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Research article

A classification of graphs of order n with at least $n-4$ Laplacian eigenvalues greater than $n-1$

Yifan Liu and Jinxing Zhao*

School of Mathematical Sciences, Inner Mongolia University, Hohhot, China

* **Correspondence:** Email: zhjxing@imu.edu.cn, bestheroest@outlook.com.

Abstract: Let G be a connected graph of order n and $m_G(I)$ be the number of Laplacian eigenvalues of G in an interval I. It is well known that the Laplacian eigenvalues of G are all in the interval $[0, n]$. Some attention has been paid to the distribution of Laplacian eigenvalues in a subinterval of $[0, n]$ of length 1. In 2022, Ahanjideh et al. [1] gave a classification for graphs with at least $n-2$ Laplacian eigenvalues in $[0, 1]$. In 2020, Wang et al. [2] proved that $m_G(n-1, n] \leq \kappa$ and $m_G(n-1, n] \leq \chi - 1$, where κ and χ are the vertex-connectivity and the chromatic number of G. Based on the above bounds for $m_G(n-1, n]$ together with some other bounds for $m_G[0, 1)$, we give a classification for connected graphs of order n with $m_G(n-1, n] \geq n-4$.

Keywords: distribution of Laplacian eigenvalue; multiplicity of eigenvalues; Laplacian matrix
Mathematics Subject Classification: 05C50

1. Introduction

Throughout this paper, we consider finite undirected simple graphs. Let G be a connected graph on n vertices. We write $u \sim v$ to mean two vertices u and v of G are adjacent, and by uv we mean the edge joining u and v. The neighbor set of a vertex v in G is defined as $N_G(v) = \{u \in V(G) : u \sim v\}$, and the degree of v is defined as $d_G(v) = |N_G(v)|$. A graph H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Further, H is called an induced subgraph of G if $\forall u, v \in V(H)$, and u, v are adjacent in H if and only if they are adjacent in G. For a subset U of $V(G)$, we denote by $G[U]$ the induced subgraph of G with vertex set U.

A subset $S \subseteq V(G)$ is called an independent set if no two of its vertices are adjacent. The independence number of G is the cardinality of a largest independent set of G and is denoted by $\alpha(G)$. A set $S \subseteq V(G)$ is a dominating set if every $v \in V(G) \setminus S$ is adjacent to a member of S. The domination number $\gamma(G)$ is the minimum size of a dominating set among all dominating sets of G. Two distinct edges in G are independent if they are not incident with a common vertex in G. A set of pairwise independent edges of G is called a matching set of G. The matching number $\beta(G)$ of G is the maximum cardinality of a matching set among all matching sets of G. It is known that $\gamma(G) \leq \beta(G)$ for all connected graphs. In a graph without isolated vertices, the matching number and edge covering number are related by the Gallai formula $\beta(G) + \rho(G) = n$. A pendant vertex of G is a vertex of degree 1. A quasi-pendant vertex of G is a vertex adjacent to a pendant vertex. Denote by $p(G)$ the number of pendant vertices of G and by $q(G)$ the number of quasi-pendant vertices. The chromatic number of a graph, which is denoted by $\chi(G)$, is the minimum number of colors needed to produce a proper coloring of a graph, which means no two adjacent vertices have the same color. The set of all vertices with the same color is called a color class. For $u, v \in V(G)$, the distance between u and v, written as $d(u, v)$, is the shortest length of paths between u and v, and the diameter of G is defined to be $\mathrm{diam}(G) = \max\{d(u, v) : u, v \in V(G)\}$. The girth of G is the size of a shortest cycle contained in G. If the graph does not contain any cycles, girth is defined as infinity. If $G = (V(G), E(G))$ and $H = (V(H), E(H))$ are two graphs, then by $G \cup H$ we mean the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$, and we call it the disjoint union of G and H. By H^k, we denote the union of k copies of H. If G and H are two graphs, then the join of G and H, denoted by $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and with the edge set consisting of all edges of G and H together with all the edges that join every vertex of G to any vertex of H. Also, we denote the complement of a graph G by $\overline{G}$. For a connected graph $G = (V, E)$, we denote $|E| - |V| + 1$ by $\theta(G)$, called the cyclomatic number of G. A tree (resp., unicyclic graph, bicyclic graph) is a connected graph with $\theta(G) = 0$ (resp., $\theta(G) = 1$, $\theta(G) = 2$). By C_n (resp., P_n, K_n, $K_{s,n-s}$, S_n), we denote a cycle (resp., a path, a complete graph, a complete bipartite graph, a star) on n vertices.

The adjacency matrix of G is the matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if vertices i and j are adjacent in G, and $a_{ij} = 0$ otherwise. The Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is a diagonal matrix with diagonal entries the degrees of the vertices of G. The eigenvalues of $L(G)$ are called the Laplacian eigenvalues of G, and we denote them in non-decreasing order by $0 = \mu_1(G) \leq \mu_2(G) \leq \ldots \leq \mu_n(G)$. The multiset of eigenvalues of $L(G)$ is called the Laplacian spectrum of G. Let $m_G(I)$ be the number of Laplacian eigenvalues of G in an interval I. If $I = \{\lambda\}$, then $m_G(I) = m_G(\lambda)$ turns out to be the multiplicity of λ as a Laplacian eigenvalue of G. It is well known that the Laplacian eigenvalues of G are all in the interval $[0, n]$.

