delta)}\Big)^{\frac{1}{2}},\quad |Z(\chi,\zeta,\tau)|_{\mathbb{B}^{m,n,r}_{\Omega}(\mathbb{B})}=\left\|\frac{\partial^{m+n+r+\eta}Z}{\partial\chi^{m}\,\partial\zeta^{n},\,\partial\tau^{r+\eta}}\right\|_{L^{2}_{\Omega}(\Delta)}.$$

Theorem 6.1. *Suppose that $z_{\mathcal{N}}(\tau)$ is the Chebyshev approximation to $z(\tau)\in C^{r}_{\varpi}(\mathbf{I})$. For $0\le k\le r<\mathcal{N}+1$, $\eta\in(0,1)$, or $(1,2)$, $\lceil\eta\rceil=n'$, one has*

$$\|{}^{c}_{0}\mathcal{D}^{k+\eta}_{\tau}(z-z_{\mathcal{N}})\|_{L^{2}_{\varpi}(\mathbf{I})}\lesssim\mathcal{N}^{-(r-k)}\,(\mathcal{N}-\eta+1)^{-\frac{3}{4}(r-k)}\,|z(\tau)|_{C^{r}_{\varpi}(\mathbf{I})}.\tag{6.1}$$

Proof. From the series representation of $z(\tau)$ and $z_{\mathcal{N}}(\tau)$ and the Stirling formula [40], one has

$$\begin{aligned}\|{}^{c}_{0}\mathcal{D}^{k+\eta}_{\tau}(z-z_{\mathcal{N}})\|^{2}_{L^{2}_{\varpi}(\mathbf{I})}&\lesssim\sum_{i=\mathcal{N}+1}^{\infty}\mathsf{z}_{i}^{2}\,\|\,{}^{c}_{0}\mathcal{D}^{k+\eta}_{\tau}\mathcal{E}_{i}\,\|^{2}_{L^{2}_{\varpi}(\mathbf{I})}\\&=\sum_{i=\mathcal{N}+1}^{\infty}\mathsf{z}_{i}^{2}\,\frac{\|{}^{c}_{0}\mathcal{D}^{k+\eta}_{\tau}\mathcal{E}_{i}\|^{2}_{L^{2}_{\varpi}(\mathbf{I})}}{\|{}^{c}_{0}\mathcal{D}^{r+\eta}_{\tau}\mathcal{E}_{i}\|^{2}_{L^{2}_{\varpi}(\mathbf{I})}}\|{}^{c}_{0}\mathcal{D}^{r+\eta}_{\tau}\mathcal{E}_{i}\|^{2}_{L^{2}_{\varpi}(\mathbf{I})}\\&\lesssim\frac{\|{}^{c}_{0}\mathcal{D}^{k+\eta}_{\tau}\mathcal{E}_{\mathcal{N}+1}\|^{2}_{L^{2}_{\varpi}(\mathbf{I})}}{\|{}^{c}_{0}\mathcal{D}^{r+\eta}_{\tau}\mathcal{E}_{\mathcal{N}+1}\|^{2}_{L^{2}_{\varpi}(\mathbf{I})}}|z(\tau)|_{C^{r}_{\varpi}(\mathbf{I})}.\end{aligned}$$

Now, evaluate the value of $\|{}^{c}_{0}\mathcal{D}^{k+\eta}_{\tau}\mathcal{E}_{\mathcal{N}+1}\|^{2}_{L^{2}_{\varpi}(\mathbf{I})}$:

$$\begin{aligned}&\|{}^{c}_{0}\mathcal{D}^{k+\eta}_{\tau}\mathcal{E}_{\mathcal{N}+1}\|^{2}_{L^{2}_{\varpi}(\mathbf{I})}=\int_{0}^{1}({}^{c}_{0}\mathcal{D}^{k+\eta}_{\tau}\mathcal{E}_{\mathcal{N}+1}(\tau))^{2}\,\varpi(\tau)\,d\tau\\&=\sum_{i=k+n'}^{\mathcal{N}+1}\sum_{j=k+n'}^{\mathcal{N}+1}\rho_{i,\mathcal{N}+1}\,\rho_{j,\mathcal{N}+1}\int_{0}^{1}\frac{\Gamma(i+1)\,\Gamma(j+1)}{\Gamma(i-k-\eta+1)\,\Gamma(j-k-\eta+1)}\tau^{i+j-2k-2\eta}\varpi(\tau)\,d\tau\\&=\sum_{i=k+n'}^{\mathcal{N}+1}\sum_{j=k+n'}^{\mathcal{N}+1}\frac{\sqrt{\pi}}{2}\rho_{i,\mathcal{N}+1}\,\rho_{j,\mathcal{N}+1}\,\frac{\Gamma(i+1)\,\Gamma(j+1)}{\Gamma(i-k-\eta+1)\,\Gamma(j-k-\eta+1)}\Big[16\frac{\Gamma(i+j-2k-2\eta+\frac{11}{2})}{\Gamma(i+j-2k-2\eta+7)}\\&-32\frac{\Gamma(i+j-2k-2\eta+\frac{9}{2})}{\Gamma(i+j-2k-2\eta+6)}+24\frac{\Gamma(i+j-2k-2\eta+\frac{7}{2})}{\Gamma(i+j-2k-2\eta+5)}-8\frac{\Gamma(i+j-2k-2\eta+\frac{5}{2})}{\Gamma(i+j-2k-2\eta+4)}\\&+\frac{\Gamma(i+j-2k-2\eta+\frac{3}{2})}{\Gamma(i+j-2k-2\eta+3)}\Big]\\&\lesssim\sum_{i=k+n'}^{\mathcal{N}+1}\sum_{j=k+n'}^{\mathcal{N}+1}\frac{\sqrt{\pi}}{2}\rho_{i,\mathcal{N}+1}\,\rho_{j,\mathcal{N}+1}\,\frac{\Gamma(i+1)\,\Gamma(j+1)\,\Gamma(i+j-2k-2\eta+\frac{11}{2})}{\Gamma(i-k-\eta+1)\,\Gamma(j-k-\eta+1)\,\Gamma(i+j-2k-2\eta+7)}\\&\lesssim\sum_{i=k+n'}^{\mathcal{N}+1}\sum_{j=k+n'}^{\mathcal{N}+1}\frac{\sqrt{\pi}}{2}\rho_{i,\mathcal{N}+1}\,\rho_{j,\mathcal{N}+1}\,(i+1)^{k+\eta}\,(j+1)^{k+\eta}\,(i+j-2k-2\eta)^{-\frac{3}{2}}\end{aligned}$$

$$\lesssim \mathcal{L}_{\eta,*}\,(\mathcal{N}+1)^{2(k+\eta)}\,(2\mathcal{N}-2k-2\eta)^{-\frac{3}{2}}\,(\mathcal{N}-k-n'+2)^{2}$$
$$\lesssim \mathcal{L}_{\eta,*}\,2^{-\frac{3}{2}}\left(\frac{\Gamma(\mathcal{N}+3)}{\Gamma(\mathcal{N}+2)}\right)^{2(k+\eta)}\left(\frac{\Gamma(\mathcal{N}-k-n'+3)}{\Gamma(\mathcal{N}-k-n'+2)}\right)^{2}(\mathcal{N}-k-\eta+1)^{-\frac{3}{2}}$$
$$\lesssim \mathcal{L}_{\eta,*}\,2^{-\frac{3}{2}}\,\mathcal{N}^{2(k+\eta+1)}\,(\mathcal{N}-k-\eta+1)^{-\frac{3}{2}},$$

where $\mathcal{L}_{\eta,*}=\max\limits_{0\le i,j\le \mathcal{N}+1}\{\sqrt{\pi}/2\,\rho_{i,\mathcal{N}+1}\,\rho_{j,\mathcal{N}+1}\}$. Similarly, one gets

$$\|{}_0^c\mathcal{D}_\tau^{r+\eta}\mathcal{E}_{\mathcal{N}+1}\|^2_{L^2_\varpi(\mathbf{I})}\lesssim \mathcal{L}_{\eta,**}\,2^{-\frac{3}{2}}\,\mathcal{N}^{2(r+\eta+1)}\,(\mathcal{N}-r-\eta+1)^{-\frac{3}{2}}.$$

