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Research article

Sum index of Cartesian product networks

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Abstract: In a graph, we assign distinct integers to the vertices, and take the sum of two integers if they are on two adjacent vertices. The sum index of a graph is defined as the minimum number of distinct such sums over all injective vertex labelings. In this paper, we establish upper bounds for the sum index of Cartesian product graphs. As applications of our results, we determine the exact values of the sum index of grid networks.

Keywords: sum index; graph labeling; Cartesian product graphs; networks
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1. Introduction

The graphs presented in this paper are all non-empty, finite and undirected simple graphs. We follow the notation and terminology of Bondy and Murty [1]. For a graph G, let $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. Furthermore, $|V(G)|$ denotes the order of graph G, and $|E(G)|$ denotes the number of edges within graph G. For an edge $e = uv \in E(G)$, the edge degree $d(e)$ of edge e is defined as

$$d(e) = d(u) + d(v).$$

The minimal edge degree of graph G is defined by

$$\min_{e\in E(G)} d(e) = \min_{uv\in E(G)} \{d(u) + d(v)\},$$

where $d(u)$ denotes the degree of vertex u. Let K_n, P_n, and C_n denote the complete graph, path, and cycle on n vertices, respectively.

The idea of a super totient number was first introduced by Mahmood and Ali in 2017 [2]. According to the paper, a positive integer n is defined as a super totient number if the set of positive integers less than n and relatively prime to n can be partitioned into two sets of equal sum. Mahmood and Ali also showed that every graph admits a super totient labeling, a concept closely related to the sum index studied in this work. The sum index has been used in studies such as [3–5]. Further relevant investigations can be found in [6–8].

Definition 1. *Let n be a positive integer. Define $R(n)$ as the set of positive integers that are less than n and relatively prime to n, expressed formally as*

$$R(n) = \{k \in \mathbb{N} \mid k < n \text{ and } \gcd(k, n) = 1\}.$$

A positive integer n is said to be a super totient number *if $R(n)$ can be partitioned into two disjoint subsets with equal sum.*

Definition 2. *An injective function $f\colon V(G) \to \mathbb{N}$ is called a* super totient labeling *of a graph G if the induced function $f^*\colon E(G) \to \mathbb{N}$, defined by*

$$f^*(uv) = f(u)f(v)$$

for every edge $uv \in E(G)$, assigns a super totient number to each edge of G. A graph G is said to be a super totient graph *if it admits a super totient labeling.*

Definition 3. *A super totient labeling f of a graph G is called a* restricted super totient labeling *if the range of f is $\{1, 2, \ldots, |V(G)|\}$. A graph G is said to be a* restricted super totient graph *if it admits a restricted super totient labeling.*

Definition 4. *The* super totient index *of a graph G is the minimum positive integer k such that a super totient labeling f of G exists where the cardinality of the range of f^* is k.*

Formally, for a graph G, we can use an injective map $f\colon V(G) \to \mathbb{Z}$ to assign distinct integers to the vertices. For a chosen map f, an arbitrary vertex $u \in V(G)$, and an arbitrary edge $vw \in E(G)$, we call $f(u)$ the rank of vertex u; and call $f(v) + f(w)$ the rank sum of edge vw. Let

$$\sigma(G, f) := \{f(v) + f(w) \mid vw \in E\},$$

which represents the set of distinct rank sums of all edges in G under the map f. Then, the sum index of G, denoted by $S(G)$, is defined as:

$$S(G) = \min_{f:V(G)\to\mathbb{Z}} \{|\sigma(G, f)|\},$$

where $\min_{f:V(G)\to\mathbb{Z}}$ means taking the minimum value over all injective maps f from $V(G)$ to $\mathbb{Z}$, and $|\sigma(G, f)|$ denotes the number of elements in the set.

Definition 5. *Let $f\colon V(G) \to \mathbb{Z}$ be an injective vertex labeling of G, and let $f^+\colon E(G) \to \mathbb{Z}$ be the induced edge labeling defined by $f^+(uv) = f(u) + f(v)$ for each edge $uv \in E(G)$. The sum index of G, denoted by $S(G)$, is the minimum positive integer k such that an injective vertex labeling f of G with $|\{f^+(uv) | uv \in E(G)\}| = k$ exists.*

Harrington and Wong [3] determined the sum indices of the complete bipartite graphs and complete graphs, with a purely graph theoretic proof.

Theorem 1.1. [3] *Let $K_{m,n}$ be the complete bipartite graph with m and n vertices in each part. Then,*

$$S(K_{m,n}) = m + n - 1.$$

Theorem 1.2. [3] *Let K_n be the complete graph with n vertices. Then,*

$$S(K_n) = 2n - 3.$$

Harrington and Wong [3] established the relationship between the super totient index and the sum index, which is given by the following theorem.

Theorem 1.3. *The super totient index of a graph G is equal to the sum index of G.*

Next, we establish some fundamental bounds for the sum index of a graph.

First, recall that two edges in a graph are *adjacent* if they share a common vertex. Since the vertex labelings are injective, it follows that the sum labels of adjacent edges must be distinct. Treating each sum label as a "color", this directly corresponds to a *proper edge coloring* of G a coloring, where the adjacent edges receive different colors. By definition of the chromatic index $\chi'(G)$ (the minimum number of colors needed for a proper edge coloring of G), the sum index of G satisfies $S(G) \geq \chi'(G)$. From Vizing's Theorem [9], we know that $\chi'(G)$ is either $\Delta(G)$ (the maximum degree of G) or $\Delta(G)+1$; thus, we have

$$S(G) \geq \min\{\Delta(G), \Delta(G) + 1\} = \Delta(G).$$

For the upper bound, consider a graph G and any of its subgraphs G'. A valid labeling for G naturally restricts to a valid labeling for G', so the sum index of G' cannot exceed that of G, i.e., $S(G') \leq S(G)$. By Theorem 1.3, we have $S(K_n) = 2n - 3$. Since every graph G of order n is a subgraph of K_n, this implies $S(G) \leq 2n - 3$ for all n-vertex graphs G.

Proposition 1.1. *Let G be a graph of order n. Then,*

$$\Delta(G) \leq \chi'(G) \leq S(G) \leq 2n - 3.$$

A further lower bound for the sum index has been established by Haslegrave, with the result published in [5]. To state this bound, we first introduce notation for vertex degrees: Given a graph G with n vertices (i.e., of order n), let its vertex degrees be arranged in a non-decreasing sequence, denoted as $\delta_1(G) \leq \delta_2(G) \leq \cdots \leq \delta_n(G)$ (where $\delta_i(G)$ represents the degree of the i-th vertex in this ordered list).

Theorem 1.4. [5] For any graph G of order n, the sum index $S(G)$ satisfies

$$S(G) \geq \max_{\substack{1\leq k\leq n-1 \\ k\in\mathbb{N}}} (\delta_k(G) + \delta_{k+1}(G) - k).$$

Theorem 1.5. *For any graph G of order n, the sum index $S(G)$ satisfies*

$$S(G) \geq \min\Big\{\delta_1(G) + \delta_2(G), \min_{e\in E(G)} \{d(e) - 1\}\Big\}.$$

Proof. Let $f\colon V(G) \to \mathbb{Z}$ be an optimal injective vertex labeling for G, so that $|\mathrm{Im}(f^{+})| = S(G)$. Let $u_{\min}$ be the vertex with the smallest label, defined as

$$f(u_{\min}) = m = \min\{f(v) \mid v \in V(G)\},$$

and Let $u_{\max}$ be the vertex with the largest label, defined as

$$f(u_{\max}) = M = \max\{f(v) \mid v \in V(G)\}.$$

Clearly, $m < M$ by injectivity. Let $d_{\min} = d(u_{\min})$ and $d_{\max} = d(u_{\max})$ denote the degrees of $u_{\min}$ and $u_{\max}$, respectively. For any edge $uv \in E(G)$, its sum label is defined as $f_{+}(uv) = f(u) + f(v)$.

