Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(9): 29774–29795.
DOI: 10.3934/math.20261181
Received: 21 April 2026
Revised: 02 August 2026
Accepted: 07 August 2026
Published: 15 September 2026

Research article

Properties of m-modified conformal vector fields on a Riemannian manifold

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Abstract: An m-modified conformal vector field ξ on a Riemannian manifold (N^k, g) is a generalization of a conformal vector field in the sense that an m-modified conformal vector field ξ becomes a conformal vector field as $m \to \infty$. An m-modified conformal vector field ξ on a Riemannian manifold (N^k, g) is said to be trivial if $\xi = 0$. In this article, we obtained two triviality results for an m-modified conformal vector field ξ on a compact and connected Riemannian manifold (N^k, g). Additionally, two results were obtained for the triviality of an m-modified conformal vector field ξ on a noncompact, complete, and connected Riemannian manifold (N^k, g). Finally, we studied the m-modified gradient conformal vector field ∇f for compact and noncompact Riemannian manifolds. Three results were obtained on triviality of an m-modified gradient conformal vector field ∇f on a compact and connected Riemannian manifold (N^k, g), and one such result was obtained for an m-modified gradient conformal vector field ∇f on a noncompact and connected Riemannian k-manifold (N^k, g).

Keywords: m-modified conformal vector field; trivial m-modified conformal vector field; Ricci curvature; Hessian operator
2020 Mathematics Subject Classification: 53C25, 53C30

1. Introduction

One of the richest structures on a Riemannian manifold (N^k, g) is provided by a conformal vector field V on (N^k, g), which is induced by the local flow of conformal transformations of (N^k, g) or equivalently described by the condition

$$\frac{1}{2}\mathcal{L}_V g = \varphi g,$$

where $\mathcal{L}_V$ is the Lie derivative with respect to V, and φ is a smooth function called the potential of the conformal vector field V. Conformal vector fields are known for their importance in geometry and physics, in particular in general relativity. The impact of a closed conformal vector field V with potential φ is well known and has been widely studied. It is worth noting that handling closed conformal vector fields on a Riemannian manifold (N^k, g) is more convenient than non-closed conformal vector fields, which is seen in [1–3].

The notion of conformal vector fields on a Riemannian manifold has recently been generalized to m-modified conformal vector fields on a Riemannian manifold by Zhang and Chen in [4]. These authors noticed that if in a Riemannian manifold $\left(N^k, g\right)$ the metric g is both Einstein as well as m-quasi-Einstein, then there exists a vector field V on (N^k, g) satisfying

$$\frac{1}{2}\mathcal{L}_V g = cg + \frac{1}{m}\eta \otimes \eta,$$

where c is a constant, $0 < m < \infty$, and η is 1-form dual to V. This paved the way of defining an m-modified conformal vector field ξ on a Riemannian manifold (N^k, g) (cf. [4]) satisfying

$$\frac{1}{2}\mathcal{L}_\xi g = \rho g + \frac{1}{m}\eta \otimes \eta,$$

where ρ is a smooth function called the potential of an m-modified conformal vector field ξ and constant m, $0 < m < \infty$. It is clear that an m-modified conformal vector field ξ on a Riemannian manifold (N^k, g) is a generalization of a conformal vector field as $m \to \infty$; we see it becomes a conformal vector field. An m-modified conformal vector field ξ on a Riemannian manifold (N^k, g) is said to be trivial if $\xi = 0$. In [4] researhers have obtained several interesting properties of an m-modified conformal vector field ξ on a Riemannian manifold (N^k, g); among them is that a compact Riemannian manifold (N^k, g) with $k \geq 2$ does not admit a nontrivial m-modified conformal vector field of constant length. Additionally, in [4], the researchers obtained several conditions for compact and noncompact Riemannian manifolds, so that an m-modified conformal vector field is trivial. For examples of m-modified conformal vector fields, we refer to [4], and they are ample on Lie groups with left invariant metric. Here, we provide an example of a noncompact Riemannian manifold (N^k, g) that admits an m-modified conformal vector field. Consider the k-sphere $S^k(a)$ of constant curvature a and has induced canonical metric g as an embedded hypersurface of the Euclidean space R^{k+1}. Upon denoting the unit normal by N and Riemannian connection with respect to g by ∇ and the Euclidean connection $\overline{\nabla}$ on R^{k+1}, we have

$$\overline{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 - \sqrt{a} g(X_1, X_2) N,$$

where X_1 and X_2 are smooth vector fields, and the shape operator $A = -\sqrt{a} I$ for the sphere $S^k(a)$. Upon choosing a nonzero parallel vector field Z on the Euclidean space R^{k+1} with Euclidean metric $\overline{g}$, we express

$$Z = \zeta + fN,$$

where $f = \overline{g}(Z, N)$ and ζ is tangential to $S^k(a)$. Differentiating the above equation with respect to a vector field X on $S^k(a)$ and equating like components, we get

$$\nabla_X \zeta = -\sqrt{a} f X, \quad \nabla f = \sqrt{a}\zeta. \tag{1.1}$$

Now, consider an open subset $N^k \subset S^k(a)$ containing those points x, where $f(x) \neq 0$, and on this Riemannian manifold (N^k, g), we define the vector field ξ by

$$\xi = \frac{-1}{f}\zeta. \tag{1.2}$$

Differentiating the above equation with respect to the smooth vector field X on N^k gives

$$\nabla_X \xi = \frac{1}{f^2} X(f)\, \zeta - \frac{1}{f} \nabla_X \zeta$$

and using Eq (1.1) in the above equation yields

$$\nabla_X \xi = \sqrt{a} g(\xi, X)\, \xi + \sqrt{a} X,$$

that is,

$$\frac{1}{2}\mathcal{L}_\xi g = \sqrt{a} g + \sqrt{a}\eta \otimes \eta.$$

This shows that ξ is an m-modified conformal vector field with $m = \frac{1}{\sqrt{a}}$ and potential $\rho = \sqrt{a}$ on the noncompact Riemannian manifold (N^k, g).

There is a skew-symmetric tensor G associated with an m-modified conformal vector field ξ on a Riemannian manifold (N^k, g) defined by

$$\frac{1}{2} d\eta\,(X_1, X_2) = g\,(GX_1, X_2)$$

called the *associated tensor* of the m-modified conformal vector field ξ, which plays an important role in our study in this article.

It is just the beginning of the study of the m-modified conformal vector field, and as conformal vector fields have important applications in General Relativity, being a generalization of conformal vector fields, m-modified conformal vector fields, too, will find applications in General Relativity. In this article, we focus on finding conditions under which an m-modified conformal vector field on a Riemannian manifold (N^k, g) is trivial. The article is organized as follows: In Section 2, we describe important properties of an m-modified conformal vector field on a Riemannian manifold (N^k, g) and prove several lemmas that are useful in the rest of the article.

In Section 3, we focus on triviality results on m-modified conformal vector fields on a compact and connected Riemannian k-manifold (N^k, g). In the first result of this section, it is shown that an m-modified conformal vector field ξ with potential ρ and the associated tensor G on a compact and connected Riemannian k-manifold (N^k, g), $k > 1$, satisfies

$$Ric\,(\xi, \xi) \geq \frac{k-1}{k}\,(\operatorname{div} \xi)^2 + \|G\|^2\,,$$

which is necessarily trivial. Similarly, in the second result, it is shown that the condition

$$Ric\,(\xi, \xi) \leq \frac{3}{m}\,\|\xi\|^2\,(\operatorname{div} \xi) + \|G\|^2$$

implies that an m-modified conformal vector field ξ on compact and connected Riemannian k-manifold (N^k, g), $k > 1$, is trivial. It should be noted that these two results should not be confused with the non-existence result in [4]. Although all three results reflect compact and connected Riemannian manifolds, the result in [4] also requires the m-modified vector field to have constant length.