Therefore,

$$\frac{\|{}_0^c\mathcal{D}_\tau^{k+\eta}\mathcal{E}_{\mathcal{N}+1}\|^2_{L^2_\varpi(\mathbf{I})}}{\|{}_0^c\mathcal{D}_\tau^{r+\eta}\mathcal{E}_{\mathcal{N}+1}\|^2_{L^2_\varpi(\mathbf{I})}}\lesssim \frac{\mathcal{L}_{\eta,*}\,2^{-\frac{3}{2}}\,\mathcal{N}^{2(k+\eta+1)}\,(\mathcal{N}-k-\eta+1)^{-\frac{3}{2}}}{\mathcal{L}_{\eta,**}\,2^{-\frac{3}{2}}\,\mathcal{N}^{2(r+\eta+1)}\,(\mathcal{N}-r-\eta+1)^{-\frac{3}{2}}}$$
$$\lesssim \mathcal{N}^{-2(r-k)}\,(\mathcal{N}-\eta+1)^{-\frac{3}{2}(r-k)}.$$

Finally, taking the square root and combining the resultant bounds leads to the desired result. □

Corollary 6.2. *If* $0\le k\le m<\mathcal{N}+1$, $0\le l\le n<\mathcal{N}+1$, $u_{\mathcal{N}}(\chi)$, *and* $w_{\mathcal{N}}(\zeta)$ *are approximations of* $u(\chi)\in\mathcal{A}^m_\varpi(\mathbf{I})$ *and* $w(\zeta)\in\mathcal{B}^n_\varpi(\mathbf{I})$, *respectively, then one has*

$$\left\|\frac{d^k(u-u_{\mathcal{N}})}{d\chi^k}\right\|_{L^2_\varpi(\mathbf{I})}\lesssim \mathcal{N}^{-(m-k)}\,(\mathcal{N}+1)^{-\frac{3}{4}(m-k)}\,|u(\chi)|_{\mathcal{A}^m_\varpi(\mathbf{I})},\tag{6.2}$$

$$\left\|\frac{d^l(w-w_{\mathcal{N}})}{d\zeta^l}\right\|_{L^2_\varpi(\mathbf{I})}\lesssim \mathcal{N}^{-(n-l)}\,(\mathcal{N}+1)^{-\frac{3}{4}(n-l)}\,|w(\zeta)|_{\mathcal{B}^n_\varpi(\mathbf{I})}.\tag{6.3}$$

Theorem 6.3. *If* $\eta\notin\mathbb{Z}$, $\lceil\eta\rceil=n'$, $0\le k\le m<\mathcal{N}+1$, $0\le l\le n<\mathcal{N}+1$, $0\le s\le r<\mathcal{N}+1$, $Z_{\mathcal{N}+1}(\chi,\zeta,\tau)$ *is a Chebyshev approximation of* $Z(\chi,\zeta,\tau)\in\mathbb{B}^{m,n,r}_\Omega(\Delta)$, *then one has*

$$\begin{aligned}\left\|\frac{\partial^k(Z-Z_{\mathcal{N}})}{\partial\chi^k}\right\|_{L^2_\varpi(\Delta)}&\lesssim\sqrt{3}\,\mathcal{N}^{-(m-k)}\,(\mathcal{N}+1)^{-\frac{3}{4}(m-k)}\,|Z(\chi,\zeta,\tau)|_{\mathbb{B}^{m,0,0}_\Omega(\Delta)},\\ \left\|\frac{\partial^l(Z-Z_{\mathcal{N}})}{\partial\zeta^l}\right\|_{L^2_\varpi(\Delta)}&\lesssim\sqrt{3}\,\mathcal{N}^{-(n-l)}\,(\mathcal{N}+1)^{-\frac{3}{4}(n-l)}\,|Z(\chi,\zeta,\tau)|_{\mathbb{B}^{0,n,0}_\Omega(\Delta)},\\ \left\|{}_0^c\mathcal{D}_\tau^{s+\eta}(Z-Z_{\mathcal{N}})\right\|_{L^2_\varpi(\Delta)}&\lesssim\sqrt{3}\,\mathcal{N}^{-(r-s)}\,(\mathcal{N}-\eta+1)^{-\frac{3}{4}(r-s)}\,|Z(\chi,\zeta,\tau)|_{\mathbb{B}^{0,0,r}_\Omega(\Delta)}.\end{aligned}\tag{6.4}$$

Proof. From the series representation of $Z(\chi,\zeta,\tau)$ and $Z_{\mathcal{N}}(\chi,\zeta,\tau)$, one has

$$\begin{aligned}\frac{\partial^k(Z(\chi,\zeta,\tau)-Z_{\mathcal{N}}(\chi,\zeta,\tau))}{\partial\chi^k}&=\sum_{i=k}^{\mathcal{N}}\sum_{j=0}^{\mathcal{N}}\sum_{k'=\mathcal{N}+1}^{\infty}\mathsf{Z}_{i,j,k'}\,\frac{\partial^k\mathbb{E}_{i,j,k'}(\chi,\zeta,\tau)}{\partial\chi^k}\\&+\sum_{i=k}^{\mathcal{N}}\sum_{j=\mathcal{N}+1}^{\infty}\sum_{k'=0}^{\infty}\mathsf{Z}_{i,j,k'}\,\frac{\partial^k\mathbb{E}_{i,j,k'}(\chi,\zeta,\tau)}{\partial\chi^k}\\&+\sum_{i=\mathcal{N}+1}^{\infty}\sum_{j=0}^{\infty}\sum_{k'=0}^{\infty}\mathsf{Z}_{i,j,k'}\,\frac{\partial^k\mathbb{E}_{i,j,k'}(\chi,\zeta,\tau)}{\partial\chi^k}.\end{aligned}$$

Applying the norm and utilizing Theorem 6.1 and Corollary 6.2, the desired results are obtained. □

Corollary 6.4. *If $k = l = s = 0$, then one has*

$$
\begin{aligned}
\|Z - Z_{\mathcal{N}}\|_{L^2_{\varpi}(\Delta)} \lesssim \max\{ &\sqrt{3}\,\mathcal{N}^{-m}\,(\mathcal{N}+1)^{-\frac{3}{4}m}\,|Z(\chi,\zeta,\tau)|_{\mathbb{B}^{m,0,0}_{\Omega}},\\
&\sqrt{3}\,\mathcal{N}^{-n}\,(\mathcal{N}+1)^{-\frac{3}{4}n}\,|Z(\chi,\zeta,\tau)|_{\mathbb{B}^{0,n,0}_{\Omega}},\\
&\sqrt{3}\,\mathcal{N}^{-r}\,(\mathcal{N}+1)^{-\frac{3}{4}r}\,|Z(\chi,\zeta,\tau)|_{\mathbb{B}^{0,0,r}_{\Omega}}\}\\
= {}&\Lambda_{\mathcal{N},m,n,r}.
\end{aligned}
\tag{6.5}
$$

Corollary 6.5. *If $2 \le m < \mathcal{N}+1, 2 \le n < \mathcal{N}+1$, $\lceil \eta \rceil \le r < \mathcal{N}+1$, $m + r \ge 2$, $n + r \ge 2$, for mixed derivatives, one can obtain the following bounds:*

$$
\begin{aligned}
&\left\| {}^{c}_{0}\mathcal{D}^{\eta}_{\tau} \frac{\partial^2 (Z - Z_{\mathcal{N}})}{\partial \chi^2} \right\|_{L^2_{\Omega}(\Delta)} \lesssim 3\,\mathcal{N}^{-(m+r-s-2)}\,(\mathcal{N}+1)^{-\frac{3}{4}(m-2)}\,(\mathcal{N}-\eta+1)^{-\frac{3}{4}(r-s)}\\
&\quad\times |Z(\chi,\zeta,\tau)|_{\mathbb{B}^{m,0,0}_{\Omega}(\Delta)}\,|Z(\chi,\zeta,\tau)|_{\mathbb{B}^{0,0,r}_{\Omega}(\Delta)},\\
&\left\| {}^{c}_{0}\mathcal{D}^{\eta}_{\tau} \frac{\partial^2 (Z - Z_{\mathcal{N}})}{\partial \zeta^2} \right\|_{L^2_{\Omega}(\Delta)} \lesssim 3\,\mathcal{N}^{-(n+r-s-2)}\,(\mathcal{N}+1)^{-\frac{3}{4}(n-2)}\,(\mathcal{N}-\eta+1)^{-\frac{3}{4}(r-s)}\\
&\quad\times |Z(\chi,\zeta,\tau)|_{\mathbb{B}^{0,n,0}_{\Omega}(\Delta)}\,|Z(\chi,\zeta,\tau)|_{\mathbb{B}^{0,0,r}_{\Omega}(\Delta)}.
\end{aligned}
\tag{6.6}
$$

