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Research article

Beltrami equations associated with β-analytic functions under different boundary conditions

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Abstract: In this paper, several fundamental boundary value problems of β-analytic functions were discussed on the unit disc. First, the fundamental properties of β-analytic functions were investigated. On this basis, the Dirichlet boundary value problems for β-analytic functions were discussed, and the solutions and solvability conditions for these problems were derived. Then, the Neumann and half-Neumann boundary value problems associated with β-analytic functions were further studied. Finally, the Rubin boundary value problems of β-analytic functions were addressed and the explicit solutions and their solvability criteria were fully characterized.

Keywords: β-analytic functions; Dirichlet problems; Neumann problems; Rubin problems
Mathematics Subject Classification: 30C45, 32A30

1. Introduction

Boundary value problems for first-order elliptic systems in the plane have a long history, yet the subject remains active, with modern developments continually uncovering new connections and challenges. The classical analytic framework cannot fully capture planar elliptic systems emerging from anisotropic elasticity, chaotic fluid motion, and other geometric physical models, which drives scholars to investigate broader families of generalized analytic functions.

The Beltrami equation, in particular, stands as a natural extension of the Cauchy-Riemann system, thereby encoding quasi conformal deformations and playing an important role in the Teichm*ü*ller theory, holomorphic dynamics, and physical field modeling [1,2]. For $|\mu| < 1$, the equation $\partial_{\bar{z}} f = \mu \partial_z f$ admits an existence theory. However, the explicit integral representations are rarely available. This state of affairs naturally prompts the search for special subclasses-those whose structure is rigid enough to yield explicit Cauchy kernels, yet is flexible enough to exhibit genuinely non-analytic behaviors.

One such subclass arises when the Beltrami coefficient takes the special form $\mu(z) = \beta \frac{z}{\bar{z}}$ in which $0 \leq \beta < 1$. Tungatarov first isolated this case and studied it systematically in the mid-1980s [3], that is, the continuously differentiable complex-valued functions satisfying

$$(\partial_{\bar{z}}^{\beta})f(z) := (\partial_{\bar{z}} - \beta\frac{z}{\bar{z}}\partial_z)f(z) = 0,$$

which is called β-analytic functions. The parameter β acts as a geometric distortion coefficient that adjusts the coupling intensity between the differential operators ∂_z and $\partial_{\bar{z}}$. If we set $\beta = 0$, then the operator equation degenerates exactly to the Cauchy-Riemann system. β-analytic functions have properties that parallel classical analytic functions in many essential respects. The corresponding Cauchy-type kernel is

$$\frac{1}{\zeta - z|z/\zeta|^{\theta}} \ \ (\theta = \frac{2\beta}{1-\beta}),$$

from which follow a β-version of the Cauchy integral, a Borel-Pompeiu formula, and Plemelj-Sokhotski relations. The mapping $\widetilde{z} = z|z|^{\theta}$ is itself β-analytic.

In 2006, Blaya, Reyes, and Pe*ñ*a [4] solved jump problems over Carleson rectifiable curves, proved the β-version Dolzhenko removable singularity theorem and the generalized Liouville theorem, and established uniqueness criteria for H*ö*lder-class solutions of homogeneous jump equations. In 2008, they systematically investigated general β-Riemann BVPs on regular curves, derived the solvability conditions, and converted associated singular integral equations into equivalent Riemann problems [5]. In 2010, Blaya, Reyes, and Vilaire [6] discussed the standard jump Riemann boundary value problem for β-analytic functions over a fractal curve. They obtained the solution to the problem and the solvability conditions. In 2012, they extended the framework to d-summable closed curves, and obtained the solvable conditions [7].

More recent work has pushed the theory of β-analytic functions in new directions. Guo and his co-authors [8–10] introduced Carleman shifts into the β-analytic Riemann-Hilbert problems. They obtained the Fredholm integral equations for the shifted boundary values and explicit solutions via decomposition into β-analytic components. This extends the classical Haseman problem and opens up the study of boundary value problems on domains with symmetry or with non-local boundary conditions. At the same time, Bory-Reyes, Barseghyan, and Schneider [1] developed a Sobolev space approach to Dirichlet type problems for β-analytic equations using orthogonal decompositions and the β-Bergman projection. Their results offer a functional analytic perspective that complements the classical H*ö*lder theory, and they also treated second-order problems involving iterated β-operators. Kats and Kats [11,12] considered generalized Beltrami coefficients of the following form:

$$\mu(z) = \beta\frac{\rho(z)}{\bar{\rho}(z)}\frac{\overline{\rho'(z)}}{\rho'(z)},$$

where ρ is analytic, and they showed that $\rho|\rho|^{2\alpha}$ solves the corresponding equation, thereby extending the β-analytic structure via conformal change of variables. Dai, Reyes, and He [13] derived the explicit solutions together with their solvability conditions for the identical-factor Riemann boundary value problem concerning poly-β-analytic and meta-β-analytic functions on d-summable closed Jordan curves.

The β-analytic function theory lies at the intersection of classical complex analyses, quasi conformal mappings, planar elasticity, and fluid models. Its explicit integral representations make it an unusually accessible testing ground for deep questions about boundary regularity, removable singularities, and the index of singular integral operators. Moreover, β-analytic functions may be used to study the boundary value problems associated with certain fractional-order differential equations. Along with the continuous development of fractional-order differential equations [14–16] and the broad applicability of partial differential equations across fluid mechanics, structural mechanics, image processing, and nonlinear dynamics [17–22], boundary value problems related to the β-analytic function have emerged as classic, powerful tools to resolve two-dimensional field equations. Although numerous elegant results concerning β-analytic functions have been established, there exist far fewer conclusions for their associated boundary value problems.

This paper aims to explore several fundamental boundary value problems for β-analytic functions on the unit disc. It intends to derive solutions and solvability conditions for their Dirichlet boundary value problems, further investigate the Neumann and half-Neumann boundary value problems related to such functions, and finally obtain the explicit solutions and complete solvability criteria for the Rubin boundary value problems of β-analytic functions.