In Section 4, we study the same question on noncompact connected Riemannian manifolds. In the first result of this section, we show that if the Ricci curvature $Ric\,(\xi, \xi)$ of a complete and connected Riemannian k-manifold (N^k, g), $k > 1$, admits an m-modified conformal vector field ξ with potential ρ and associated tensor G has a specific expression, then ξ is necessarily trivial. The proof involves the arclength and Energy of the integral curves of ξ. In [4], the researchers show that on a compact and connected (N^k, g), $k \geq 2$, an m-modified conformal vector field ξ of constant length is necessarily trivial. This naturally raises the question about the status of an m-modified conformal vector field ξ of constant length noncompact (N^k, g). In the second result of this section, we answer this question.

In the last section of this article, we study m-modified gradient conformal vector field ξ on compact as well as noncompact Riemannian manifolds. It is worth noting, in this case, that the associated tensor $G = 0$, which makes the study easier than a non-gradient m-modified conformal vector field. If ξ is an m-modified gradient conformal vector field on (N^k, g), then $\xi = \nabla f$ and the defining equation of m-modified gradient conformal vector field ∇f is

$$Hess(f) = \rho g + \frac{1}{m}\eta \otimes \eta.$$

We obtain four results in this section: The first three entail the triviality of m-modified gradient conformal vector field ∇f on compact and connected (N^k, g), and the last result deals with the triviality of m-modified gradient conformal vector field ∇f on a connected (N^k, g).

2. Preliminaries

The curvature tensor R of a Riemannian k-manifold (N^k, g) is given by

$$R(X_1, X_2)X_3 = \nabla_{X_1}\nabla_{X_2}X_3 - \nabla_{X_2}\nabla_{X_1}X_3 - \nabla_{[X_1,X_2]}X_3, \tag{2.1}$$

for $X_i \in \mathfrak{X}\left(N^k\right)$ the Lie algebra of vector fields, $i = 1, 2, 3$ and ∇ is the Riemannian connection on (N^k, g). We also have the Ricci tensor Ric of (N^k, g), which is given by

$$Ric\,(X_1, X_2) = \sum_j g\left(R\left(e_j, X_1\right)X_2, e_j\right), \quad X_1, X_2 \in \mathfrak{X}(N^k), \tag{2.2}$$

and this is a symmetric tensor, where $\{e_1, .., e_k\}$ is a local frame and the associated Ricci operator Q of (N^k, g) is related to the Ricci tensor by

$$Ric\,(X_1, X_2) = g\,(QX_1, X_2)\,, \quad X_1, X_2 \in \mathfrak{X}(N^k). \tag{2.3}$$

The scalar curvature τ of the Riemannian manifold (N^k, g) is given by $\tau = Tr\,Q$, which has gradient given by (cf. [3, 5])

$$\frac{1}{2}\nabla\tau = \sum_j \left(\nabla_{ej}Q\right)\left(e_j\right), \tag{2.4}$$

where

$$(\nabla_{X_1} Q)(X_2) = \nabla_{X_1} Q X_2 - Q(\nabla_{X_1} X_2).$$

Let ξ be an m-modified conformal vector field on a Riemannian manifold (N^k, g), which satisfies (cf. [4, 6])

$$\frac{1}{2}\mathcal{L}_{\xi} g = \rho g + \frac{1}{m}\eta \otimes \eta, \tag{2.5}$$

where $m > 0$ is a constant, η is 1-form dual to ξ, and ρ is a smooth function called the potential of the m-modified conformal vector field ξ. The Hessian operator H^{ρ} of the function ρ is defined by

$$H^{\rho}(X) = \nabla_X \nabla \rho, \quad X \in \mathfrak{X}(N^k), \tag{2.6}$$

which has trace $TrH^{\rho} = \Delta\rho$ for the Laplace operator acting on ρ. The divergence of ξ is defined by

$$\operatorname{div}\xi = \sum_j g\left(\nabla_{e_j}\xi, e_j\right),$$

which, on account of Eq (2.5), implies

$$\operatorname{div}\xi = k\rho + \frac{1}{m}\|\xi\|^2. \tag{2.7}$$

Using the exterior derivative $d\eta$, we define a skew-symmetric operator G on the Riemannian manifold $\left(N^k, g\right)$ possessing an m-modified conformal vector field ξ, defined by

$$\frac{1}{2}d\eta(X_1, X_2) = g(GX_1, X_2), \quad X_1, X_2 \in \mathfrak{X}(N^k) \tag{2.8}$$

and we call G the associated tensor of the m-modified conformal vector field ξ. Combining Eqs (2.5) and (2.8) with Koszul's formula, we obtain

$$\nabla_X \xi = \rho X + \frac{1}{m}\eta(X)\xi + GX, \quad X \in \mathfrak{X}(N^k). \tag{2.9}$$

The squared length of the tensor G is given by

$$\|G\|^2 = \sum_j g\left(Ge_j, Ge_j\right).$$

Upon differentiating Eq (2.9), we have

$$\begin{aligned}\nabla_{X_1}\nabla_{X_2}\xi &= X_1(\rho)X_2 + \frac{1}{m}g\left(X_2, \rho X_1 + \frac{1}{m}\eta(X_1)\xi + GX_1\right)\xi + \nabla_{\nabla_{X_1}X_2}\xi \\ &\quad + \frac{1}{m}\eta(X_2)\left(\rho X_1 + \frac{1}{m}\eta(X_1)\xi + GX_1\right) + (\nabla_{X_1}G)(X_2),\end{aligned}$$

which in view of Eq (2.1) gives

$$\begin{aligned}R(X_1, X_2)\xi &= (X_1(\rho)X_2 - X_2(\rho)X_1) + \frac{\rho}{m}(\eta(X_2)X_1 - \eta(X_1)X_2) + \frac{2}{m}g(GX_1, X_2)\xi \\ &\quad + \frac{1}{m}(\eta(X_2)GX_1 - \eta(X_1)GX_2) + (\nabla_{X_1}G)(X_2) - (\nabla_{X_2}G)(X_1).\end{aligned} \tag{2.10}$$

Next, we prove the following useful result:

Lemma 2.1. *Let ξ be an m-modified conformal vector field with potential ρ and associated tensor G on a Riemannian manifold (N^k, g). Then the covariant derivative of the associated tensor is given by*

$$\begin{aligned}(\nabla_{X_1}G)(X_2) &= \frac{1}{m}(2\eta(X_1)GX_2 + \rho g(X_1,X_2)\xi - \rho\eta(X_2)X_1 - g(GX_1,X_2)\xi) \\ &+\frac{1}{m}\eta(X_2)GX_1 + X_2(\rho)X_1 - g(X_1,X_2)\nabla\rho + R(X_1,\xi)X_2\end{aligned}$$

for $X_1, X_2 \in \mathfrak{X}(N^k)$.

Proof. Note that the differential form $\frac{1}{2}d\eta$ in Eq (2.8) is closed, and we have

$$g((\nabla_{X_1}G)(X_2), X_3) + g((\nabla_{X_2}G)(X_3), X_1) + g((\nabla_{X_3}G)(X_1), X_2) = 0.$$

Since G is skew-symmetric, the above equation could be rearranged as

$$g((\nabla_{X_1}G)(X_2) - (\nabla_{X_2}G)(X_1), X_3) + g((\nabla_{X_3}G)(X_1), X_2) = 0. \tag{2.11}$$

The Eq (2.10), implies

$$\begin{aligned}(\nabla_{X_1}G)(X_2) - (\nabla_{X_2}G)(X_1) &= (X_2(\rho)X_1 - X_1(\rho)X_2) + \frac{\rho}{m}(\eta(X_1)X_2 - \eta(X_2)X_1) \\ &-\frac{2}{m}g(GX_1,X_2)\xi + \frac{1}{m}(\eta(X_1)GX_2 - \eta(X_2)GX_1) \\ &+R(X_1,X_2)\xi.\end{aligned}$$

Inserting Eq (2.10) in the above equation yields

$$\begin{aligned}g((\nabla_{X_3}G)(X_1), X_2) &= g\Big((X_1(\rho)X_2 - X_2(\rho)X_1) + \frac{\rho}{m}(\eta(X_2)X_1 - \eta(X_1)X_2), X_3\Big) \\ &+g\left(\frac{2}{m}g(GX_1,X_2)\xi + \frac{1}{m}(\eta(X_2)GX_1 - \eta(X_1)GX_2), X_3\right) \\ &-g(R(X_1,X_2)\xi, X_3).\end{aligned}$$

Transvecting the slot at X_2 in the above equation yields the desired result. □

Next, we use the above lemma to find the expression for the vector field $Q(\xi)$, which follows directly on taking the trace in the equation appearing in the statement of the above lemma.