A lot of attention has been paid to the distribution of Laplacian eigenvalues in certain subinterval I of $[0, n]$. In 1985, Faria [3] first considered the multiplicities of Laplacian eigenvalues, and showed that $m_G(1) \geq p(G) - q(G)$, with $p(G)$ and $q(G)$ being the number of pendant vertices and the number of quasi-pendant vertices of G. In 1990, Grone et al. [4] proved the following:

$$\begin{aligned} m_G[0,1] &\geq p(G), & m_G[0,1) &\geq q(G), \\ m_G[1,n] &\geq p(G), & m_G(1,n] &\geq q(G), \\ m_T(2,n] &\geq \lfloor \tfrac{d}{2} \rfloor, & m_T(0,2) &\geq \lfloor \tfrac{d}{2} \rfloor, \end{aligned}$$

where d is the diameter of tree T. In 1991, Merris [5] showed that if $n > 2q(G)$, then $m_G(2, n] \geq q(G)$. In 2001, Guo and Tan [6] proved that if $n > 2\beta(G)$, then $m_G(2, n] \geq \beta(G)$ with $\beta(G)$ being the matching number of G. In 2011, Guo et al. [7] proved that $m_G[1, n] \geq \rho(G)$ with $\rho(G)$ being the edge covering number for a connected graph G of order n. As a corollary, they proved that $q(G) \leq m_G[0, 1) \leq \beta(G)$ for a connected graph G of order n. In 2016, Hedetniemi et al. [8] improved the above upper bound by proving that $m_G[0, 1) \leq \gamma$ for a graph G with domination number γ. In 2020 and 2021, Jacobs et al. [9] and Sin [10] independently proved that $m_G(0, 2 - \frac{2}{n}) \geq \frac{n}{2}$ if G is a tree of order n, which was conjectured in [11]. In 2020, Wang et al. [2] proved that $m_G(n-1, n] \leq \kappa(G)$ and $m_G(n-1, n] \leq \chi(G) - 1$ for a connected graph G of order n, where $\kappa(G)$ and $\chi(G)$ are the vertex-connectivity and the chromatic number of G. In 2022, Ahanjideh et al. [1] showed that $m_G(n - \alpha(G), n] \leq n - \alpha(G)$ and $m_G(n - d(G) + 3, n] \leq n - d(G) - 1$, where $\alpha(G)$ and $d(G)$ are the independence number and the diameter of G, respectively. In addition, Ahanjideh et al. [1] proved that if G is a connected triangle-free or quadrangle-free graph of order n, then $m_G(n-1, n] \leq 1$. In 2023, Xu and Zhou [12] showed that $m_G[n-d(G)+2, n] \leq n-d(G)$, which was conjectured in [1], and they gave a complete characterization of graphs for which the conjectured bound is attained. Related structural investigations of special graph classes have also been considered. For example, [13] studied decomposition problems for circulant graphs using algorithmic approaches.

We are motivated by another result obtained by [1], which gave a classification for graphs with at least $n-2$ Laplacian eigenvalues in $[0, 1]$.

Proposition 1.1. ([1], Theorem 5.2) *Let G be a connected graph of order n. Then*
(a) $m_G[0, 1] = n - 1$ *if and only if G is the star S_n ;*
(b) $m_G[0, 1] = n - 2$ *if and only if G is obtained from a star S_n by adding an edge joining two pendant vertices of S_n, or $G \cong H(s, 2, t)$ for some positive integers s, t such that $s + t = n - 2$, where $H(s, 2, t)$ is a tree obtained by attaching s pendant vertices to one end point of P_2, and t pendant vertices to its other end point.*

This proposition encourages us to consider another extremal case: How about the graphs with most Laplacian eigenvalues in $(n-1, n]$?

For a connected graph G of order n, it is easy to see that $m_G(n-1, n] = n-1$ if and only if G is the complete graph K_n. Indeed, if $G = K_n$, then the Laplacian spectrum of G consists of 0 and n (with multiplicity $n-1$), and thus $m_G(n-1, n] = n-1$. Conversely, if $m_G(n-1, n] = n-1$, then the chromatic number of G must be n (by Lemma 2.4), forcing $G = K_n$. Moreover, if $m_G(n-1, n] = n-2$, then the chromatic number of G must be $n-1$, thus G is obtained from K_n by deleting some edges joining a given vertex v and some other vertices of K_n. If at least two such edges, say vu, vw, are deleted from K_n, then the deletion of all vertices in $V(G) \setminus \{v, u, w\}$ results in a disconnected graph, which implies that $n - 2 = m_G(n-1, n] \leq \kappa(G) \leq n-3$ (by Lemma 2.3), a contradiction. Thus G is obtained from K_n by deleting an edge. That is to say that $G = K_{n-2} \vee K_1^2$. Conversely, if $G = K_{n-2} \vee K_1^2$, then the Laplacian spectrum of G is $\{0, 2, n^{(n-2)}\}$, and thus $m_G(n-1, n] = n-2$. Consequently, $m_G(n-1, n] = n-2$ if and only if $G = K_{n-2} \vee K_1^2$. A further problem is the following: For a positive integer $k \geq 3$, how about the connected graphs of order n with $m_G(n-1, n] = n-k$?

In the present paper, we aim to give an answer to the above problem for $k = 3$ or $k = 4$. The following two theorems are our main results, giving a classification for connected graphs G of order n with $m_G(n-1, n] = n-3$ or $m_G(n-1, n] = n-4$.

Theorem 1.2. *Let G be a connected graph of order $n \geq 4$. Then $m_G(n-1, n] = n-3$ if and only if G is one of the following types:*
(i) $G = K_{n-4} \vee P_4$*;*
(ii) $G = K_{n-4} \vee C_4$*;*
(iii) $G = K_{n-3} \vee (K_1^3)$*;*
(iv) $G = K_{n-3} \vee (K_1 \cup P_2)$.

Theorem 1.3. *Let G be a connected graph of order $n \geq 6$. Then $m_G(n-1, n] = n-4$ if and only if $G = K_{n-s} \vee H_i$, where H_i is one of the 17 graphs in Figure 1 with order $s = 4, 5$, or 6.*