The outline of the article is as follows: Section 2 presents the fundamental definitions and essential lemmas that serve as the theoretical basis for the subsequent research; Section 3 first elaborates on several basic properties of β-analytic functions, and further investigates the corresponding Dirichlet boundary value problems of such functions on the unit disc; in Section 4, the Neumann and half-Neumann boundary value problems for β-analytic functions are systematically explored; and Section 5 focuses on the research of Rubin boundary value problems for β-analytic functions defined on the unit disc.

Throughout this paper, $H^{\mu}(\overline{G})$ represents the set of Hölder continuous functions with an exponent μ $(0 < \mu \leq 1)$ on G. $C^{(n)}(G)$ represents the set of all complex valued functions on G that have continuous partial derivatives up to n-order, and $\widetilde{z} = z|z|^{\theta}$ with $\theta = \frac{2\beta}{1-\beta}$ and $\beta \in [0, 1)$.

2. Some definitions and lemmas

To get the main results, we need the following definitions and lemmas.

Definition 2.1. *[3] Let Ω be an open set of $\mathbb{C}$, and let $f(z)$ be a continuously differentiable complex-valued function on Ω. $f(z)$ is called a β-analytic function on Ω if it is the solution of the Beltrami equation*

$$(\partial_{\bar{z}}^{\beta})f(z) := (\partial_{\bar{z}} - \beta\frac{z}{\bar{z}}\partial_z)f(z) = 0, \quad z \in \Omega, \quad 0 \leq \beta < 1,$$

where

$$\partial_{\bar{z}} = \frac{1}{2}(\partial x + i\partial y), \quad \partial_z = \frac{1}{2}(\partial x - i\partial y).$$

Obviously, $z|z|^{\theta}$ is an example of β-analytic functions where $\theta = \frac{2\beta}{1-\beta}$. For $\beta = 0$, β-analytic functions are well-known analytic functions.

Lemma 2.2. *[5] Let Γ be a rectifiable positively oriented Jordan closed curve, which divides the entire complex plane into a bounded connected region G^+ containing the origin and an unbounded region G^- containing ∞. Let F being β-analytic in G^+ and continuous on $\overline{G^+}$ with $\beta \in [0, 1)$; then,*

$$\frac{1}{2(1-\beta)\pi i}\int_{\Gamma}\frac{F(\zeta)d\zeta}{\zeta - z|z/\zeta|^{\theta}} + \frac{\beta}{2(1-\beta)\pi i}\int_{\Gamma}\frac{\zeta F(\zeta)d\bar{\zeta}}{\bar{\zeta}(\zeta - z|z/\zeta|^{\theta})} = \begin{cases} F(z), & z \in G^+, \\ 0, & z \in G^-, \end{cases}$$

where $\theta = \frac{2\beta}{1-\beta}$.

Lemma 2.3. *[5] Let* Γ *be a rectifiable positively oriented Jordan closed curve, and let* $f \in H^{\mu}(\Gamma)$*; then,*

$$(C^{\beta}_{\Gamma}f)^{\pm}(t) = \frac{1}{2}[(S^{\beta}_{\Gamma}f)(t) \pm f(t)], \quad t \in \Gamma,$$

where $(C^{\beta}_{\Gamma}f)(z)$ *is the* β*-Cauchy type integral:*

$$(C^{\beta}_{\Gamma}f)(z) = \frac{1}{2(1-\beta)\pi i}\int_{\Gamma}\frac{f(\zeta)d\zeta}{\zeta - z|z/\zeta|^{\theta}} + \frac{\beta}{2(1-\beta)\pi i}\int_{\Gamma}\frac{\zeta f(\zeta)d\bar{\zeta}}{\bar{\zeta}(\zeta - z|z/\zeta|^{\theta})}, \quad z \notin \Gamma,$$

and $(S^{\beta}_{\Gamma}f)(t)$ *is the* β*-Cauchy singular integral:*

$$(S^{\beta}_{\Gamma}f)(t) = \frac{1}{(1-\beta)\pi i}\int_{\Gamma}\frac{[f(\zeta) - f(t)]d\zeta}{\zeta - z|z/\zeta|^{\theta}} + \frac{\beta}{(1-\beta)\pi i}\int_{\Gamma}\frac{\zeta[f(\zeta) - f(t)]d\bar{\zeta}}{\bar{\zeta}(\zeta - z|z/\zeta|^{\theta})} + f(t), \quad t \in \Gamma.$$

Obviously, $(C^{\beta}_{\Gamma}f)(z)$ is β-analytic in $\mathbb{C} \setminus \Gamma$, and $(C^{\beta}_{\Gamma}f)(\infty) = 0$. Setting $\widetilde{z} = z|z|^{\theta}$, we get that

$$d\widetilde{\zeta} = \frac{|\zeta|^{\theta}}{1-\beta}d\zeta + \frac{\beta}{1-\beta}|\zeta|^{\theta}\frac{\zeta}{\bar{\zeta}}d\bar{\zeta},$$

and

$$(C^{\beta}_{\Gamma}f)(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\widetilde{\zeta} - \widetilde{z}}d\widetilde{\zeta}, \quad (S^{\beta}_{\Gamma}f)(t) = \frac{1}{\pi i}\int_{\Gamma}\frac{f(\zeta) - f(t)}{\widetilde{\zeta} - \widetilde{t}}d\widetilde{\zeta} + f(t).$$

Lemma 2.4. *[5] Let G be a bounded domain in* $\mathbb{C}$*, whose boundary is a rectifiable positively oriented Jordan closed curve* Γ*. Let* $f \in C^{(1)}(G) \bigcap C(\overline{G})$*; then,*

$$\frac{1}{2(1-\beta)\pi i}\int_{\Gamma}\frac{f(\zeta)d\zeta}{\zeta - z|z/\zeta|^{\theta}} + \frac{\beta}{2(1-\beta)\pi i}\int_{\Gamma}\frac{\zeta f(\zeta)d\bar{\zeta}}{\bar{\zeta}(\zeta - z|z/\zeta|^{\theta})} - \frac{1}{(1-\beta)\pi}\int_{G}\frac{\partial^{\beta}_{\bar{\zeta}}f(\zeta)d\sigma_{\zeta}}{\zeta - z|z/\zeta|^{\theta}}$$
$$= \begin{cases} f(z), & z \in G, \\ 0, & z \in \mathbb{C} \setminus \overline{G}, \end{cases}$$

where $\theta = \frac{2\beta}{1-\beta}$, $\beta \in [0, 1)$*, and* $\partial^{\beta}_{\bar{\zeta}}f(\zeta) = (\partial_{\bar{\zeta}} - \beta\frac{\zeta}{\bar{\zeta}}\partial_{\zeta})f(\zeta)$.