Let $e_0 = uv \in E(G)$ be an edge with minimum edge degree. Then, $d(e_0) = d(u) + d(v)$. We consider two cases,

Case 1. $u_{\min}u_{\max} \notin E(G)$.

Let $N(u_{\min}) = \{a_1, a_2, \ldots, a_{d_{\min}}\}$ be the set of all neighbors of $u_{\min}$. The set of edge sums for edges incident to $u_{\min}$ is

$$S_{\min} = \{f_{+}(u_{\min}a_i) \mid i = 1, 2, \ldots, d_{\min}\} = \{m + f(a_i) \mid i = 1, 2, \ldots, d_{\min}\}.$$

Since f is injective, all values $f(a_i)$ for $1 \le i \le d_{\min}$ are distinct, so $S_{\min}$ contains exactly $d_{\min}$ distinct elements. Furthermore, since $f(a_i) < M$ for all $1 \le i \le d_{\min}$, every elements in $S_{\min}$ is strictly less than $m + M$.

Let $N(u_{\max}) = \{b_1, b_2, \ldots, b_{d_{\max}}\}$ be the set of neighbors of $u_{\max}$. The set of edge sums for edges incident to $u_{\max}$ is

$$S_{\max} = \Big\{f_{+}(u_{\max}b_j) \mid j = 1, 2, \ldots, d_{\max}\Big\} = \Big\{M + f(b_j) \mid j = 1, 2, \ldots, d_{\max}\Big\}.$$

Similarly, injectivity ensures $f(b_1), f(b_2), \ldots, f(b_{d_{\max}})$ are distinct, so $S_{\max}$ contains $d_{\max}$ distinct edge sums. Since $f(b_j) > m$, every sum in $S_{\max}$ satisfies $M + f(b_j) > m + M$.

All elements in $S_{\min}$ are strictly smaller than $m + M$ and all elements in $S_{\max}$ are strictly larger than $m + M$. Hence $S_{\min} \cap S_{\max} = \emptyset$, and therefore

$$|S_{\min} \cup S_{\max}| = d_{\min} + d_{\max}.$$

Notice that $S_{\min}$ and $S_{\max}$ are both subsets of the set of all edge sums $\mathrm{Im}(f^{+})$. Consequently, $S_{\min} \cup S_{\max} \subseteq \mathrm{Im}(f^{+})$, which yields

$$|\,\mathrm{Im}(f^{+})| \ge |S_{\min} \cup S_{\max}|.$$

By definition, $S(G) = |\,\mathrm{Im}(f^{+})|$. Since $\delta_1(G)$ and $\delta_2(G)$ are the minimum and second-minimum vertex degrees of G, we have

$$S(G) \ge d_{\min} + d_{\max} \ge \delta_1(G) + \delta_2(G).$$

Case 2. $u_{\min}u_{\max} \in E(G)$.

The neighbor set $N(u_{\min})$ includes $u_{\max}$, so the set of edge sums for edges incident to $u_{\min}$ is:

$$S_{\min} = \{m + M\} \cup \{m + f(a_i) \mid a_i \in N(u_{\min}) \setminus \{u_{\max}\}\}.$$

Since f is injective, we have $f(a_i) \neq M$ for all $a_i \neq u_{\max}$, and all $f(a_i)$ are distinct. Hence, $S_{\min}$ contains exactly $d_{\min}$ distinct edge sums, one sum from the edge $u_{\min}u_{\max}$, and $d_{\min} - 1$ sums from the remaining neighbors.

The neighbor set $N(u_{\max})$ includes $u_{\min}$, so the set of edge sums for edges incident to $u_{\max}$ is

$$S_{\max} = \{m + M\} \cup \Big\{M + f(b_j) \mid b_j \in N(u_{\max}) \setminus \{u_{\min}\}\Big\}.$$

Similarly, injectivity of f implies $f(b_j) \neq m$ for all $b_j \neq u_{\min}$ and all $f(b_j)$ are distinct. Hence, $S_{\max}$ contains exactly $d_{\max}$ distinct edge sums, one sum from the edge $u_{\min}u_{\max}$ and $d_{\max} - 1$ sums from the remaining neighbors.

The only common edge sum between $S_{\min}$ and $S_{\max}$ is $m + M$, which comes from the shared edge $u_{\min}u_{\max}$. The total number of distinct edge sums is

$$|S_{\min} \cup S_{\max}| = |S_{\min}| + |S_{\max}| - |S_{\min} \cap S_{\max}| = d_{\min} + d_{\max} - 1.$$

Since $u_{\min}u_{\max} \in E(G)$, its edge degree is $d_{\min} + d_{\max}$. By minimality of $d(e_0)$, we have $d_{\min} + d_{\max} \geq d(e_0)$. It follows that the number of distinct edge sums is at least $d(e_0) - 1$, so $S(G) \geq d(e_0) - 1$.

Combining both cases, the sum index of G satisfies

$$S(G) \geq \min\Big\{\delta_1(G) + \delta_2(G), \min_{e \in E(G)} \{d(e) - 1\}\Big\}.$$

This completes the proof. □

Theorem 1.6. [4] Let $n \geq 3$ be an integer. Then, $S(C_n) = 3$.

The n-dimensional hypercube Q_n is defined recursively via Cartesian product: $Q_1 = P_2$, and $Q_n = Q_{n-1} \square P_2$ for $n \geq 2$. This recursive product structure makes hypercubes a typical representative of Cartesian product networks.

Desai and Wang [7] obtained the exact sum index of hypercubes by making full use of the recursive decomposition property of Cartesian products.

Theorem 1.7. [7] For the n-dimensional hypercube Q_n, the sum index satisfies

$$S(Q_n) = 2n - 1.$$

We organize this paper as follows: In Section 2, we establish upper and lower bounds for the sum indices of the Cartesian product of two graphs G and H. In Section 3, we apply these results to determine the sum indices of grid networks.

2. Cartesian product networks

The Cartesian product of two graphs G and H, written as $G \square H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $(v, v') \in E(H)$, or $v = v'$ and $(u, u') \in E(G)$. By symmetry, we have $G \square H = H \square G$. For more details on product networks, we refer to [10, 11].

Let G and H be two connected graphs with $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$, respectively. Then, we have

$$V(G\square H) = \{(u_i, v_j) \mid 1 \le i \le n, 1 \le j \le m\}.$$

For $v \in V(H)$, we use $G(v)$ to denote the subgraph of $G\square H$ induced by the vertex set $\{(u_i, v) \mid 1 \le i \le n\}$. Similarly, for $u \in V(G)$, we use $H(u)$ to denote the subgraph of $G\square H$ induced by the vertex set $\{(u, v_j) \mid 1 \le j \le m\}$.