Lemma 2.2. *Let ξ be an m-modified conformal vector field with potential ρ and associated tensor G on a Riemannian k-manifold (N^k, g). Then we have*

$$Q(\xi) = -(k-1)\nabla\rho + \frac{3}{m}G\xi + \frac{k-1}{m}\rho\xi - \sum_j \left(\nabla_{e_j}G\right)\left(e_j\right),$$

where $\{e_1, .., e_k\}$ is a local orthonormal frame on (N^k, g).

We are interested in invoking the second Lie derivative of the m-modified conformal vector field ξ on a Riemannian k-manifold (N^k, g), as considered in [7]. Thus, we first prove:

Lemma 2.3. *Let ξ be an m-modified conformal vector field with potential ρ and associated tensor G on a Riemannian manifold (N^k, g). Then the 1-form η dual to ξ obeys*

$$\mathcal{L}_\xi(\eta\otimes\eta) = 4\left(\rho + \frac{1}{m}\|\xi\|^2\right)\eta\otimes\eta.$$

Proof. Using Eq (2.9) in

$$\left(\mathcal{L}_\xi(\eta\otimes\eta)\right)(X_1, X_2) = \xi\left(\eta(X_1)\,\eta(X_2)\right) - \eta\left([\xi, X_1]\right)\eta(X_2) - \eta(X_1)\,\eta\left([\xi, X_2]\right),$$

after some trivial calculations involving skew symmetry of G, we arrive at

$$\left(\mathcal{L}_\xi(\eta\otimes\eta)\right)(X_1, X_2) = 4\rho\eta(X_1)\,\eta(X_2) + \frac{4}{m}\|\xi\|^2\,\eta(X_1)\,\eta(X_2).$$

This proves the result. □

Note that the Riemannian curvature tensor on a Riemannian manifold (N^k, g) is given by

$$R(X_1, X_2; X_3, X_4) = g\left(R(X_1, X_2)X_3, X_4\right)$$

for $X_i \in \mathfrak{X}(N^k)$, $i = 1, \ldots, 4$.

Following the techniques of the second Lie derivative described in [7], we have the following:

Lemma 2.4. *Let ξ be an m-modified conformal vector field with potential ρ and associated tensor G on a Riemannian manifold (N^k, g) with dual 1-form η. Then*

$$\begin{aligned}\frac{1}{2}\left(\mathcal{L}_\xi\mathcal{L}_\xi\right)(X_1, X_2) \;=\;& R(\xi, X_1; \xi, X_2) + \frac{1}{2}\left(\mathcal{L}_{\nabla_\xi\xi}g\right)(X_1, X_2) + \rho^2 g(X_1, X_2)\\ &+\left(\frac{2}{m}\rho + \frac{1}{m^2}\|\xi\|^2\right)\eta(X_1)\,\eta(X_2) + \frac{1}{m}\eta(X_1)\,\eta(GX_2)\\ &+\frac{1}{m}\eta(X_2)\,\eta(GX_1) + g(GX_1, GX_2),\end{aligned}$$

for $X_1, X_2 \in \mathfrak{X}(N^k)$.

Proof. Using the techniques in [7], we have

$$\frac{1}{2}\left(\mathcal{L}_\xi\mathcal{L}_\xi\right)(X_1, X_2) = R(\xi, X_1; \xi, X_2) + \frac{1}{2}\left(\mathcal{L}_{\nabla_\xi\xi}g\right)(X_1, X_2) + g\left(\nabla_{X_1}\xi, \nabla_{X_2}\xi\right). \tag{2.12}$$

Now, using Eq (2.9), we have

$$g\left(\nabla_{X_1}\xi, \nabla_{X_2}\xi\right) = g\left(\rho X_1 + \frac{1}{m}\eta(X_1)\,\xi + GX_1, \rho X_2 + \frac{1}{m}\eta(X_2)\,\xi + GX_2\right),$$

which, on using the skew symmetry of the operator G, yields

$$\begin{aligned}g\left(\nabla_{X_1}\xi, \nabla_{X_2}\xi\right) \;=\;& \rho^2 g(X_1, X_2) + \frac{2}{m}\rho\eta(X_1)\,\eta(X_2) + \frac{1}{m^2}\eta(X_1)\,\eta(X_2)\,\|\xi\|^2\\ &+\frac{1}{m}\eta(X_1)\,\eta(GX_2) + \frac{1}{m}\eta(X_2)\,\eta(GX_1) + g(GX_1, GX_2).\end{aligned}$$

Inserting the above equation in Eq (2.12) gives the desired result. □

Using Eq (2.9) and the skew symmetry of the associated tensor G, we see that

$$\operatorname{div}\xi = k\rho + \frac{1}{m}\|\xi\|^2. \tag{2.13}$$

Let ξ be an m-modified conformal vector field with potential ρ and associated tensor G on a Riemannian manifold (N^k, g). Then, we define the half-squared length function h of ξ by

$$h = \frac{1}{2}\|\xi\|^2.$$

Differentiating the above equation and using Eq (2.9), we find the gradient of h given by

$$\nabla h = \rho\xi + \frac{1}{m}\|\xi\|^2\xi - G\xi. \tag{2.14}$$

The Hessian operator $\mathcal{H}^h$ of h is defined by

$$\mathcal{H}^h X = \nabla_X \nabla h, \quad X \in \mathfrak{X}(N^k) \tag{2.15}$$

and the Laplace operator acting on h, that is Δh is given by

$$\Delta h = Tr\,\mathcal{H}^h = \sum_j g\left(\mathcal{H}^h e_j, e_j\right). \tag{2.16}$$

Lemma 2.5. *Let ξ be an m-modified conformal vector field on a Riemannian manifold (N^k, g) with potential ρ and associated tensor G. Then*

$$\begin{aligned}\mathcal{H}^h X \;=\;& \left(\rho^2 + \frac{2}{m}\rho\|\xi\|^2 - \xi(\rho)\right)X + \left(X(\rho) + \frac{3}{m}\eta(GX) + \frac{2}{m}\rho\eta(X)\right)\xi \\ &+ \frac{3}{m^2}\|\xi\|^2\eta(X)\xi - \frac{3}{m}\eta(X)G\xi + \eta(X)\nabla\rho - G^2X - R(X,\xi)\xi.\end{aligned}$$

Proof. Using Eqs (2.14) and (2.15), we have

$$\begin{aligned}\mathcal{H}^h X \;=\;& X(\rho)\xi + \rho\nabla_X\xi + \frac{2}{m}g(\nabla_X\xi,\xi)\xi + \frac{1}{m}\|\xi\|^2\nabla_X\xi \\ &- (\nabla_X G)(\xi) - G\nabla_X\xi.\end{aligned}$$

Inserting Eq (2.9) in the above equation and using Lemma 2.1, after some simple computations, we get the desired result. □

3. m-modified conformal vector fields on compact manifolds

Recall that an m-modified conformal vector field ξ on a Riemannian k-manifold (N^k, g) is said to be trivial if $\xi = 0$. One of the interesting questions is to find conditions under which an m-modified conformal vector field on a Riemannian k-manifold (N^k, g) is trivial (cf. [4,6]). Compact Riemannian manifolds provide a better space to study such questions due to the availability of many tools. Our first result in this section is the following:

Theorem 3.1. *An m–modified conformal vector ξ on a compact and connected Riemannian k-manifold (N^k, g), $k > 1$, satisfying*

$$Ric\,(\xi, \xi) \geq \frac{k-1}{k}\,(\operatorname{div}\xi)^2 + \|G\|^2$$

is trivial.