If $\mathcal{U}_{\mathcal{N}}(\chi,\zeta,\tau)$ is the approximate solution to the given problem, then it satisfies the following equation:

$$
\begin{aligned}
\mathcal{R}_{\mathcal{N}}(\chi,\zeta,\tau) = {}&\alpha_1\,{}^{c}_{0}\mathcal{D}^{\mu_1}_{\tau}\mathcal{U}_{\mathcal{N}}(\chi,\zeta,\tau) + \alpha_2\,{}^{c}_{0}\mathcal{D}^{\mu_2}_{\tau}\mathcal{U}_{\mathcal{N}}(\chi,\zeta,\tau) + \beta_1\,{}^{c}_{0}\mathcal{D}^{\nu_1}_{\tau}\mathcal{U}_{\mathcal{N}}(\chi,\zeta,\tau) + \beta_2\,{}^{c}_{0}\mathcal{D}^{\nu_2}_{\tau}\mathcal{U}_{\mathcal{N}}(\chi,\zeta,\tau)\\
&+ \eta_1\,\frac{\partial \mathcal{U}_{\mathcal{N}}(\chi,\zeta,\tau)}{\partial \tau} + \eta_2\,\mathcal{U}_{\mathcal{N}}(\chi,\zeta,\tau) - \eta_3\,\frac{\partial^2 \mathcal{U}_{\mathcal{N}}(\chi,\zeta,\tau)}{\partial \chi^2} - \eta_4\,\frac{\partial^2 \mathcal{U}_{\mathcal{N}}(\chi,\zeta,\tau)}{\partial \zeta^2}\\
&- \gamma_1\,{}^{c}_{0}\mathcal{D}^{\iota}_{\tau}\frac{\partial^2 \mathcal{U}_{\mathcal{N}}(\chi,\zeta,\tau)}{\partial \chi^2} - \gamma_2\,{}^{c}_{0}\mathcal{D}^{\iota}_{\tau}\frac{\partial^2 \mathcal{U}_{\mathcal{N}}(\chi,\zeta,\tau)}{\partial \zeta^2} - \mathsf{f}(\chi,\zeta,\tau).
\end{aligned}
\tag{6.7}
$$

Subtracting (6.7) from (3.1) and using the definition of error function as $e_{\mathcal{N}} = \mathcal{U} - \mathcal{U}_{\mathcal{N}}$, one finds the following error equation:

$$
\begin{aligned}
\mathcal{R}_{\mathcal{N}}(\chi,\zeta,\tau) = {}&\alpha_1\,{}^{c}_{0}\mathcal{D}^{\mu_1}_{\tau}e_{\mathcal{N}}(\chi,\zeta,\tau) + \alpha_2\,{}^{c}_{0}\mathcal{D}^{\mu_2}_{\tau}e_{\mathcal{N}}(\chi,\zeta,\tau) + \beta_1\,{}^{c}_{0}\mathcal{D}^{\nu_1}_{\tau}e_{\mathcal{N}}(\chi,\zeta,\tau) + \beta_2\,{}^{c}_{0}\mathcal{D}^{\nu_2}_{\tau}e_{\mathcal{N}}(\chi,\zeta,\tau)\\
&+ \eta_1\,\frac{\partial e_{\mathcal{N}}(\chi,\zeta,\tau)}{\partial \tau} + \eta_2\,e_{\mathcal{N}}(\chi,\zeta,\tau) - \eta_3\,\frac{\partial^2 e_{\mathcal{N}}(\chi,\zeta,\tau)}{\partial \chi^2} - \eta_4\,\frac{\partial^2 e_{\mathcal{N}}(\chi,\zeta,\tau)}{\partial \zeta^2}\\
&- \gamma_1\,{}^{c}_{0}\mathcal{D}^{\iota}_{\tau}\frac{\partial^2 e_{\mathcal{N}}(\chi,\zeta,\tau)}{\partial \chi^2} - \gamma_2\,{}^{c}_{0}\mathcal{D}^{\iota}_{\tau}\frac{\partial^2 e_{\mathcal{N}}(\chi,\zeta,\tau)}{\partial \zeta^2} - \mathsf{f}(\chi,\zeta,\tau).
\end{aligned}
\tag{6.8}
$$

Taking the norm from (6.8) and utilizing the obtained error bounds, one gets

$$
\begin{aligned}
\|\mathcal{R}_{\mathcal{N}}\|_{L^2_{\Omega}(\Delta)} \lesssim {}&\sqrt{3}\,|\alpha_1|\,\mathcal{N}^{-r}\,(\mathcal{N}-\mu_1+1)^{-\frac{3}{4}r}\,|Z(\chi,\zeta,\tau)|_{\mathbb{B}^{0,0,r}_{\Omega}(\Delta)}\\
&+ \sqrt{3}\,|\alpha_2|\,\mathcal{N}^{-r}\,(\mathcal{N}-\mu_2+1)^{-\frac{3}{4}r}\,|Z(\chi,\zeta,\tau)|_{\mathbb{B}^{0,0,r}_{\Omega}(\Delta)}\\
&+ \sqrt{3}\,|\beta_1|\,\mathcal{N}^{-r}\,(\mathcal{N}-\nu_1+1)^{-\frac{3}{4}r}\,|Z(\chi,\zeta,\tau)|_{\mathbb{B}^{0,0,r}_{\Omega}(\Delta)}\\
&+ \sqrt{3}\,|\beta_2|\,\mathcal{N}^{-r}\,(\mathcal{N}-\nu_2+1)^{-\frac{3}{4}r}\,|Z(\chi,\zeta,\tau)|_{\mathbb{B}^{0,0,r}_{\Omega}(\Delta)}\\
&+ \sqrt{3}\,|\eta_1|\,\mathcal{N}^{-(r-1)}\,(\mathcal{N}+1)^{-\frac{3}{4}(r-1)}\,|Z(\chi,\zeta,\tau)|_{\mathbb{B}^{0,0,r}_{\Omega}(\Delta)} + |\eta_2|\,\Lambda_{\mathcal{N},m,n,r}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sqrt{3}\,|\eta_3|\,\mathcal{N}^{-(m-2))}\,(\mathcal{N}+1)^{-\frac{3}{4}(m-2))}\,|Z(\chi,\zeta,\tau)|_{\mathbb{B}_{\Omega}^{m,0,0}(\Delta)}\\
&+ \sqrt{3}\,|\eta_4|\,\mathcal{N}^{-(n-2))}\,(\mathcal{N}+1)^{-\frac{3}{4}(n-2))}\,|Z(\chi,\zeta,\tau)|_{\mathbb{B}_{\Omega}^{0,n,0}(\Delta)}\\
&+ 3\mathcal{N}^{-(m+r-s-2)}\,(\mathcal{N}+1)^{-\frac{3}{4}(m-2)}\,(\mathcal{N}-\iota+1)^{-\frac{3}{4}(r-s)}|Z(\chi,\zeta,\tau)|_{\mathbb{B}_{\Omega}^{m,0,0}(\Delta)}\,|Z(\chi,\zeta,\tau)|_{\mathbb{B}_{\Omega}^{0,0,r}(\Delta)}\\
&+ 3\,\mathcal{N}^{-(n+r-s-2)}\,(\mathcal{N}+1)^{-\frac{3}{4}(n-2)}\,(\mathcal{N}-\iota+1)^{-\frac{3}{4}(r-s)}\,|Z(\chi,\zeta,\tau)|_{\mathbb{B}_{\Omega}^{0,n,0}(\Delta)}\,|Z(\chi,\zeta,\tau)|_{\mathbb{B}_{\Omega}^{0,0,r}(\Delta)}.
\end{aligned}
$$

By selecting an appropriate value of $\mathcal{N}$, the right-hand side of the above inequality can be sufficiently small.

7. Illustrated examples

In this section, the proposed algorithm is applied to time-fractional multiterm diffusion–wave equations to illustrate its accuracy and computational efficiency. The numerical results are reported in several tables and figures. The maximum absolute errors (MAEs) are evaluated, and the performance of the proposed method is compared with the results presented in [13, 41], where spectral element and Legendre spectral methods were employed to solve the same problems.