By applying Lemma 2.4, we can get the following conclusion.

Lemma 2.5. *Let D be the unit disc in* $\mathbb{C}$*, whose boundary is L. Let* $f \in C^{(1)}(D) \bigcap C(\overline{D})$*; then,*

$$\frac{1}{2\pi i}\int_{L}\frac{f(\zeta)d\zeta}{\zeta - z|z|^{\theta}} - \frac{1}{(1-\beta)\pi}\int_{D}\frac{\partial^{\beta}_{\bar{\zeta}}f(\zeta)d\sigma_{\zeta}}{\zeta - z|z/\zeta|^{\theta}} = \begin{cases} f(z), & z \in D, \\ 0, & z \in \mathbb{C} \setminus \overline{D}, \end{cases} \tag{2.1}$$

and

$$\frac{-1}{2\pi i}\int_{L}\frac{f(\zeta)d\bar{\zeta}}{\bar{\zeta} - \bar{z}|z|^{\theta}} - \frac{1}{(1-\beta)\pi}\int_{D}\frac{\partial^{\beta}_{\zeta}f(\zeta)d\sigma_{\zeta}}{\bar{\zeta} - \bar{z}|\bar{z}/\bar{\zeta}|^{\theta}} = \begin{cases} f(z), & z \in D, \\ 0, & z \in \mathbb{C} \setminus \overline{D}, \end{cases} \tag{2.2}$$

where $\theta = \frac{2\beta}{1-\beta}$, $\beta \in [0, 1)$*, and*

$$\partial^{\beta}_{\bar{\zeta}}f(\zeta) = (\partial_{\bar{\zeta}} - \beta\frac{\zeta}{\bar{\zeta}}\partial_{\zeta})f(\zeta), \quad \partial^{\beta}_{\zeta}f(\zeta) = (\partial_{\zeta} - \beta\frac{\bar{\zeta}}{\zeta}\partial_{\bar{\zeta}})f(\zeta).$$

Proof. For $\zeta \in L$,

$$\frac{d\zeta}{1-\beta} + \frac{\beta}{1-\beta}\frac{\zeta d\overline{\zeta}}{\overline{\zeta}} = d\zeta.$$

Then, (2.1) holds by Lemma 2.4. Taking the complex conjugate of both sides of Eq (2.1) and substituting f for $\bar{f}$ yields Eq (2.2). □

Lemma 2.6. *[23] Let $w \in C^{(1)}(G;\mathbb{C}) \bigcap C(\overline{G};\mathbb{C})$, where G is a regular domain in the complex plane; then,*

$$\int_G w_{\bar{z}}(z)d\sigma_z = \frac{1}{2i}\int_{\partial G} w(z)dz, \int_G w_z(z)d\sigma_z = -\frac{1}{2i}\int_{\partial G} w(z)d\bar{z}, \tag{2.3}$$

and

$$w(z) = \frac{1}{2\pi i}\int_{\partial G} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_G w_{\bar{\zeta}}(\zeta)\frac{d\sigma_\zeta}{\zeta - z}, \tag{2.4}$$

$$w(z) = \frac{-1}{2\pi i}\int_{\partial G} w(\zeta)\frac{d\overline{\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int_G w_{\zeta}(\zeta)\frac{d\sigma_\zeta}{\overline{\zeta - z}}. \tag{2.5}$$

Lemma 2.7. *Let G be a bounded smooth domain in the complex plane, and let $g \in L_1(G;\mathbb{C})$ and*

$$T_\beta g(z) = \frac{-1}{(1-\beta)\pi}\int_G \frac{|\zeta|^\theta g(\zeta)}{\zeta|\zeta|^\theta - z|z|^\theta}d\sigma_\zeta. \tag{2.6}$$

Then, for all $\varphi \in C^{(1)}(G;\mathbb{C})$ with $\varphi(\zeta) = 0$ ($\zeta \in \partial G$),

$$\int_G g(z)|z|^\theta\varphi(z)d\sigma_z + \int_G T_\beta g(z)|z|^\theta\partial^\beta_{\bar{z}}\varphi(z)d\sigma_z = 0 \tag{2.7}$$

and

$$\partial^\beta_{\bar{z}}T_\beta g(z) = g(z), \tag{2.8}$$

where $\theta = \frac{2\beta}{1-\beta}$, $\beta \in [0, 1)$ and

$$\partial^\beta_{\bar{z}}\varphi(z) = (\partial_{\bar{z}} - \beta\frac{z}{\bar{z}}\partial_z)\varphi(z).$$

Proof. As $\theta = \frac{2\beta}{1-\beta}$, then

$$\partial^\beta_{\bar{z}}|z|^\theta = (\partial_{\bar{z}} - \beta\frac{z}{\bar{z}}\partial_z)(z^{\frac{\theta}{2}}\bar{z}^{\frac{\theta}{2}}) = \frac{\theta}{2}z^{\frac{\theta}{2}}\bar{z}^{\frac{\theta}{2}-1} - \beta\frac{z}{\bar{z}}\cdot\frac{\theta}{2}z^{\frac{\theta}{2}-1}\bar{z}^{\frac{\theta}{2}} = \beta\frac{|z|^\theta}{\bar{z}} = |z|^\theta\partial_z\frac{\beta z}{\bar{z}}.$$