Some well-known interconnection networks, e.g., hypercubes, meshes, tori, k-ary n-cubes, and generalized hypercubes, are product networks [12–14]cankaowenxian.

Theorem 2.1. *Let G be a graph of order n and H be a graph of order m. Then,*

$$S(G\square H) \le \min\{S(G)m + S(H)n, 2mn - 3\}.$$

Proof. By definition of the sum index, an injective vertex labeling f_G: $V(G) \to \mathbb{Z}$ exists such that $|\mathrm{Im}(f_G^+)| = S(G)$, where $f_G^+(uu') = f_G(u) + f_G(u')$ for all $uu' \in E(G)$. Similarly, an injective vertex labeling f_H: $V(H) \to \mathbb{Z}$ exists such that $|\mathrm{Im}(f_H^+)| = S(H)$, where $f_H^+(vv') = f_H(v) + f_H(v')$ for all $vv' \in E(H)$.

We define a vertex labeling F: $V(G\square H) \to \mathbb{Z}$ as

$$F(u, v) = M \cdot f_G(u) + f_H(v) \quad \forall (u, v) \in V(G\square H),$$

where M denotes a positive integer satisfying

$$M > \max\{2|f_H(v) - f_H(v')| \mid v, v' \in V(H)\}.$$

Suppose $F(u_1, v_1) = F(u_2, v_2)$ for two vertices $(u_1, v_1), (u_2, v_2) \in V(G\square H)$. Then,

$$M \cdot (f_G(u_1) - f_G(u_2)) = f_H(v_2) - f_H(v_1).$$

By the choice of M, the right-hand side satisfies $|f_H(v_2) - f_H(v_1)| < M$. The left-hand side is an integer multiple of M, so the only solution is $f_G(u_1) = f_G(u_2)$ and $f_H(v_1) = f_H(v_2)$. Since f_G and f_H are injective, this implies $u_1 = u_2$ and $v_1 = v_2$, so F is injective.

By the definition of the Cartesian product, the edge set $E(G\square H)$ can be partitioned into two disjoint subsets:

$$E_G = \{(u_1, v)(u_2, v) \mid u_1u_2 \in E(G),\ v \in V(H)\},$$
$$E_H = \{(u, v_1)(u, v_2) \mid v_1v_2 \in E(H),\ u \in V(G)\}.$$

We analyze the induced sum labeling F^+ on E_G and E_H.

Case 1. Induced labeling F^+ on E_G.

For any edge $(u_1, v)(u_2, v) \in E_G$. Compute F^+ as follows:

$$\begin{aligned} F^+((u_1, v)(u_2, v)) &= F(u_1, v) + F(u_2, v) \\ &= (M \cdot f_G(u_1) + f_H(v)) + (M \cdot f_G(u_2) + f_H(v)) \\ &= M \cdot \underbrace{(f_G(u_1) + f_G(u_2))}_{f_G^+(u_1u_2)} + 2f_H(v). \end{aligned}$$

For each fixed $v \in V(H)$, define

$$\mathcal{A}_v = \{M \cdot s + 2f_H(v) \mid s \in \mathrm{Im}(f_G^+)\}.$$

Since the map $s \mapsto M \cdot s + 2f_H(v)$ is injective, we have $|\mathcal{A}_v| = |\mathrm{Im}(f_G^+)| = S(G)$. Moreover, $\mathcal{A}_v \cap \mathcal{A}_{v'} = \emptyset$ for $v \neq v' \in V(H)$. Suppose for contradiction that there are distinct $v, v' \in V(H)$ with $\mathcal{A}_v \cap \mathcal{A}_{v'} \neq \emptyset$. Then, $s_1, s_2 \in \mathrm{Im}(f_G^+)$ exist such that

$$M \cdot s_1 + 2f_H(v) = M \cdot s_2 + 2f_H(v').$$

Then, we have

$$M \cdot (s_1 - s_2) = 2\left(f_H(v') - f_H(v)\right).$$

Since $|2\left(f_H(v') - f_H(v)\right)| < M$, the left-hand side is an integer multiple of M, we obtain $s_1 = s_2$ and $f_H(v) = f_H(v')$. By the injectivity of f_H, we have $v = v'$, a contradiction.

It follows that the image of F^+ restricted to E_G is the disjoint union $\bigcup_{v \in V(H)} A_v$, so the cardinality satisfies

$$\left|\bigcup_{v \in V(H)} \mathcal{A}_v\right| = \sum_{v \in V(H)} |\mathcal{A}_v| = m \cdot S(G).$$

Case 2. Induced labeling F^+ on E_H.

For any edge $(u, v_1)(u, v_2) \in E_H$. We compute F^+ as follows:

$$\begin{aligned} F^+\left((u, v_1)(u, v_2)\right) &= F(u, v_1) + F(u, v_2) \\ &= (M \cdot f_G(u) + f_H(v_1)) + (M \cdot f_G(u) + f_H(v_2)) \\ &= 2M \cdot f_G(u) + f_H^+(v_1 v_2). \end{aligned}$$

For each vertex $u \in V(G)$, let

$$\mathcal{B}_u = \{2M \cdot f_G(u) + t \mid t \in \mathrm{Im}(f_H^+)\}.$$

Since $t \mapsto 2M \cdot f_G(u) + t$ is injective, we have

$$|\mathcal{B}_u| = |\mathrm{Im}(f_H^+)| = S(H).$$

Moreover, $\mathcal{B}_u \cap \mathcal{B}_{u'} = \emptyset$ whenever $u \neq u' \in V(G)$. Suppose for contradiction that there are distinct $u, u' \in V(G)$ with $\mathcal{B}_u \cap \mathcal{B}_{u'} \neq \emptyset$. Then, $t_1, t_2 \in \mathrm{Im}(f_H^+)$ exists such that

$$2Mf_G(u) + t_1 = 2Mf_G(u') + t_2.$$

Then, we have

$$2M \cdot (f_G(u) - f_G(u')) = t_2 - t_1.$$

Since $|t_2 - t_1| < M$, we have $f_G(u) = f_G(u')$ and $t_1 = t_2$, implying $u = u'$, which is a contradiction.

Thus, we have

$$|\bigcup_{u \in V(G)} \mathcal{B}_u| = \sum_{u \in V(G)} |\mathcal{B}_u| = n \cdot S(H).$$

The total range of F^+ is the union of two sets:

$$\mathrm{Im}(F^+) = \left(\bigcup_{v\in V(H)} \mathcal{A}_v\right) \cup \left(\bigcup_{u\in V(G)} \mathcal{B}_u\right).$$

Therefore, the cardinality satisfies

$$|\mathrm{Im}(F^+)| \leq \left|\bigcup \mathcal{A}_v\right| + \left|\bigcup \mathcal{B}_u\right| = mS(G) + nS(H).$$

By the definition of $S(G\Box H)$, we have $S(G\Box H) \leq |\mathrm{Im}(F^+)|$. Since

$$|\,\mathrm{Im}(F^+)| \leq mS(G) + nS(H),$$

it follows that

$$S(G\Box H) \leq S(G)m + S(H)n.$$

From Proposition 1.1, we have $S(G\Box H) \leq 2mn - 3$. Combining the two inequalities, we conclude that

$$S(G\Box H) \leq \min\{S(G)m + S(H)n, 2mn - 3\}.$$

This completes the proof. □

Lemma 2.1. *Let G be a connected graph of order $n \geq 3$. Then, an injective vertex labeling f_G: $V(G) \to \mathbb{Z}$ exists such that $|\,\mathrm{Im}(f_G^+)| = S(G)$. Moreover,*

$$|\,\mathrm{Im}(f_G^+) \cap \mathrm{Im}(-f_G^+)| \geq 2,$$

where

$$f_G^+(uu') = f_G(u) + f_G(u')$$

for every edge $uu' \in E(G)$.