Example 7.1. *Consider Eq* (1.1) *as follows:*

$$
{}_0^c\mathcal{D}_\tau^\mu \mathcal{U}(\chi,\zeta,\tau)+\frac{\partial \mathcal{U}(\chi,\zeta,\tau)}{\partial\tau}+{}_0^c\mathcal{D}_\tau^\nu \mathcal{U}(\chi,\zeta,\tau)+\mathcal{U}(\chi,\zeta,\tau)-\frac{\partial^2 \mathcal{U}(\chi,\zeta,\tau)}{\partial\chi^2}-\frac{\partial^2 \mathcal{U}(\chi,\zeta,\tau)}{\partial\zeta^2}=\mathsf{f}(\chi,\zeta,\tau), \quad (7.1)
$$

where $(\chi,\zeta,\tau)\in\Delta=[0,1]^3$*, and the exact solution is* $\mathcal{U}(\chi,\zeta,\tau)=\tau^{\mu+\nu}\sin(\chi)\sin(\zeta)$*. The initial and boundary conditions are as follows:*

$$
\begin{aligned}
&\mathcal{U}(\chi,\zeta,0)=\frac{\partial \mathcal{U}(\chi,\zeta,0)}{\partial\tau}=0,\quad \mathcal{U}(0,\zeta,\tau)=0,\quad \mathcal{U}(1,\zeta,\tau)=\sin(1)\,\tau^{\mu+\nu}\sin(\zeta),\\
&\mathcal{U}(\chi,0,\tau)=0,\quad \mathcal{U}(\chi,1,\tau)=\sin(1)\,\tau^{\mu+\nu}\sin(\chi).
\end{aligned}
$$

Equation (7.1) *is solved by applying the suggested method for diverse choices of* $\mathcal{N},\mu$*, and* ν*. The values of maximum absolute errors (MAEs) are seen in Table 1 for* $\mathcal{N}=2,\tau=0.5$*, and different values of* μ *and* ν*. Values of absolute error of the obtained solutions are calculated at the points* $\chi_k=\zeta_k=0.1,0.3,\ldots,0.9$ *for* $\tau=0.5,\mathcal{N}=2$*, and different choices of* μ,ν *in Table 2. The three-dimensional figures of absolute error functions are depicted in Figure 1 for* $\mathcal{N}=2,3,4$*,* $(\mu,\nu)=(1.8,0.2)$*, and* $\tau=0.5$*. Overall, the numerical results demonstrate that the proposed method produces accurate and reliable solutions, as evidenced by the small maximum absolute errors and their consistent improvement with increasing* $\mathcal{N}$*, confirming the effectiveness of the approach. The equation in* (7.1) *was solved using spectral element and Legendre spectral methods in [13, 41], respectively, and the reported absolute errors are* 4.4709×10^{-5} *(for* $(N,M)=(2,4)$*) and* 9.4589×10^{-5} *(for* $(M,N_1,N_2)=(6,6,6)$*) for* $(\mu,\nu)=(1.8,0.3)$*, respectively*

Table 1. MAEs for diverse choices of μ,ν, and $\mathcal{N}=2$ in Example 7.1.

(μ,ν)	$(1.8,0.2)$	$(1.8,0.3)$	$(1.6,0.5)$	$(1.4,0.8)$	$(1.1,0.6)$
MAE	1.4744×10^{-5}	5.6905×10^{-5}	2.9173×10^{-5}	4.3609×10^{-5}	1.6111×10^{-4}

Table 2. Absolute errors of approximate solutions at selected points for $\mathcal{N} = 2$ and diverse choices of μ, ν in Example 7.1.

$\chi_k = \zeta_k$	$(1.1, 0.6)$	$(1.8, 0.2)$	$(1.8, 0.3)$	$(1.6, 0.5)$
0.1	1.4740×10^{-8}	2.2202×10^{-9}	6.5050×10^{-9}	6.0919×10^{-10}
03	1.1512×10^{-8}	1.5901×10^{-8}	1.2727×10^{-8}	2.2538×10^{-8}
0.5	2.5907×10^{-6}	5.8315×10^{-7}	3.5536×10^{-7}	2.6303×10^{-7}
0.7	1.0391×10^{-5}	3.2270×10^{-6}	1.9816×10^{-6}	1.0537×10^{-6}
0.9	4.8679×10^{-5}	1.3508×10^{-7}	1.4011×10^{-5}	1.0040×10^{-5}

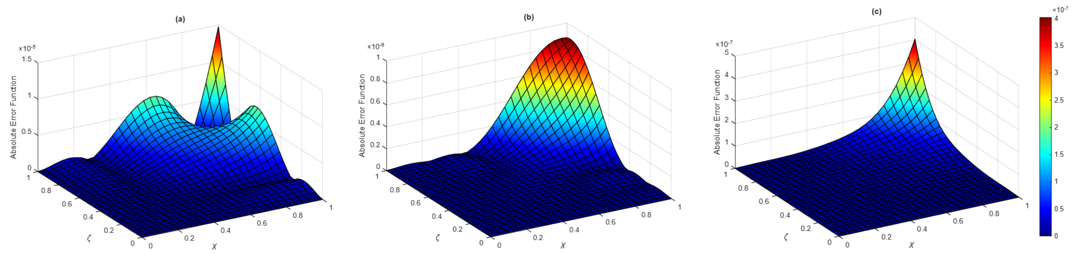

Figure 1. Absolute error functions for (a) $\mathcal{N} = 2$, (b) $\mathcal{N} = 3$, (c) $\mathcal{N} = 4$, $(\mu, \nu) = (1.8, 0.2)$, $\tau = 0.5$ in Example 7.1.

Example 7.2. *Consider Eq* (1.1) *as follows:*

$$\begin{aligned} {}_0^c\mathcal{D}_\tau^\mu \mathcal{U}(\chi,\zeta,\tau) + \frac{\partial \mathcal{U}(\chi,\zeta,\tau)}{\partial \tau} + {}_0^c\mathcal{D}_\tau^\nu \mathcal{U}(\chi,\zeta,\tau) + \mathcal{U}(\chi,\zeta,\tau) - {}_0^c\mathcal{D}_\tau^\gamma \frac{\partial^2 \mathcal{U}(\chi,\zeta,\tau)}{\partial \chi^2} \\ - {}_0^c\mathcal{D}_\tau^\gamma \frac{\partial^2 \mathcal{U}(\chi,\zeta,\tau)}{\partial \zeta^2} - \frac{\partial^2 \mathcal{U}(\chi,\zeta,\tau)}{\partial \chi^2} - \frac{\partial^2 \mathcal{U}(\chi,\zeta,\tau)}{\partial \zeta^2} = \mathrm{f}(\chi,\zeta,\tau), \end{aligned} \tag{7.2}$$

where $(\chi, \zeta, \tau) \in \Delta = [0, 1]^3$, *and the exact solution is* $\mathcal{U}(\chi, \zeta, \tau) = (\tau^3 + 1)\sin(\chi)\sin(\zeta)$. *The initial and boundary conditions are below:*

$$\begin{aligned} &\mathcal{U}(\chi,\zeta,0) = \sin(\chi)\sin(\zeta), \quad \frac{\partial \mathcal{U}(\chi,\zeta,0)}{\partial \tau} = 0, \quad \mathcal{U}(0,\zeta,\tau) = 0, \\ &\mathcal{U}(1,\zeta,\tau) = \sin(1)(\tau^3+1)\sin(\zeta), \quad \mathcal{U}(\chi,0,\tau) = 0, \quad \mathcal{U}(\chi,1,\tau) = \sin(1)(\tau^3+1)\sin(\chi). \end{aligned}$$

Equation (7.2) *is solved by applying the suggested method for diverse choices of* $\mathcal{N}, \mu, \nu$, *and* γ. *The values of MAEs are seen in Table 3 for* $\mathcal{N} = 2, 3, 4, 5, \tau = 0.5$, *and* $(\mu, \nu, \gamma) = (1.6, 0.7, 0.8), (1.5, 0.2, 0.3)$. *Values of absolute error of the obtained solutions are calculated at the points* $\chi_k = \zeta_k = 0.1, 0.3, \ldots, 0.9$ *for* $\tau = 0.5, \mathcal{N} = 5$, *and* $(\mu, \nu, \gamma) = (1.6, 0.7, 0.8), (1.5, 0.2, 0.3)$ *in Table 4. The three-dimensional figures of the exact solution, the approximate solution, and the absolute error function are plotted in Figure 2 for* $\tau = 0.5$, $\mathcal{N} = 4$, *and* $(\mu, \nu, \gamma) = (1.6, 0.7, 0.8)$. *The 2D heat maps of absolute error functions are depicted in Figure 3 for* $\mathcal{N} = 2, 3, 4, 5$, $(\mu, \nu, \gamma) = (1.5, 0.2, 0.3)$, *and* $\tau = 0.5$. *The numerical results confirm the accuracy and robustness of the proposed method. The reported MAEs decrease consistently as the value of* $\mathcal{N}$ *increases, and the close agreement between the exact and approximate solutions, illustrated by the three-dimensional plots and heat* *maps, demonstrates the effectiveness and convergence of the approach for different choices of the fractional orders. Equation* (7.2) *was solved applying the spectral element and Legendre spectral methods [13, 41], and the obtained maximum absolute errors are* 8.0586×10^{-6} *(*$N = M = 4$*) and* 2.6460×10^{-3} *(*$M = N = \mathcal{N} = 4$*), respectively, for* $(\mu, \nu) = (0.7, 0.8)$*.*

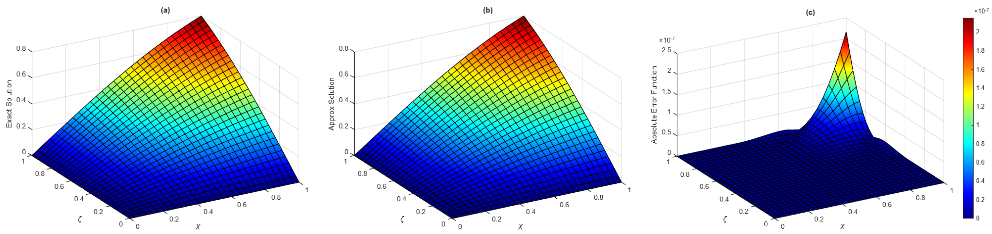

Figure 2. 3D plots of the (a) exact solution, (b) approximate solution, (c) absolute error function for $\tau = 0.5$, $\mathcal{N} = 4$, and $(\mu, \nu, \gamma) = (1.6, 0.7, 0.8)$ in Example 7.2.