Thus,

$$\partial^\beta_{\bar{z}}[T_\beta g(z)|z|^\theta\varphi(z)] = \partial^\beta_{\bar{z}}(T_\beta g(z))|z|^\theta\varphi(z) + T_\beta g(z)|z|^\theta\partial^\beta_{\bar{z}}\varphi(z) + T_\beta g(z)\varphi(z)|z|^\theta\partial_z\frac{\beta z}{\bar{z}}.$$

Therefore, by applying the Gauss formula (2.3) and the Pompeiu formula (2.5) in Lemma 2.6, we get that

$$\begin{aligned}
&\int_G \partial^\beta_{\bar{z}}(T_\beta g(z))|z|^\theta\varphi(z)d\sigma_z + \int_G T_\beta g(z)|z|^\theta\partial^\beta_{\bar{z}}\varphi(z)d\sigma_z \\
&= \int_G \partial^\beta_{\bar{z}}[T_\beta g(z)|z|^\theta\varphi(z)]d\sigma_z - \int_G T_\beta g(z)\varphi(z)|z|^\theta\partial_z\frac{\beta z}{\bar{z}}d\sigma_z \\
&= \int_G \partial_{\bar{z}}[T_\beta g(z)|z|^\theta\varphi(z)]d\sigma_z - \int_G \beta\frac{z}{\bar{z}}\partial_z[T_\beta g(z)|z|^\theta\varphi(z)]d\sigma_z - \int_G T_\beta g(z)\varphi(z)|z|^\theta\partial_z\frac{\beta z}{\bar{z}}d\sigma_z \\
&= -\int_G \partial_z[T_\beta g(z)|z|^\theta\varphi(z)\beta z]\frac{d\sigma_z}{\overline{z-0}} = 0.
\end{aligned} \tag{2.9}$$

As $\varphi(\zeta) = 0$ ($\zeta \in \partial G$), Lemma 2.4 yields that

$$\varphi(z) = \frac{-1}{(1-\beta)\pi}\int_G \frac{\partial^\beta_{\bar\zeta}\varphi(\zeta)d\sigma_\zeta}{\zeta - z|z/\zeta|^\theta} = \frac{-1}{(1-\beta)\pi}\int_G \frac{|\zeta|^\theta\partial^\beta_{\bar\zeta}\varphi(\zeta)d\sigma_\zeta}{\zeta|\zeta|^\theta - z|z|^\theta}. \tag{2.10}$$

(2.6) and (2.10) lead to the following:

$$\begin{aligned}\int_G T_\beta g(z)|z|^\theta\partial^\beta_{\bar z}\varphi(z)d\sigma_z &= \int_G\Big[\frac{-1}{(1-\beta)\pi}\int_G\frac{|\zeta|^\theta g(\zeta)}{\zeta|\zeta|^\theta - z|z|^\theta}d\xi d\eta\Big]|z|^\theta\partial^\beta_{\bar z}\varphi(z)d\sigma_z\\ &= \int_G|\zeta|^\theta g(\zeta)\Big[\frac{-1}{(1-\beta)\pi}\int_G\partial^\beta_{\bar z}\varphi(z)\frac{|z|^\theta d\sigma_z}{\zeta|\zeta|^\theta - z|z|^\theta}\Big]d\sigma_\zeta\\ &= -\int_G|\zeta|^\theta g(\zeta)\varphi(\zeta)d\sigma_\zeta,\end{aligned}$$

which follows (2.7). Comparing (2.7) and (2.9), we obtain the following:

$$\int_G g(z)|z|^\theta\varphi(z)d\sigma_z = \int_G \partial^\beta_{\bar z}(T_\beta g(z))|z|^\theta\varphi(z)d\sigma_z.$$

Since φ is arbitrary, (2.8) follows. □

3. Dirichlet problems

In this section, we discuss the Dirichlet problems for β-analytic functions on the unit disc D. Let L be the boundary of D. Before that, we first discuss several properties of β-analytic functions.

Theorem 3.1. *$f(z)$ being β-analytic on $K = \{z : |z|z|^\theta - \widetilde{a}| < R\}$ is equivalent to $\widetilde{f}(\widetilde{z})$ being analytic on $\widetilde{K} = \{\widetilde{z} : |\widetilde{z} - \widetilde{a}| < R\}$, where $\widetilde{z} = z|z|^\theta$ ($\theta = \frac{2\beta}{1-\beta}$, $0 \le \beta < 1$) and $\widetilde{f}(\widetilde{z}) = f(z)$.*

Proof. (i) If $\widetilde{f}(\widetilde{z})$ is analytic on $\widetilde{K} = \{\widetilde{z} : |\widetilde{z} - \widetilde{a}| < R\}$, then $\partial_{\bar{\widetilde{z}}}\widetilde{f}(\widetilde{z}) = 0$. Thus,

$$\partial^\beta_{\bar z}f(z) = \partial^\beta_{\bar z}\widetilde{f}(\widetilde{z}) = \partial_{\widetilde{z}}\widetilde{f}(\widetilde{z})\cdot\partial^\beta_{\bar z}\widetilde{z} + \partial_{\bar{\widetilde{z}}}\widetilde{f}(\widetilde{z})\cdot\partial^\beta_{\bar z}\bar{\widetilde{z}} = 0.$$

Therefore, $f(z)$ is β-analytic on K.