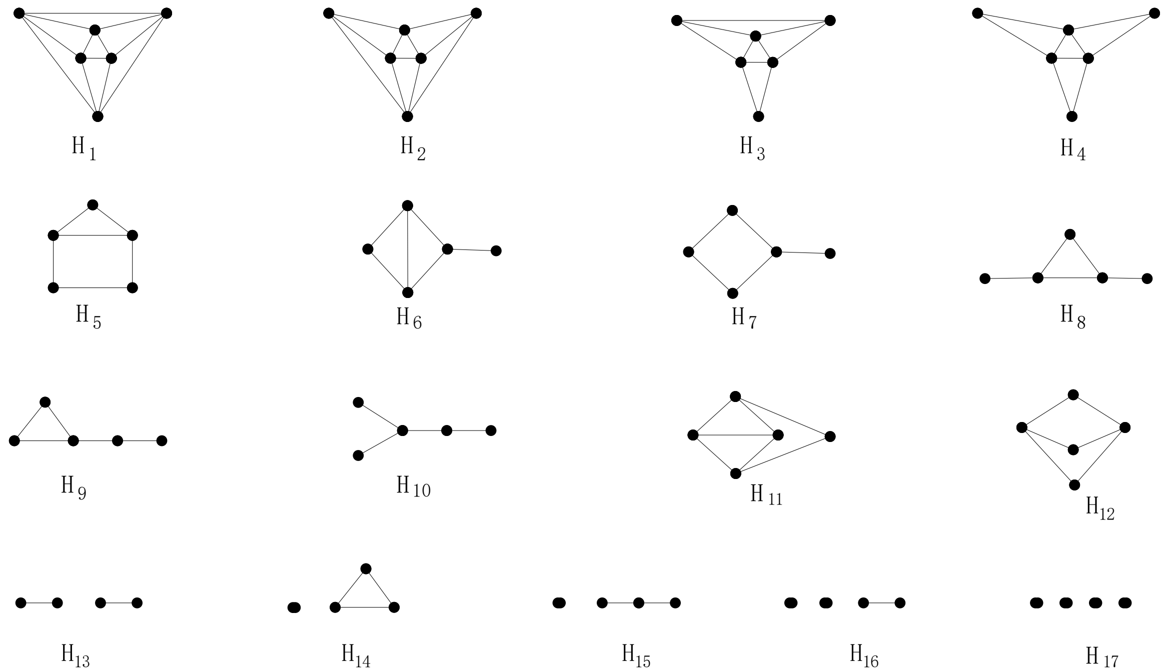


Figure 1. 17 graphs about $H_1, H_2, \ldots, H_{17}$.

In Section 2, we introduce some lemmas on the distribution of Laplacian eigenvalues for later use. The proof of Theorem 1.2 is given in Section 3. The proof of Theorem 1.3 is given in Section 4.

2. Preliminaries

In this section, we introduce some lemmas on the distribution of Laplacian eigenvalues of graphs, each of which will be applied to the proof of our main results.

Lemma 2.1. ([14], Theorem 3.2) *Let $G = (V, E)$ be a graph with edge set E, and let $e \in E$. Then the Laplacian eigenvalues satisfy:*

$$0 = \mu_1(G-e) = \mu_1(G) \leq \mu_2(G-e) \leq \mu_2(G) \leq \ldots \leq \mu_{n-1}(G-e) \leq \mu_{n-1}(G) \leq \mu_n(G-e) \leq \mu_n(G).$$

Lemma 2.2. ([15], Lemma 13.1.3) *Let G be a graph on n vertices with the Laplacian eigenvalues arranged in nondecreasing order as*

$$0 = \mu_1(G) \leq \mu_2(G) \leq \ldots \leq \mu_n(G).$$

If $2 \leq i \leq n$, then $\mu_i(\overline{G}) = n - \mu_{n-i+2}(G)$.

Lemma 2.3. ([2], Theorem 2.1) *Let G be a connected graph of order n with vertex-connectivity $\kappa(G)$. Then $m_G(n-1, n] \leq \kappa(G)$.*

Lemma 2.4. ([2], Theorem 3.1) *Let G be a connected graph of order n with chromatic number $\chi(G)$. Then $m_G(n-1, n] \leq \chi(G) - 1$.*

Lemma 2.5. ([7], Corollary 6) *Let G be a connected graph of order n. Then $m_G[0,1) \leq \beta(G)$.*

Lemma 2.6. ([8], Theorem 1) *For a graph G with domination number γ, $m_G[0,1) \leq \gamma$.*

Lemma 2.7. ([4], Theorem 3.11) *Let G be a connected graph of order n with $q(G)$ the number of quasi-pendant vertices of G. Then $m_G[0,1) \geq q(G)$.*

Lemma 2.8. ([8], Lemma 4) *Let $G = (V,E), H = (V,F)$ be graphs with $F \subseteq E$. Then for any a, $m_H[0,a) \geq m_G[0,a)$.*

Lemma 2.9. ([16], Pages 15–16) *(1) If G is the cycle with n vertices, then the Laplacian eigenvalues of G are $2 - 2\cos\frac{2k\pi}{n}, k = 0,1,2,\ldots,n-1$.*
(2) If G is the path with n vertices, then the Laplacian eigenvalues of G are $2 - 2\cos\frac{k\pi}{n}, k = 0,1,2,\ldots,n-1$.

The following lemma was obtained by M. Fiedler [17].

Lemma 2.10. ([16], Proposition 1.8.2) *Let G be a graph with vertex set V. Suppose $D \subset V$ is a set of vertices such that the subgraph induced by G on $V \setminus D$ is disconnected. Then $\mu_2(G) \leq |D|$.*

3. Proof of Theorem 1.2

Before giving a proof for Theorem 1.2, we introduce a lemma for use.

Lemma 3.1. *Let G be a graph obtained from $\Gamma = K_{n-4} \vee C_4$ by deleting at least one edge joining vertices of K_{n-4} and vertices of C_4. Then $m_G(n-1,n] \leq n-4$.*

Proof. Let $e_1, e_2, \ldots, e_k$ be the deleted edges, where each e_i joins a vertex of K_{n-4} and a vertex of C_4. Denoted by $G_i = \Gamma - \{e_1, e_2, \ldots, e_i\}, 1 \leq i \leq k$. Then $G_k = G$. For G_1, its complement is $K_1^{n-5} \cup P_3 \cup P_2$, whose Laplacian spectrum is $\{0^{(n-3)}, 1, 2, 3\}$. By Lemma 2.2, the Laplacian spectrum of G_1 is $\{n^{(n-4)}, n-1, n-2, n-3, 0\}$. Hence $m_{G_1}(n-1,n] = n-4$. For $2 \leq i \leq k$, G_i is obtained from G_{i-1} by deleting one edge. By Lemma 2.1, $m_G(n-1,n] = m_{G_k}(n-1,n] \leq \cdots \leq m_{G_1}(n-1,n] = n-4$. □

Based on the lemmas shown in Section 2 and Lemma 3.1, we give a proof for Theorem 1.2.

Proof of Theorem 1.2

Proof. Let G be a connected graph of order n with $m_G(n-1,n] = n-3$. Suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$. By Lemma 2.4, the chromatic number of G is at least $n-2$. If $\chi(G) = n$, then $G = K_n$ and $m_G(n-1,n] = n-1$, leading to a contradiction.

