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Research article

Complexification of z-semicircular systems and orientation-averaged z-circular laws

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Abstract: In this work, we study whether taking the complexification of a z-semicircular system naturally gives rise to a z-circular system. Starting from a semicircular family (s_r), we form complex variables $c_r = \frac{1}{\sqrt{2}}(s_{2r-1} + i\, s_{2r})$ and analyze their joint $*$-moments using the combinatorics of pairings with oriented crossings. In the classical case where the deformation parameter is real $q \in (-1, 1)$, this construction yields a q-circular system, recovering the known correspondence between semicircular and circular variables. However, we show that this correspondence fails for general complex parameters z when $|z| < 1$. The obstruction is the additional averaging over balanced orientations, which agrees with the standard z-circular law only when z is real. We show that the complexification of a z-semicircular system introduces a new class of distributions, which we call orientation-averaged z-circular systems, defined by averaging the contributions of oriented crossings over all admissible orientations. When $z \in \mathbb{R}$, these systems coincide with the classical z-circular systems. However, they are fundamentally different in the non-real case. As a result, we get a rigidity finding indicating that the only z-circular systems formed by complexifying semicircular systems have a real deformation parameter.

Keywords: z-circular systems; z-semicircular systems; free probability; operator theory; noncommutative probability
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1. Introduction

The study of noncommutative distributions has played a central role in free probability since the pioneering work of Voiculescu [1, 2]. In particular, semicircular and circular systems provide basic models for noncommutative Gaussian random variables. The q-Gaussian framework was developed in [3, 4], and q-circular systems were studied by Mingo and Nica [5]. These developments have led to a fruitful interplay among operator algebras, combinatorics, and probability theory.

The complex-parameter circular systems referred to here as z-circular systems were studied by Mingo and Nica, who constructed an asymptotic model using random unitaries in noncommutative tori and derived moment formulas involving oriented crossings [5]. The broader theory of q-Gaussian variables and generalized Brownian motions has been developed through noncommutative central limit theorems [6]. Foundational constructions of generalized Brownian motions and q-Gaussian processes were developed in [3, 4]. Fock-space representations of the q-commutation relations were studied in [7], while deformed commutation relations and related operator-space structures were investigated in [8]. Wick-product expansions were developed in [9].

Several complementary approaches have since emerged. Random-matrix models for q-Gaussian variables and related many-particle spectral limits were developed by Śniady [10, 11]. Nou constructed asymptotic matricial models for q-Araki-Woods von Neumann algebras and established their quotient weak expectation property (QWEP) [12]. Noninjectivity was studied in [13], factoriality in [14], and exactness of the Fock-space representation in [15]. Dual and conjugate systems for q-Gaussians were constructed in [16], and the factoriality problem for q-Araki-Woods von Neumann algebras was resolved via conjugate variables in [17].

Two-parameter deformations and related limit theorems were studied in [18–20], and Coxeter-type deformations associated with groups of type B were considered in [21, 22]. Deformed convolutions were investigated in [23]. Convolutions related to q-deformed commutativity were studied in [24], and a limit theorem for the q-convolution was established in [25]. Feynman-diagram formulas for q-Wick products and crossing statistics in partitions were studied in [9, 26], respectively. The free infinite divisibility of the q-Gaussian distribution for $0 \le q \le 1$ was proved in [27].

Not all of these works deal directly with z-circular systems. They do, however, place the orientation-sensitive pairing formulas studied here within the wider development of deformed noncommutative probability, Fock-space models, random-matrix approximations, and crossing combinatorics.

For a broader numerical and computational context, we also mention recent developments in neural-network-based methods for partial differential equations [28], high-order compact finite-difference methods based on radial basis functions [29], radial basis function-generated finite difference (RBF-FD) methods [30], and iterative methods for nonlinear systems [31]. Although these works address problems different from those considered here, they provide relevant examples of recent computational developments in applied mathematics.

The combinatorial framework used in this paper is based on moment formulas indexed by pair partitions in noncommutative probability [3, 6]. For general background on combinatorial methods in free probability, see [32]. Mingo and Nica introduced z-circular systems and described their moments using pairings, permutations, and oriented crossings [5]. The notion of a z-semicircular system was introduced later in [33], with moment formulas that also involve balanced orientation maps. In both z-settings, crossings are distinguished as positive or negative [5, 33].

A key structural result in this setting is that every z-circular system gives rise to a z-semicircular system by taking normalized self-adjoint parts. More precisely, if $(c_m)_{m=1}^{s}$ is a z-circular system, then the family

$$\widehat{s_m} = \frac{c_m + c_m^*}{\sqrt{2}}, \qquad m = 1, \ldots, s$$

forms a z-semicircular system; see [33].

This result establishes a natural passage from z-circular systems to z-semicircular systems. A natural question was explicitly posed as Conjecture 5.4.4 in [33]. The conjecture may be stated as follows:

If $(s_r)_{r=1}^{2s}$ is a z-semicircular system, and

$$c_m = \frac{1}{\sqrt{2}}\,(s_{2m-1} + i\, s_{2m}), \qquad m = 1, \ldots, s,$$

then $(c_1, \ldots, c_s)$ is a z-circular system.

In this paper, we show that this conjecture holds when z is real, but fails when z is nonreal.

This study aims to provide a complete answer to this problem. Our analysis reveals a sharp contrast between the real and complex cases: Although the converse is valid in the q-case, it fails for nonreal complex parameters z. Moreover, we identify the structure that arises in place of the z-circularity.

The main contributions of this paper can be summarized as follows. First, we show that in the q-deformed setting the converse construction does hold: The complexification of a q-semicircular system produces a q-circular system, thereby giving a complete characterization in the real case. We then turn to the complex deformation framework and derive a general moment formula for the complexification of a z-semicircular system. This formula reveals that the resulting family is governed by a pairing expansion in which one averages over all balanced orientation maps associated with each pairing. Building on this description, we construct an explicit counterexample showing that for nonreal values of z, the complexified family is not z-circular in general, thereby disproving the conjectured converse in the complex setting. This finding motivates the introduction of a new class of noncommutative distributions, which we call orientation-averaged z-circular systems. These arise naturally as the distributions of complexified z-semicircular systems and are characterized by a modified moment formula that incorporates an averaging over all possible orientations. Finally, we examine the structural relationship between these new systems and the classical z-circular systems, establishing that the two notions agree when $z \in \mathbb{R}$ but they differ fundamentally in the nonreal case.

Conceptually, our results reveal that the passage from semicircular to circular systems within the z-framework is governed not only by the combinatorics of pairings but also by an additional layer of orientation information. In z-circular systems, this orientation is fully preserved, whereas in the complexification of z-semicircular systems, it is effectively averaged out, giving rise to a new class of distributions. This difference highlights a rigidity phenomenon specific to the complex deformation setting and suggests new directions for exploring orientation-dependent structures in noncommutative probability.

The paper is organized as follows. In Section 2, we recall the necessary combinatorial and probabilistic background. Section 3 treats the q-case and establishes the converse result in the real setting. In Section 4, we derive the general moment formula for complexified z-semicircular systems. Section 5 contains the counterexample showing the failure of the conjectured converse. In Section 6, we introduce and study orientation-averaged z-circular systems. Finally, Section 7 summarizes the main results and presents several open problems.

2. Preliminaries

In this section, we recall the pairing formulas used to describe circular and semicircular systems. In the q-case, each pairing π contributes the factor $q^{\mathrm{cr}(\pi)}$ [3, 5, 6]. In the z-case, the contribution of a crossing also depends on its orientation. This orientation-sensitive framework was introduced for z-circular systems in [5] and extended to z-semicircular systems in [33].

Definition 2.1. A $*$-probability space is a pair (A, ψ), where A is a unital $*$-algebra, and $\psi : A \to \mathbb{C}$ is a unital positive linear functional.

Definition 2.2. A partition of $\{1, \dots, n\}$ is a family π of disjoint nonempty subsets (called blocks) whose union is $\{1, \dots, n\}$.

Definition 2.3. A partition π is called a pair partition if every block has cardinality 2. The set of all pair partitions of $\{1, \dots, 2k\}$ is denoted by $P_2(2k)$.

Pair partitions encode the contraction structure of Gaussian-type moment formulas and play a central role in noncommutative central limit theorems [6].

Definition 2.4. Let $\pi \in P_2(2k)$. Two blocks $\{a, b\}$ and $\{c, d\}$ (with $a < b$ and $c < d$) are said to be crossing if

$$a < c < b < d \quad \text{or} \quad c < a < d < b.$$

The number of such crossings is denoted by $\mathrm{cr}(\pi)$.

Definition 2.5. Let $t : \{1, \dots, n\} \to I$ be a map. The kernel partition of t, denoted by $\ker(t)$, is the partition of $\{1, \dots, n\}$ whose blocks are the level sets of t, namely

$$a \sim_{\ker(t)} b \quad \Longleftrightarrow \quad t(a) = t(b).$$

Equivalently, for any partition π of $\{1, \dots, n\}$, $\pi \leq \ker(t)$ if and only if t is constant on each block of π.

Definition 2.6. A direction map is a function $\varepsilon : \{1, \dots, 2k\} \to \{1, *\}$.