Proof. By definition of $S(G)$, there is an injective vertex labeling g_G: $V(G) \to \mathbb{Z}$ satisfying $|\,\mathrm{Im}(g_G^+)| = S(G)$. Write $g_G(v_i) = a_i$ for $i = 1, \ldots, n$, where $\{a_1, \ldots, a_n\}$ is a set of distinct integers.

Since G is connected with $n \geq 3$, we have $\Delta(G) \geq 2$. Thus, there is a vertex $v \in V(G)$ with at least two distinct neighbors u_1, u_2. By the injectivity of g_G, $g_G(u_1) \neq g_G(u_2)$, which implies

$$g_G(v) + g_G(u_1) \neq g_G(v) + g_G(u_2).$$

Therefore, the edge-sum set admits at least two distinct values, thus $|\,\mathrm{Im}(g_G^+)| \geq 2$.

Without loss of generality, let $s < t$ be the two smallest distinct elements of $\mathrm{Im}(g_G^+)$. Choose the edges $uv, wz \in E(G)$ such that

$$s = g_G(u) + g_G(v), \qquad t = g_G(w) + g_G(z).$$

We define a transformed labeling f_G on each vertex $v_i \in V(G)$ by

$$f_G(v_i) = 4g_G(v_i) - (s + t) = 4a_i - (s + t).$$

For any edge $xx' \in E(G)$, direct computation yields

$$\begin{aligned} f_G^+(xx') &= f_G(x) + f_G(x') \\ &= [4g_G(x) - (s+t)] + [4g_G(x') - (s+t)] \\ &= 4(g_G(x) + g_G(x')) - 2(s+t) \\ &= 4g_G^+(xx') - 2(s+t). \end{aligned}$$

The map $\phi\colon x \mapsto 4x - 2(s+t)$ is a bijection on $\mathbb{Z}$, so

$$|\operatorname{Im}(f_G^+)| = |\operatorname{Im}(g_G^+)| = S(G).$$

Claim 1. *f_G is injective.*

Proof. If $f_G(v_i) = f_G(v_j)$, then, $4a_i - (s+t) = 4a_j - (s+t)$, which gives $a_i = a_j$. The injectivity of g_G forces $v_i = v_j$. □

Claim 2. $|\operatorname{Im}(f_G^+) \cap \operatorname{Im}(-f_G^+)| \geq 2$.

Proof. Evaluate the edge sums corresponding to the selected edges uv and wz as follows:

$$\begin{aligned} f_G^+(uv) &= 4s - 2(s+t) = 2(s-t), \\ f_G^+(wz) &= 4t - 2(s+t) = 2(t-s) = -f_G^+(uv). \end{aligned}$$

This yields a pair of mutually opposite nonzero edge sums

$$\{2(s-t),\ -2(s-t)\} \subseteq \operatorname{Im}(f_G^+) \cap \operatorname{Im}(-f_G^+),$$

which directly implies $|\operatorname{Im}(f_G^+) \cap \operatorname{Im}(-f_G^+)| \geq 2$. □

The above claims complete the proof. □

Observation 2.1. Let f_G be an injective vertex labeling $V(G) \to \mathbb{Z}$ such that $|\operatorname{Im}(f_G^+)| = S(G)$, where $f_G^+(uu') = f_G(u) + f_G(u')$ for all $uu' \in E(G)$. Let $a, b \in \mathbb{Z}$ with $a \neq 0$. Then, $g_G = af_G + b$ is an injective vertex labeling such that $|\operatorname{Im}(g_G^+)| = S(G)$, where $g_G^+(uu') = g_G(u) + g_G(u')$ for all $uu' \in E(G)$.

Theorem 2.2. *Let G and H be connected bipartite graphs of order at least* 3. *Then,*

$$S(G \square H) \leq 2S(G) + 2S(H) - 2.$$

Proof. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be bipartite graphs with bipartitions $V(G) = U_1 \cup U_2$ and $V(H) = V_1 \cup V_2$, respectively.

By Observation 2.1, starting from some optimal injective vertex labeling of G, after applying a suitable affine transformation $af_G + b$ ($a \neq 0$), we obtain an optimal injective labeling, which we denote as f_G, satisfying $f_G(u_1) + f_G(u_2) \neq 0$ for any two distinct vertices $u_1, u_2 \in V(G)$.

Similarly, by Lemma 2.1, there is an injective vertex labeling $f_H\colon V(H) \to \mathbb{Z}$. Let $f_H^+(vv') = f_H(v) + f_H(v')$ for all $vv' \in E(H)$. This labeling satisfies

$$|\operatorname{Im}(f_H^+)| = S(H) \quad \text{and} \quad |\operatorname{Im}(f_H^+) \cap \operatorname{Im}(-f_H^+)| \geq 2.$$

Let
$$S_G = \{f_G(u) + f_G(u') \mid uu' \in E(G)\} \quad \text{and} \quad S_H = \{f_H(v) + f_H(v') \mid vv' \in E(H)\}$$
be the sets of edge sums induced by f_G and f_H, respectively. By the definition of the sum index, $|S_G| = S(G)$ and $|S_H| = S(H)$.

Choose an integer k sufficiently large so that
$$k > \max\left\{|f_H(v) \pm f_H(v')| \;\Big|\; v, v' \in V(H)\right\}.$$
Define $F\colon V(G\square H) \to \mathbb{Z}$ by
$$F(u, v) = \begin{cases} kf_G(u) + f_H(v), & u \in U_1,\ v \in V_1, \\ -kf_G(u) + f_H(v), & u \in U_1,\ v \in V_2, \\ kf_G(u) - f_H(v), & u \in U_2,\ v \in V_1, \\ -kf_G(u) - f_H(v), & u \in U_2,\ v \in V_2. \end{cases}$$

Claim 3. *The labeling F is injective.*

Proof. Suppose for contradiction that $F(u_1, v_1) = F(u_2, v_2)$ with $(u_1, v_1) \neq (u_2, v_2)$. We consider all cases according to the membership of vertices in the bipartitions.

Case 1. $u_1, u_2 \in U_1$, $v_1, v_2 \in V_1$.
$$kf_G(u_1) + f_H(v_1) = kf_G(u_2) + f_H(v_2) \implies k\big(f_G(u_1) - f_G(u_2)\big) = f_H(v_2) - f_H(v_1).$$
If $u_1 \neq u_2$, then $|k(f_G(u_1) - f_G(u_2))| \geq k$, while $|f_H(v_2) - f_H(v_1)| < k$, which is a contradiction. Hence, $u_1 = u_2$, and also $v_1 = v_2$, which is a contradiction.

Case 2. $u_1, u_2 \in U_1$, $v_1, v_2 \in V_2$.
$$-kf_G(u_1) + f_H(v_1) = -kf_G(u_2) + f_H(v_2) \implies -k\big(f_G(u_1) - f_G(u_2)\big) = f_H(v_2) - f_H(v_1).$$
If $u_1 \neq u_2$, then $|k(f_G(u_1) - f_G(u_2))| \geq k$, while $|f_H(v_2) - f_H(v_1)| < k$, which is a contradiction. Hence, $u_1 = u_2$ and $v_1 = v_2$, which is a contradiction.