(1) $\chi(G) = n-1$.

Since $\chi(G) = n-1$, G is obtained from K_n by deleting some edges joining a given vertex v and some other vertices of K_n. If G is obtained from K_n by deleting exactly one edge, then $G = K_{n-2} \vee K_1^2$ and thus $m_G(n-1,n] = n-2$, a contradiction. We claim that G is obtained from K_n by deleting exactly two edges incident to v. Otherwise, $\overline{G} = K_1^{n-k} \cup S_k$ with $k \geq 4$, whose spectrum is $\{0^{(n-k+1)}, 1^{(k-2)}, k\}$, thus G has spectrum $\{n^{(n-k)}, (n-1)^{(k-2)}, n-k, 0\}$, proving that $m_G(n-1,n] = n-k \leq n-4$, a contradiction. Consequently, $G = K_{n-3} \vee (K_1 \cup P_2)$. If $G = K_{n-3} \vee (K_1 \cup P_2)$, then $\overline{G} = K_1^{n-3} \cup S_3$ with Laplacian spectrum $\{0^{(n-2)}, 1, 3\}$, thus G has Laplacian spectrum $\{n^{(n-3)}, n-1, n-3, 0\}$ and $m_G(n-1,n] = n-3$. In this case, $m_G(n-1,n] = n-3$ if and only if $G = K_{n-3} \vee (K_1 \cup P_2)$.

In what follows, we assume $\chi(G) = n-2$. Let $f : V(G) \to X$ with $|X| = n-2$ be a surjective mapping assigning each vertex v of G a color $f(v)$ such that $f(v) \neq f(u)$ if v, u are adjacent. Two cases may occur.

(2) Three vertices of G, say v_1, v_2, v_3, are assigned a common color and all other vertices are assigned distinct colors. More precisely, $f(v_1) = f(v_2) = f(v_3)$ and $f(v_i) \neq f(v_j)$ for $3 \leq i \neq j \leq n$.

In this case, we aim to prove that $G = K_{n-3} \vee K_1^3$. Since $f(v_1) = f(v_2) = f(v_3)$, $\{v_1, v_2, v_3\}$ forms an independent set of G. Let $W = V(G) \setminus \{v_1, v_2, v_3\}$. As $\chi(G) = n-2$, it is easy to see that $G[W]$ forms a complete subgraph of G. Indeed, if two vertices v_i, v_j in W are not adjacent, then the mapping f' from $V(G)$ to $X \setminus \{f(v_i)\}$ with $f'(v_k) = f(v_k)$ for $k \neq i$ and $f'(v_i) = f(v_j)$ gives a proper coloring for G with one fewer colors than $n-2$, leading to a contradiction. To prove $G = K_{n-3} \vee K_1^3$, it suffices to prove that each vertex in $\{v_1, v_2, v_3\}$ is adjacent to all vertices in W. Suppose for a contradiction that a vertex v_i, with $1 \leq i \leq 3$, is not adjacent to a vertex $v_s \in W$ with $s \geq 4$. Then the deletion of all vertices in $W \setminus \{v_s\}$ from G will turn G into a disconnected graph with v_i an isolated vertex. Consequently, $\kappa(G) \leq n-4$, leading to $\kappa(G) < m_G(n-1, n]$, which contradicts Lemma 2.3. Consequently, $G = K_{n-3} \vee K_1^3$. Conversely, if $G = K_{n-3} \vee K_1^3$, the complement of G is $K_1^{n-3} \cup K_3$, whose Laplacian spectrum is $\{0^{(n-2)}, 3, 3\}$. By Lemma 2.2, the Laplacian spectrum of G is $\{n^{(n-3)}, n-3, n-3, 0\}$, proving that $m_G(n-1, n] = n-3$.

(3) $f(v_1) = f(v_3)$, $f(v_2) = f(v_4)$, and $f(v_i) \neq f(v_j)$ for $3 \leq i \neq j \leq n$.

In this case, we aim to prove that $m_G(n-1, n] = n-3$ if and only if either $G = K_{n-4} \vee C_4$ or $G = K_{n-4} \vee P_4$. Let $U = V(G) \setminus \{v_1, v_2, v_3, v_4\}$.

■ $G[U]$ is a complete graph.

If two vertices v_i, v_j in U are not adjacent, then the mapping f' from $V(G)$ to $X \setminus \{f(v_i)\}$ with $f'(v_k) = f(v_k)$ for $k \neq i$ and $f'(v_i) = f(v_j)$ gives a proper coloring for G with one fewer color than $n-2$, a contradiction.

■ $G[v_1, v_2, v_3, v_4]$ is connected.

Otherwise, the deletion of U from G results in a disconnected graph, thus $\kappa(G) \leq n-4 < m_G(n-1, n]$, a contradiction to Lemma 2.3.

If $G[v_1, v_2, v_3, v_4] = C_4$, then each vertex of C_4 must be adjacent to each vertex of K_{n-4} (by Lemma 3.1), thus $G = K_{n-4} \vee C_4$. The complement of $K_{n-4} \vee C_4$ is $K_1^{n-4} \cup P_2 \cup P_2$, whose Laplacian spectrum is $\{0^{(n-2)}, 2, 2\}$. By applying Lemma 2.2, the Laplacian spectrum of G is $\{n^{(n-3)}, n-2, n-2, 0\}$, proving that $m_G(n-1, n] = n-3$.

If $G[v_1, v_2, v_3, v_4] = P_4$, then each vertex of P_4 must be adjacent to each vertex of K_{n-4}, and thus $G = K_{n-4} \vee P_4$. If $G[v_1, v_2, v_3, v_4] = P_4$, then each vertex of P_4 must be adjacent to every vertex of K_{n-4}. Otherwise, let G' be the graph obtained from $K_{n-4} \vee C_4$ by deleting precisely the same cross edges that are absent in G. Since at least one cross edge is deleted, Lemma 3.1 gives $m_{G'}(n-1, n] \leq n-4$. Moreover, G is obtained from G' by deleting one edge of the C_4. Hence, $m_G(n-1, n] \leq m_{G'}(n-1, n] \leq n-4$ (by Lemma 2.1), a contradiction. Conversely, if $G = K_{n-4} \vee P_4$, the complement of G is $K_1^{n-4} \cup P_4$, whose Laplacian spectrum is $\{0^{(n-3)}, 2-\sqrt{2}, 2, 2+\sqrt{2}\}$ (by Lemma 2.9). By Lemma 2.2, the Laplacian spectrum of G is $\{n^{(n-4)}, n-2+\sqrt{2}, n-2, n-2-\sqrt{2}, 0\}$, proving that $m_G(n-1, n] = n-3$. □

4. Proof of Theorem 1.3

A main technique for proving Theorem 1.3 is to reduce the problem about graphs with $m_G(n-1, n] = n-4$ to that about graphs with $m_G[0, 1) = n-3$, then Lemmas 2.5–2.7 are applied to a classification of graphs with $m_G[0, 1) = n-3$.