We use the convention $a^1 = a$ and $a^* = a^*$.

Definition 2.7. Let $r : \{1, \dots, 2k\} \to [s]$ and ε be a direction map. A pairing $\pi \in P_2(2k)$ is said to be admissible if for every block $\{a, b\} \in \pi$, one has

$$r(a) = r(b) \quad \text{and} \quad \varepsilon(a) \neq \varepsilon(b).$$

We denote this set by $P_2(r, \varepsilon)$.

Definition 2.8. Let $\pi \in P_2(2k)$. A map

$$\delta : \{1, \dots, 2k\} \to \{1, *\}$$

is called π-balanced if for every block $\{a, b\} \in \pi$, one has

$$\delta(a) \neq \delta(b).$$

Equivalently, each pair of π contains exactly one occurrence of 1 and one occurrence of $*$.

Remark 2.9. Throughout the paper, the symbol δ denotes a π-balanced orientation map unless stated otherwise. Thus, for every block $\{a, b\} \in \pi$, one has

$$\delta(a) \neq \delta(b).$$

We now introduce the refinement of crossings, which is essential in the z-deformed setting.

Definition 2.10. Let $\pi \in P_2(2k)$ be admissible. For a block $\{a, b\}$ with $a < b$, we define its orientation:
(i) forward if $(\varepsilon(a), \varepsilon(b)) = (1, *)$,
(ii) backward if $(\varepsilon(a), \varepsilon(b)) = (*, 1)$.

Definition 2.11. Let $\pi \in P_2(2k)$ be admissible, and consider two crossing blocks $\{a, b\}$ and $\{c, d\}$ with $a < c < b < d$.
(i) The crossing is called positive if both blocks have the same orientation.
(ii) It is called negative if the orientations differ.

Definition 2.12. Let $\pi = \{V_1, \dots, V_k\} \in P_2(2k)$ be a pairing written in standard order, where

$$V_i = \{a_i, b_i\}, \qquad a_i < b_i,$$

and

$$a_1 < a_2 < \cdots < a_k.$$

Let $\sigma \in S_k$, and let

$$\delta : \{1, \dots, 2k\} \to \{1, *\}$$

be π-balanced. A pair of blocks (V_i, V_j) with $i < j$ is called a crossing if

$$a_i < a_j < b_i < b_j.$$

For such a crossing, define its sign as follows.
(i) If $\sigma^{-1}(i) < \sigma^{-1}(j)$, then the crossing is positive if

$$\delta(a_i) = \delta(a_j)$$

and negative otherwise.
(ii) If $\sigma^{-1}(i) > \sigma^{-1}(j)$, then the crossing is positive if

$$\delta(a_i) \neq \delta(a_j)$$

and negative otherwise.

We denote by $\mathrm{cr}_+(\pi, \delta, \sigma)$ the number of positive crossings and by $\mathrm{cr}_-(\pi, \delta, \sigma)$ the number of negative crossings.

The following Figure 1 illustrates the distinction between positive and negative crossings for a fixed block order $V \prec_\sigma W$.

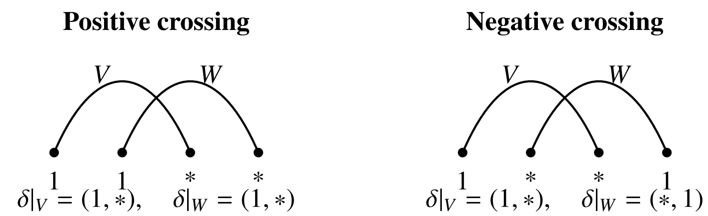

Figure 1. Positive-and negative-oriented crossings for a fixed block order $V \prec_\sigma W$. Equal orientations give a positive crossing, whereas opposite orientations give a negative crossing.

Definition 2.13. Let $q \in [-1, 1]$. A family $(c_1, \ldots, c_s) \subset A$ is a q-circular system if

$$\psi\left(c_{r(1)}^{\varepsilon(1)} \cdots c_{r(2k)}^{\varepsilon(2k)}\right) = \sum_{\pi \in P_2(r,\varepsilon)} q^{\mathrm{cr}(\pi)},$$

and all odd moments vanish.

Definition 2.14. Let $q \in [-1, 1]$. A family $(s_r)_{r=1}^{2s} \subset A$ is a q-semicircular system if

$$\psi\left(s_{t(1)} \cdots s_{t(2k)}\right) = \sum_{\pi \in P_2(2k): \pi \leq \ker(t)} q^{\mathrm{cr}(\pi)},$$

and all odd moments vanish.

Remark 2.15. The q-deformation depends only on the total number of crossings and is insensitive to orientation.

We next recall the definitions of z-circular systems introduced by Mingo and Nica [5], together with the definition of z-semicircular systems developed in [33]. In contrast with the q-case, the moment formulas depend not only on the pairing and the direction map but also on an ordering of the blocks, which is encoded by permutations in S_k.

Let $\pi \in P_2(2k)$, and let $\varepsilon : \{1, \ldots, 2k\} \to \{1, *\}$ be such that $\pi \in P_2(r, \varepsilon)$. Given a permutation $\sigma \in S_k$, one defines the numbers

$$\mathrm{cr}_+(\pi, \varepsilon, \sigma) \quad \text{and} \quad \mathrm{cr}_-(\pi, \varepsilon, \sigma),$$

which count the positive and negative crossings of π with respect to the direction map ε and the ordering of blocks induced by σ (see [5, 33]).

Definition 2.16. Let (A, ψ) be a $*$-probability space, and let $z \in \mathbb{C}$ with $|z| < 1$. A family $(c_1, \ldots, c_s) \subset A$ is called a z-circular system if the following hold:

- for every odd integer $n \geq 1$, every map $r : [n] \to [s]$, and every map $\varepsilon : [n] \to \{1, *\}$, one has

$$\psi\left(c_{r(1)}^{\varepsilon(1)} \cdots c_{r(n)}^{\varepsilon(n)}\right) = 0;$$

- for every even integer $n = 2k \geq 2$, every map $r : [2k] \to [s]$, and every map $\varepsilon : [2k] \to \{1, *\}$, one has

$$\psi\left(c_{r(1)}^{\varepsilon(1)} \cdots c_{r(2k)}^{\varepsilon(2k)}\right) = \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{\pi \in P_2(r,\varepsilon)} z^{\mathrm{cr}_+(\pi,\varepsilon,\sigma)} \bar{z}^{\mathrm{cr}_-(\pi,\varepsilon,\sigma)}.$$

Definition 2.17. Let (A, ψ) be a $*$-probability space, and let $z \in \mathbb{C}$ with $|z| < 1$. A family $(s_r)_{r=1}^{2s} \subset A$ is called a z-semicircular system if the following hold:

- each s_r is self-adjoint;
- for every odd integer $n \geq 1$ and every map $t : [n] \to [2s]$, one has

$$\psi\left(s_{t(1)} \cdots s_{t(n)}\right) = 0;$$

- for every even integer $n = 2k \geq 2$ and every map $t : [2k] \to [2s]$, one has

$$\psi\left(s_{t(1)} \cdots s_{t(2k)}\right) = \frac{1}{k!\,2^k} \sum_{\sigma \in S_k} \sum_{\pi \in P_2(2k): \pi \leq \ker(t)} \sum_{\substack{\delta:\{1,\ldots,2k\} \to \{1,*\} \\ \delta \text{ is } \pi\text{-balanced}}} z^{\mathrm{cr}_+(\pi,\delta,\sigma)} \bar{z}^{\mathrm{cr}_-(\pi,\delta,\sigma)}.$$

Remark 2.18. The essential difference from the q-framework is that the moment formulas depend on a refinement of the crossing structure into positive and negative crossings, together with an averaging over permutations of the blocks. In the semicircular case, one must in addition average over all π-balanced direction maps.

3. The converse in the q-case

In this section, we establish the converse result in the real case, corresponding to the q-deformation. This result confirms that unlike the complex z-setting, the passage from semicircular to circular systems is reversible when the deformation parameter is real. The statement proved in this section corresponds to Conjecture 5.2.5 in [33].

Theorem 3.1. *Let (A, ψ) be a $*$-probability space, and let $q \in [-1, 1]$. Suppose that $(s_r)_{r=1}^{2s}$ is a q-semicircular system. For $m = 1, \ldots, s$, define*

$$c_m := \frac{1}{\sqrt{2}}(s_{2m-1} + i s_{2m}).$$

Then, $(c_1, \ldots, c_s)$ is a q-circular system.