Case 3. $u_1, u_2 \in U_2$, $v_1, v_2 \in V_1$.
$$kf_G(u_1) - f_H(v_1) = kf_G(u_2) - f_H(v_2) \implies k\big(f_G(u_1) - f_G(u_2)\big) = f_H(v_1) - f_H(v_2).$$
If $u_1 \neq u_2$, then $|k(f_G(u_1) - f_G(u_2))| \geq k$, while $|f_H(v_1) - f_H(v_2)| < k$, which is a contradiction. Hence, $u_1 = u_2$ and $v_1 = v_2$, which is a contradiction.

Case 4. $u_1, u_2 \in U_2$, $v_1, v_2 \in V_2$.
$$-kf_G(u_1) - f_H(v_1) = -kf_G(u_2) - f_H(v_2) \implies k\big(f_G(u_2) - f_G(u_1)\big) = f_H(v_1) - f_H(v_2).$$
If $u_1 \neq u_2$, then $|k(f_G(u_2) - f_G(u_1))| \geq k$, while $|f_H(v_1) - f_H(v_2)| < k$, which is a contradiction. Hence, $u_1 = u_2$ and $v_1 = v_2$, which is a contradiction.

Case 5. $u_1, u_2 \in U_1$, $v_1 \in V_1$, $v_2 \in V_2$.

$$kf_G(u_1) + f_H(v_1) = -kf_G(u_2) + f_H(v_2) \implies k(f_G(u_1) + f_G(u_2)) = f_H(v_2) - f_H(v_1).$$

If $f_G(u_1) + f_G(u_2) \neq 0$, then $\left|k(f_G(u_1) + f_G(u_2))\right| \geq k$. Since $|f_H(v_2) - f_H(v_1)| < k$, which is a contradiction. If $f_G(u_1) + f_G(u_2) = 0$, then $f_H(v_2) - f_H(v_1) = 0$; that is, $f_H(v_1) = f_H(v_2)$. By our assumption that $f_G(x) + f_G(y) \neq 0$ for any pair of distinct vertices $x, y \in V(G)$, we conclude that $u_1 = u_2$. The injectivity of f_H then yields $v_1 = v_2$, which contradicts $(u_1, v_1) \neq (u_2, v_2)$.

Case 6. $u_1, u_2 \in U_2$, $v_1 \in V_1$, $v_2 \in V_2$.

$$kf_G(u_1) - f_H(v_1) = -kf_G(u_2) - f_H(v_2) \implies k(f_G(u_1) + f_G(u_2)) = f_H(v_1) - f_H(v_2).$$

If $f_G(u_1) + f_G(u_2) \neq 0$, then $\left|k(f_G(u_1) + f_G(u_2))\right| \geq k$. Since $|f_H(v_1) - f_H(v_2)| < k$, which is a contradiction. If $f_G(u_1) + f_G(u_2) = 0$, then $f_H(v_1) - f_H(v_2) = 0$; that is, $f_H(v_1) = f_H(v_2)$. By our assumption that $f_G(x) + f_G(y) \neq 0$ for any pair of distinct vertices $x, y \in V(G)$, we conclude that $u_1 = u_2$. The injectivity of f_H then yields $v_1 = v_2$, which contradicts $(u_1, v_1) \neq (u_2, v_2)$.

Case 7. $u_1 \in U_1$, $u_2 \in U_2$, $v_1, v_2 \in V_1$.

$$kf_G(u_1) + f_H(v_1) = kf_G(u_2) - f_H(v_2) \implies k(f_G(u_1) - f_G(u_2)) = -f_H(v_1) - f_H(v_2).$$

Since u_1, u_2 are distinct vertices of G, we have $|k(f_G(u_1) - f_G(u_2))| \geq k$, while $|f_H(v_1) + f_H(v_2)| < k$, which is a contradiction.

Case 8. $u_1 \in U_1$, $u_2 \in U_2$, $v_1, v_2 \in V_2$.

$$-kf_G(u_1) + f_H(v_1) = -kf_G(u_2) - f_H(v_2) \implies k(f_G(u_2) - f_G(u_1)) = -f_H(v_1) - f_H(v_2).$$

Since u_1, u_2 are distinct vertices of G, we have $|k(f_G(u_2) - f_G(u_1))| \geq k$, while $|f_H(v_1) + f_H(v_2)| < k$, which is a contradiction.

Case 9. $u_1 \in U_1$, $u_2 \in U_2$, $v_1 \in V_1$, $v_2 \in V_2$.

$$kf_G(u_1) + f_H(v_1) = -kf_G(u_2) - f_H(v_2) \implies k(f_G(u_1) + f_G(u_2)) = -f_H(v_1) - f_H(v_2).$$

Since u_1, u_2 are distinct vertices of G, we have $f_G(u_1) + f_G(u_2) \neq 0$, so $|k(f_G(u_1) + f_G(u_2))| \geq k$, but $|f_H(v_1) + f_H(v_2)| < k$, which is a contradiction.

Case 10. $u_1 \in U_2$, $u_2 \in U_1$, $v_1 \in V_1$, $v_2 \in V_2$.

$$kf_G(u_1) - f_H(v_1) = -kf_G(u_2) + f_H(v_2) \implies k(f_G(u_1) + f_G(u_2)) = f_H(v_1) + f_H(v_2).$$

Since u_1, u_2 are distinct vertices of G, we have $f_G(u_1) + f_G(u_2) \neq 0$, so $|k(f_G(u_1) + f_G(u_2))| \geq k$, but $|f_H(v_1) + f_H(v_2)| < k$, which is a contradiction.

All cases lead to a contradiction unless $(u_1, v_1) = (u_2, v_2)$. Therefore, the claim holds. □

We now estimate the number of distinct edge sums of F. The edges of $G \square H$ are partitioned into two types: Horizontal edges $(u, v)(u', v)$ induced by G and vertical edges $(u, v)(u, v')$ induced by H.

For each horizontal edge $(u, v)(u', v) \in E(G \square H)$, the vertices u, u' belong to opposite partition classes of G. Direct computation yields

$$F(u, v) + F(u', v) = \pm k(f_G(u) + f_G(u')).$$

Define $\mathcal{A}_+ = \{k \cdot s \mid s \in S_G\}$ and $\mathcal{A}_- = \{-k \cdot s \mid s \in S_G\}$. Clearly,

$$|\mathcal{A}_+ \cup \mathcal{A}_-| \leq |\mathcal{A}_+| + |\mathcal{A}_-| = 2|S_G| = 2S(G).$$

All horizontal edge sums belong to $\mathcal{A}_+ \cup \mathcal{A}_-$.