Proof. Using Lemma 2.2, we see that

$$Ric\,(\xi, \xi) = -\,(k-1)\,\xi\,(\rho) + \frac{(k-1)}{m}\rho\,\|\xi\|^2 - \sum_j g\left(\xi, \left(\nabla_{e_j} G\right)\left(e_j\right)\right), \tag{3.1}$$

where $\{e_1, .., e_k\}$ is a local frame on $\left(N^k, g\right)$. We use Lemma 2.1, the skew symmetry of G, and Eq (2.10) in finding

$$\begin{aligned} \operatorname{div} G\xi &= \sum_j g\left(\nabla_{e_j} G\xi, e_j\right) = \sum_j g\left(\left(\nabla_{e_j} G\right)(\xi) + G\left(\rho e_j + \frac{1}{m}\eta\left(e_j\right)\xi + Ge_j\right), e_j\right) \\ &= -\,\|G\|^2 - \sum_j g\left(\xi, \left(\nabla_{e_j} G\right)\left(e_j\right)\right). \end{aligned}$$

Inserting the above equation in Eq (3.1) yields

$$Ric\,(\xi, \xi) = -\,(k-1)\,\xi\,(\rho) + \frac{(k-1)}{m}\rho\,\|\xi\|^2 + \operatorname{div} G\xi + \|G\|^2\,.$$

Integrating the above equation and using Stokes's theorem, we arrive at

$$\int_{N^k}\left(Ric\,(\xi, \xi) + (k-1)\,\xi\,(\rho) - \frac{(k-1)}{m}\rho\,\|\xi\|^2 - \|G\|^2\right) = 0. \tag{3.2}$$

Additionally, upon employing Eq (2.13), we have

$$\operatorname{div}\,(\rho\xi) = \xi\,(\rho) + \rho\left(k\rho + \frac{1}{m}\,\|\xi\|^2\right)$$

and inserting it in Eq (3.2) gives

$$\int_{N^k}\left(Ric\,(\xi, \xi) - (k-1)\,\rho\left(k\rho + \frac{1}{m}\,\|\xi\|^2\right) - \frac{(k-1)}{m}\rho\,\|\xi\|^2 - \|G\|^2\right) = 0.$$

The above equation simplifies to

$$\int_{N^k}\left(Ric\,(\xi, \xi) - k\,(k-1)\,\rho^2 - \frac{2\,(k-1)}{m}\rho\,\|\xi\|^2 - \|G\|^2\right) = 0,$$

that is,

$$\int_{N^k}\left(Ric\,(\xi, \xi) - \frac{(k-1)}{k}\left(k\rho + \frac{1}{m}\,\|\xi\|^2\right)^2 + \frac{(k-1)}{m^2 k}\,\|\xi\|^4 - \|G\|^2\right) = 0.$$

Thus, on using Eq (2.13), we have

$$\frac{(k-1)}{m^2 k}\int_{N^k}\|\xi\|^4 = \int_{N^k}\left(\frac{(k-1)}{k}\,(\operatorname{div}\xi)^2 + \|G\|^2 - Ric\,(\xi, \xi)\right),$$

which together with $k > 1$ and the bound on $Ric\,(\xi, \xi)$ in the statement yields $\xi = 0$. ◻

In the next result, we use the half-squared length of the m-modified conformal vector on a compact Riemannian manifold to get the following triviality result:

Theorem 3.2. *An m–modified conformal vector ξ on a compact and connected Riemannian k-manifold (N^k, g), $k > 1$, satisfying*

$$Ric\,(\xi,\xi) \le \frac{3}{m}\,\|\xi\|^2\,(\operatorname{div}\xi) + \|G\|^2$$

is trivial.

Proof. Using Eq (2.16) in Lemma 2.5, we see that

$$\begin{aligned}\Delta h \;&=\; k\left(\rho^2 + \frac{2}{m}\rho\,\|\xi\|^2 - \xi\,(\rho)\right) + 2\xi\,(\rho) + \frac{2}{m}\rho\,\|\xi\|^2 + \frac{3}{m^2}\,\|\xi\|^4 \\ &\quad + \|G\|^2 - Ric\,(\xi,\xi)\,,\end{aligned}$$

that is,

$$\begin{aligned}\Delta h &= -\,(k-2)\,\xi\,(\rho) + k\rho^2 + 2\,(k+1)\,\frac{1}{m}\rho\,\|\xi\|^2 + \frac{3}{m^2}\,\|\xi\|^4 \\ &\quad + \|G\|^2 - Ric\,(\xi,\xi)\,.\end{aligned} \tag{3.3}$$

Using Eq (2.13), we derive

$$\operatorname{div}\,(\rho\xi) = \xi\,(\rho) + \rho\left(k\rho + \frac{1}{m}\,\|\xi\|^2\right)$$

and inserting it in Eq (3.3) and integrating the resulting equation, upon simplification, yields

$$\int_{N^k}\left(k\,(k-1)\,\rho^2 + \frac{3}{m}\,\|\xi\|^2\left(k\rho + \frac{1}{m}\,\|\xi\|^2\right) + \|G\|^2 - Ric\,(\xi,\xi)\right) = 0,$$

that is,

$$\int_{N^k}\left(\frac{3}{m}\,\|\xi\|^2\left(k\rho + \frac{1}{m}\,\|\xi\|^2\right) + \|G\|^2 - Ric\,(\xi,\xi)\right) = -k\,(k-1)\int_{N^k}\rho^2.$$

Using Eq (2.13) in the above equation, it takes the form

$$\int_{N^k}\left(\frac{3}{m}\,\|\xi\|^2\,(\operatorname{div}\xi) + \|G\|^2 - Ric\,(\xi,\xi)\right) = -k\,(k-1)\int_{N^k}\rho^2, \tag{3.4}$$

which in view of the inequality in the statement gives

$$k\,(k-1)\int_{N^k}\rho^2 \le 0.$$

Thus, in view of $k > 1$, we conclude $\rho = 0$, and Eq (3.4) reduces to

$$\int_{N^k}\left(\frac{3}{m}\,\|\xi\|^2\,(\operatorname{div}\xi) + \|G\|^2 - Ric\,(\xi,\xi)\right) = 0. \tag{3.5}$$

With $\rho = 0$, the Eq (2.13) is

$$\operatorname{div}\xi = \frac{1}{m}\,\|\xi\|^2 \tag{3.6}$$

and Lemma 2.2 gives

$$Ric\,(\xi,\xi) = -\sum_j g\left(\xi, \left(\nabla_{e_j} G\right)\left(e_j\right)\right). \tag{3.7}$$

Additionally, using Eq (2.9) with $\rho = 0$, we find

$$\operatorname{div}\,(G\xi) = \sum_j g\left(\nabla_{e_j} G\xi, e_j\right) = \sum_j g\left(\left(\nabla_{e_j} G\right)(\xi) + G\left(\frac{1}{m}\eta\left(e_j\right)\xi + Ge_j\right), e_j\right),$$

which, owing to the skew symmetry of G, implies

$$\operatorname{div}\,(G\xi) = -\sum_j g\left(\xi, \left(\nabla_{e_j} G\right)\left(e_j\right)\right) + \frac{1}{m} g\,(G\xi, \xi) - \|G\|^2,$$

that is,

$$-\sum_j g\left(\xi, \left(\nabla_{e_j} G\right)\left(e_j\right)\right) = \operatorname{div}\,(G\xi) + \|G\|^2. \tag{3.8}$$