Proof. Fix $k \in \mathbb{N}$, a map $r : \{1, \ldots, 2k\} \to [s]$, and a map $\varepsilon : \{1, \ldots, 2k\} \to \{1, *\}$. We must compute

$$\psi\Big(c_{r(1)}^{\varepsilon(1)} \cdots c_{r(2k)}^{\varepsilon(2k)}\Big).$$

For each j, define

$$\lambda_j := \begin{cases} 1, & \varepsilon(j) = 1, \\ -1, & \varepsilon(j) = *. \end{cases}$$

Then,

$$c_{r(j)}^{\varepsilon(j)} = \frac{1}{\sqrt{2}}(s_{2r(j)-1} + i\lambda_j\, s_{2r(j)}).$$

Expanding the product, we obtain

$$\psi\left(\prod_{j=1}^{2k} c_{r(j)}^{\varepsilon(j)}\right) = 2^{-k} \sum_{\chi:\{1,\ldots,2k\} \to \{0,1\}} \left(\prod_{j:\chi(j)=1} i\,\lambda_j\right) \times \psi\left(s_{t_\chi(1)} \cdots s_{t_\chi(2k)}\right), \tag{3.1}$$

where

$$t_\chi(j) := 2r(j) - 1 + \chi(j) \in [2s].$$

Because (s_r) is a q-semicircular system, its moments are given by

$$\psi\left(s_{t_\chi(1)} \cdots s_{t_\chi(2k)}\right) = \sum_{\pi \in P_2(2k),\ \pi \leq \ker(t_\chi)} q^{\mathrm{cr}(\pi)}.$$

Substituting this into (3.1) yields

$$\psi\left(\prod_{j=1}^{2k} c_{r(j)}^{\varepsilon(j)}\right) = 2^{-k} \sum_{\pi\in P_2(2k)} q^{\mathrm{cr}(\pi)} \sum_{\chi:\pi\leq\ker(t_\chi)} \prod_{j:\chi(j)=1} i\lambda_j.$$

Fix a pairing $\pi \in P_2(2k)$, and consider the inner sum. Let $\{a, b\}$ be a block of π. The condition $\pi \leq \ker(t_\chi)$ means that

$$t_\chi(a) = t_\chi(b),$$

which is equivalent to the two conditions

$$r(a) = r(b), \qquad \chi(a) = \chi(b).$$

Thus, the admissible maps χ are constant on each block of π. The inner sum therefore factorizes over the blocks of π:

$$\sum_{\chi:\pi\leq\ker(t_\chi)} \prod_{j:\chi(j)=1} i\,\lambda_j = \prod_{\{a,b\}\in\pi} (1 + (i\,\lambda_a)(i\,\lambda_b)) = \prod_{\{a,b\}\in\pi} (1 - \lambda_a\,\lambda_b)\,.$$

Now, we observe

$$\lambda_a\,\lambda_b = \begin{cases} 1, & \varepsilon(a) = \varepsilon(b), \\ -1, & \varepsilon(a) \neq \varepsilon(b). \end{cases}$$

Hence,

$$1 - \lambda_a\,\lambda_b = \begin{cases} 0, & \varepsilon(a) = \varepsilon(b), \\ 2, & \varepsilon(a) \neq \varepsilon(b). \end{cases}$$

It follows that the contribution of π is nonzero if and only if for every block $\{a, b\} \in \pi$, one has

$$r(a) = r(b) \quad \text{and} \quad \varepsilon(a) \neq \varepsilon(b),$$

that is, if and only if $\pi \in P_2(r, \varepsilon)$. In that case, each block contributes a factor 2, so the inner sum equals 2^k. Therefore,

$$\psi\left(\prod_{j=1}^{2k} c_{r(j)}^{\varepsilon(j)}\right) = 2^{-k} \sum_{\pi\in P_2(r,\varepsilon)} q^{\mathrm{cr}(\pi)}\, 2^k = \sum_{\pi\in P_2(r,\varepsilon)} q^{\mathrm{cr}(\pi)}.$$

This is exactly the defining moment formula of a q-circular system (see, e.g., [3]). Moreover, odd moments vanish because no pairings exist in that case. This completes the proof. □

Remark 3.2. The above result shows that in the real case $q \in [-1, 1]$, the passage from semicircular to circular systems via complexification preserves the full combinatorial structure of moments. In particular, no additional averaging phenomenon occurs. This stands in contrast with the complex z-case, where such averaging cannot, in general, be avoided.

4. Complexification of z-semicircular systems

We now turn to the complex deformation setting. In contrast with the q-case treated in the previous section, the passage from semicircular to circular variables is no longer described solely by ordinary crossing numbers. The additional orientation data present in the z-theory lead to a more delicate moment formula. The goal of this section is to carry out a complete and general computation of the mixed moments of the complexified family associated with a z-semicircular system.

The result proved below is the central technical statement of the paper. It identifies the precise distributional law satisfied by the family obtained by complexifying a z-semicircular system, and it is this formula that will later allow us to compare the resulting distribution with the usual z-circular law. The argument is entirely combinatorial and rests on the moment formula for z-semicircular systems recalled in Section 2.

Lemma 4.1. *Fix $k \in \mathbb{N}$, a map $r : \{1, \dots, 2k\} \to [s]$, and a map $\varepsilon : \{1, \dots, 2k\} \to \{1, *\}$. For each $j \in \{1, \dots, 2k\}$, set*

$$\lambda_j := \begin{cases} 1, & \varepsilon(j) = 1, \\ -1, & \varepsilon(j) = *. \end{cases}$$

For a pairing $\pi \in P_2(2k)$ and a map $\chi : \{1, \dots, 2k\} \to \{0, 1\}$, define

$$t_\chi(j) := 2r(j) - 1 + \chi(j)$$

and

$$\Sigma(\pi) := \sum_{\chi : \pi \leq \ker(t_\chi)} \prod_{j : \chi(j) = 1} i\lambda_j.$$

Then,

$$\Sigma(\pi) = \begin{cases} 2^k, & \pi \in P_2(r, \varepsilon), \\ 0, & \pi \notin P_2(r, \varepsilon). \end{cases}$$

Proof. Let $\{a, b\}$ be a block of π. The condition $\pi \leq \ker(t_\chi)$ means that $t_\chi(a) = t_\chi(b)$ for every block $\{a, b\} \in \pi$. Because

$$t_\chi(j) = 2r(j) - 1 + \chi(j),$$

the equality $t_\chi(a) = t_\chi(b)$ is equivalent to

$$2r(a) - 1 + \chi(a) = 2r(b) - 1 + \chi(b).$$

Because

$$2(r(a) - r(b)) = \chi(b) - \chi(a),$$

the left-hand side is even, whereas the right-hand side belongs to $\{-1, 0, 1\}$. Hence, both sides must be zero, so $r(a) = r(b)$, and $\chi(a) = \chi(b)$. The converse is immediate. Therefore,

$$t_\chi(a) = t_\chi(b) \quad \Longleftrightarrow \quad r(a) = r(b), \quad \text{and} \quad \chi(a) = \chi(b).$$

Hence, $\pi \leq \ker(t_\chi)$ holds exactly when for every block $\{a, b\} \in \pi$,

$$r(a) = r(b), \qquad \text{and} \qquad \chi(a) = \chi(b).$$

If $r(a) \neq r(b)$ for some block $\{a, b\} \in \pi$, then no admissible map χ exists, and the corresponding contribution is zero.

Assume now that $r(a) = r(b)$ for every block $\{a, b\} \in \pi$. The condition above then forces χ to be constant on each block of π. Thus, for each block $\{a, b\} \in \pi$, there are exactly two admissible choices. If $\chi(a) = \chi(b) = 0$, the block contributes 1, whereas if $\chi(a) = \chi(b) = 1$, it contributes $(i\lambda_a)(i\lambda_b)$. Because the blocks of π are disjoint, these choices can be made independently from one block to another. Consequently, the sum over all admissible maps χ factorizes as

$$\Sigma(\pi) = \prod_{\{a,b\}\in\pi} \left(\sum_{\eta\in\{0,1\}} (i\lambda_a)^\eta \, (i\lambda_b)^\eta \right) = \prod_{\{a,b\}\in\pi} (1 + (i\lambda_a)\,(i\lambda_b))\,.$$

Because $(i\lambda_a)(i\lambda_b) = -\lambda_a\lambda_b$, we obtain

$$\Sigma(\pi) = \prod_{\{a,b\}\in\pi} (1 - \lambda_a\,\lambda_b)\,.$$

We now examine the factor $1 - \lambda_a\lambda_b$. By definition of λ_a and λ_b,

$$\lambda_a\,\lambda_b = \begin{cases} 1, & \varepsilon(a) = \varepsilon(b), \\ -1, & \varepsilon(a) \neq \varepsilon(b). \end{cases}$$

Consequently,

$$1 - \lambda_a\,\lambda_b = \begin{cases} 0, & \varepsilon(a) = \varepsilon(b), \\ 2, & \varepsilon(a) \neq \varepsilon(b). \end{cases}$$

This shows that $\Sigma(\pi) \neq 0$ if and only if every block $\{a, b\}$ of π satisfies

$$r(a) = r(b) \qquad \text{and} \qquad \varepsilon(a) \neq \varepsilon(b).$$

By definition, this is equivalent to saying that $\pi \in P_2(r, \varepsilon)$. Moreover, for such a pairing, every block contributes the factor 2, and hence,

$$\Sigma(\pi) = 2^k \qquad \text{for every } \pi \in P_2(r, \varepsilon).$$

□

Theorem 4.2. *Let (A, ψ) be a $*$-probability space, let $z \in \mathbb{C}$ with $|z| < 1$, and let $(s_r)_{r=1}^{2s}$ be a z-semicircular system. For $m = 1, \ldots, s$, define*

$$c_m := \frac{1}{\sqrt{2}}(s_{2m-1} + is_{2m}).$$

Then, for every $k \in \mathbb{N}$, every map $r : \{1, \ldots, 2k\} \to [s]$, and every map $\varepsilon : \{1, \ldots, 2k\} \to \{1, *\}$, one has

$$\psi\left(c_{r(1)}^{\varepsilon(1)} \cdots c_{r(2k)}^{\varepsilon(2k)}\right) = \frac{1}{k!\,2^k} \sum_{\sigma\in S_k} \sum_{\pi\in P_2(r,\varepsilon)} \sum_{\delta \text{ is } \pi\text{-balanced}} z^{\mathrm{cr}_+(\pi,\delta,\sigma)} \bar{z}^{\mathrm{cr}_-(\pi,\delta,\sigma)}. \tag{4.1}$$

In particular, all odd moments of $(c_1, \ldots, c_s)$ vanish.