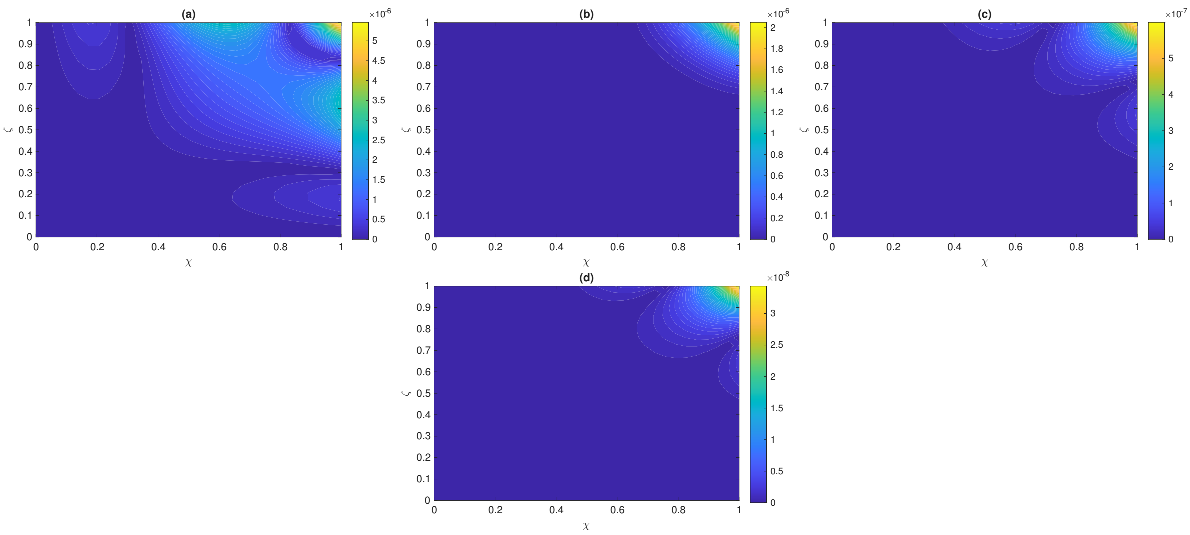

Figure 3. 2D heat maps of absolute error functions for (a) $\mathcal{N} = 2$, (b) $\mathcal{N} = 3$, (c) $\mathcal{N} = 4$, (d) $\mathcal{N} = 5$ for $\tau = 0.5$ and $(\mu, \nu, \gamma) = (1.5, 0.2, 0.3)$ in Example 7.2.

Table 3. MAEs for diverse choices of μ, ν, and $\mathcal{N}$ in Example 7.2.

(μ, ν, γ)	$\mathcal{N} = 2$	$\mathcal{N} = 3$	$\mathcal{N} = 4$	$\mathcal{N} = 5$
$(1.6, 0.7, 0.8)$	6.2431×10^{-6}	1.3365×10^{-6}	2.9496×10^{-7}	2.1565×10^{-7}
$(1.5, 0.2, 0.3)$	5.5684×10^{-6}	2.0883×10^{-6}	6.1076×10^{-7}	3.5034×10^{-8}

Table 4. Absolute errors of approximate solutions at selected points for $\mathcal{N} = 2, 5$ and diverse choices of μ, ν in Example 7.2.

	$(\mu, \nu, \gamma) =$	$(1.6, 0.7, 0.8)$	$(\mu, \nu, \gamma) =$	$(1.5, 0.2, 0.3)$
$\chi_k = \zeta_k$	$\mathcal{N} = 2$	$\mathcal{N} = 5$	$\mathcal{N} = 2$	$\mathcal{N} = 5$
0.1	5.8404×10^{-10}	4.1507×10^{-14}	5.8596×10^{-10}	9.6951×10^{-15}
03	9.3341×10^{-9}	1.5220×10^{-12}	9.1986×10^{-9}	4.0745×10^{-13}
0.5	2.8491×10^{-7}	1.3157×10^{-10}	2.9136×10^{-7}	2.9140×10^{-12}
0.7	1.1229×10^{-6}	6.2982×10^{-7}	1.2383×10^{-6}	6.3589×10^{-10}
0.9	1.0225×10^{-6}	6.5570×10^{-8}	5.1118×10^{-7}	9.5199×10^{-9}

Example 7.3. *Consider Eq* (7.2) *with the exact solution as* $\mathcal{U}(\chi, \zeta, \tau) = (\tau^3 + 1)(1 - \chi^2 - \zeta^2)$ *and the initial and boundary conditions below:*

$$\mathcal{U}(\chi, \zeta, 0) = (1 - \chi^2 - \zeta^2), \quad \frac{\partial \mathcal{U}(\chi, \zeta, 0)}{\partial \tau} = 0, \quad \mathcal{U}(0, \zeta, \tau) = (\tau^3 + 1)(1 - \zeta^2),$$
$$\mathcal{U}(1, \zeta, \tau) = -(\tau^3 + 1)\zeta^2, \quad \mathcal{U}(\chi, 0, \tau) = (\tau^3 + 1)(1 - \chi^2), \quad \mathcal{U}(\chi, 1, \tau) = -(\tau^3 + 1)\chi^2.$$

The equation is solved by applying the suggested method for $(\mu, \nu, \gamma) = (1.6, 0.7, 0.8), (1.2, 0.5, 0.1)$*. The three-dimensional figures of the exact solutions, the approximate solutions, and the absolute error functions are plotted in Figures 4 and 5 for* $\tau = 0.5$*,* $\mathcal{N} = 3$*,* $(\mu, \nu, \gamma) = (1.6, 0.7, 0.8)$*, and* $(\mu, \nu, \gamma) = (1.2, 0.5, 0.1)$*, respectively. Results confirm that the approximate solutions are in good agreement with the exact one.*

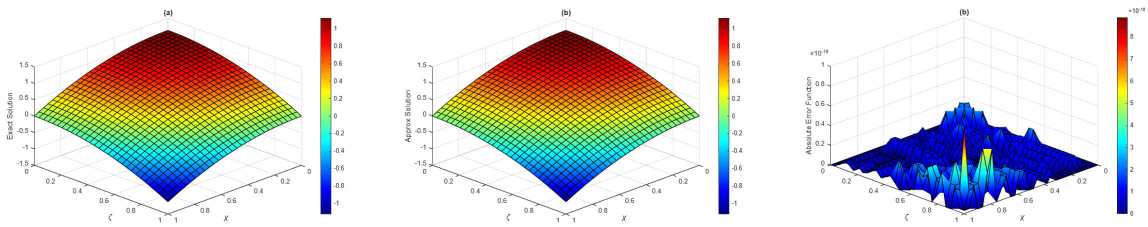

Figure 4. 3D plots of the (a) exact solution, (b) approximate solution, (c) absolute error function for $\tau = 0.5$, $\mathcal{N} = 3$, and $(\mu, \nu, \gamma) = (1.6, 0.7, 0.8)$ in Example 7.3.