For each vertical edge $(u, v)(u, v') \in E(G \square H)$, the terms involving $k f_G(u)$ cancel out, and thus

$$F(u, v) + F(u, v') = \pm(f_H(v) + f_H(v')).$$

Set $\mathcal{B}_+ = \{t \mid t \in S_H\}$ and $\mathcal{B}_- = \{-t \mid t \in S_H\}$. By Lemma 2.1, $|S_H \cap (-S_H)| \geq 2$, so $|\mathcal{B}_+ \cap \mathcal{B}_-| \geq 2$. Therefore,

$$|\mathcal{B}_+ \cup \mathcal{B}_-| \leq |\mathcal{B}_+| + |\mathcal{B}_-| - 2 = 2|S_H| - 2 = 2S(H) - 2.$$

By the choice of k, every element in $\mathcal{A}_+ \cup \mathcal{A}_-$ has an absolute value of at least k, while every element in $\mathcal{B}_+ \cup \mathcal{B}_-$ has an absolute value of less than k. This implies

$$(\mathcal{A}_+ \cup \mathcal{A}_-) \cap (\mathcal{B}_+ \cup \mathcal{B}_-) = \emptyset.$$

The whole set of edge sums of F is contained in $(\mathcal{A}_+ \cup \mathcal{A}_-) \cup (\mathcal{B}_+ \cup \mathcal{B}_-)$. Consequently,

$$|\operatorname{Im}(F^+)| \leq \left|\mathcal{A}_+ \cup \mathcal{A}_-\right| + \left|\mathcal{B}_+ \cup \mathcal{B}_-\right| \leq 2S(G) + 2S(H) - 2.$$

Since the sum index $S(G \square H)$ is the minimum possible size of $\operatorname{Im}(F^+)$ over all injective labelings, we obtain

$$S(G \square H) \leq 2S(G) + 2S(H) - 2.$$

The proof is complete. □

This bound is particularly effective for analyzing special bipartite graphs, including grid graphs, hypercubes, and complete bipartite graphs, for which the sum index of Cartesian products depends crucially on the sizes of the vertex partitions.

Corollary 2.3. *Let G be a bipartite graph and H be a bipartite graph. Suppose there is an injective vertex labeling $f\colon V(G) \to \mathbb{Z}$ such that $|\operatorname{Im}(f^+)| = S(G)$, $\operatorname{Im}(f^+) = -\operatorname{Im}(f^+)$ and $f(u) + f(u') \neq 0$ for any distinct $u, u' \in V(G)$. Suppose there is also an injective vertex labeling $g\colon V(H) \to \mathbb{Z}$, such that $|\operatorname{Im}(g^+)| = S(H)$ and $\operatorname{Im}(g^+) = -\operatorname{Im}(g^+)$. Then,*

$$S(G \square H) \leq S(G) + S(H),$$

and there is an injective labeling of $G \square H$ satisfying $\operatorname{Im}(F^+) = -\operatorname{Im}(F^+)$.

Proof. Let $G = (U_1 \cup U_2, E_G)$ and $H = (V_1 \cup V_2, E_H)$ be bipartite graphs, where U_1, U_2 and V_1, V_2 are the bipartitions of G and H, respectively.

By assumption, there is an injective vertex labeling $f\colon V(G) \to \mathbb{Z}$ satisfying

$$|\operatorname{Im}(f^+)| = S(G), \quad \operatorname{Im}(f^+) = -\operatorname{Im}(f^+), \quad f(u) + f(u') \neq 0,\ \forall u \neq u' \in V(G),$$

where $f^+(uu') = f(u) + f(u')$ for all $uu' \in E_G$. Similarly, an injective vertex labeling $g\colon V(H) \to \mathbb{Z}$ exists such that

$$|\operatorname{Im}(g^+)| = S(H), \quad \operatorname{Im}(g^+) = -\operatorname{Im}(g^+),$$

where $g^+(vv') = g(v) + g(v')$ for all $vv' \in E_H$.

Write

$$S_G = \{f(u) + f(u') \mid uu' \in E_G\}, \qquad S_H = \{g(v) + g(v') \mid vv' \in E_H\}.$$

Then, $|S_G| = S(G)$, $|S_H| = S(H)$, and $S_G = -S_G$, $S_H = -S_H$.

Choose an integer k that is sufficiently large such that

$$\max_{v_1, v_2 \in V(H)} |g(v_1) \pm g(v_2)| < k.$$

Define a vertex labeling $F\colon V(G \square H) \to \mathbb{Z}$ by

$$F(u, v) = \begin{cases} kf(u) + g(v), & u \in U_1,\ v \in V_1, \\ -kf(u) + g(v), & u \in U_1,\ v \in V_2, \\ kf(u) - g(v), & u \in U_2,\ v \in V_1, \\ -kf(u) - g(v), & u \in U_2,\ v \in V_2. \end{cases}$$

Claim 4. *The labeling F is injective.*

Proof. Suppose $F(u_1, v_1) = F(u_2, v_2)$. The case analysis is entirely analogous to the proof of the preceding theorem. In each case, we obtain an equality of the form $k\alpha = \beta$, where $\alpha = f(u_1) \pm f(u_2)$. By the injectivity of f and the condition $f(u) + f(u') \neq 0$ for all distinct vertices u, u', we have $\alpha \neq 0$. Since all labels are integers, $|\alpha| \geq 1$, so $|k\alpha| \geq k$. On the other hand, β is a combination of values of g, and $|\beta| < k$ by the choice of k. This yields a contradiction. Therefore, $(u_1, v_1) = (u_2, v_2)$, and F is injective. □

We now analyze the set of edge sums of F. The edges of $G \square H$ consist of horizontal edges $(u, v)(u', v)$ induced by G and vertical edges $(u, v)(u, v')$ induced by H.

For each horizontal edge $(u, v)(u', v) \in E(G \square H)$, we have

$$F(u, v) + F(u', v) = \pm k\big(f(u) + f(u')\big).$$

Let $\mathcal{A} = \{ks \mid s \in S_G\} \cup \{-ks \mid s \in S_G\}$. Since $S_G = -S_G$, we have $\{-ks \mid s \in S_G\} \subseteq \{ks \mid s \in S_G\}$. Hence, $\mathcal{A} = \{ks \mid s \in S_G\}$ and $|\mathcal{A}| = |S_G| = S(G)$. All horizontal edge sums belong to $\mathcal{A}$.

For each vertical edge $(u, v)(u, v') \in E(G \square H)$, we have

$$F(u, v) + F(u, v') = \pm\big(g(v) + g(v')\big).$$

Let $\mathcal{B} = \{t \mid t \in S_H\} \cup \{-t \mid t \in S_H\}$. Since $S_H = -S_H$, we obtain $\mathcal{B} = S_H$ and $|\mathcal{B}| = |S_H| = S(H)$. All vertical edge sums belong to $\mathcal{B}$.

By the choice of k, every element in $\mathcal{A}$ has an absolute value of at least k, and every element in $\mathcal{B}$ has an absolute value of less than k. Thus, $\mathcal{A} \cap \mathcal{B} = \emptyset$. Note that $|\mathcal{A}| = |\mathrm{Im}(f^+)|$ and $|\mathcal{B}| = |\mathrm{Im}(g^+)|$, and $|\mathrm{Im}(F^+)| = |\mathrm{Im}(f^+)| + |\mathrm{Im}(g^+)|$. Recall that $\mathrm{Im}(f^+) = -\mathrm{Im}(f^+)$ and $\mathrm{Im}(g^+) = -\mathrm{Im}(g^+)$. Hence, $\mathcal{A} = -\mathcal{A}$ and $\mathcal{B} = -\mathcal{B}$, which yields

$$\mathrm{Im}(F^+) = \mathcal{A} \cup \mathcal{B} = -(\mathcal{A} \cup \mathcal{B}) = -\mathrm{Im}(F^+).$$

Consequently, $\mathrm{Im}(F^+) = -\mathrm{Im}(F^+)$.