Inserting Eq (3.7) in the above equation and integrating the resulting equation yields

$$\int_{N^k} Ric\,(\xi,\xi) = \int_{N^k} \|G\|^2. \tag{3.9}$$

Using Eqs (3.7) and (3.9) in Eq (3.5), we conclude

$$\frac{3}{m^2}\int_{N^k} \|\xi\|^4 = 0,$$

which completes the proof. □

Remark 3.1. *By [4], a compact and connected Riemannian manifold (N^k, g) with $k \geq 2$ admits no nontrivial m-modified conformal vector field of constant length, so the conclusions of Theorems 3.1 and 3.2 are, as triviality statements, already contained in that result. The value of Theorems 3.1 and 3.2 lies in providing two alternative and self-contained proofs by different techniques, and in exhibiting explicit curvature and integral conditions under which the identities established in Section 2 directly yield $\xi = 0$. The same identities are indispensable in Sections 4 and 5, where compactness is no longer assumed and the corresponding triviality results do not follow from [4].*

4. m-modified conformal vector fields on complete manifolds

In the previous section, we found conditions for an m–modified conformal vector ξ on a compact and connected Riemannian k-manifold (N^k, g) to be trivial. In this section, we study the same question on noncompact Riemannian manifolds. We prove the following:

Theorem 4.1. *An m–modified conformal vector ξ on a complete and connected Riemannian k-manifold (N^k, g), $k > 1$, with potential ρ and associated tensor G satisfying*

$$Ric\,(\xi,\xi) = \left(\operatorname{div} \nabla_\xi \xi\right) + \|G\|^2 - \frac{2}{m}\left(3\rho + \frac{2}{m}\|\xi\|^2\right)\|\xi\|^2$$

is trivial.

Proof. We take Lie derivative with respect to ξ in Eq (2.5) and arrive at

$$\mathcal{L}_\xi\mathcal{L}_\xi g = 2\xi(\rho)g + 2\rho\left(2\rho g + \frac{2}{m}\eta\otimes\eta\right) + \frac{2}{m}\mathcal{L}_\xi(\eta\otimes\eta),$$

that is,

$$\frac{1}{2}\mathcal{L}_\xi\mathcal{L}_\xi g = \xi(\rho)g + 2\rho^2 g + \frac{2}{m}\rho\eta\otimes\eta + \frac{1}{m}\mathcal{L}_\xi(\eta\otimes\eta).$$

Inserting from Lemma 2.3, in the above equation yields

$$\frac{1}{2}\mathcal{L}_\xi\mathcal{L}_\xi g = \xi(\rho)g + 2\rho^2 g + \frac{2}{m}\rho\eta\otimes\eta + \frac{4}{m}\left(\rho + \frac{1}{m}\|\xi\|^2\right)\eta\otimes\eta,$$

that is,

$$\frac{1}{2}\mathcal{L}_\xi\mathcal{L}_\xi g = \xi(\rho)g + 2\rho^2 g + \frac{2}{m}\left(3\rho + \frac{2}{m}\|\xi\|^2\right)\eta\otimes\eta.$$

Taking the trace in above equation yields

$$\frac{1}{2}Tr\left(\mathcal{L}_\xi\mathcal{L}_\xi g\right) = k\left(\xi(\rho) + 2\rho^2\right) + \frac{2}{m}\left(3\rho + \frac{2}{m}\|\xi\|^2\right)\|\xi\|^2. \tag{4.1}$$

Additionally, taking trace in the expression of Lemma 2.4, we have

$$\frac{1}{2}Tr\left(\mathcal{L}_\xi\mathcal{L}_\xi g\right) = -Ric(\xi,\xi) + \left(\operatorname{div}\nabla_\xi\xi\right) + k\rho^2 + \|G\|^2, \tag{4.2}$$

where we use

$$\sum_j \eta\left(e_j\right)\eta\left(Ge_j\right) = -\sum_j g\left(\xi, e_j\right)g\left(G\xi, e_j\right) = g(G\xi,\xi) = 0.$$

Comparing Eqs (4.1) and (4.2), we arrive at

$$k\left(\xi(\rho) + \rho^2\right) = -Ric(\xi,\xi) + \left(\operatorname{div}\nabla_\xi\xi\right) + \|G\|^2 - \frac{2}{m}\left(3\rho + \frac{2}{m}\|\xi\|^2\right)\|\xi\|^2,$$

which, in view of the condition in the statement, yields

$$\xi(\rho) = -\rho^2. \tag{4.3}$$

Let $\alpha(t)$ be an integral curve of the m-modified conformal vector field ξ, and we put $\xi(t) = \xi(\alpha(t))$. For the potential, we put $\rho(t) = (\rho\circ\alpha)(t)$. Then, Eq (4.3) gives the following ordinary differential equation

$$\frac{d\rho}{dt} = -\rho^2(t)$$

with initial condition $\rho(0) = \rho_0$. Assume that $\rho_0 \neq 0$, with this, we see the unique solution of the above differential equation is

$$\rho(t) = \frac{\rho_0}{1 + t\rho_0}. \tag{4.4}$$

Now, using Eq (2.9), we compute

$$\frac{d}{dt}\left\|\xi(t)\right\| = \frac{1}{\left\|\xi(t)\right\|} g\left(\rho(t)\,\xi(t) + \frac{1}{m}\left\|\xi(t)\right\|^2 \xi(t) + G(\xi(t)), \xi(t)\right),$$

which, by virtue of Eq (4.4), gives the Bernoulli's equation

$$\frac{d}{dt}\left\|\xi(t)\right\| - \frac{\rho_0}{1+t\rho_0}\left\|\xi(t)\right\| = \frac{1}{m}\left\|\xi(t)\right\|^3. \tag{4.5}$$

Notice that solution (4.4) blows at the point $t_* = -\frac{1}{\rho_0}$. Solving Bernoulli's Eq (4.5), we get

$$\left\|\xi(t)\right\|^2 = \frac{3m(1+t\rho_0)^2\left\|\xi(0)\right\|^2}{2t_*\left\|\xi(0)\right\|^2(1+t\rho_0)^3 + \left(3m - 2t_*\left\|\xi(0)\right\|^2\right)}. \tag{4.6}$$

Note that the arc-length $L_0^{t_*}$ and the Energy $E_0^{t_*}$of $\alpha(t)$ from 0 to t_* are given by (cf. [5])

$$L_0^{t_*} = \int_0^{t_*}\left\|\xi(t)\right\| dt \quad \text{and} \quad E_0^{t_*} = \int_0^{t_*}\left\|\xi(t)\right\|^2 dt.$$

Using Eq (4.6), we have

$$E_0^{t_*} = \frac{m}{2t_*\rho_0}\int_0^{t_*}\frac{6t_*(1+t\rho_0)^2\left\|\xi(0)\right\|^2}{2t_*\left\|\xi(0)\right\|^2(1+t\rho_0)^3 + \left(3m - 2t_*\left\|\xi(0)\right\|^2\right)}dt,$$

that is,

$$\begin{aligned} E_0^{t_*} &= -\frac{m}{2}\left[\ln\left(2t_*\left\|\xi(0)\right\|^2(1+t\rho_0)^3 + \left(3m - 2t_*\left\|\xi(0)\right\|^2\right)\right)\right]_0^{t_*} \\ &= \frac{m}{2}\left[\ln(3m) - \ln\left(3m - 2t_*\left\|\xi(0)\right\|^2\right)\right] < \infty. \end{aligned} \tag{4.7}$$

Now, using inequality (cf. [5], page 194), we have

$$\left(L_0^{t_*}\right)^2 \leq t_* E_0^{t_*} < \infty.$$

As N^k is complete, there exists a point $x_* \in N^k$ such that $\alpha(t_*) = x_*$, we have as $t \to t_*$, $\alpha(t) \to x_*$. Thus, upon using Eq (4.4), we have

$$\rho(x_*) = \lim_{t\to t_*}\rho(\alpha(t)) = \lim_{t\to t_*}\frac{\rho_0}{1+t\rho_0} = \infty,$$

contradicting the fact that ρ is continuous. Note that the assumption $\rho_0 \neq 0$ leads to the contradiction. Hence, $\rho_0 = 0$ and consequently, Eq (4.4) confirms $\rho(\alpha(t)) = 0$ for each integral curve of ξ. Since, N^k is complete and connected, we conclude $\rho = 0$.