(ii) If $f(z)$ is β-analytic on K, then $\partial_{\bar z}f(z) = \frac{\beta z}{\bar z}\partial_z f(z)$. Therefore,

$$\frac{\partial\widetilde{f}(\widetilde{z})}{\partial\bar{\widetilde{z}}} = \frac{\partial f(z)}{\partial\bar{\widetilde{z}}} = \frac{\partial f(z)}{\partial z}\cdot\frac{\partial z}{\partial\bar{\widetilde{z}}} + \frac{\partial f(z)}{\partial\bar z}\cdot\frac{\partial\bar z}{\partial\bar{\widetilde{z}}} = \frac{\partial f(z)}{\partial z}\Big[\frac{\partial z}{\partial\bar{\widetilde{z}}} + \frac{\beta z}{\bar z}\frac{\partial\bar z}{\partial\bar{\widetilde{z}}}\Big]. \tag{3.1}$$

From

$$\widetilde{z} = z|z|^\theta = z^{\frac{\theta}{2}+1}\bar z^{\frac{\theta}{2}}, \quad \bar{\widetilde{z}} = z^{\frac{\theta}{2}}\bar z^{\frac{\theta}{2}+1}$$

we can get that

$$z = \widetilde{z}^{\frac{\theta+2}{2(\theta+1)}}\bar{\widetilde{z}}^{\frac{-\theta}{2(\theta+1)}}, \quad \bar z = \widetilde{z}^{\frac{-\theta}{2(\theta+1)}}\bar{\widetilde{z}}^{\frac{\theta+2}{2(\theta+1)}}. \tag{3.2}$$

Plugging (3.2) into (3.1), we obtain that

$$
\begin{aligned}
\frac{\partial \widetilde{f}(\widetilde{z})}{\partial \overline{\widetilde{z}}} &= \frac{\partial f(z)}{\partial z}\Big[\frac{\partial z}{\partial \overline{\widetilde{z}}} + \frac{\beta z}{\bar{z}}\frac{\partial \bar{z}}{\partial \overline{\widetilde{z}}}\Big] \\
&= \frac{\partial f(z)}{\partial z}\Big[\widetilde{z}^{\frac{\theta+2}{2(\theta+1)}}\overline{\widetilde{z}}^{\frac{-\theta}{2(\theta+1)}-1} \cdot \frac{-\theta}{2(\theta+1)} + \beta \widetilde{z}^{\frac{2\theta+2}{2(\theta+1)}}\overline{\widetilde{z}}^{\frac{-2\theta-2}{2(\theta+1)}} \cdot \widetilde{z}^{\frac{-\theta}{2(\theta+1)}}\overline{\widetilde{z}}^{\frac{\theta+2}{2(\theta+1)}-1} \cdot \frac{\theta+2}{2(\theta+1)}\Big] \\
&= \frac{\partial f(z)}{\partial z}\widetilde{z}^{\frac{\theta+2}{2(\theta+1)}}\overline{\widetilde{z}}^{\frac{-\theta}{2(\theta+1)}-1}\Big[\frac{-\theta}{2(\theta+1)} + \beta\frac{\theta+2}{2(\theta+1)}\Big] \\
&= 0,
\end{aligned}
$$

which means that $\widetilde{f}(\widetilde{z})$ is analytic on $\widetilde{K}$.

The above (i) and (ii) indicate that $f(z)$ being β-analytic on K is equivalent to $\widetilde{f}(\widetilde{z})$ being analytic on $\widetilde{K}$. □

Theorem 3.2. *Let $f(z)$ be a β-analytic function on the domain G, and let $|f(z)| = C$, where $0 \leq \beta < 1$ and C is a constant. Then, $f(z)$ is a constant.*

Proof. Let $f(z) = u(x, y) + iv(x, y)$. Since $f(z)$ is β-analytic on G, then

$$(\partial_{\bar{z}} - \beta\frac{z}{\bar{z}}\partial_z)f(z) = 0;$$

thus,

$$\bar{z}(\partial x + i\partial y)(u + iv) - \beta z(\partial x - i\partial y)(u + iv) = 0,$$

which leads to

$$\begin{cases} (1-\beta)(xu_x + yu_y) + (1+\beta)(yv_x - xv_y) = 0, \\ (1+\beta)(yu_x - xu_y) + (\beta - 1)(xv_x + yv_y) = 0. \end{cases} \tag{3.3}$$

As $|f(z)| = C$, then $u^2 + v^2 = C$, which follows that

$$v_x = \frac{-u}{v}u_x, \quad v_y = \frac{-u}{v}u_y. \tag{3.4}$$

Plugging (3.4) into (3.3), we get that

$$\begin{cases} [(1-\beta)vx - (1+\beta)uy]u_x + [(1-\beta)vy + (1+\beta)ux]u_y = 0, \\ (1+\beta)vyu_x + (1-\beta)uxu_x + [(1-\beta)uy - (1+\beta)vx]u_y = 0; \end{cases}$$

therefore,

$$u_x = u_y = v_x = v_y = 0.$$

Thus, $f(z)$ is a constant. □

Theorem 3.3. *Let G being a bounded smooth domain in the complex plane, and $f(z)$ being β-analytic on G with $0 \leq \beta < 1$. Then, for $K = \{z : |\widetilde{z} - \widetilde{a}| < R\} \subset G$ with $\widetilde{a} \in D$, $f(z)$ can be expanded into*

$$f(z) = \sum_{n=0}^{\infty} c_n(\widetilde{z} - \widetilde{a})^n$$

in K, where

$$c_n = \frac{1}{2\pi i}\int_{\widetilde{\Gamma_\rho}} \frac{f(\zeta)}{(\widetilde{\zeta}-\widetilde{a})^{n+1}} d\widetilde{\zeta} \ \ (n=0,1,\cdots),$$

in which $\widetilde{\Gamma_\rho} = \{\widetilde{\zeta} : |\widetilde{\zeta}-\widetilde{a}| = \rho\}, \ \ \rho \in (0,R).$

Proof. For $z \in K, \widetilde{z}$ is inside $\widetilde{K} = \{\widetilde{z} : |\widetilde{z}-\widetilde{a}| < R\}$. Then, there exists ρ ($\rho \in (0,R)$) such that $\widetilde{z}$ is inside $\{\widetilde{z} : |\widetilde{z}-\widetilde{a}| < \rho\}$, which means that z is inside $\{z : |z|z|^\theta - \widetilde{a}| < \rho\}$. Let

$$\Gamma_\rho = \{\zeta : |\zeta|\zeta|^\theta - \widetilde{a}| = \rho\}, \quad \widetilde{\Gamma_\rho} = \{\widetilde{\zeta} : |\widetilde{\zeta}-\widetilde{a}| = \rho\}.$$