Proof. Fix $k \in \mathbb{N}$, a map $r : \{1, \dots, 2k\} \to [s]$, and a map $\varepsilon : \{1, \dots, 2k\} \to \{1, *\}$. We compute the mixed moment

$$\psi\left(c_{r(1)}^{\varepsilon(1)} \cdots c_{r(2k)}^{\varepsilon(2k)}\right).$$

For each $j \in \{1, \dots, 2k\}$, set

$$\lambda_j := \begin{cases} 1, & \varepsilon(j) = 1, \\ -1, & \varepsilon(j) = *. \end{cases}$$

Then, by the definition of c_m,

$$c_{r(j)}^{\varepsilon(j)} = \frac{1}{\sqrt{2}}\,(s_{2r(j)-1} + i\lambda_j\, s_{2r(j)}).$$

Indeed, if $\varepsilon(j) = 1$, this is simply the definition of $c_{r(j)}$, whereas if $\varepsilon(j) = *$, we have

$$c_{r(j)}^{*} = \frac{1}{\sqrt{2}}(s_{2r(j)-1} - i s_{2r(j)}),$$

which is again the same formula with $\lambda_j = -1$. Expanding the product, we obtain

$$\psi\left(\prod_{j=1}^{2k} c_{r(j)}^{\varepsilon(j)}\right) = 2^{-k} \sum_{\chi:\{1,\dots,2k\}\to\{0,1\}} \left(\prod_{j:\chi(j)=1} i\lambda_j\right) \psi\left(s_{t_\chi(1)} \cdots s_{t_\chi(2k)}\right), \tag{4.2}$$

where

$$t_\chi(j) := 2r(j) - 1 + \chi(j) \in [2s].$$

Thus, for each position j, the choice $\chi(j) = 0$ selects the variable $s_{2r(j)-1}$, whereas the choice $\chi(j) = 1$ selects the variable $s_{2r(j)}$ and contributes the scalar factor $i\lambda_j$.

Because $(s_r)_{r=1}^{2s}$ is a z-semicircular system, the moment formula from Definition 2.17 yields

$$\psi\left(s_{t_\chi(1)} \cdots s_{t_\chi(2k)}\right) = \frac{1}{k!\,2^k} \sum_{\sigma\in S_k} \sum_{\substack{\pi\in P_2(2k)\\ \pi\le\ker(t_\chi)}} \sum_{\delta \text{ is } \pi\text{-balanced}} z^{\mathrm{cr}_+(\pi,\delta,\sigma)}\bar{z}^{\mathrm{cr}_-(\pi,\delta,\sigma)}.$$

Substituting this into (4.2) gives

$$\begin{aligned}\psi\left(\prod_{j=1}^{2k} c_{r(j)}^{\varepsilon(j)}\right) &= \frac{1}{k!\,2^{2k}} \sum_{\chi:\{1,\dots,2k\}\to\{0,1\}} \left(\prod_{j:\chi(j)=1} i\lambda_j\right) \sum_{\sigma\in S_k} \sum_{\substack{\pi\in P_2(2k)\\ \pi\le\ker(t_\chi)}} \sum_{\delta \text{ is } \pi\text{-balanced}} z^{\mathrm{cr}_+(\pi,\delta,\sigma)}\bar{z}^{\mathrm{cr}_-(\pi,\delta,\sigma)} \\ &= \frac{1}{k!\,2^{2k}} \sum_{\sigma\in S_k} \sum_{\pi\in P_2(2k)} \sum_{\delta \text{ is } \pi\text{-balanced}} z^{\mathrm{cr}_+(\pi,\delta,\sigma)}\bar{z}^{\mathrm{cr}_-(\pi,\delta,\sigma)} \sum_{\chi:\pi\le\ker(t_\chi)} \prod_{j:\chi(j)=1} i\lambda_j. \end{aligned} \tag{4.3}$$

Here, we have simply interchanged the finite sums and rewritten the condition $\pi \le \ker(t_\chi)$ as a restriction on the innermost sum over χ.

We now fix $\sigma \in S_k$, $\pi \in P_2(2k)$ and a π-balanced map δ, and we write

$$\Sigma(\pi) := \sum_{\chi:\pi\le\ker(t_\chi)} \prod_{j:\chi(j)=1} i\lambda_j.$$

The value of $\Sigma(\pi)$ does not depend on σ or δ, only on π, r, and ε. By Lemma 4.1,

$$\Sigma(\pi) = \begin{cases} 2^k, & \pi \in P_2(r, \varepsilon), \\ 0, & \pi \notin P_2(r, \varepsilon). \end{cases}$$

Substituting this evaluation of the inner sum into (4.3), we obtain

$$\begin{aligned} \psi\left(c_{r(1)}^{\varepsilon(1)} \cdots c_{r(2k)}^{\varepsilon(2k)}\right) &= \frac{1}{k!2^{2k}} \sum_{\sigma \in S_k} \sum_{\pi \in P_2(r,\varepsilon)} \sum_{\delta \text{ is } \pi\text{-balanced}} z^{\mathrm{cr}_+(\pi,\delta,\sigma)} \bar{z}^{\mathrm{cr}_-(\pi,\delta,\sigma)} 2^k \\ &= \frac{1}{k!2^k} \sum_{\sigma \in S_k} \sum_{\pi \in P_2(r,\varepsilon)} \sum_{\delta \text{ is } \pi\text{-balanced}} z^{\mathrm{cr}_+(\pi,\delta,\sigma)} \bar{z}^{\mathrm{cr}_-(\pi,\delta,\sigma)}. \end{aligned}$$

This is exactly the claimed formula (4.1). Finally, if one considers a product of odd length, then the same expansion leads to moments of odd order in the z-semicircular family, and these vanish by definition. Equivalently, there are no pairings of an odd number of points. Hence, all odd moments of $(c_1, \ldots, c_s)$ vanish, as well. □

The following Figure 2 illustrates the averaging mechanism for a single crossing between two blocks with a fixed block order.

$(\delta\|_V, \delta\|_W)$	Crossing weight
$((1,*),(1,*))$	z
$((1,*),(*,1))$	$\bar{z}$
$((*,1),(1,*))$	$\bar{z}$
$((*,1),(*,1))$	z

$$\Longrightarrow \quad \frac{1}{4}(z + \bar{z} + \bar{z} + z) = \frac{z + \bar{z}}{2}.$$

Figure 2. Averaging over the four balanced orientations of two crossing blocks. For a fixed block order, two orientations contribute z, and two contribute $\bar{z}$.

Remark 4.3. The formula in Theorem 4.2 should be compared with the defining moment formula of a z-circular system recalled in Section 2. The crucial distinction is that the right-hand side of (4.1) contains an additional average over all π-balanced maps δ. Thus, after complexification, the external adjoint pattern ε determines the admissible pairings $\pi \in P_2(r, \varepsilon)$, but it does not determine the orientation weights themselves; these are averaged over all balanced orientations. This averaging phenomenon has no analog in the q-case and is the basic mechanism behind the failure of the naive converse in the complex setting.

Remark 4.4. When the deformation parameter is real, the distinction between positive and negative crossings disappears at the level of weights, as $z = \bar{z} = q$. In that case, the orientation sum contributes only through the total number of crossings, and the averaging mechanism collapses to the usual q-pairing formula. Section 3 may therefore be viewed as the real shadow of the more general computation carried out here.

5. Failure of the converse in the z-case

In the real q-case considered in Section 3, the complexification of a q-semicircular pair recovers a q-circular variable. We now show that the analogous converse fails for general complex parameters. More precisely, if $z \in \mathbb{C}$ is nonreal, then the complexification of a z-semicircular system does not, in general, satisfy the defining moment formula of a z-circular system.

The mechanism behind this failure is already visible in the lowest nontrivial mixed moment: For a crossing pairing, the z-semicircular moment formula averages over all balanced orientations, whereas the z-circular formula uses the single orientation canonically determined by the word ε. For a pairing with a single crossing, averaging over all balanced orientations replaces the oriented weight z by

$$\frac{(z+\bar{z})}{2} = \Re z.$$

Proposition 5.1. *Let (A, ψ) be a $*$-probability space, let $z \in \mathbb{C}$ with $|z| < 1$, and let (s_1, s_2) be a z-semicircular system. Define*

$$c := \frac{1}{\sqrt{2}}(s_1 + is_2).$$

Then, the following statements hold.