Now, we proceed to complete the proof. With $\rho = 0$, Eq (2.9) implies

$$\nabla_\xi \xi = \frac{1}{m}\left\|\xi\right\|^2 \xi + G\xi. \tag{4.8}$$

In the same situation, we have Eq (3.8), which gives

$$\operatorname{div}(G\xi) = -\sum_j g\left(\xi, \left(\nabla_{e_j} G\right)\left(e_j\right)\right) - \|G\|^2 . \tag{4.9}$$

Additionally, using Eqs (2.7) and (4.8), with $\rho = 0$, we have

$$\begin{aligned} \operatorname{div}\left(\|\xi\|^2 \xi\right) &= 2g\left(\frac{1}{m}\|\xi\|^2 \xi + G\xi, \xi\right) + \|\xi\|^2 \left(\frac{1}{m}\|\xi\|^2\right) \\ &= \frac{3}{m}\|\xi\|^4 . \end{aligned}$$

Plugging this equation and Eq (4.9) with Eq (4.8), we get

$$\operatorname{div}\left(\nabla_\xi \xi\right) = \frac{3}{m^2}\|\xi\|^4 - \sum_j g\left(\xi, \left(\nabla_{e_j} G\right)\left(e_j\right)\right) - \|G\|^2 . \tag{4.10}$$

Now, Lemma 2.2 with $\rho = 0$ implies

$$Ric(\xi, \xi) = -\sum_j g\left(\xi, \left(\nabla_{e_j} G\right)\left(e_j\right)\right). \tag{4.11}$$

The given condition in the statement with $\rho = 0$ is

$$Ric(\xi, \xi) = \left(\operatorname{div} \nabla_\xi \xi\right) + \|G\|^2 - \frac{4}{m^2}\|\xi\|^4$$

and inserting Eqs (4.10) and (4.11) into the above equation, we have

$$-\sum_j g\left(\xi, \left(\nabla_{e_j} G\right)\left(e_j\right)\right) = \frac{3}{m^2}\|\xi\|^4 - \sum_j g\left(\xi, \left(\nabla_{e_j} G\right)\left(e_j\right)\right) - \|G\|^2 + \|G\|^2 - \frac{4}{m^2}\|\xi\|^4 ,$$

that is,

$$\frac{1}{m^2}\|\xi\|^4 = 0.$$

This completes the proof. □

The researchers in [4] proved that there are no nontrivial m–modified conformal vector fields of constant length on a compact and connected Riemannian k-manifold (N^k, g) for $k \geq 2$; in other words, their result states that an m–modified conformal vector field of constant length on a compact (N^k, g) is trivial. This naturally raises a question about the status of m–modified conformal vector fields of constant length on a connected Riemannian manifold (N^k, g). In this direction, we prove the following:

Theorem 4.2. *An m–modified conformal vector field ξ of constant length with potential ρ and associated tensor G on a connected Riemannian k-manifold, (N^k, g), $k > 1$, satisfying*

$$Ric(\xi, \xi) \geq \|G\|^2$$

is trivial.

Proof. Using Eq (2.9) in $Xg(\xi,\xi) = 0$, we get

$$g\left(\rho X + \frac{1}{m}\eta(X)\xi + GX, \xi\right) = 0, \quad X \in \mathfrak{X}\left(N^k\right),$$

which yields

$$\rho\xi + \frac{1}{m}\|\xi\|^2 \xi = G\xi. \tag{4.12}$$

Upon taking the inner product with ξ on both sides of the above equation, we get

$$\left(\rho + \frac{1}{m}\|\xi\|^2\right)\|\xi\|^2 = 0. \tag{4.13}$$

Let U be the open set defined by

$$U = \left\{p \in N^k : \xi(p) \neq 0\right\}.$$

Thus, on U, we have $\left(\rho + \frac{1}{m}\|\xi\|^2\right) = 0$, and as ξ has constant length, it follows that ρ is a constant and by Eq (4.12) that

$$G\xi = 0. \tag{4.14}$$

Differentiating the above equation, while using Eq (2.9), we have

$$(\nabla_X G)(\xi) + G\left(\rho X + \frac{1}{m}\eta(X)\xi + GX\right) = 0$$

and using Lemma 2.1 with ρ a constant, and

$$\rho = -\frac{1}{m}\|\xi\|^2, \tag{4.15}$$

we arrive at

$$R(X,\xi)\xi = -\frac{1}{m}\rho\eta(X)\xi - \rho^2 X - G^2 X.$$

Taking the trace of the above equation yields

$$Ric(\xi,\xi) = -\frac{1}{m}\rho\|\xi\|^2 - k\rho^2 + \|G\|^2,$$

which, upon being treated with Eq (4.15), yields

$$Ric(\xi,\xi) - \|G\|^2 = -(k-1)\rho^2.$$

Using the condition in the statement as well as $k > 1$, we get $\rho = 0$, which in view of Eq (4.15) is $\xi = 0$. Since U is empty, we have $\xi = 0$, and this completes the proof. □

5. m-modified gradient conformal vector fields

An m–modified conformal vector field ξ on a connected Riemannian manifold (N^k, g) is said to be an m–modified gradient conformal vector field if $\xi = \nabla f$ for a smooth function f on N^k. It is worth noting that for an m–modified gradient conformal vector field ∇f being closed, the associated tensor $G = 0$, and therefore studying the geometry of a Riemannian manifold (N^k, g) admitting an m–modified gradient conformal vector field ∇f becomes easier than studying those admitting a non-closed m–modified conformal vector field. In this section, we investigate conditions under which an m–modified gradient conformal vector field ∇f on a Riemannian manifold (N^k, g) becomes trivial.

First, note that for an m–modified gradient conformal vector field ∇f with potential ρ on a Riemannian manifold (N^k, g), using Eq (2.16), Eq (2.10) assumes the form

$$\mathcal{H}^f X = \rho X + \frac{1}{m} X(f)\nabla f, \quad X \in \mathfrak{X}\left(N^k\right) \tag{5.1}$$

and taking trace in the above equation, we have

$$\Delta f = k\rho + \frac{1}{m}\|\nabla f\|^2 . \tag{5.2}$$

Our first result is the following:

Theorem 5.1. *An m–modified gradient conformal vector field* ∇f *with potential* ρ *on a compact and connected Riemannian k-manifold* (N^k, g)*,* $k > 1$*, satisfying*

$$\int_{N^k} Ric(\nabla f, \nabla f) \leq \frac{2}{m}(k-1)\int_{N^k} \rho\|\nabla f\|^2$$

is trivial.