By Lemma 2.2, we get that

$$\begin{aligned} f(z) &= \frac{1}{2(1-\beta)\pi i}\int_{\Gamma_\rho}\frac{f(\zeta)d\zeta}{\zeta - z|z/\zeta|^\theta} + \frac{\beta}{2(1-\beta)\pi i}\int_{\Gamma_\rho}\frac{\zeta f(\zeta)d\bar{\zeta}}{\bar{\zeta}(\zeta - z|z/\zeta|^\theta)} \\ &= \frac{1}{2\pi i}\int_{\widetilde{\Gamma_\rho}}\frac{f(\zeta)}{\widetilde{\zeta}-\widetilde{z}}d\widetilde{\zeta}, \end{aligned}$$

where

$$d\widetilde{\zeta} = \frac{|\zeta|^\theta}{1-\beta}d\zeta + \frac{\beta}{1-\beta}|\zeta|^\theta\frac{\zeta}{\bar{\zeta}}d\bar{\zeta}.$$

Since $\widetilde{z}$ is inside $\{\widetilde{z} : |\widetilde{z}-\widetilde{a}| < \rho\}$, then $|\frac{\widetilde{z}-\widetilde{a}}{\widetilde{\zeta}-\widetilde{a}}| < 1$. Thus,

$$\frac{f(\zeta)}{\widetilde{\zeta}-\widetilde{z}} = \frac{f(\zeta)}{\widetilde{\zeta}-\widetilde{a}}\cdot\frac{1}{1-\frac{\widetilde{z}-\widetilde{a}}{\widetilde{\zeta}-\widetilde{a}}} = \frac{f(\zeta)}{\widetilde{\zeta}-\widetilde{a}}\sum_{n=0}^{\infty}\Big(\frac{\widetilde{z}-\widetilde{a}}{\widetilde{\zeta}-\widetilde{a}}\Big)^n.$$

The boundedness of $\dfrac{f(\zeta)}{\widetilde{\zeta}-\widetilde{a}}$ and the uniform convergence of $\displaystyle\sum_{n=0}^{\infty}\Big(\frac{\widetilde{z}-\widetilde{a}}{\widetilde{\zeta}-\widetilde{a}}\Big)^n$ on $\widetilde{\Gamma_\rho}$ follow that $\displaystyle\frac{f(\zeta)}{\widetilde{\zeta}-\widetilde{a}}\sum_{n=0}^{\infty}\Big(\frac{\widetilde{z}-\widetilde{a}}{\widetilde{\zeta}-\widetilde{a}}\Big)^n$ being uniformly convergent on $\widetilde{\Gamma_\rho}$. Therefore,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\widetilde{\Gamma_\rho}}\frac{f(\zeta)}{\widetilde{\zeta}-\widetilde{z}}d\widetilde{\zeta} = \frac{1}{2\pi i}\int_{\widetilde{\Gamma_\rho}}\frac{f(\zeta)}{\widetilde{\zeta}-\widetilde{a}}\sum_{n=0}^{\infty}\Big(\frac{\widetilde{z}-\widetilde{a}}{\widetilde{\zeta}-\widetilde{a}}\Big)^n d\widetilde{\zeta} \\ &= \sum_{n=0}^{\infty}(\widetilde{z}-\widetilde{a})^n\cdot\frac{1}{2\pi i}\int_{\widetilde{\Gamma_\rho}}\frac{f(\zeta)}{(\widetilde{\zeta}-\widetilde{a})^{n+1}}d\widetilde{\zeta} \\ &= \sum_{n=0}^{\infty}c_n(\widetilde{z}-\widetilde{a})^n, \end{aligned}$$

where

$$c_n = \frac{1}{2\pi i}\int_{\widetilde{\Gamma_\rho}} \frac{f(\zeta)}{(\widetilde{\zeta}-\widetilde{a})^{n+1}} d\widetilde{\zeta} \ \ (n=0,1,\cdots).$$

□

Theorem 3.4. *Let $f(z)$ being β-analytic on $K = \{z : |\widetilde{z}-\widetilde{a}| < R\}$ with $0 \le \beta < 1$. Assuming that there exists a column of zeros $\{z_n\}$ ($z_n \neq a$) of $f(z)$ inside K converges to a, then $f(z) \equiv 0$ ($z \in K$).*

Proof. Let $\widetilde{f(\widetilde{z})} = f(z)$ in which $\widetilde{z} = z|z|^{\theta}$ ($\theta = \frac{2\beta}{1-\beta}$). By Theorem 3.1, $f(z)$ being β-analytic on K is equivalent to $\widetilde{f(\widetilde{z})}$ being analytic on $\widetilde{K} = \{\widetilde{z} : |\widetilde{z} - \widetilde{a}| < R\}$. The column of zeros $\{z_n\}$ of $f(z)$ inside K converging to a means that the column of zeros $\{\widetilde{z_n}\}$ ($\widetilde{z_n} \neq \widetilde{a}$) of $\widetilde{f(\widetilde{z})}$ inside $\widetilde{K}$ converge to $\widetilde{a}$. By applying the isolation of zero points for analytic functions, we get $\widetilde{f(\widetilde{z})} \equiv 0$ ($z \in \widetilde{K}$), so $f(z) \equiv 0$ ($z \in K$). □

By Theorem 3.4, we can get the following condition.