(a) *The sixth moment of c is given by*

$$\psi\left(c^3(c^*)^3\right) = 1 + (z+\bar{z}) + \frac{1}{2}(z^2+\bar{z}^2) + z\bar{z} + \frac{1}{8}(z^3+\bar{z}^3) + \frac{3}{8}\left(z^2\bar{z} + z\bar{z}^2\right). \tag{5.1}$$

By comparison, the corresponding moment of a genuine z-circular variable is

$$\psi_{z\text{-cir}}\left(c^3(c^*)^3\right) = 1 + (z+\bar{z}) + \frac{2}{3}\left(z^2 + z\bar{z} + \bar{z}^2\right) + \frac{1}{6}(z^3+\bar{z}^3) + \frac{1}{3}\left(z^2\bar{z} + z\bar{z}^2\right). \tag{5.2}$$

Thus, the two sixth moments differ in general.

(b) *The difference between the corresponding eighth moments is*

$$\psi_{z\text{-cir}}\left(c^4(c^*)^4\right) - \psi\left(c^4(c^*)^4\right) = \frac{(z-\bar{z})^2(z+\bar{z}+2)^2\left(z^2 + 2z\bar{z} + 2z + \bar{z}^2 + 2\bar{z} + 4\right)}{48}. \tag{5.3}$$

Consequently,

$$\psi_{z\text{-cir}}\left(c^4(c^*)^4\right) \neq \psi\left(c^4(c^*)^4\right)$$

for every nonreal z with $|z| < 1$.

In particular, for every nonreal z with $|z| < 1$, the variable

$$c = \frac{1}{\sqrt{2}}(s_1 + is_2)$$

is not a z-circular variable.

Proof of part (a). For the sixth moment, set

$$\alpha_j = \begin{cases} i, & j = 1,\ 2,\ 3, \\ -i, & j = 4,\ 5,\ 6. \end{cases}$$

Then,

$$c_j = \begin{cases} c, & j = 1,\ 2,\ 3, \\ c^*, & j = 4,\ 5,\ 6, \end{cases} \qquad \text{and} \qquad c_j = \frac{1}{\sqrt{2}}\,(s_1 + \alpha_j\, s_2).$$

Hence,

$$\psi(c^3(c^*)^3) = \frac{1}{2^3} \sum_{r:[6]\to\{1,2\}} \left(\prod_{j=1}^{6} \beta_j(r(j)) \right) \psi\left(s_{r(1)} \cdots s_{r(6)}\right),$$

where $\beta_j(1) = 1$, and $\beta_j(2) = \alpha_j$. Applying the z-semicircular moment formula, for each pair partition $\pi \in P_2(6)$, only those colorings r with $\pi \le \ker(r)$ contribute. For a block $\{p, q\} \in \pi$, the color sum contributes

$$\frac{1}{2} \sum_{t=1}^{2} \beta_p(t)\beta_q(t) = \frac{1}{2}(1 + \alpha_p\, \alpha_q).$$

Now, $\alpha_p \alpha_q = 1$ exactly when p or q lies in $\{1, 2, 3\}$ and the other in $\{4, 5, 6\}$, and $\alpha_p\alpha_q = -1$ otherwise. Thus,

$$\frac{1}{2}(1 + \alpha_p\, \alpha_q) = \begin{cases} 1, & \text{if } \{p, q\} \text{ joins a } c \text{ to a } c^*, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, only pair partitions that survive as follows:

$$\begin{aligned} \pi_1 &= (1,4)(2,5)(3,6), \quad & \pi_2 &= (1,4)(2,6)(3,5), \\ \pi_3 &= (1,5)(2,4)(3,6), \quad & \pi_4 &= (1,5)(2,6)(3,4), \\ \pi_5 &= (1,6)(2,4)(3,5), \quad & \pi_6 &= (1,6)(2,5)(3,4). \end{aligned}$$

Define

$$A(\pi) := \frac{1}{3!\, 2^3} \sum_{\sigma \in S_3} \sum_{\varepsilon\ \pi\text{-balanced}} z^{\mathrm{cr}_+(\pi,\varepsilon,\sigma)} \bar{z}^{\mathrm{cr}_-(\pi,\varepsilon,\sigma)}.$$

Then,

$$\psi(c^3(c^*)^3) = \sum_{\nu=1}^{6} A(\pi_\nu).$$

Step 1: The noncrossing pairing.

$$\pi_6 = (1,6)(2,5)(3,4)$$

has no crossings. Hence,

$$A(\pi_6) = 1.$$

Step 2: The one-crossing pairings.

$$\pi_4 = (1,5)(2,6)(3,4), \qquad \pi_5 = (1,6)(2,4)(3,5)$$

each has one crossing. Exactly half of the $3! \cdot 2^3 = 48$ choices of (σ, ε) contribute z, and half contribute $\bar{z}$; hence,

$$A(\pi_4) = A(\pi_5) = \frac{1}{2}(z + \bar{z}).$$

Therefore,

$$A(\pi_4) + A(\pi_5) = z + \bar{z}.$$

Step 3: The two-crossing pairings.

$$\pi_2 = (1,4)(2,6)(3,5), \qquad \pi_3 = (1,5)(2,4)(3,6)$$

each has two crossings. Among the 48 choices of (σ, ε), the distribution is

$$12 \text{ times } (2,0), \qquad 24 \text{ times } (1,1), \qquad 12 \text{ times } (0,2).$$

Hence,

$$A(\pi_2) = A(\pi_3) = \frac{1}{4}(z^2 + 2z\bar{z} + \bar{z}^2).$$

Thus,

$$A(\pi_2) + A(\pi_3) = \frac{1}{2}(z^2 + \bar{z}^2) + z\bar{z}.$$

Step 4: The three-crossing pairing.

$$\pi_1 = (1,4)(2,5)(3,6)$$

has three crossings. Among the 48 choices of (σ, ε), the distribution is

$$6 \text{ times } (3,0), \qquad 18 \text{ times } (2,1), \qquad 18 \text{ times } (1,2), \qquad 6 \text{ times } (0,3).$$

Therefore,

$$A(\pi_1) = \frac{1}{8}(z^3 + 3z^2\bar{z} + 3z\bar{z}^2 + \bar{z}^3) = \frac{1}{8}(z^3 + \bar{z}^3) + \frac{3}{8}(z^2\bar{z} + z\bar{z}^2).$$

Summing all six contributions gives the following:

$$\psi(c^3(c^*)^3) = 1 + (z + \bar{z}) + \frac{1}{2}(z^2 + \bar{z}^2) + z\bar{z} + \frac{1}{8}(z^3 + \bar{z}^3) + \frac{3}{8}(z^2\bar{z} + z\bar{z}^2).$$

On the other hand, the sixth moment of a z-circular variable is

$$1 + (z + \bar{z}) + \frac{2}{3}(z^2 + z\bar{z} + \bar{z}^2) + \frac{1}{6}(z^3 + \bar{z}^3) + \frac{1}{3}(z^2\bar{z} + z\bar{z}^2).$$

These differ in general. For instance, at $z = i/2$, one obtains

$$\psi(c^3(c^*)^3) = 1, \qquad \psi_{z\text{-cir}}(c^3(c^*)^3) = \frac{5}{6}.$$

Hence, the conjecture fails for complex, nonreal z.

Proof of part (b). We next consider the eighth moment. Put

$$M_8^{\mathrm{cir}}(z) := \psi_{z\text{-cir}}(c^4(c^*)^4)$$

and

$$M_8^{\mathrm{semi}}(z) := \psi(c^4(c^*)^4).$$

The complete enumeration is given in the Appendix. In that appendix, the 24 admissible pairings are parameterized by permutations $\tau \in S_4$, the 2^4 balanced direction maps are parameterized explicitly, and the numbers of configurations producing each pair of crossing counts are recorded. The resulting calculation gives

$$M_8^{\mathrm{cir}}(z) - M_8^{\mathrm{semi}}(z) = \frac{(z-\bar{z})^2(z+\bar{z}+2)^2\left(z^2+2z\bar{z}+2z+\bar{z}^2+2\bar{z}+4\right)}{48},$$

as shown in (A.21). This proves (5.3). It remains to show that this difference is nonzero whenever $z \notin \mathbb{R}$, and $|z| < 1$. Write $z = x + iy,\ x, y \in \mathbb{R}$. Because $z \notin \mathbb{R}$, one has $y \neq 0$, and therefore,

$$(z-\bar{z})^2 = (2iy)^2 = -4y^2 \neq 0.$$

Moreover,

$$z + \bar{z} + 2 = 2(x+1).$$

The assumption $|z| < 1$ implies $x > -1$, and hence,

$$(z+\bar{z}+2)^2 = 4(x+1)^2 > 0.$$

Finally,

$$\begin{aligned} z^2 + 2z\bar{z} + 2z + \bar{z}^2 + 2\bar{z} + 4 &= (z+\bar{z})^2 + 2(z+\bar{z}) + 4 \\ &= 4x^2 + 4x + 4 \\ &= 4\left(\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}\right) > 0. \end{aligned}$$

All three factors are therefore nonzero, so

$$M_8^{\mathrm{cir}}(z) \neq M_8^{\mathrm{semi}}(z) \qquad \text{for every } z \notin \mathbb{R}, \quad |z| < 1.$$

Consequently,

$$c = \frac{1}{\sqrt{2}}(s_1 + is_2)$$

does not satisfy the defining moment formula of a z-circular variable for any nonreal z with $|z| < 1$.