Moreover,

$$|\mathrm{Im}(F^+)| = |\mathcal{A} \cup \mathcal{B}| = |\mathcal{A}| + |\mathcal{B}| = S(G) + S(H).$$

By the definition of the sum index $S(G\square H)$ as the minimum cardinality of $\mathrm{Im}(F^+)$ over all injective labelings, we conclude that

$$S(G\square H) \leq S(G) + S(H).$$

The proof is complete. □

Theorem 2.4. *Let G be a bipartite graph of order n and H be a graph of order m. Then,*

$$S(G\square H) \leq S(G) + nS(H).$$

Proof. Let $G = (U_1 \cup U_2, E(G))$ be a bipartite graph of order n, whose vertex set is partitioned into two independent sets U_1 and U_2 satisfying $|U_1| = n_1$, $|U_2| = n_2$ and $n_1 + n_2 = n$. Let $H = (V(H), E(H))$ be a graph of order m with vertex set $V(H) = \{v_1, v_2, \ldots, v_m\}$.

By the definition of the sum index, there is an optimal injective vertex labeling $f_G\colon V(G) \to \mathbb{Z}$ with $|\mathrm{Im}(f_G^+)| = S(G)$, where $f_G^+(uu') = f_G(u) + f_G(u')$ for every edge $uu' \in E(G)$.

Similarly, take an optimal injective labeling f_H for H with $|\mathrm{Im}(f_H^+)| = S(H)$.

Define a vertex labeling $F\colon V(G\square H) \to \mathbb{Z}$ for the Cartesian product $G\square H$ as follows:

$$F(u,v) = \begin{cases} k \cdot f_G(u) + f_H(v), & \text{if } u \in U_1,\ v \in V_H, \\ k \cdot f_G(u) - f_H(v), & \text{if } u \in U_2,\ v \in V_H, \end{cases}$$

where k is a sufficiently large positive constant satisfying

$$k > 2\max\{|f_H(v)| \,\big|\, v \in V(H)\}.$$

Claim 5. *F is injective.*

Proof. Suppose for contradiction that F is not injective. Then there are two distinct vertices $(u_1, v_1) \neq (u_2, v_2) \in V(G\square H)$ such that $F(u_1, v_1) = F(u_2, v_2)$. We consider the following cases.

Case 1. $u_1, u_2 \in U_1$ or $u_1, u_2 \in U_2$, with $v_1, v_2 \in V(H)$.

If $u_1, u_2 \in U_1$, then

$$k \cdot f_G(u_1) + f_H(v_1) = k \cdot f_G(u_2) + f_H(v_2).$$

Rearranging, we have

$$k \cdot (f_G(u_1) - f_G(u_2)) = f_H(v_2) - f_H(v_1).$$

If $u_1 \neq u_2$, then the absolute value of the left-hand side is at least k, whereas $|f_H(v_2) - f_H(v_1)| < k$, which is a contradiction. Thus, $u_1 = u_2$. Then, $f_H(v_2) = f_H(v_1)$. By injectivity of f_H, we have $v_1 = v_2$, which contradicts $(u_1, v_1) \neq (u_2, v_2)$.

The case $u_1, u_2 \in U_2$ follows by an entirely symmetric argument, replacing $+f_H(\cdot)$ with $-f_H(\cdot)$.

Case 2. $u_1 \in U_1$,$u_2 \in U_2$ and $v_1, v_2 \in V(H)$.

By the definition of F, we have

$$k \cdot f_G(u_1) + f_H(v_1) = k \cdot f_G(u_2) - f_H(v_2).$$

Rearranging, we obtain

$$k \cdot (f_G(u_1) - f_G(u_2)) = -f_H(v_1) - f_H(v_2).$$

Since $u_1 \neq u_2$ and f_G is injective, we have $f_G(u_1) \neq f_G(u_2)$. Thus, the left-hand side is a nonzero integer multiple of k, so its absolute value is at least k. However, by the choice of $k > 2\max\{|f_H(v)| : v \in V(H)\}$, we have

$$|-f_H(v_1) - f_H(v_2)| \leq |f_H(v_1)| + |f_H(v_2)| < k.$$

This is a contradiction.

The case with $u_1 \in U_2$, $u_2 \in U_1$, and $v_1, v_2 \in V(H)$ follows by a similar argument. Therefore, F is injective. □

Claim 6. $|\mathrm{Im}(F^+)| \leq S(G) + nS(H)$.

Proof. The edge set of $G\square H$ can be partitioned into two disjoint families: Edges inherited from G, and edges inherited from H. We compute the cardinality of the edge sum set induced by F.

Case 1. Edges inherited from G (fixing vertices in H).

These edges have the form $(u_1, v)(u_2, v) \in E(G\square H)$, where $u_1u_2 \in E(G)$. Since G is bipartite, we have $u_1 \in U_1$, $u_2 \in U_2$, and $v \in V(H)$. The corresponding edge sum is $F(u_1, v) + F(u_2, v) = k \cdot (f_G(u_1) + f_G(u_2))$. For each fixed $v \in V(H)$, define $\mathcal{A}_v = \{k \cdot s \mid s \in S(G)\}$. Since multiplication by the nonzero constant k is injective, we have$|A_v| = |S(G)| = S(G)$.

Moreover, all sets A_v coincide for every $v \in V(H)$, so

$$\left|\bigcup_{v\in V(H)} A_v\right| = |\mathcal{A}_v| = S(G).$$

Case 2. Edges inherited from H (fixing vertices in G).

These edges have the form $(u, v_1)(u, v_2) \in E(G\square H)$ with $v_1v_2 \in E(H)$ and $u \in V(G)$.

If $u \in U_1$, the edge sum satisfies

$$\begin{aligned} F^+((u, v_1)(u, v_2)) &= F(u, v_1) + F(u, v_2) \\ &= (k \cdot f_G(u) + f_H(v_1)) + (k \cdot f_G(u) + f_H(v_2)) \\ &= 2k \cdot f_G(u) + f_H^+(v_1v_2). \end{aligned}$$

For each $u \in U_1$, let $\mathcal{B}_u^+ = \{2k \cdot f_G(u) + t \mid t \in \mathrm{Im}(f_H^+)\}$. The map $t \mapsto 2kf_G(u) + t$ is injective, so $|\mathcal{B}_+| = |\mathrm{Im}(f_H^+)| = S(H)$. For distinct vertices $u \neq u' \in U_1$, we claim $\mathcal{B}_u \cap \mathcal{B}_{u'} = \emptyset$. Suppose, for contradiction, that $2k \cdot f_G(u) + t_1 = 2k \cdot f_G(u') + t_2$. which rearranges to $2k \cdot (f_G(u) - f_G(u')) = t_2 - t_1$. By the choice of k, $|t_2 - t_1| < k$. Since $u \neq u'$, the absolute value of the left-hand side is at least $2k > k$, which is a contradiction.