Lemma 4.1. *Let G be a connected graph of order n. Suppose $2 \leq k \leq n-1$, then $m_G(n-1, n] = n-k$ if and only if $m_{\overline{G}}[0, 1) = n-k+1$. In particular, $m_G(n-1, n] = n-4$ if and only if $m_{\overline{G}}[0, 1) = n-3$.*

Proof. Assume the Laplacian eigenvalues of G are arranged in nondecreasing order as

$$0 = \mu_1(G) \leq \mu_2(G) \leq \ldots \leq \mu_n(G).$$

By Lemma 2.2, $\mu_i(\overline{G}) = n - \mu_{n-i+2}(G)$ for $2 \leq i \leq n$. One can easily see that

$$\begin{array}{cc} m_G(n-1, n] = n-k & \Leftrightarrow \\ \mu_{k+1}(G) > n-1, \ \ \mu_k \leq n-1 & \Leftrightarrow \\ \mu_{n-k+1}(\overline{G}) < 1, \ \ \mu_{n-k+2}(\overline{G}) \geq 1 & \Leftrightarrow \\ m_{\overline{G}}[0, 1) = n-k+1. & \end{array}$$

□

In view of Lemma 4.1, to give a classification for graphs with $m_G(n-1, n] = n-4$, it suffices to give a classification for graphs with $m_G[0, 1) = n-3$. When we proceed with the classification for graphs with $m_G[0, 1) = n-3$, we find that the number of isolated vertices in G, denoted by $\sigma(G)$, contributes a lot to $m_G[0, 1)$.

Lemma 4.2. *Let G be a graph of order n. Then $m_G[0, 1) \leq \lfloor \frac{n+\sigma(G)}{2} \rfloor$.*

Proof. Let H be the subgraph of G obtained by deleting all isolated vertices from G. Then $m_G[0, 1) = \sigma(G) + m_H[0, 1)$. Let $H_1, \ldots, H_t$ be the connected components of H. Since H has no isolated vertices, Lemma 2.5 can be applied to each H_i. Hence $m_H[0, 1) = \sum_{i=1}^{t} m_{H_i}[0, 1) \leq \sum_{i=1}^{t} \beta(H_i) = \beta(H)$. Clearly, $\beta(H) \leq \lfloor \frac{|V(H)|}{2} \rfloor = \lfloor \frac{n-\sigma(G)}{2} \rfloor$. Consequently, $m_G[0, 1) \leq \lfloor \frac{n+\sigma(G)}{2} \rfloor$. □

Lemma 4.3. *Let G be a graph of order n with $\sigma(G)$ isolated vertices. Then $n-2k \leq \sigma(G) \leq n-k-1$ if $m_G[0, 1) = n-k$ with $1 \leq k \leq n-1$. In particular, $n-6 \leq \sigma(G) \leq n-4$ if $m_G[0, 1) = n-3$.*

Proof. Assume that $m_G[0, 1) = n-k$ with $1 \leq k \leq n-1$. If $\sigma(G) < n-2k$, then $m_G[0, 1) \leq \lfloor \frac{n+\sigma(G)}{2} \rfloor < n-k$, a contradiction to the assumption, proving that $n-2k \leq \sigma(G)$. Observe that $m_G[0, 1) = \sigma(G) + m_H[0, 1)$, where H is the subgraph of G obtained by deleting all isolated vertices. If $\sigma(G) \geq n-k$, then we have $m_G[0, 1) \geq n-k+1$ (since $m_H[0, 1) \geq 1$), which is a contradiction to the assumption for $m_G[0, 1)$. Hence, $\sigma(G) \leq n-k-1$. □

Suppose G is a graph of order n with $n-6 \leq \sigma \leq n-4$ isolated vertices and H is the subgraph of G obtained by deleting all isolated vertices from G. If $m_G[0, 1) = n-3$, then H is a graph on $n-\sigma$ vertices with $m_H[0, 1) = n-3-\sigma$. More precisely,

$$\begin{array}{ll} m_H[0, 1) = 3 & \text{if } \sigma = n-6; \\ m_H[0, 1) = 2 & \text{if } \sigma = n-5; \\ m_H[0, 1) = 1 & \text{if } \sigma = n-4. \end{array}$$

This tells us that the classification of graphs of order n with $m_G(n-1, n] = n-4$ can be reduced to the classification of the following graphs:

- Graphs of order 6 without isolated vertices such that $m_H[0, 1) = 3$;
- Graphs of order 5 without isolated vertices such that $m_H[0, 1) = 2$;
- Graphs of order 4 without isolated vertices such that $m_H[0, 1) = 1$.

Before proceeding with the classification of the above three types of graphs, we introduce a lemma that can simplify our proof.

Lemma 4.4. *Let $G = (V, E)$, $Q = (V, E_1)$, and $H = (V, E_2)$ be graphs with $E_2 \subseteq E_1 \subseteq E$. If $m_G[0, 1) = m_H[0, 1) = k$, then $m_Q[0, 1) = k$.*

Proof. It follows from Lemma 2.8 that

$$k = m_G[0, 1) \leq m_Q[0, 1) \leq m_H[0, 1) = k,$$

which proves that $m_Q[0, 1) = k$. □

To make the completeness of the classification transparent, we proceed by the number of connected components and, in the connected case, refine the discussion according to the cyclomatic number.

Lemma 4.5. *Let Q be a graph of order* 6 *without isolated vertices. Then $m_Q[0, 1) = 3$ if and only if Q is one of the graphs shown in Figure 2.*

Figure 2. Graphs of order 6 with $m_Q[0, 1) = 3$.