Corollary 3.5. *Let $f_1(z)$ and $f_2(z)$ be β-analytic on the domain G with $0 \leq \beta < 1$. Assuming that there exists a point column $\{z_n\}$ ($z_n \neq a$) in G which converges to a such that $f_1(z_n) = f_2(z_n)$, then $f_1(z) \equiv f_2(z)$ ($z \in G$).*

Theorem 3.6. *Let $f(z)$ be β-analytic on the domain G with $0 \leq \beta < 1$, then $|f(z)|$ cannot reach its maximum value at any point within G unless $f(z)$ is a constant.*

Proof. Suppose that $|f(z)| \leq M$, and there exists $\widetilde{z_0} \in G$ such that $|f(\widetilde{z_0})| = M$. Let $\{z : |z|z|^{\theta} - \widetilde{z_0}| \leq R\} \subset G$, and let $\widetilde{\Gamma} : |\widetilde{\zeta} - \widetilde{z_0}| = R$. By Lemma 2.2,

$$f(z) = \frac{1}{2\pi i}\int_{\widetilde{\Gamma}} \frac{f(\zeta)}{\widetilde{\zeta} - \widetilde{z}} d\widetilde{\zeta} = \frac{1}{2\pi i}\int_{\widetilde{\Gamma}} \frac{\widetilde{f(\widetilde{\zeta})}}{\widetilde{\zeta} - \widetilde{z}} d\widetilde{\zeta},$$

where $\widetilde{f(\widetilde{\zeta})} = f(\zeta)$ and

$$d\widetilde{\zeta} = \frac{|\zeta|^{\theta}}{1-\beta} d\zeta + \frac{\beta}{1-\beta}|\zeta|^{\theta}\frac{\zeta}{\bar{\zeta}} d\bar{\zeta}.$$

For $\widetilde{\zeta} \in \widetilde{\Gamma}$, let $\widetilde{\zeta} = \widetilde{z_0} + Re^{i\alpha}$ ($\alpha \in [0, 2\pi]$). Therefore,

$$f(\widetilde{z_0}) = \frac{1}{2\pi i}\int_{\widetilde{\Gamma}} \frac{\widetilde{f(\widetilde{\zeta})}}{Re^{i\alpha}} d(Re^{i\alpha}) = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{f}(\widetilde{z_0} + Re^{i\alpha}) d\alpha,$$

which follows that

$$M = |f(\widetilde{z_0})| \leq \frac{1}{2\pi}\int_0^{2\pi} |\widetilde{f}(\widetilde{z_0} + Re^{i\alpha})| d\alpha = \frac{1}{2\pi}\int_0^{2\pi} |f(\zeta)| d\alpha \leq M.$$

Thus, $|f(z)| = M$ for $|z|z|^{\theta} - \widetilde{z_0}| \leq R$. By Theorem 3.2, $f(z)$ is a constant in $\{z : |z|z|^{\theta} - \widetilde{z_0}| \leq R\}$. By applying Corollary 3.5, $f(z)$ is a constant on G. □

In this section, we discuss the Dirichlet problem for β-analytic functions on the unit disc D whose boundary is L.

Theorem 3.7. *Let $\gamma \in C(\partial D, \mathbb{C})$ and $0 \leq \beta < 1$; then, the Dirichlet Problem D1*

$$\partial_{\bar{z}}^{\beta} f(z) = 0 \ \ (z \in D), \quad f(\zeta) = \gamma(\zeta) \ \ (\zeta \in L)$$

is solvable if and only if

$$\frac{1}{2\pi i}\int_L \frac{\bar{z}|z|^{\theta}\gamma(\zeta)d\zeta}{1 - \bar{z}|z|^{\theta}\zeta} = 0 \quad (|z| < 1), \tag{3.5}$$

and the solution is given by

$$f(z) = \frac{1}{2\pi i}\int_L \frac{\gamma(\zeta)d\zeta}{\zeta - z|z|^{\theta}}, \tag{3.6}$$

where $\theta = \frac{2\beta}{1-\beta}$.

Proof. (Necessity) Suppose that Problem D1 is solvable, and $f(z)$ is the solution. By applying Lemma 2.2, we get that

$$f(z) = \frac{1}{2(1-\beta)\pi i}\int_L \frac{f(\zeta)d\zeta}{\zeta - z|z/\zeta|^\theta} + \frac{\beta}{2(1-\beta)\pi i}\int_L \frac{\zeta f(\zeta)d\bar{\zeta}}{\bar{\zeta}(\zeta - z|z/\zeta|^\theta)} = \frac{1}{2\pi i}\int_L \frac{\gamma(\zeta)d\zeta}{\zeta - z|z|^\theta},$$

that is, $f(z)$ can be expressed as (3.6). Furthermore, it can be concluded that $f(\infty) = 0$ and $f(z)$ is β-analytic in $\mathbb{C} \setminus L$. In addition, for $|z| > 1$, we have $|\frac{1}{\bar{z}}| < 1$. By (3.6),

$$f(\frac{1}{\bar{z}}) = \frac{1}{2\pi i}\int_L \frac{\gamma(\zeta)d\zeta}{\zeta - \frac{1}{\bar{z}}|\frac{1}{\bar{z}}|^\theta} = \frac{1}{2\pi i}\int_L \frac{\bar{z}|z|^\theta\gamma(\zeta)}{\bar{z}|z|^\theta - \bar{\zeta}}\cdot\frac{d\zeta}{\zeta} = \frac{-1}{2\pi i}\int_L \frac{\overline{\overline{\widetilde{z}}}\gamma(\zeta)}{\overline{\zeta - \widetilde{z}}}\cdot\frac{d\zeta}{\zeta}, \tag{3.7}$$

where $\widetilde{z} = z|z|^\theta$ ($\theta = \frac{2\beta}{1-\beta}$). For $|z| > 1$ and $\zeta \in L$, we have that $\frac{1}{\bar{z}} \to \zeta$ if $z \to \zeta$. Thus, $\lim\limits_{z\to\zeta} f(\frac{1}{\bar{z}})$ exists, that is, $\lim\limits_{z\to\zeta} f(z)$ exists for $|z| > 1$. From

$$\begin{aligned} f(z) - f(\frac{1}{\bar{z}}) &= \frac{1}{2\pi i}\int_L \frac{\gamma(\zeta)d\zeta}{\zeta - z|z|^\theta} - \frac{-1}{2\pi i}\int_L \frac{\overline{\overline{\widetilde{z}}}\gamma(\zeta)}{\overline{\zeta - \widetilde{z}}}\cdot\frac{d\zeta}{\zeta} \\ &= \frac{1}{2\pi i}\int_L \gamma(\zeta)\Big(\frac{\zeta}{\zeta - \widetilde{z}} + \frac{\bar{\zeta}}{\overline{\zeta - \widetilde{z}}} - 1\Big)\frac{d\zeta}{\zeta} \end{aligned}$$

and the properties of the Poisson kernel on L, we get that

$$\lim_{z\to\zeta,|z|<1} f(z) - \lim_{z\to\zeta,|z|>1} f(z) = \gamma(\zeta).$$

f being the solution of the Problem D1 indicates that

$$\lim_{z\to\zeta,|z|<1} f(z) = \gamma(\zeta).$$

Thus,

$$\lim_{z\to\zeta,|z|>1} f(z) = 0.$$

As $f(\infty) = 0$, by applying Theorem 3.6, we get $f(z) \equiv 0$ for $|z| > 1$. Combining (3.7), we get (3.5).