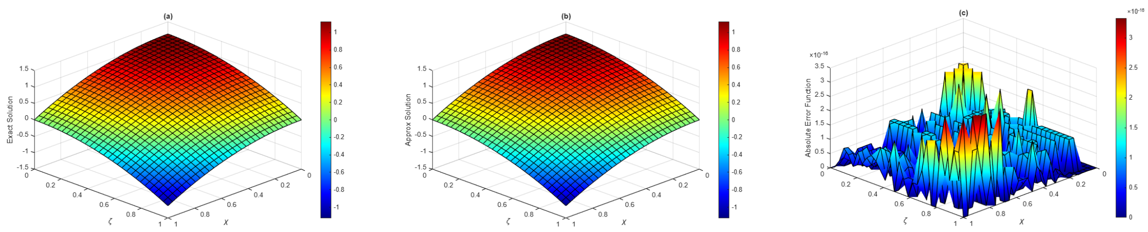

Figure 5. 3D plots of the (a) exact solution, (b) approximate solution, (c) absolute error function for $\tau = 0.5$, $\mathcal{N} = 3$, and $(\mu, \nu, \gamma) = (1.2, 0.5, 0.1)$ in Example 7.3.

Example 7.4. *Consider Eq* (1.1) *as follows:*

$$\begin{aligned}&{}_{0}^{c}\mathcal{D}_{\tau}^{\mu_1}\mathcal{U}(\chi,\zeta,\tau)+{}_{0}^{c}\mathcal{D}_{\tau}^{\mu_2}\mathcal{U}(\chi,\zeta,\tau)+\frac{\partial\mathcal{U}(\chi,\zeta,\tau)}{\partial\tau}+{}_{0}^{c}\mathcal{D}_{\tau}^{\nu_1}\mathcal{U}(\chi,\zeta,\tau)+{}_{0}^{c}\mathcal{D}_{\tau}^{\nu_2}\mathcal{U}(\chi,\zeta,\tau)\\&\quad+\mathcal{U}(\chi,\zeta,\tau)-\frac{\partial^2\mathcal{U}(\chi,\zeta,\tau)}{\partial\chi^2}-\frac{\partial^2\mathcal{U}(\chi,\zeta,\tau)}{\partial\zeta^2}=\mathsf{f}(\chi,\zeta,\tau),\end{aligned}\tag{7.3}$$

where $(\chi,\zeta,\tau)\in\Delta=[0,1]^3$*, and the exact solution is* $\mathcal{U}(\chi,\zeta,\tau)=\tau^2\chi\,\zeta\,(\exp(1)-\exp(\chi))\,(\exp(1)-\exp(\zeta))$*. The initial and boundary conditions are as follows:*

$$\mathcal{U}(\chi,\zeta,0)=\frac{\partial\mathcal{U}(\chi,\zeta,0)}{\partial\tau}=0,\quad \mathcal{U}(0,\zeta,\tau)=\mathcal{U}(1,\zeta,\tau)=0,\quad \mathcal{U}(\chi,0,\tau)=\mathcal{U}(\chi,1,\tau)=0.$$

Equation (7.3) *is solved by applying the suggested method for diverse choices of* $\mathcal{N}$*, and* $\mu_k,\nu_k,k=1,2$*. The values of MAEs are seen in Table 5 for* $\mathcal{N}=2,3,4,\tau=0.5$*, and different values of* $\mu_k,\nu_k,k=1,2$*. Values of absolute error of the obtained solutions are calculated at the points* $\chi_k=\zeta_k=0.1,0.3,\ldots,0.9$ *for* $\tau=0.5,\mathcal{N}=4$ *in Table 6. The three-dimensional figures of the absolute error functions are seen in Figure 6 for* $\tau=0.5$*,* $\mathcal{N}=4$*, and* $(\mu_1,\mu_2,\nu_1,\nu_2)=(1.1,1.8,0.4,0.7),(1.6,1.2,0.5,0.4),(1.3,1.4,0.1,0.2),(1.1,1.5,0.3,0.6)$*. The figures of the exact and approximate solutions and the absolute error functions are plotted in Figure 7 for* $\mathcal{N}=4$*,* $(\mu_1,\mu_2,\nu_1,\nu_2)=(1.1,1.5,0.3,0.6)$*, and* $\tau=0.5$*. The numerical results demonstrate that the proposed method yields accurate approximations for various choices of the fractional orders, with the MAEs decreasing as* $\mathcal{N}$ *increases. This behavior confirms the convergence and effectiveness of the method, as further supported by the close agreement between the exact and approximate solutions shown in the graphical results.*

Table 5. MAEs for diverse choices of μ_k,ν_k, and $\mathcal{N}$ in Example 7.4.

$(\mu_1,\mu_2,\nu_1,\nu_2)$	$\mathcal{N}=2$	$\mathcal{N}=3$	$\mathcal{N}=4$
$(1.1,1.8,0.4,0.7)$	5.9733×10^{-3}	6.4283×10^{-4}	1.6386×10^{-4}
$(1.6,1.2,0.5,0.4)$	4.0334×10^{-3}	6.4865×10^{-4}	8.1122×10^{-5}
$(1.1,1.6,0.3,0.8)$	4.0547×10^{-3}	6.7293×10^{-4}	8.1118×10^{-5}
$(1.3,1.4,0.1,0.2)$	1.1112×10^{-3}	5.0965×10^{-4}	7.5476×10^{-5}
$(1.1,1.5,0.3,0.6)$	1.8604×10^{-3}	6.2702×10^{-4}	3.2498×10^{-5}
$(1.3,1.6,0.2,0.6)$	4.6851×10^{-3}	6.5850×10^{-4}	6.7188×10^{-4}

Table 6. Absolute errors of approximate solutions at selected points for $\mathcal{N}=4$ and diverse choices of μ_k,ν_k in Example 7.4.

$\chi_k=\zeta_k$	$(1.1,1.8,0.4,0.7)$	$(1.6,1.2,0.5,0.4)$	$(1.3,1.4,0.1,0.2)$	$(1.1,1.5,0.3,0.6)$
0.1	1.8008×10^{-8}	4.8281×10^{-8}	1.2702×10^{-8}	1.5084×10^{-8}
03	5.5098×10^{-7}	1.5100×10^{-6}	4.0480×10^{-7}	4.7874×10^{-7}
0.5	7.3774×10^{-7}	9.1090×10^{-7}	1.0245×10^{-6}	1.0086×10^{-6}
0.7	4.8546×10^{-6}	9.3291×10^{-7}	4.4901×10^{-6}	3.7759×10^{-6}
0.9	2.6054×10^{-5}	1.3721×10^{-5}	1.8099×10^{-5}	8.2235×10^{-6}

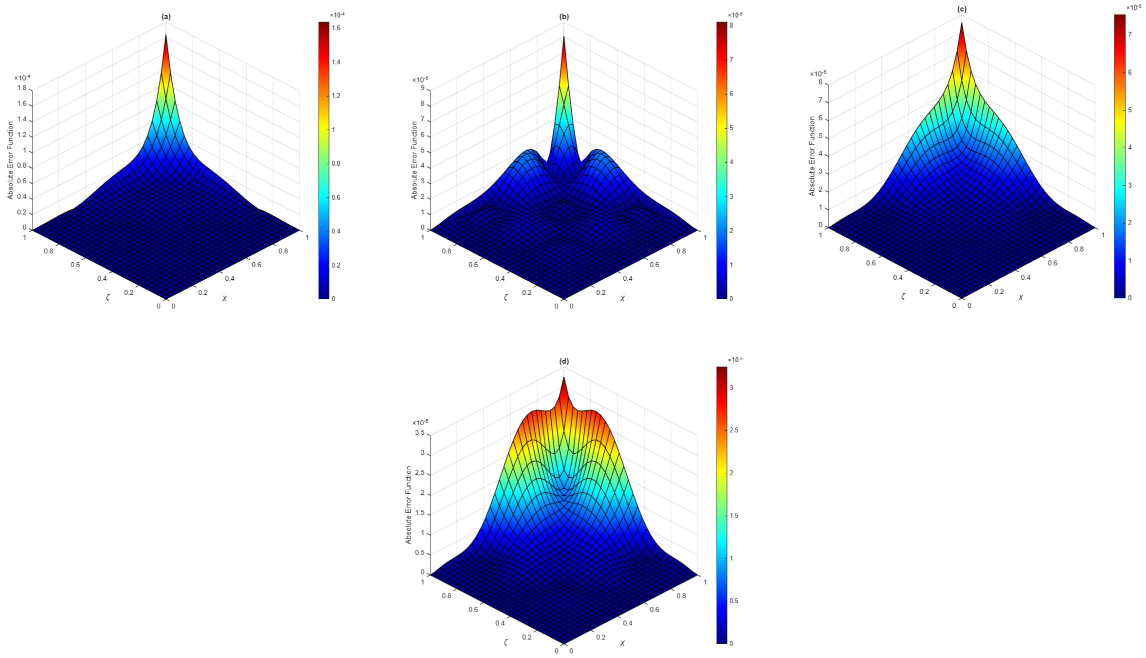

Figure 6. 3D plots of absolute error functions for (a) $(\mu_1, \mu_2, \nu_1, \nu_2) = (1.1, 18, 0.4, 0.7)$, (b) $(\mu_1, \mu_2, \nu_1, \nu_2) = (1.6, 1.2, 0.5, 0.4)$,, (c) $(\mu_1, \mu_2, \nu_1, \nu_2) = (1.3, 1.4, 0.1, 0.2)$, (d) $(\mu_1, \mu_2, \nu_1, \nu_2) = (1.1, 1.5, 0.3, 0.6)$ for $\tau = 0.5$, and $\mathcal{N} = 4$ in Example 7.4.