Proof. The Laplacian spectrum of Q_1 is $\{0^{(3)}, 2^{(3)}\}$, thus $m_{Q_1}[0, 1) = 3$. Observing that Q_4 has three quasi-pendant vertices and its matching number is three, by Lemmas 2.5 and 2.7, we have $m_{Q_4}[0, 1) = 3$. Applying Lemma 4.4, we have $m_{Q_2}[0, 1) = m_{Q_3}[0, 1) = 3$.

Conversely, suppose Q is a graph of order 6 without isolated vertices and $m_Q[0, 1) = 3$. Since Q has no isolated vertices, it contains at most three components.

If Q has three components, then $Q = P_2^3$, which is the graph Q_1 in Figure 2.

If Q has two components, one component must be P_2 (otherwise, each component of Q has three vertices, and it is either P_3 or K_3, leading to $m_Q[0, 1) = 2$, a contradiction), the other component, say H, must have four vertices. If a vertex of H has degree 3, then $\gamma(H) \leq 1$, leading to $m_Q[0, 1) = 1 + m_H[0, 1) \leq 2$, which is a contradiction. Hence, each vertex of H has degree at most 2, forcing H to be C_4 or P_4. Since $m_{C_4}[0, 1) = 1$ and $m_{P_4}[0, 1) = 2$, H must be P_4. Thus $Q = P_2 \cup P_4$, which is the graph Q_2 in Figure 2.

Next, we consider the case when Q is connected. If Q is a tree, since $3 = m_Q[0, 1) \leq \gamma(Q)$, the domination number of Q must be 3, from which it follows that Q must be the tree Q_3 shown in Figure 2.

If Q is a unicyclic graph, it must be Q_4, which is obtained from Q_3 by adding an edge e joining two vertices of Q_3 of degree 2. Indeed, if the added edge e joins a pendant vertex of Q_3 and a vertex of degree at most 2, then the resultant graph has P_6 as a spanning subgraph, and thus $m_{P_6}[0, 1) \geq m_Q[0, 1) = 3$, a contradiction to Lemma 2.9. If e joins a pendant vertex of Q_3 and the 3-degree vertex, then the resultant graph has domination number 2, and thus $m_Q[0, 1) \leq 2$, a contradiction.

Now, we confirm that the case $\theta(Q) \geq 2$ cannot happen. Suppose that $\theta(Q) \geq 2$. By deleting cycle edges if necessary, we obtain a spanning bicyclic subgraph B of Q. By Lemma 2.8, $m_B[0, 1) \geq m_Q[0, 1) = 3$. On the other hand, since B has order 6 and Lemma 2.5, $m_B[0, 1) \leq \beta(B) \leq 3$. Hence $m_B[0, 1) = 3$. Deleting a cycle edge from B gives a spanning unicyclic graph U. Similarly, $3 \leq m_U[0, 1) \leq \beta(U) \leq 3$, and hence $m_U[0, 1) = 3$. By the preceding unicyclic case, $U = Q_4$. Therefore B is obtained from Q_4 by adding one edge. Let the unique triangle of Q_4 be $x_1x_2x_3x_1$ and y_i be the pendant neighbor of x_i for $i = 1, 2, 3$. Under the sense of isomorphism, an edge not contained in Q_4 is either y_1y_2 or y_1x_2. If $e = y_1y_2$, then $y_3x_3x_1y_1y_2x_2$ is a spanning path P_6. If $e = y_1x_2$, then $y_2x_2y_1x_1x_3y_3$ is a spanning path P_6. Thus every bicyclic graph obtained from Q_4 by adding an edge contains P_6 as a spanning subgraph. Consequently, $m_{P_6}[0, 1) \geq m_B[0, 1) = 3$, contradicting Lemma 2.9. □

Lemma 4.6. *Let Q be a graph of order 5 without isolated vertices. Then $m_Q[0, 1) = 2$ if and only if Q is one of the graphs shown in Figure 3.*

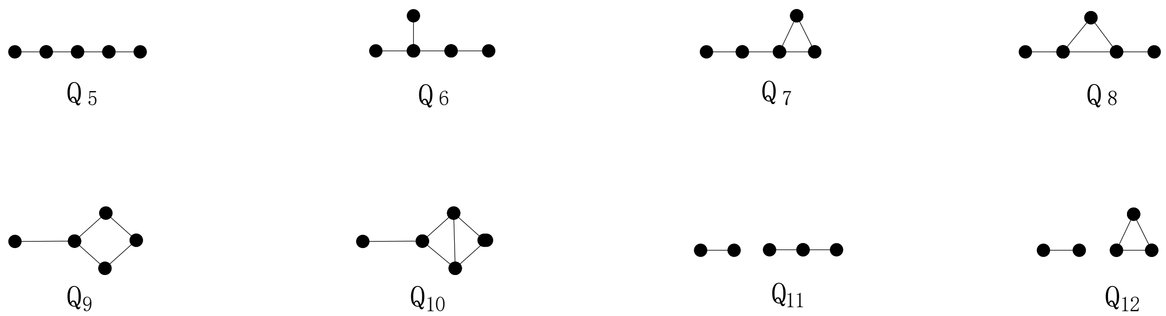


Figure 3. Graphs of order 5 with $m_Q[0, 1) = 2$.

Proof. Considering the edge set of each graph in Figure 3, one can see that $E(Q_{11}) \subseteq E(Q_i) \subseteq E(Q_{10})$ for each i with $5 \leq i \leq 12$. By Lemma 4.4, if we can prove that $m_{Q_{10}}[0, 1) = m_{Q_{11}}[0, 1) = 2$, then $m_{Q_i}[0, 1) = 2$ for each Q_i with $5 \leq i \leq 12$.

Observing that Q_{11} has spectrum $\{0^{(2)}, 1, 2, 3\}$, thus $m_{Q_{11}}[0, 1) = 2$.

The deletion of the quasi-pendant vertex from Q_{10} turns the graph into a disconnected graph, thus $\mu_2(Q_{10}) \leq 1$ (by Lemma 2.10). However, 1 is not a Laplacian eigenvalue of Q_{10} by using Mathematica, showing that $\mu_2(Q_{10}) < 1$. Hence $m_{Q_{10}}[0, 1) \geq 2$. Observing the matching number of Q_{10} is 2, we have $m_{Q_{10}}[0, 1) \leq 2$ (by Lemma 2.5). Hence, $m_{Q_{10}}[0, 1) = 2$.