If $u \in U_2$, the edge sum satisfies

$$F^+((u, v_1)(u, v_2)) = (kf_G(u) - f_H(v_1)) + (kf_G(u) - f_H(v_2)) = 2kf_G(u) - f_H^+(v_1v_2).$$

For each $u \in V(G)$, let $\mathcal{B}_u^- = \{2k \cdot f_G(u) - t \mid t \in \mathrm{Im}(f_H^+)\}$. The map $t \mapsto 2k \cdot f_G(u) - t$ is injective, so $|\mathcal{B}_-| = |\mathrm{Im}(f_H^+)| = S(H)$. For distinct vertices $u \neq u' \in U_2$, we claim$\mathcal{B}_u \cap \mathcal{B}_{u'} = \emptyset$. Suppose, for contradiction, that $2k \cdot f_G(u) - t_1 = 2k \cdot f_G(u') - t_2$. which rearranges to $2k \cdot (f_G(u) - f_G(u')) = t_1 - t_2$. By the choice of k, $|t_2 - t_1| < k$. Since $u \neq u'$, the absolute value of the left-hand side is at least $2k > k$, which is a contradiction.

The total cardinality of the edge sum set of $G\square H$ induced by the injective labeling F satisfies

$$\begin{aligned} \left|\mathrm{Im}(F^+)\right| &= \left|\Big(\bigcup_{v\in V(H)} A_v\Big) \cup \Big(\bigcup_{u\in U_1} \mathcal{B}_u^+ \cup \bigcup_{u\in U_2} \mathcal{B}_u^-\Big)\right| \\ &\le \left|\bigcup_{v\in V(H)} A_v\right| + \sum_{u\in U_1} |\mathcal{B}_u^+| + \sum_{u\in U_2} |\mathcal{B}_u^-| \\ &= S(G) + nS(H). \end{aligned}$$

This completes the proof. □

By the definition of the sum index $S(G\square H)$, which is the minimum cardinality of the edge sum set over all injective vertex labelings of $G\square H$, we conclude

$$S(G\square H) \le S(G) + nS(H).$$

This completes the proof. □

Corollary 2.5. *Let G be a graph of order n and H be a bipartite graph of order m. Then,*

$$S(G\square H) \le mS(G) + S(H).$$

3. Applications

In this section, we illustrate the effectiveness of our theoretical results by applying them to one typical class of Cartesian product networks, namely the m-dimensional grid networks.

m-dimensional grid networks. An m-dimensional grid graph is defined as the Cartesian product $P_{n_1}\square P_{n_2}\square \ldots \square P_{n_m}$ of the path graphs, where each path P_{n_i} has $n_i \ge 3$ vertices for all $1 \le i \le m$. For more details about grid graphs, we refer to [15].

Lemma 3.1. *Let P_n be a path graph of order $n \ge 3$. Then, $S(P_n) = 2$, and an injective vertex labeling $F\colon V(P_n) \to \mathbb{Z}$ exists such that $|\mathrm{Im}(F^+)| = S(P_n)$, $0 \notin \mathrm{Im}(F^+)$ and $\mathrm{Im}(F^+) = -\mathrm{Im}(F^+)$, where $F^+(ww') = F(w) + F(w')$ for every edge $ww' \in E(P_n)$.*

Proof. For the lower bound. By Proposition 1.1, we have $S(P_n) \geq 2$.

Let $P_n = w_1w_2\dots w_n$. Define a vertex labeling $F(w_i) = (-1)^i \cdot i$. Since $1, 2, \dots, n$ are mutually distinct integers, the labeling F is injective. For each edge w_iw_{i+1}, we have

$$F(w_i) + F(w_{i+1}) = (-1)^i i + (-1)^{i+1}(i+1) = (-1)^{i+1}.$$

The edge sum set is $\mathrm{Im}(F^+) = \{1, -1\}$. Thus, $|\mathrm{Im}(F^+)| = 2$, $\mathrm{Im}(F^+) = -\mathrm{Im}(F^+)$ and $0 \notin \mathrm{Im}(F^+)$. Hence, $S(P_n) \leq 2$. Combining the lower bound and upper bound, we have $S(P_n) = 2$. The labeling F satisfies all the required conditions. □

Theorem 3.1. *Let integer $m \geq 2$ and $n_i \geq 3$ for each $1 \leq i \leq m$. Then,*

$$S(P_{n_1} \square P_{n_2} \square \cdots \square P_{n_m}) = 2m.$$

Proof. We proceed by induction on the dimension m. Denote $G_m = P_{n_1} \square P_{n_2} \square \cdots \square P_{n_m}$.

For $m = 2$. Both P_{n_1} and P_{n_2} are bipartite graphs. By Lemma 3.1, we have each path admits a symmetric optimal labeling satisfying the hypotheses of Corollary 2.3. Applying Corollary 2.3, we have

$$S(P_{n_1} \square P_{n_2}) \leq S(P_{n_1}) + S(P_{n_2}) = 2 + 2 = 4 = 2 \cdot 2.$$

The maximum degree of the two-dimensional grid G_2 is 4, so $S(G_2) \geq 4$. Consequently, $S(G_2) = 4$. Moreover, Corollary 2.3 guarantees that G_2 also has a symmetric optimal labeling. The case $m = 2$ holds.

Assume that for the $(m-1)$-dimensional grid

$$G_{m-1} = P_{n_1} \square P_{n_2} \square \cdots \square P_{n_{m-1}},$$

we have $S(G_{m-1}) = 2(m-1)$, and G_{m-1} satisfies the labeling condition required in Corollary 2.3.

Write the m-dimensional grid as the Cartesian product

$$G_m = G_{m-1} \square P_{n_m} = P_{n_m} \square G_{m-1}.$$

G_{m-1} and P_{n_m} are both bipartite graphs equipped with symmetric optimal labelings. By Corollary 2.3,

$$S(G_m) \leq S(G_{m-1}) + S(P_{n_m}) = 2(m-1) + 2 = 2m.$$

On the other hand, the maximum degree of the m-dimensional grid satisfies $\Delta(G_m) = 2m$, which implies the lower bound $S(G_m) \geq 2m$.

Combining the upper bound and lower bound, we conclude that $S(G_m) = 2m$. □

4. Conclusions

In this work, we study the sum index of Cartesian product graphs, which serve as topological models for interconnection networks. Several upper bound inequalities for $S(G \square H)$ are established under different structural assumptions. When both factor graphs are bipartite, we obtain enhanced bounds by utilizing bipartite partition structures. As an application, we derive the exact sum index value for m-dimensional grid networks: $S(P_{n_1} \square P_{n_2} \square \cdots \square P_{n_m}) = 2m$ for all integers $n_i \geq 3$.

Our analysis focuses primarily on upper bound estimates, while tight lower bound results for $S(G\square H)$ remain limited in existing literature. The labeling constructions we adopt rely on affine transformations with large scaling constants for theoretical derivation, yet they do not yield labelings with small integer ranges. Our bounds become sharp for bipartite product networks such as multi-dimensional grids, which benefit from the symmetric optimal labeling property of path graphs. However, such favorable labelings may not exist for non-bipartite networks. For example, it remains open whether torus networks constructed from odd length cycles admit symmetric optimal labelings, which restricts direct application of our bipartite oriented techniques.

Some open problems remain for future research. It is worthwhile to develop tighter lower bounds for the sum index of general Cartesian product graphs. Constructing minimal range integer labelings achieving the sum index and computing exact sum index values for torus networks constitute interesting directions. Moreover, extending the sum index analysis to strong products, lexicographic products, tensor products, directed graphs, and weighted graphs deserves further attention.

Author contributions

Shuo Liu: formal analysis, investigation, writing–original draft; Zhao Wang: conceptualization, methodology, supervision, writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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