6. Orientation-averaged z-circular systems

In Section 4, we identified the exact moment structure satisfied by the complexification of a z-semicircular system. The resulting formula differs from the defining relation of a z-circular system by the presence of an additional averaging over all balanced orientation maps. In Section 5, we showed that this averaging prevents, in general, the recovery of a z-circular system when z is nonreal.

The goal of this section is to take a closer look at this phenomenon and to introduce a new class of noncommutative distributions that emerge naturally from it. These systems, which we call orientation-averaged z-circular systems, provide the correct structural counterpart of z-semicircular systems under complexification.

The notation for pairings, balanced orientation maps, and oriented crossings follows the framework of Mingo and Nica [5]. The corresponding z-semicircular framework used here is taken from [33].

Definition 6.1. Let (A, ψ) be a $*$-probability space, and let $z \in \mathbb{C}$ with $|z| < 1$. A family $(c_1, \ldots, c_s) \subset A$ is called an orientation-averaged z-circular system if the following conditions hold:

(1) For every odd integer $n \geq 1$, every map $r : [n] \to [s]$, and every map $\varepsilon : [n] \to \{1, *\}$, one has

$$\psi\left(c_{r(1)}^{\varepsilon(1)} \cdots c_{r(n)}^{\varepsilon(n)}\right) = 0.$$

(2) For every even integer $n = 2k \geq 2$, every map $r : [2k] \to [s]$, and every map $\varepsilon : [2k] \to \{1, *\}$, one has

$$\psi\left(c_{r(1)}^{\varepsilon(1)} \cdots c_{r(2k)}^{\varepsilon(2k)}\right) = \frac{1}{k!2^k} \sum_{\sigma \in S_k} \sum_{\pi \in P_2(r,\varepsilon)} \sum_{\substack{\delta:\{1,\ldots,2k\} \to \{1,*\} \\ \delta \text{ is } \pi\text{-balanced}}} z^{\mathrm{cr}_+(\pi,\delta,\sigma)} \bar{z}^{\mathrm{cr}_-(\pi,\delta,\sigma)}. \tag{6.1}$$

Remark 6.2. The defining feature of orientation-averaged z-circular systems is the presence of the additional averaging over all π-balanced maps δ. In contrast with the usual z-circular systems, the orientation of each pairing is not fixed by the external pattern ε; instead, all admissible orientations contribute equally.

It is frequently helpful to rewrite the moment formula by associating an effective weight to each pairing.

Definition 6.3. Let $\pi \in P_2(2k)$ and $\sigma \in S_k$. We define the averaged weight

$$W(\pi, \sigma) := \frac{1}{2^k} \sum_{\delta \text{ is } \pi\text{-balanced}} z^{\mathrm{cr}_+(\pi,\delta,\sigma)} \bar{z}^{\mathrm{cr}_-(\pi,\delta,\sigma)}. \tag{6.2}$$

With this notation, the moment formula (6.1) can be rewritten as

$$\psi\left(c_{r(1)}^{\varepsilon(1)} \cdots c_{r(2k)}^{\varepsilon(2k)}\right) = \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{\pi \in P_2(r,\varepsilon)} W(\pi, \sigma). \tag{6.3}$$

Remark 6.4. The function $W(\pi, \sigma)$ can be viewed as an effective pairing weight obtained by averaging the oriented contributions over all admissible directions. In particular, in contrast with the usual z-circular case, the weight no longer depends on the external pattern ε but only on the intrinsic combinatorics of (π, σ).

We now show that the notion introduced above is not purely formal but emerges naturally from the complexification of z-semicircular systems.

Theorem 6.5. *Let (A, ψ) be a $*$-probability space, let $z \in \mathbb{C}$ with $|z| < 1$, and let $(s_r)_{r=1}^{2s}$ be a z-semicircular system. For $m = 1, \ldots, s$, define*

$$c_m := \frac{1}{\sqrt{2}}(s_{2m-1} + i s_{2m}).$$

Then, $(c_1, \ldots, c_s)$ is an orientation-averaged z-circular system.

Proof. This is an immediate consequence of Theorem 4.2. Indeed, the moment formula established there coincides exactly with the defining relation (6.1) of an orientation-averaged z-circular system. Therefore, the family $(c_1, \ldots, c_s)$ satisfies the required conditions. □

We conclude by comparing the new notion with the classical one.

Proposition 6.6. *Let $(c_1, \ldots, c_s)$ be an orientation-averaged z-circular system.*

- *If $z \in \mathbb{R}$, then $(c_1, \ldots, c_s)$ is a z-circular system.*
- *If $z \notin \mathbb{R}$, then in general, $(c_1, \ldots, c_s)$ is not a z-circular system.*

Proof. If $z \in \mathbb{R}$, then $z = \bar{z}$, and hence, for every (π, δ, σ), one has

$$z^{\mathrm{cr}_+(\pi,\delta,\sigma)}\,\bar{z}^{\mathrm{cr}_-(\pi,\delta,\sigma)} = z^{\mathrm{cr}_+(\pi,\delta,\sigma)+\mathrm{cr}_-(\pi,\delta,\sigma)} = z^{\mathrm{cr}(\pi)}.$$

Thus, the sum over δ becomes trivial, and the averaged formula reduces to

$$\psi(\cdots) = \frac{1}{k!}\sum_{\sigma\in S_k}\sum_{\pi\in P_2(r,\varepsilon)} z^{\mathrm{cr}(\pi)},$$

which is precisely the q-circular formula with $q = z$.

The second statement follows from Proposition 5.1, which provides an explicit counterexample. □

Remark 6.7. The above result shows that orientation-averaged z-circular systems interpolate naturally between the q-circular theory and the fully oriented z-circular theory. In the real case, the averaging mechanism plays no visible role, whereas in the genuinely complex case, it leads to a loss of phase information, as already seen in Section 5.

Remark 6.8. The preceding theorem clarifies the precise scope of the complexification procedure in the z-deformed setting. Starting from a z-semicircular system, the variables

$$c_m = \frac{1}{\sqrt{2}}(s_{2m-1} + is_{2m})$$

always form an orientation-averaged z-circular system, whose mixed $*$-moments are given by the averaging formula of Theorem 4.2. The counterexample of Section 5 shows that in general, this distribution does not coincide with the usual z-circular law whenever $z \notin \mathbb{R}$. Thus, the classical converse correspondence between semicircular and circular systems persists exactly in the real case. For nonreal deformation parameters, the complexification procedure necessarily loses the full directional information present in the z-circular moment formula, retaining only its balanced average over admissible orientations. In this sense, orientation-averaged z-circular systems provide the canonical complex counterpart of z-semicircular systems in the general complex regime.

Theorem 6.5 gives a natural realization of an orientation-averaged z-circular system through the complexification of a z-semicircular system. Nevertheless, a direct construction in terms of creation and annihilation operators, random matrices, or an independent asymptotic model is not currently known. We briefly describe a possible route toward such a realization and explain the main difficulty.

Let

$$\pi = \{V_1, \ldots, V_k\} \in P_2(2k).$$

For each block V_j, choose an independent random sign $\eta_j \in \{-1, 1\}$ with equal probabilities, and use η_j to select one of the two possible orientations of V_j. Denote the resulting π-balanced orientation map by δ_η. Because every π-balanced map arises exactly once in this way, the averaged weight can be written as

$$W(\pi, \sigma) = \mathbb{E}_\eta \left[z^{\mathrm{cr}_+(\pi,\delta_\eta,\sigma)} \, \bar{z}^{\mathrm{cr}_-(\pi,\delta_\eta,\sigma)} \right].$$

Thus, at the combinatorial level, orientation averaging may be interpreted as an expectation over independent two-state orientation variables.

This observation suggests enlarging a Fock-type construction by an auxiliary two-dimensional orientation space so that each crossing contributes either z or $\bar{z}$ according to the corresponding orientation states. The main obstruction is that these orientation variables are attached to the blocks of a pairing, whereas the blocks become visible only after the Wick contractions have been selected. Consequently, there is no immediate way to assign the required orientation states to the operators before the moment expansion is performed. A successful operator model would therefore need to preserve the orientation information throughout the contraction process, possibly through a matrix-valued or dynamically varying braiding operator.

A second possible approach is asymptotic. One could introduce random orientation labels into a random-matrix or noncommutative-torus model and then average over those labels. To recover the required moment formula, one would need to prove that the surviving terms are indexed precisely by admissible pairings and that their crossing weights reproduce $W(\pi, \sigma)$.

Although these observations do not yet provide a complete construction, they identify natural operator-theoretic and asymptotic routes toward one, together with the principal obstruction that any concrete realization must overcome.