Next, we prove that Q is one of the graphs in Figure 3; if Q is a graph of order 5 without isolated vertices such that $m_Q[0, 1) = 2$.

If Q is not connected, then it has exactly two components; one component is P_2, and the other component is P_3 or C_3. In this case, Q is Q_{11} or Q_{12}. In what follows, we suppose Q is connected. Clearly, Q has no 4-degree vertex. Otherwise, Q has domination number 1, and thus $m_Q[0, 1) \leq 1$ (by Lemma 2.6), a contradiction.

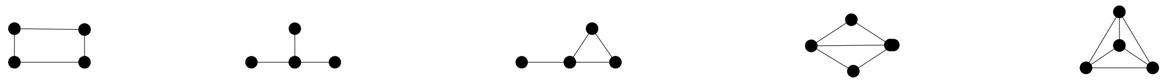
When Q is a tree, if Q has a 3-degree vertex, then Q is the graph Q_6 in Figure 3. If Q has no 3-degree vertices, then Q is P_5, which is the graph Q_5.

Now, we consider the case when Q is a unicyclic graph. If the girth of Q is 3, then Q is either Q_7 or Q_8. If the girth of Q is 4, then Q is Q_9. The girth of Q can not be 5, otherwise $Q = C_5$ with $m_Q[0, 1) = 1$, a contradiction.

Next, we consider the case when Q is a bicyclic graph. By Lemma 2.8, Q is obtained from one of the graphs in $\{Q_7, Q_8, Q_9\}$ by adding an edge. If Q has no pendant vertex, either Q is $K_{3,2}$ or Q has a Hamilton cycle passing all vertices of Q (recalling that no vertex in Q has degree 4). Since $K_{3,2}$ has Laplacian spectrum $\{0, 2, 2, 3, 5\}$, the former case cannot occur. If the latter case happens, then $m_{C_5}[0, 1) \geq m_Q[0, 1) = 2$, a contradiction. Hence, Q has a pendant vertex. Since Q has no 4-degree vertex, Q must be the graph Q_{10}.

Finally, Suppose that $\theta(Q) \geq 3$. By deleting edges lying on cycles, we obtain a spanning graph R with $\theta(R) = 3$. By Lemma 2.8, $m_R[0, 1) \geq m_Q[0, 1) = 2$, while $m_R[0, 1) \leq \beta(R) \leq 2$ by Lemma 2.5. Hence $m_R[0, 1) = 2$. Deleting a cycle edge from R gives a spanning bicyclic graph B. The same argument gives $m_B[0, 1) = 2$, and hence, by the preceding bicyclic case, $B = Q_{10}$. Therefore R is obtained from Q_{10} by adding one edge. Recall that R has no vertex of degree 4. Since Q_{10} has three vertices of degree 3, one vertex of degree 2, and one pendant vertex, the added edge must join the pendant vertex and the vertex of degree 2. The resulting graph has C_5 as a spanning subgraph. Hence, by Lemma 2.8, $m_{C_5}[0, 1) \geq m_R[0, 1) = 2$, which contradicts $m_{C_5}[0, 1) = 1$. Therefore, $\theta(Q) \leq 2$. □

Lemma 4.7. *Let Q be a graph of order 4 without isolated vertices. Then $m_Q[0, 1) = 1$ if and only if Q is one of the graphs shown in Figure 4.*



Q_{13} Q_{14} Q_{15} Q_{16} Q_{17}

Figure 4. Graphs of order 4 with $m_Q[0, 1) = 1$.

Proof. Except for Q_{13}, each graph in Figure 4 has a vertex of degree 3 and hence has domination number 1, forcing that $m_{Q_i}[0, 1) = 1$ for $14 \leq i \leq 17$. Since C_4 has Laplacian spectrum $\{0, 2, 2, 4\}$, we have $m_{Q_{13}}[0, 1) = 1$.

Next, we prove that Q is one of the graphs in Figure 4: if Q is a graph of order 4 without isolated vertices such that $m_Q[0, 1) = 1$. Since $m_Q[0, 1) = m_Q(0) = 1$, the graph has only one connected component. So, Q is connected. If Q is a tree, it is P_4 or S_4. However, P_4 has Laplacian spectrum $\{0, 2 - \sqrt{2}, 2, 2 + \sqrt{2}\}$ with $m_{P_4}[0, 1) = 2$, forcing $Q = S_4$, which is the graph Q_{14}. If Q is not a tree, then Q has at least four edges. If $|E(Q)| = 4$, then Q is either C_4 or a triangle with a pendant vertex. If $|E(Q)| = 5$, then $Q = K_4 - e$, and if $|E(Q)| = 6$, then $Q = K_4$. These graphs are precisely Q_{13}, Q_{15}, Q_{16}, and Q_{17} shown in Figure 4. □

For convenience, Table 1 lists the correspondence between each graph H_i in Figure 1 and its complement Q_i appearing in Figures 2–4.

Table 1. The correspondence between the graphs H_i and their complements Q_i.

Order	Graph H_i	Complement $Q_i = \overline{H_i}$
6	H_1	Q_1
	H_2	Q_2
	H_3	Q_3
	H_4	Q_4
5	H_5	Q_5
	H_6	Q_6
	H_7	Q_7
	H_8	Q_8
	H_9	Q_9
	H_{10}	Q_{10}
	H_{11}	Q_{11}
	H_{12}	Q_{12}
4	H_{13}	Q_{13}
	H_{14}	Q_{14}
	H_{15}	Q_{15}
	H_{16}	Q_{16}
	H_{17}	Q_{17}

Now, we are ready to give a proof for Theorem 1.3.

Proof of Theorem 1.3

Proof. If $G = K_{n-s} \vee H_i$, where H_i is one of the graphs in Figure 1, then the complement of G is $\overline{G} = K_1^{n-s} \cup Q_i$, where $Q_i = \overline{H_i}$ is one of the graphs shown in Figure 2, Figure 3, or Figure 4. By Lemmas 4.5–4.7, $m_{\overline{G}}[0, 1) = n - 3$. Then by Lemma 4.1, we have $m_G(n - 1, n] = n - 4$.