Proof. Use Eq (5.1) to conclude

$$\left(\mathcal{H}^f - \frac{1}{k}\Delta f I\right)X = \left(\rho - \frac{1}{k}\Delta f\right)X + \frac{1}{m}X(f)\nabla f, X \in \mathfrak{X}\left(N^k\right).$$

Thus, upon taking a local frame $\{e_1, ..e_k\}$, we have

$$\begin{aligned}\left\|\mathcal{H}^f - \frac{1}{k}\Delta f I\right\|^2 &= \sum_j g\left(\left(\rho - \frac{1}{k}\Delta f\right)e_j + \frac{1}{m}e_j(f)\nabla f, \left(\rho - \frac{1}{k}\Delta f\right)e_j + \frac{1}{m}e_j(f)\nabla f\right)\\ &= k\left(\rho - \frac{1}{k}\Delta f\right)^2 + \frac{1}{m^2}\|\nabla f\|^4 + \frac{2}{m}\left(\rho - \frac{1}{k}\Delta f\right)\|\nabla f\|^2 .\end{aligned} \tag{5.3}$$

Also, we have

$$\begin{aligned}\left\|\mathcal{H}^f - \frac{1}{k}\Delta f I\right\|^2 &= \left\|\mathcal{H}^f\right\|^2 + \frac{1}{k}(\Delta f)^2 - \frac{2}{k}\sum_j \Delta f g\left(\mathcal{H}^f e_j, e_j\right)\\ &= \left\|\mathcal{H}^f\right\|^2 - \frac{1}{k}(\Delta f)^2 ,\end{aligned}$$

which, upon combining with Eq (5.3), gives

$$\left\|\mathcal{H}^f\right\|^2 - \frac{1}{k}(\Delta f)^2 = k\left(\rho - \frac{1}{k}\Delta f\right)^2 + \frac{1}{m^2}\|\nabla f\|^4 + \frac{2}{m}\left(\rho - \frac{1}{k}\Delta f\right)\|\nabla f\|^2. \tag{5.4}$$

The Eq (5.2) implies

$$\frac{1}{k}\Delta f - \rho = \frac{1}{mk}\|\nabla f\|^2$$

and inserting it in Eq (5.4) yields

$$\begin{aligned}\left\|\mathcal{H}^f\right\|^2 - \frac{1}{k}(\Delta f)^2 &= \frac{1}{m^2 k}\|\nabla f\|^4 + \frac{1}{m^2}\|\nabla f\|^4 - \frac{2}{m^2 k}\|\nabla f\|^4 \\ &= \frac{(k-1)}{m^2 k}\|\nabla f\|^4. \end{aligned} \tag{5.5}$$

Now, we recall the following integral formula from [8],

$$\int_{N^k}\left(Ric(\xi,\xi) + \frac{1}{2}\left|\mathcal{L}_\xi g\right|^2 - \|\nabla\xi\|^2 - (\operatorname{div}\xi)^2\right) = 0,$$

which upon using $\xi = \nabla f$, that is, $\operatorname{div}\xi = \Delta f$,

$$\frac{1}{2}\mathcal{L}_\xi g = \mathcal{H}^f \quad \text{and} \quad \|\nabla\xi\|^2 = \left\|\mathcal{H}^f\right\|^2$$

yields

$$\int_{N^k}\left(Ric(\nabla f,\nabla f) + \left\|\mathcal{H}^f\right\|^2 - (\Delta f)^2\right) = 0.$$

Rearranging the above equation, we have

$$\int_{N^k}\left(Ric(\nabla f,\nabla f) + \left(\left\|\mathcal{H}^f\right\|^2 - \frac{1}{k}(\Delta f)^2\right) - \frac{(k-1)}{k}(\Delta f)^2\right) = 0$$

and inserting Eqs (5.2) and (5.5), we get

$$\int_{N^k}\left(Ric(\nabla f,\nabla f) + \frac{(k-1)}{m^2 k}\|\nabla f\|^4 - \frac{(k-1)}{k}\left(k\rho + \frac{1}{m}\|\nabla f\|^2\right)^2\right) = 0.$$

Thus, we have

$$\int_{N^k} Ric(\nabla f,\nabla f) - \frac{2(k-1)}{m}\int_{N^k}\rho\|\nabla f\|^2 = k(k-1)\int_{N^k}\rho^2,$$

which upon using the condition in the statement and $k > 1$, implies $\rho = 0$. In this case, Eq (5.2) transforms to

$$\Delta f = \frac{1}{m}\|\nabla f\|^2,$$

which, by virtue of Stokes's theorem, confirms $\nabla f = 0$, which completes the proof. □

Our next result is the following:

Theorem 5.2. *An m–modified gradient conformal vector field* ∇f *with potential* ρ *on a compact and connected Riemannian k-manifold* (N^k, g)*,* $k > 1$*, satisfying*

$$Q(\nabla f) = -(k-1)\nabla\rho$$

is trivial.

Proof. We use Eq (5.1) in computing

$$\left(\nabla_{X_1}\mathcal{H}^f\right)(X_2) = X_1(\rho)X_2 + \frac{1}{m}g\left(\mathcal{H}^f X_1, X_2\right)\nabla f + \frac{1}{m}X_1(f)\mathcal{H}^f X_2$$

and inserting it in the identity

$$R(X_1, X_2)\nabla f = \left(\nabla_{X_1}\mathcal{H}^f\right)(X_2) - \left(\nabla_{X_2}\mathcal{H}^f\right)(X_1)$$

and using symmetry of the Hessian operator $\mathcal{H}^f$, we have

$$R(X_1, X_2)\nabla f = (X_1(\rho)X_2 - X_2(\rho)X_1) + \frac{1}{m}\left(X_1(f)\mathcal{H}^f X_2 - X_2(f)\mathcal{H}^f X_1\right).$$

Taking trace in the above equation and using Eq (2.2), we get

$$Ric(X_2, \nabla f) = -(k-1)X_2(\rho) + \frac{1}{m}\left(g\left(\mathcal{H}^f\nabla f, X_2\right) - (\Delta f)X_2(f)\right),$$

that is,

$$Q(\nabla f) = -(k-1)\nabla\rho + \frac{1}{m}\mathcal{H}^f\nabla f - \frac{1}{m}(\Delta f)\nabla f. \tag{5.6}$$

Using Eqs (5.1) and (5.2), the above equation changes to

$$Q(\nabla f) = -(k-1)\nabla\rho - \frac{(k-1)}{m}\rho\nabla f.$$

The above equation, together with the condition in the statement, implies

$$\frac{(k-1)}{m}\rho\nabla f = 0.$$

As $k > 1$, we have two options: (i) $\rho = 0$ or (ii) ∇f is trivial. Now, if $\rho = 0$, the Eq (5.2) on compact (N^k, g) implies $\nabla f = 0$. This finishes the proof. □

In the next result, we consider a m–modified gradient conformal vector field ∇f on a compact and connected Riemannian manifold (N^k, g) with Ricci curvature $Ric(\nabla f, \nabla f)$ having an upper bound involving the scalar curvature τ. We prove:

Theorem 5.3. *An m–modified gradient conformal vector field* ∇f *with potential* ρ *on a compact and connected Riemannian k-manifold* (N^k, g)*,* $k > 2$*, with nonzero scalar curvature* τ *such that* ρ *and* τ *have the same sign on* N^k *and Ricci curvature* $Ric(\nabla f, \nabla f)$ *satisfying*

$$Ric(\nabla f, \nabla f) \le \frac{\tau}{2m}\|\nabla f\|^2$$

is trivial.