Remark 6.9. Although a direct Fock-space, random-matrix, or independent asymptotic realization is not yet known, the orientation-averaged z-circular law should not be viewed as merely a combinatorial label. Indeed, Theorem 4.2 shows that whenever a z-semicircular system is complexified by setting

$$c_m = \frac{1}{\sqrt{2}}(s_{2m-1} + i s_{2m}),$$

the resulting joint $*$-distribution is necessarily given by the orientation-averaged moment formula. Consequently, this law is the unique joint $*$-distribution produced by the complexification procedure. It therefore defines a natural and well-established mathematical object within the ambient $*$-probability space, even in the absence of a separate Fock-space or random-matrix realization.

7. Conclusions and open problems

In this paper, we have investigated the relationship between z-semicircular and z-circular systems from a combinatorial perspective. Our approach builds on the framework of oriented pairings developed by Mingo and Nica for z-circular systems [5] and its extension to z-semicircular systems in [33]. We have obtained a complete answer to the converse problem relating these two classes.

Our main findings can be summarized as follows. First, we showed that in the real case $q \in [-1, 1]$, the complexification of a q-semicircular system yields a q-circular system, thereby establishing the converse in that setting and recovering the classical behavior known from the theory of q-Gaussian variables [3]. Second, we proved that in the general complex case, the situation is fundamentally different: The complexification of a z-semicircular system leads to a new moment formula involving an additional averaging over all balanced orientations. This yields a natural class of distributions, which we have termed orientation-averaged z-circular systems. Third, we showed that this averaging mechanism leads to a loss of phase information, and as a consequence, the naïve converse fails for nonreal values of z.

From a conceptual point of view, our results highlight a structural dichotomy between the real and complex deformation regimes. In the q-case, the combinatorics of pairings is governed solely by crossing numbers, and the passage from semicircular to circular systems preserves the full moment structure. In contrast, in the z-case, the presence of oriented crossings introduces additional degrees of freedom, and the averaging over these orientations produces a genuinely new class of noncommutative distributions. This phenomenon may be viewed as a form of loss of orientation information, which has no analog in the classical or q-deformed settings.

The results obtained here suggest several natural directions for further research.

(1) Operator models and Fock space realizations. The discussion in Section 6 concerning concrete realizations identifies possible Fock-space and asymptotic approaches to constructing orientation-averaged z-circular systems. Establishing such a direct realization remains an open problem.

(2) Limit theorems and probabilistic models. Another natural direction is the development of limit theorems leading to orientation-averaged z-circular systems, in analogy with the free central limit theorem, whose limiting law is semicircular [32], and with the q-Gaussian framework [3] and its realization through a noncommutative central limit theorem [6]. In particular, one may ask whether there exist sequences of noncommutative random variables whose normalized sums converge, in moments, to the distributions described in Section 6. Such results would provide a probabilistic interpretation of the averaging mechanism and connect the present work to broader themes in noncommutative limit theory.

(3) Reconstruction of orientation data. The failure of the converse in the z-case is ultimately due to the averaging over all balanced orientations. This raises the question of whether the lost orientation information can be recovered by enriching the underlying structure. For instance, one may ask whether additional observables, multiparameter deformations, or refined combinatorial data allow one to reconstruct the original z-circular distribution from an orientation-averaged one.

(4) Relations with other notions of independence. It would also be of interest to investigate the relation between orientation-averaged z-circular systems and other notions of noncommutative independence, such as Boolean, monotone, or conditionally free independence. One may ask whether orientation averaging admits a meaningful interpretation in any of these settings or gives rise to new hybrid forms of independence.

(5) Boundary cases and analytic properties. For systems obtained by complexifying a z-semicircular system, positivity follows from the ambient $*$-probability space. An open problem is to find a direct Fock-space, bounded-operator, or random-matrix realization that does not rely on this complexification.

We expect that the notion of orientation-averaged z-circular systems introduced in this work provides a useful framework for exploring these questions and for developing a deeper understanding of orientation-sensitive structures in noncommutative probability.

Use of Generative-AI tools declaration

The author used the generative AI tools ChatGPT (OpenAI) to improve the language and readability of the manuscript. The author takes full responsibility for the content, interpretation, and conclusions of this work.

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Conflict of interest

The author declares no conflict of interest.

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Appendix

In this appendix, we describe explicitly the finite enumeration used to obtain the formulas for

$$M_8^{\mathrm{cir}}(z) \qquad \text{and} \qquad M_8^{\mathrm{semi}}(z)$$

appearing in Proposition 5.1.

The counts reported below were verified computationally using Python.

Consider the direction pattern

$$\varepsilon = (1, 1, 1, 1, *, *, *, *).$$

Every admissible pairing must join one point in

$$\{1, 2, 3, 4\}$$

to one point in

$$\{5, 6, 7, 8\}.$$

Consequently, every admissible pairing is uniquely indexed by a permutation $\tau \in S_4$ and has the form

$$\pi_\tau = \{\{j,\ 4 + \tau(j)\} : j = 1,\ 2,\ 3,\ 4\}. \tag{A.1}$$

Thus, there are exactly

$$4! = 24$$

admissible pairings.

For $j = 1, 2, 3, 4$, write

$$V_j = \{j, 4 + \tau(j)\}.$$

Because the left endpoints satisfy $1 < 2 < 3 < 4$, and every right endpoint belongs to $\{5, 6, 7, 8\}$, two blocks V_i and V_j, with $i < j$, cross if and only if

$$i < j < 4 + \tau(i) < 4 + \tau(j).$$

Equivalently,

$$V_i \text{ and } V_j \text{ cross} \quad \Longleftrightarrow \quad \tau(i) < \tau(j). \tag{A.2}$$

Therefore,

$$\mathrm{cr}(\pi_\tau) = \#\{(i, j) : 1 \le i < j \le 4,\ \tau(i) < \tau(j)\}. \tag{A.3}$$

The 24 admissible pairings may also be classified according to their ordinary crossing number. If

$$\mathrm{inv}(\tau) = \#\{(i, j) : 1 \le i < j \le 4,\ \tau(i) > \tau(j)\}$$

denotes the inversion number of τ, then

$$\mathrm{cr}(\pi_\tau) = 6 - \mathrm{inv}(\tau).$$

Using the inversion-number distribution on S_4, the admissible pairings are distributed as follows:

r	0	1	2	3	4	5	6
$\#\{\tau \in S_4 : \mathrm{cr}(\pi_\tau) = r\}$	1	3	5	6	5	3	1

Thus, the 24 admissible pairings split into seven classes according to their total number of crossings. Within each class, the sums over σ and in the orientation-averaged case, over the 2^4 balanced maps, are evaluated using the sign rules described below.

A1. Enumeration of the z-circular moment

For the external direction map

$$\varepsilon = (1, 1, 1, 1, *, *, *, *),$$

the left endpoint of every block V_j has direction 1. According to the definition of the oriented crossing statistics, for a crossing pair V_i, V_j with $i < j$, the crossing is positive if

$$\sigma^{-1}(i) < \sigma^{-1}(j)$$

and negative if

$$\sigma^{-1}(i) > \sigma^{-1}(j).$$

For $\tau, \sigma \in S_4$, define

$$C_+^{\mathrm{cir}}(\tau, \sigma) := \#\left\{(i, j) : 1 \le i < j \le 4,\ \tau(i) < \tau(j),\ \sigma^{-1}(i) < \sigma^{-1}(j)\right\}, \tag{A.4}$$

$$C_-^{\mathrm{cir}}(\tau, \sigma) := \#\left\{(i, j) : 1 \le i < j \le 4,\ \tau(i) < \tau(j),\ \sigma^{-1}(i) > \sigma^{-1}(j)\right\}. \tag{A.5}$$

Clearly,

$$C_+^{\mathrm{cir}}(\tau, \sigma) + C_-^{\mathrm{cir}}(\tau, \sigma) = \mathrm{cr}(\pi_\tau).$$

The defining moment formula of a z-circular variable therefore gives

$$M_8^{\mathrm{cir}}(z) = \frac{1}{4!} \sum_{\tau \in S_4} \sum_{\sigma \in S_4} z^{C_+^{\mathrm{cir}}(\tau,\sigma)} \bar{z}^{C_-^{\mathrm{cir}}(\tau,\sigma)}. \tag{A.6}$$

Thus, the computation consists of

$$4! \cdot 4! = 576$$

evaluations of the oriented crossing statistics. For $a, b \ge 0$, define

$$N_{a,b}^{\mathrm{cir}} := \#\left\{(\tau, \sigma) \in S_4 \times S_4 : C_+^{\mathrm{cir}}(\tau, \sigma) = a,\ C_-^{\mathrm{cir}}(\tau, \sigma) = b\right\}. \tag{A.7}$$

Then, (A.6) becomes

$$M_8^{\mathrm{cir}}(z) = \frac{1}{24} \sum_{a,b \ge 0} N_{a,b}^{\mathrm{cir}}\, z^a\, \bar{z}^b. \tag{A.8}$$

A2. Enumeration of the complexified z-semicircular moment

For a fixed pairing π_τ, there are 2^4 π_τ-balanced direction maps. We parameterize them by

$$\eta = (\eta_1, \eta_2, \eta_3, \eta_4) \in \{0, 1\}^4.$$

Define

$$\iota(0) = 1, \qquad \iota(1) = *,$$

and let $\delta_\eta : \{1, \ldots, 8\} \to \{1, *\}$ be given by

$$\delta_\eta(j) = \iota(\eta_j), \qquad \delta_\eta(4 + \tau(j)) = \iota(1 - \eta_j), \qquad j = 1, 2, 3, 4. \tag{A.9}$$

The two endpoints of every block V_j receive opposite directions, so δ_η is π_τ-balanced. Conversely, every π_τ-balanced direction map is obtained uniquely in this way.