Conversely, assume that $m_G(n - 1, n] = n - 4$. By Lemma 4.1, $m_{\overline{G}}[0, 1) = n - 3$. Let Q be the graph obtained from $\overline{G}$ by deleting the isolated vertices. Then Q has order $6, 5$, or 4 and $m_Q[0, 1)$ is $3, 2$, or 1, respectively. If Q has order 6 and $m_Q[0, 1) = 3$, then Q is one of the graphs in Figure 2. In this case, G is one of the graphs in $\{K_{n-6} \vee H_i : i = 1, 2, 3, 4\}$, where H_i is the complement of Q_i for $1 \leq i \leq 4$. If Q has order 5 and $m_Q[0, 1) = 2$, then Q is one of the graphs in Figure 3. In this case, G is one of the graphs in $\{K_{n-5} \vee H_j : j = 5, 6, \ldots, 12\}$, where H_j is the complement of Q_j for $5 \leq j \leq 12$. If Q has order 4 and $m_Q[0, 1) = 1$, then Q is one of the graphs in Figure 4. In this case, G is one of the graphs in $\{K_{n-4} \vee H_k : k = 13, 14, 15, 16, 17\}$, where H_k is the complement of Q_k for $13 \leq k \leq 17$. □

5. Conclusions

In this paper, we have characterized the connected graphs of order n with at least $n - 4$ Laplacian eigenvalues in the interval $(n - 1, n]$. In particular, we have obtained complete classifications for the cases $m_G(n - 1, n] = n - 3$ and $m_G(n - 1, n] = n - 4$. The former consists of four graph families, while the latter is described by 17 graphs of orders 4, 5, and 6, each joined with a complete graph of the appropriate order. Together with the elementary characterizations of the cases $m_G(n - 1, n] = n - 1$ and $m_G(n - 1, n] = n - 2$, these results complete the classification for the range considered here.

The main approach is to pass to the complement and translate the distribution of Laplacian eigenvalues in $(n-1,n]$ into that of eigenvalues in $[0,1)$. Bounds involving the matching number, domination number, and the number of isolated vertices then reduce the problem to a finite collection of small graphs. This reduction provides a systematic way to obtain the classification and illustrates the usefulness of combining spectral inequalities with structural properties of graphs. The same perspective may also be useful in investigating the corresponding classification problem for $m_G(n-1,n]=n-k$ with larger values of k.

Author contributions

Yifan Liu: Writing-original draft, conceptualization, data curation and formal analysis; Jinxing Zhao: Funding acquisition, investigation, methodology, project administration, software, resources, supervision, validation, visualization, and writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

References

1. M. Ahanjideh, S. Akbari, M. H. Fakharan, V. Trevisan, Laplacian eigenvalue distribution and graph parameters, *Linear Algebra Appl.*, **632** (2022), 1–14. https://doi.org/10.1016/j.laa.2021.09.012
2. L. Wang, C. Yan, X. Fang, X. Geng, F. Tian, Vertex-connectivity, chromatic number, domination number, maximum degree and Laplacian eigenvalue distribution, *Linear Algebra Appl.*, **607** (2020), 307–318. https://doi.org/10.1016/j.laa.2020.08.011
3. I. Faria, Permanental roots and the star degree of a graph, *Linear Algebra Appl.*, **64** (1985), 255–265. https://doi.org/10.1016/0024-3795(85)90281-2
4. R. Grone, R. Merris, V. S. Sunder, The Laplacian spectrum of a graph, *SIAM J. Matrix Anal. Appl.*, **11** (1990), 218–238. https://doi.org/10.1137/0611016
5. R. Merris, The number of eigenvalues greater than two in the Laplacian spectrum of a graph, *Port. Math.*, **48** (1991), 345–349.

6. J. M. Guo, S. W. Tan, A relation between the matching number and Laplacian spectrum of a graph, *Linear Algebra Appl.*, **325** (2001), 71–74. https://doi.org/10.1016/S0024-3795(00)00333-5
7. J. M. Guo, X. L. Wu, J. M. Zhang, K. F. Fang, On the distribution of Laplacian eigenvalues of a graph, *Acta Math. Sinica*, **27** (2011), 2259–2268. https://doi.org/10.1007/s10114-011-8624-y
8. S. T. Hedetniemi, D. P. Jacobs, V. Trevisan, Domination number and Laplacian eigenvalue distribution, *European J. Combin.*, **53** (2016), 66–71. https://doi.org/10.1016/j.ejc.2015.11.005
9. D. P. Jacobs, E. R. Oliveira, V. Trevisan, Most Laplacian eigenvalues of a tree are small, *J. Comb. Theory, Ser. B*, **146** (2021), 1–33. https://doi.org/10.1016/j.jctb.2020.07.003
10. C. Sin, On the number of Laplacian eigenvalues of trees less than the average degree, *Discrete Math.*, **343** (2020), 111986. https://doi.org/10.1016/j.disc.2020.111986
11. V. Trevisan, J. B. Carvalho, R. R. Del Vecchio, C. T. M. Vinagre, Laplacian energy of diameter 3 trees, *Appl. Math. Lett.*, **24** (2011), 918–923. https://doi.org/10.1016/j.aml.2010.12.050
12. L. Xu, B. Zhou, Proof of a conjecture on distribution of Laplacian eigenvalues and diameter, and beyond, *Linear Algebra Appl.*, **678** (2023), 92–106. https://doi.org/10.1016/j.laa.2023.08.013
13. A. El-Mesady, Y. S. Hamed, H. Shabana, On the decomposition of circulant graphs using algorithmic approaches, *Alex. Eng. J.*, **61** (2022), 8263–8275. https://doi.org/10.1016/j.aej.2022.01.049
14. B. Mohar, The Laplacian spectrum of graphs, In: *Graph theory, combinatorics, and applications*, Wiley, 1991, 871–898.
15. C. Godsil, G. Royle, *Algebraic graph theory*, New York: Springer, 2001. https://doi.org/10.1007/978-1-4613-0163-9
16. A. E. Brouwer, W. H. Haemers, *Spectra of graphs*, New York: Springer, 2012. https://doi.org/10.1007/978-1-4614-1939-6
17. M. Fiedler, Algebraic connectivity of graphs, *Czechoslovak Math. J.*, **23** (1973), 298–305. https://doi.org/10.21136/CMJ.1973.101168