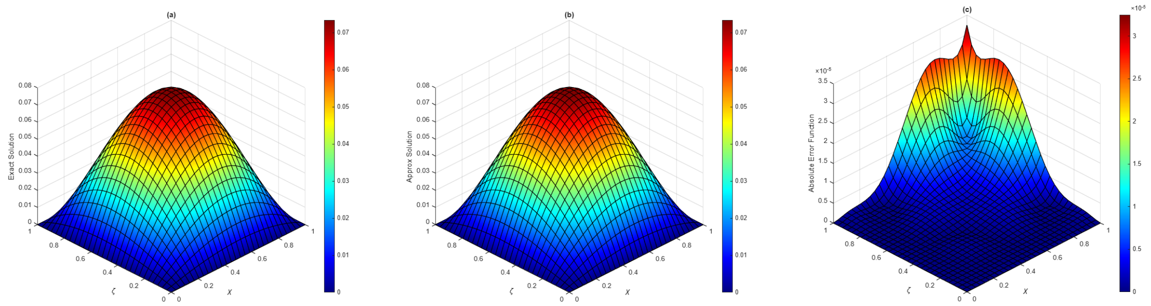

Figure 7. 3D plots of the (a) exact solution, (b) approximate solution, (c) absolute error function for $\tau = 0.5$, $\mathcal{N} = 4$, and $(\mu_1, \mu_2, \nu_1, \nu_2) = (1.1, 1.5, 0.3, 0.6)$ in Example 7.4.

Example 7.5. *Consider a nonlinear form of Eq* (1.1) *as follows:*

$$\begin{aligned} {}_0^c\mathcal{D}_\tau^{\mu_1}\mathcal{U}(\chi,\zeta,\tau) + {}_0^c\mathcal{D}_\tau^{\mu_2}\mathcal{U}(\chi,\zeta,\tau) + \frac{\partial \mathcal{U}(\chi,\zeta,\tau)}{\partial \tau} + {}_0^c\mathcal{D}_\tau^{\nu_1}\mathcal{U}(\chi,\zeta,\tau) + {}_0^c\mathcal{D}_\tau^{\nu_2}\mathcal{U}(\chi,\zeta,\tau) \\ + \mathcal{U}(\chi,\zeta,\tau) - \frac{\partial^2 \mathcal{U}(\chi,\zeta,\tau)}{\partial \chi^2} - \frac{\partial^2 \mathcal{U}(\chi,\zeta,\tau)}{\partial \zeta^2} - \mathcal{U}^2(\chi,\zeta,\tau) = \mathsf{f}(\chi,\zeta,\tau), \end{aligned} \tag{7.4}$$

where $(\chi, \zeta, \tau) \in \Delta = [0, 1]^3$, *and the exact solution is* $\mathcal{U}(\chi,\zeta,\tau) = \tau^{\mu_2} \sin(\chi) \sin(\zeta)$. *The initial and boundary conditions are as follows:*

$$\mathcal{U}(\chi,\zeta,0) = \frac{\partial \mathcal{U}(\chi,\zeta,0)}{\partial \tau} = 0, \quad \mathcal{U}(0,\zeta,\tau) = 0, \quad \mathcal{U}(1,\zeta,\tau) = \sin(1)\,\tau^{\mu_2} \sin(\zeta),$$

$$\mathcal{U}(\chi, 0, \tau) = 0, \quad \mathcal{U}(\chi, 1, \tau) = \sin(1)\,\tau^{\mu_2}\,\sin(\chi).$$

Equation (7.4) *is solved using the proposed method for various choices of* $\mathcal{N}$ *and the fractional orders* μ_k, ν_k, $k = 1, 2$. *The absolute errors of the computed solutions are evaluated at the grid points* $\chi_k = \zeta_k = 0.1, 0.3, \ldots, 0.9$ *for* $\tau = 0.5$ *and* $\mathcal{N} = 3$, *considering the parameter sets* $(\mu_1, \mu_2, \nu_1, \nu_2) = (1.2, 1.8, 0.3, 0.6)$ *and* $(1.1, 1.7, 0.3, 0.4)$*, as reported in Table 7. The three-dimensional plots of the exact and approximate solutions, along with the corresponding absolute error functions, are illustrated in Figures 8 and 9 for* $\tau = 0.5$ *and* $\mathcal{N} = 3$ *for the same sets of fractional orders. It should be noted that the term* $\mathcal{U}^2$ *introduces a nonlinear reaction mechanism into the model. Consequently, the resulting collocation equations are nonlinear in the unknown coefficients. The numerical results indicate that the proposed pseudo-operational framework effectively handles this nonlinearity while preserving a high level of accuracy.*

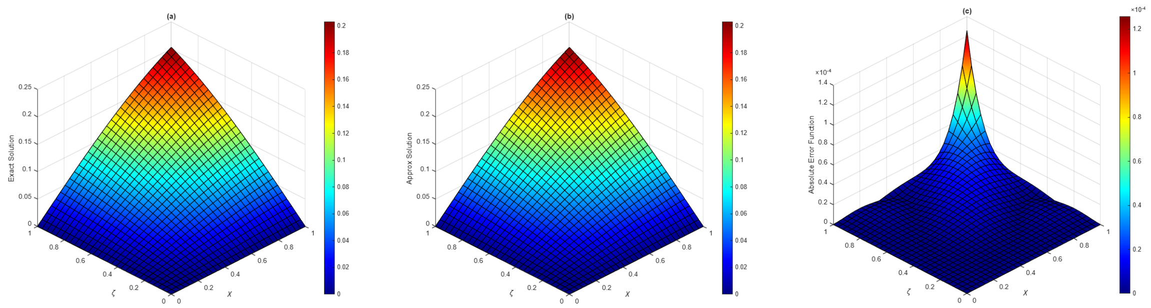

Figure 8. 3D plots of the (a) exact solution, (b) approximate solution, (c) absolute error function for $\tau = 0.5$, $\mathcal{N} = 3$, and $(\mu_1, \mu_2, \nu_1, \nu_2) = (1.2, 1.8, 0.3, 0.6)$ in Example 7.5.

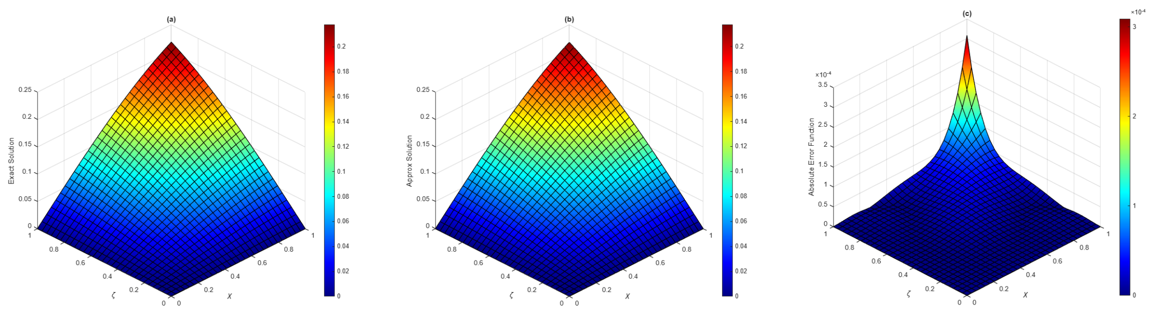

Figure 9. 3D plots of the (a) exact solution, (b) approximate solution, (c) absolute error function for $\tau = 0.5$, $\mathcal{N} = 4$, and $(\mu_1, \mu_2, \nu_1, \nu_2) = (1.1, 1.7, 0.3, 0.4)$ in Example 7.5.

Table 7. Absolute errors of approximate solutions at selected points for $\mathcal{N} = 3$ and $\tau = 0.5$ in Example 7.5.