Consider a crossing pair V_i, V_j with $i < j$. By the definition of the oriented crossing statistics, the crossing is positive precisely in one of the following two cases:

(1) $\sigma^{-1}(i) < \sigma^{-1}(j)$, and $\eta_i = \eta_j$;
(2) $\sigma^{-1}(i) > \sigma^{-1}(j)$, and $\eta_i \neq \eta_j$.

Accordingly, define

$$\begin{gathered} C_+^{\mathrm{semi}}(\tau, \sigma, \eta) := \#\Big\{(i, j) : \ 1 \le i < j \le 4, \ \tau(i) < \tau(j), \\ [\sigma^{-1}(i) < \sigma^{-1}(j) \text{ and } \eta_i = \eta_j] \quad \text{or} \quad [\sigma^{-1}(i) > \sigma^{-1}(j) \text{ and } \eta_i \neq \eta_j]\Big\}. \end{gathered} \tag{A.10}$$

We then set

$$C_-^{\mathrm{semi}}(\tau, \sigma, \eta) = \mathrm{cr}(\pi_\tau) - C_+^{\mathrm{semi}}(\tau, \sigma, \eta). \tag{A.11}$$

The moment arising from the complexification of the z-semicircular pair is therefore

$$M_8^{\mathrm{semi}}(z) = \frac{1}{4!\, 2^4} \sum_{\tau \in S_4} \sum_{\sigma \in S_4} \sum_{\eta \in \{0,1\}^4} z^{C_+^{\mathrm{semi}}(\tau,\sigma,\eta)} \bar{z}^{C_-^{\mathrm{semi}}(\tau,\sigma,\eta)}. \tag{A.12}$$

Thus, this computation consists of $4! \cdot 4! \cdot 2^4 = 9216$ evaluations. For $a, b \ge 0$, define

$$N_{a,b}^{\mathrm{semi}} := \#\Big\{(\tau, \sigma, \eta) \in S_4 \times S_4 \times \{0, 1\}^4 : \ C_+^{\mathrm{semi}}(\tau, \sigma, \eta) = a, \ C_-^{\mathrm{semi}}(\tau, \sigma, \eta) = b\Big\}. \tag{A.13}$$

Then, (A.12) becomes

$$M_8^{\mathrm{semi}}(z) = \frac{1}{384} \sum_{a,b \ge 0} N_{a,b}^{\mathrm{semi}} z^a \bar{z}^b. \tag{A.14}$$

A3. Coefficient counts

The coefficient counts obtained from the preceding enumeration are listed below. The symmetry of these counts can be seen by reversing the ordering of the blocks. More precisely, for each $\sigma \in S_4$, let $\sigma^{\mathrm{rev}} \in S_4$ be the permutation determined by

$$(\sigma^{\mathrm{rev}})^{-1}(i) = 5 - \sigma^{-1}(i), \qquad i = 1, \ldots, 4.$$

Then, for every pair $i < j$,

$$\sigma^{-1}(i) < \sigma^{-1}(j) \quad \Longleftrightarrow \quad (\sigma^{\mathrm{rev}})^{-1}(i) > (\sigma^{\mathrm{rev}})^{-1}(j).$$

Thus, every positive crossing becomes negative, and every negative crossing becomes positive. This defines an involution on the relevant finite sets and proves that

$$N^{\mathrm{cir}}_{a,b} = N^{\mathrm{cir}}_{b,a}, \qquad N^{\mathrm{semi}}_{a,b} = N^{\mathrm{semi}}_{b,a}. \tag{A.15}$$

It is consequently sufficient to list the entries for which $a \geq b$.

The coefficient counts for the two eighth-moment computations are listed in Table A. The row $N^{\mathrm{cir}}_{a,b}$ gives the counts for the z-circular moment, while $N^{\mathrm{semi}}_{a,b}$ gives the corresponding counts for the complexified z-semicircular moment.

Table A. Coefficient counts for the eighth-moment enumeration.

(a,b)	$(0,0)$	$(1,0)$	$(1,1)$	$(2,0)$	$(2,1)$	$(2,2)$	$(3,0)$	$(3,1)$	$(3,2)$	$(3,3)$	$(4,0)$	$(4,1)$	$(4,2)$	$(5,0)$	$(5,1)$	$(6,0)$
$N^{\mathrm{cir}}_{a,b}$	24	36	44	38	42	32	30	28	18	6	16	12	5	6	3	1
$N^{\mathrm{semi}}_{a,b}$	384	576	960	480	864	768	288	448	384	128	128	144	88	48	32	8

Entries with $a < b$ are obtained from (A.15). As consistency checks, the complete coefficient counts satisfy

$$\sum_{a,b\geq 0} N^{\mathrm{cir}}_{a,b} = 576 \tag{A.16}$$

and

$$\sum_{a,b\geq 0} N^{\mathrm{semi}}_{a,b} = 9216. \tag{A.17}$$

Substituting the circular counts into (A.8) gives

$$\begin{aligned} M_8^{\mathrm{cir}}(z) = 1 &+ \frac{3}{2}(z+\bar{z}) + \frac{19}{12}(z^2+\bar{z}^2) + \frac{11}{6}z\bar{z} + \frac{5}{4}(z^3+\bar{z}^3) \\ &+ \frac{7}{4}(z^2\bar{z}+z\bar{z}^2) + \frac{2}{3}(z^4+\bar{z}^4) + \frac{1}{2}(z^4\bar{z}+z\bar{z}^4) \\ &+ \frac{4}{3}z^2\bar{z}^2 + \frac{7}{6}(z^3\bar{z}+z\bar{z}^3) + \frac{1}{4}(z^5+\bar{z}^5) \\ &+ \frac{1}{8}(z^5\bar{z}+z\bar{z}^5) + \frac{3}{4}(z^3\bar{z}^2+z^2\bar{z}^3) + \frac{5}{24}(z^4\bar{z}^2+z^2\bar{z}^4) \\ &+ \frac{1}{4}z^3\bar{z}^3 + \frac{1}{24}(z^6+\bar{z}^6). \end{aligned} \tag{A.18}$$

Similarly, substituting the semicircular counts into (A.14) gives

$$\begin{aligned} M_8^{\mathrm{semi}}(z) = 1 &+ \frac{3}{2}(z+\bar{z}) + \frac{5}{4}(z^2+\bar{z}^2) + \frac{5}{2}z\bar{z} \\ &+ \frac{3}{4}(z^3+\bar{z}^3) + \frac{9}{4}(z^2\bar{z}+z\bar{z}^2) + \frac{1}{3}(z^4+\bar{z}^4) \\ &+ \frac{3}{8}(z^4\bar{z}+z\bar{z}^4) + 2z^2\bar{z}^2 + \frac{7}{6}(z^3\bar{z}+z\bar{z}^3) \end{aligned}$$

$$
+\frac{1}{8}(z^5+\bar{z}^5)+\frac{1}{12}(z^5\bar{z}+z\bar{z}^5)+(z^3\bar{z}^2+z^2\bar{z}^3)
$$
$$
+\frac{11}{48}(z^4\bar{z}^2+z^2\bar{z}^4)+\frac{1}{3}z^3\bar{z}^3+\frac{1}{48}(z^6+\bar{z}^6). \tag{A.19}
$$

A4. Factorization of the difference

For completeness, we give a direct verification of the factorization of the difference. Set

$$
u := z+\bar{z}, \qquad v := z\bar{z}.
$$

Subtracting (A.19) from (A.18) and rewriting the resulting symmetric polynomial in terms of u and v, we obtain

$$
\begin{aligned}
48(M_8^{\mathrm{cir}}(z)-M_8^{\mathrm{semi}}(z)) &= u^6+6u^5+16u^4+24u^3+16u^2\\
&\quad -4u^4v-24u^3v-64u^2v-96uv-64v\\
&= (u^2-4v)\,(u^4+6u^3+16u^2+24u+16)\\
&= (u^2-4v)\,(u+2)^2\,(u^2+2u+4).
\end{aligned} \tag{A.20}
$$

Because $u^2-4v=(z-\bar{z})^2$, $u+2=z+\bar{z}+2$, and $u^2+2u+4=z^2+2z\bar{z}+2z+\bar{z}^2+2\bar{z}+4$, Eq (A.20) becomes

$$
M_8^{\mathrm{cir}}(z)-M_8^{\mathrm{semi}}(z)=\frac{(z-\bar{z})^2(z+\bar{z}+2)^2\left(z^2+2z\bar{z}+2z+\bar{z}^2+2\bar{z}+4\right)}{48}. \tag{A.21}
$$

This proves the factorization used in Proposition 5.1.