Proof. We compute the divergence of the vector field $Q(\nabla f)$ and find

$$\operatorname{div} Q(\nabla f) = \sum_j g\left(\nabla_{e_j} Q(\nabla f), e_j\right) = \sum_j g\left(\left(\nabla_{e_j} Q\right)(\nabla f) + Q\left(\mathcal{H}^f e_j\right), e_j\right).$$

Using symmetry of Q and Eqs (2.4) and (5.1) in the above equation yields

$$\begin{aligned}\operatorname{div} Q(\nabla f) &= \sum_j g\left(\nabla f, \left(\nabla_{e_j} Q\right) e_j\right) + \sum_j g\left(\rho e_j + \frac{1}{m} e_j(f) \nabla f, Q e_j\right) \\ &= \frac{1}{2} g(\nabla f, \nabla \tau) + \rho\tau + Ric(\nabla f, \nabla f).\end{aligned} \tag{5.7}$$

Observe that, when using Eq (5.2), we have

$$\operatorname{div}(\tau \nabla f) = g(\nabla f, \nabla \tau) + \tau\left(k\rho + \frac{1}{m}\|\nabla f\|^2\right)$$

and inserting it in Eq (5.7) yields

$$\operatorname{div} Q(\nabla f) = \frac{1}{2}\operatorname{div}(\tau\nabla f) - \frac{\tau}{2}\left(k\rho + \frac{1}{m}\|\nabla f\|^2\right) + \rho\tau + Ric(\nabla f, \nabla f).$$

Integration in the above equation brings

$$\int_{N^k}\left(Ric(\nabla f, \nabla f) - \frac{\tau}{2m}\|\nabla f\|^2\right) = \frac{k-2}{2}\int_{N^k}\rho\tau.$$

Note that $\rho\tau \geq 0$ and the left-hand integrand is non-positive. Thus, the above integral yields

$$Ric(\nabla f, \nabla f) = \frac{\tau}{2m}\|\nabla f\|^2 \quad \text{and} \quad \rho\tau = 0.$$

Since $\tau \neq 0$ on connected N^k, we should have $\rho = 0$. Now, Eq (5.2) on compact N^k yields $\nabla f = 0$, which completes the proof. □

Finally, we consider an m-modified gradient conformal vector field of constant length on a noncompact Riemannian manifold (N^k, g) and find conditions under which it is trivial.

Theorem 5.4. *An m–modified gradient conformal vector field* ∇f *of constant length with potential* ρ *on a connected Riemannian k-manifold* (N^k, g), $k > 1$, *with non-negative Ricci curvature Ric* $(\nabla f, \nabla f)$, *is trivial.*

Proof. If $\|\nabla f\| = c$ is a constant, then we have $X\left(\|\nabla f\|^2\right) = 0$ for a smooth vector field X. Thus, we have

$$g\left(\mathcal{H}^f X, \nabla f\right) = 0, \quad X \in X(N^k),$$

which implies

$$\mathcal{H}^f \nabla f = 0. \tag{5.8}$$

Differentiating the above equation, we get

$$\left(\nabla_X \mathcal{H}^f\right)(\nabla f) + \left(\mathcal{H}^f\right)^2(X) = 0$$

and on taking trace in the above equation and noticing $\mathcal{H}^f$ is a symmetric operator, we get

$$\sum_j g\left(\nabla f, \left(\nabla_{e_j}\mathcal{H}^f\right)\left(e_j\right)\right) + \left\|\mathcal{H}^f\right\|^2 = 0. \tag{5.9}$$

Next, we use the identity

$$R\left(X_1, X_2\right)\nabla f = \left(\nabla_{X_1}\mathcal{H}^f\right)\left(X_2\right) - \left(\nabla_{X_2}\mathcal{H}^f\right)\left(X_1\right)$$

in computing

$$\begin{aligned} X\left(\Delta f\right) &= \sum_j Xg\left(\mathcal{H}^f e_j, e_j\right) = \sum_j g\left(\left(\nabla_X\mathcal{H}^f\right)\left(e_j\right), e_j\right) \\ &= \sum_j g\left(R\left(X, e_j\right)\nabla f + \left(\nabla_{e_j}\mathcal{H}^f\right)\left(X\right), e_j\right), \end{aligned}$$

where we have used the local frame of normal coordinates. Using symmetry of $\mathcal{H}^f$, we conclude

$$X\left(\Delta f\right) = -Ric\left(X, \nabla f\right) + \sum_j g\left(X, \left(\nabla_{e_j}\mathcal{H}^f\right)\left(e_j\right)\right), \quad X \in X\left(N^k\right).$$

Thus, we have

$$\sum_j g\left(\nabla f, \left(\nabla_{e_j}\mathcal{H}^f\right)\left(e_j\right)\right) = Ric\left(\nabla f, \nabla f\right) + \nabla f\left(\Delta f\right).$$

Inserting this equation in Eq (5.9), we have

$$Ric\left(\nabla f, \nabla f\right) + \nabla f\left(\Delta f\right) + \left\|\mathcal{H}^f\right\|^2 = 0. \tag{5.10}$$

Now using Eq (5.1), we find

$$\left\|\mathcal{H}^f\right\|^2 = k\rho^2 + \frac{1}{m^2}\left\|\nabla f\right\|^4 + \frac{2}{m}\rho\left\|\nabla f\right\|^2 \tag{5.11}$$

and using Eqs (5.2) and (5.8) gives

$$\nabla f\left(\Delta f\right) = kg\left(\nabla\rho, \nabla f\right) + \frac{2}{m}g\left(\mathcal{H}^f\nabla f, \nabla f\right) = kg\left(\nabla\rho, \nabla f\right). \tag{5.12}$$

Inserting Eqs (5.11) and (5.12) into Eq (5.10), we conclude

$$Ric\left(\nabla f, \nabla f\right) + kg\left(\nabla\rho, \nabla f\right) + k\rho^2 + \frac{1}{m^2}\left\|\nabla f\right\|^4 + \frac{2}{m}\rho\left\|\nabla f\right\|^2 = 0. \tag{5.13}$$

Now, using Eqs (5.1) and (5.8), we have

$$\left(\rho + \frac{1}{m}\left\|\nabla f\right\|^2\right)\nabla f = 0,$$

which implies either ∇f is trivial or

$$\rho = -\frac{1}{m}\left\|\nabla f\right\|^2. \tag{5.14}$$

The above equation implies ρ is a constant (as ∇f has a constant length). Using $\nabla\rho = 0$ and Eq (5.14) in Eq (5.1), we find

$$Ric(\nabla f, \nabla f) + \frac{k}{m^2}\|\nabla f\|^4 + \frac{1}{m^2}\|\nabla f\|^4 - \frac{2}{m^2}\|\nabla f\|^4 = 0,$$

that is,

$$Ric(\nabla f, \nabla f) = -\frac{(k-1)}{m^2}\|\nabla f\|^4.$$

Using $Ric(\nabla f, \nabla f) \geq 0$ in the above equation, proves $\nabla f = 0$, completing the proof. □

6. Conclusions

The notion of m-modified conformal vector fields on a Riemannian manifold was defined recently by H. Zhang and A. Chen in [4], which generalized the important classical notion of conformal vector fields. In this paper, we investigated m-modified conformal vector fields and proved two triviality results for an m-modified conformal vector field ξ on a compact and connected Riemannian k-manifold (N^k, g). Additionally, two more results were established for the triviality of an m-modified conformal vector field ξ on a noncompact, complete, and connected Riemannian manifold (N^k, g). Finally, we studied the m-modified gradient conformal vector field ∇f for compact or noncompact Riemannian manifolds; we obtained three additional results on the triviality of an m-modified gradient conformal vector field ∇f on a compact and connected Riemannian manifold (N^k, g), and one such result for an m-modified gradient conformal vector field ∇f on a noncompact and connected Riemannian k-manifold (N^k, g).

Author contribution

Sharief Deshmukh: Conceptualization, methodology, software, validation, visualization, writing-original draft, methodology, validation, investigation, writing and editing; Nasser Bin Turki: conceptualization; methodology, visualization, validation, investigation, writing and editing, funding acquisition; Bang-Yen Chen: Conceptualization, methodology, validation, investigation, writing and editing Tanveer Fatima: Conceptualization. methodology, software, validation, investigation, writing and editing, project administration. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Funding

This project was supported by the Ongoing Research Funding Program (ORF-2026-413), King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare no conflicts of interest.

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