$\chi_k = \zeta_k$	$(1.2, 1.8, 0.3, 0.6)$	$(1.1, 1.2, 0.3, 0.4)$
0.1	4.3315×10^{-10}	5.9281×10^{-9}
03	7.9963×10^{-7}	1.6142×10^{-6}
0.5	7.4444×10^{-6}	1.4641×10^{-5}
0.7	1.6601×10^{-5}	3.1772×10^{-5}
0.9	3.5080×10^{-5}	8.8972×10^{-5}

8. Conclusions

This study dealt with designing a pseudo-operational collocation approach with the use of the eighth-kind Chebyshev polynomials as basis functions to solve 2D multiterm time-fractional diffusion–wave equations. For the nature of the given problem, tri-variable Chebyshev polynomials of the eighth kind were constructed using the Kronecker product of shifted EKCPs in one variable. pseudo-operational matrices of the integration of the integer and fractional orders, corresponding to the time and space variables, were derived utilizing the Kronecker product of the obtained pseudo-operational matrices related to the one-variable basis vector. The resultant matrices and approximations converted the problem under study into a linear and nonlinear system of $(\mathcal{N} + 1)^3$ algebraic equations; solving it led to an approximate solution with acceptable accuracy for a high-dimensional multiterm time-fractional PDE. The existence and uniqueness of the solution of the problem were established using Krasnoselskii's fixed-point theorem. Some upper bounds of the obtained approximate solutions were estimated in a Chebyshev weighted space, which showed that choosing a finite number of basis functions provided a sufficiently accurate approximate solution. In summary, the numerical results reveal a close agreement between the exact and approximate solutions, with small absolute errors across the computational domain, demonstrating the accuracy and effectiveness of the proposed method for different choices of the fractional orders. The novelty of the present work lies in the development of a pseudo-operational collocation method based on TVEKCPs for solving TDMTTFDWEs. Unlike many existing spectral and collocation approaches, the proposed framework combines multidimensional Chebyshev basis functions with an efficient operational formulation, leading to a computationally attractive numerical scheme. In addition, the existence and uniqueness of the solution are established using Krasnoselskii's fixed-point theorem, and rigorous residual error estimates are derived in a suitable weighted space. These theoretical and computational features provide an effective framework for the accurate numerical treatment of high-dimensional fractional partial differential equations.

The proposed pseudo-operational collocation method exhibits high accuracy for the class of multiterm time-fractional diffusion–wave equations considered in this work. Nevertheless, several limitations should be noted: The convergence analysis is established under the regularity assumptions imposed on the exact solution. The computational cost may increase significantly for problems involving very complex geometries or extremely high-dimensional domains. The present formulation is restricted to the selected class of fractional diffusion–wave models and may require modifications for strongly nonlinear systems or problems with irregular boundary conditions. Future research may focus on adaptive basis selection strategies, nonlinear extensions, domain decomposition techniques, and applications to more general classes of fractional-order PDEs.

Author contributions

The contributions of the authors are as follows: Data curation, Formal analysis, Investigation, Methodology, Software, and Original draft preparation by K. Sadri; Original draft preparation and editing by D. Amilo; Editing and supervision by E. Hincal and M.A. Zaky. The authors declare that the study was conducted collaboratively, with responsibilities clearly defined. All authors reviewed and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article. The authors take full responsibility for the content, interpretation, and conclusions of this work.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have influenced the work reported in this article.

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Appendix

I. Put $\zeta = 1$ in (5.5), and use the boundary condition $\mathcal{U}(\chi, 1, \tau) = w_2(\chi, \tau)$ to find an approximation to $\partial^3\mathcal{U}(\chi, 0, \tau)/\partial\chi^2\,\partial\zeta$:

$$\begin{aligned}\frac{\partial^3\mathcal{U}(\chi,0,\tau)}{\partial\chi^2\,\partial\zeta} \approx{}& \frac{\partial^2 w_2(\chi,\tau)}{\partial\chi^2} - \tau^2\,\mathbb{U}^\top\,\mathbb{B}^0_\tau\,\mathbb{B}^1_\tau\,\mathbb{B}^0_\zeta\,\mathbb{B}^1_\zeta\,\mathbf{\Psi}(\chi,1,\tau) - \frac{\partial^2\mathsf{g}_2(\chi,1)}{\partial\chi^2}\tau + \frac{\partial^2\mathsf{g}_2(\chi,0)}{\partial\chi^2}\tau + \frac{\partial^3\mathsf{g}_2(\chi,0)}{\partial\chi^2\,\partial\zeta}\tau\\ &- \frac{\partial^2\mathsf{g}_1(\chi,1)}{\partial\chi^2} + \frac{\partial^2\mathsf{g}_1(\chi,0)}{\partial\chi^2} + \frac{\partial^3\mathsf{g}_1(\chi,0)}{\partial\chi^2\,\partial\zeta} - \frac{\partial^2 w_1(\chi,\tau)}{\partial\chi^2} = Z_1(\chi,\tau).\end{aligned}$$

II. Put $\chi = 1$ in (5.7), and use the boundary condition $\mathcal{U}(1, \zeta, \tau) = \mathsf{h}_2(\zeta, \tau)$ so that one has an approximation of $\partial\mathcal{U}(0, \zeta, \tau)/\partial\chi$ as follows:

$$\begin{aligned}\frac{\partial\mathcal{U}(0,\zeta,\tau)}{\partial\chi} \approx{}& \mathsf{h}_2(\zeta,\tau) - \zeta^2\,\tau^2\,\mathbb{U}^\top\,\mathbb{B}^0_\tau\,\mathbb{B}^1_\tau\,\mathbb{B}^0_\zeta\,\mathbb{B}^1_\zeta 0\,\mathbb{B}^1_\zeta\,\mathbb{B}^0_\chi\,\mathbb{B}^1_\chi\,\mathbf{\Psi}(1,\zeta,\tau) - \mathsf{g}_2(1,\zeta)\,\tau + \mathsf{g}_2(0,\zeta)\,\tau\\ &+ \frac{\partial\mathsf{g}_2(0,\zeta)}{\partial\chi}\tau + \mathsf{g}_2(1,0)\,\tau - \mathsf{g}_2(0,0)\,\tau - \frac{\partial\mathsf{g}_2(0,0)}{\partial\chi}\tau + \frac{\partial\mathsf{g}_2(1,0)}{\partial\zeta}\zeta\,\tau - \frac{\partial\mathsf{g}_2(0,0)}{\partial\zeta}\zeta\,\tau\end{aligned}$$

$$+\frac{\partial^2 \mathsf{g}_2(0,0)}{\partial\chi\,\partial\zeta}\zeta\,\tau + \mathsf{g}_1(1,\zeta) + \mathsf{g}_1(0,\zeta) + \frac{\partial \mathsf{g}_1(0,\zeta)}{\partial\chi} + \mathsf{g}_1(1,0) - \mathsf{g}_1(0,0) + \frac{\partial \mathsf{g}_1(0,0)}{\partial\chi}$$
$$-\frac{\partial \mathsf{g}_1(1,0)}{\partial\zeta}\zeta + \frac{\partial \mathsf{g}_1(0,0)}{\partial\zeta}\zeta = Z_2(\zeta,\tau).$$

III. Putting $\zeta = 1$ into (5.13) and utilizing the boundary condition $\mathcal{U}(\chi,1,\tau) = w_2(\chi,\tau)$ leads to an approximation of $\partial^3\mathcal{U}(\chi,0,\tau)/\partial\zeta\,\partial\tau$:

$$\begin{aligned}\frac{\partial^3\mathcal{U}(\chi,0,\tau)}{\partial\zeta\,\partial\tau} \approx{}& \frac{\partial^2 w_2(\chi,\tau)}{\partial\tau^2} - \chi^2\,\mathbb{U}^\top\,\mathbb{B}^0_\chi\,\mathbb{B}^1_\chi\,\mathbb{B}^0_\zeta\,\mathbb{B}^1_\zeta\,\mathbf{\Psi}(\chi,1,\tau) - \frac{\partial^2 Z_2(1,\tau)}{\partial\tau^2}\chi \\ &+ \frac{\partial^2 Z_2(0,\tau)}{\partial\tau^2}\chi + \frac{\partial^3 Z_2(0,\tau)}{\partial\zeta\,\partial\tau^2} - \frac{\partial^2 \mathsf{h}_1(1,\tau)}{\partial\tau^2} + \frac{\partial^2 \mathsf{h}_1(0,\tau)}{\partial\tau^2} \\ &+ \frac{\partial^3 \mathsf{h}_1(0,\tau)}{\partial\zeta\,\partial\tau^2} - \frac{\partial^2 w_1(\chi,\tau)}{\partial\tau^2} = Z_3(\chi,\tau).\end{aligned}$$