(Sufficiency) Obviously, $f(z)$ expressed by (3.6) is β-analytic on D. Furthermore, (3.5) and (3.6) derive that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_L \frac{\gamma(\zeta)d\zeta}{\zeta - z|z|^\theta} + \frac{1}{2\pi i}\int_L \frac{\bar{z}|z|^\theta\gamma(\zeta)d\zeta}{1 - \bar{z}|z|^\theta\zeta} \\ &= \frac{1}{2\pi i}\int_L \frac{\gamma(\zeta)\zeta}{\zeta - z|z|^\theta}\frac{d\zeta}{\zeta} + \frac{1}{2\pi i}\int_L \frac{\bar{z}|z|^\theta\gamma(\zeta)}{\bar{\zeta} - \bar{z}|z|^\theta}\frac{d\zeta}{\zeta} \\ &= \frac{1}{2\pi i}\int_L \gamma(\zeta)\Big(\frac{\zeta}{\zeta - \widetilde{z}} + \frac{\bar{\zeta}}{\overline{\zeta - \widetilde{z}}} - 1\Big)\frac{d\zeta}{\zeta}. \end{aligned}$$

By the properties of the Poisson kernel on L, we obtain that

$$\lim_{z\to\zeta,|z|<1} f(z) = \gamma(\zeta),$$

that is, $f(z)$ satisfies $f = \gamma$ on L. Therefore, (3.6) is the solution of Problem D1 if (3.5) holds. □

Theorem 3.8. *Let $\gamma \in C(\partial D, \mathbb{C})$, $g \in L_1(D, \mathbb{C})$, and $0 \le \beta < 1$; then, the inhomogeneous Dirichlet Problem D2*

$$\partial_{\bar z}^{\beta} f(z) = g(z) \ \ (z \in D), \quad f(\zeta) = \gamma(\zeta) \ \ (\zeta \in L) \tag{3.8}$$

is solvable if and only if

$$\frac{1}{2\pi i}\int_L \gamma(\zeta)\frac{\bar{\tilde{z}}d\zeta}{1-\bar{\tilde{z}}\zeta} = \frac{1}{(1-\beta)\pi}\int_D |\zeta|^{\theta} g(\zeta)\frac{\bar{\tilde{z}}d\sigma_{\zeta}}{1-\widetilde{\bar{\tilde{z}}\zeta}} \quad (|z|<1), \tag{3.9}$$

and the solution is given by

$$f(z) = \frac{1}{2\pi i}\int_L \frac{\gamma(\zeta)d\zeta}{\zeta - \tilde{z}} - \frac{1}{(1-\beta)\pi}\int_D \frac{g(\zeta)d\sigma_{\zeta}}{\zeta - z|z/\zeta|^{\theta}}, \tag{3.10}$$

where $(\partial_{\bar z}^{\beta}) f(z) = (\partial_{\bar z} - \beta\frac{z}{\bar z}\partial_z) f(z)$, $\theta = \frac{2\beta}{1-\beta}$.

Proof. (Necessity) Suppose that Problem D2 is solvable. As $f = \gamma$ on L, by Lemma 2.5, $f(z)$ can be expressed as (3.10). Now, we prove the uniqueness of the solution.

If both f_1 and f_2 are solutions to Problem D2, then it follows from (3.8) that

$$\partial_{\bar z}^{\beta}(f_1 - f_2)(z) = 0 \ \ (z \in D), \quad (f_1 - f_2)(\zeta) = 0 \ \ (\zeta \in L).$$

By applying Theorem 3.7, we get $f_1 - f_2 = 0 \ \ (z \in D)$, i.e., $f_1 = f_2 \ \ (z \in D)$, which demonstrates that the solution to the problem is unique.

In addition, by (3.8) and Lemma 2.7, we get $\partial_{\bar z}^{\beta} f(z) = g(z) = \partial_{\bar z}^{\beta} T_{\beta} g(z)$, which leads to $\partial_{\bar z}^{\beta}(f - T_{\beta}g)(z) = 0$. Moreover, from (3.8), we have $(f - T_{\beta}g)(\zeta) = \gamma(\zeta) - T_{\beta}g(\zeta)$. By Theorem 3.7,

$$\partial_{\bar z}^{\beta}(f - T_{\beta}g)(z) = 0, \quad (f - T_{\beta}g)(\zeta) = \gamma(\zeta) - T_{\beta}g(\zeta)$$

is solvable if and only if

$$\frac{1}{2\pi i}\int_L [\gamma(\zeta) - T_{\beta}g(\zeta)]\frac{\bar z|z|^{\theta}d\zeta}{1-\bar z|z|^{\theta}\zeta} = 0 \quad (|z|<1). \tag{3.11}$$

Furthermore, since

$$\begin{aligned}
\frac{-1}{2\pi i}\int_L T_{\beta}g(\zeta)\frac{\bar z|z|^{\theta}d\zeta}{1-\bar z|z|^{\theta}\zeta} &= \frac{1}{2\pi i}\int_L \Big[\frac{1}{(1-\beta)\pi}\int_D \frac{|\tau|^{\theta}g(\tau)}{\tau|\tau|^{\theta} - \zeta|\zeta|^{\theta}}d\sigma_{\tau}\Big]\frac{\bar z|z|^{\theta}d\zeta}{1-\bar z|z|^{\theta}\zeta} \\
&= \frac{1}{(1-\beta)\pi}\int_D |\tau|^{\theta}g(\tau)\Big[\frac{1}{2\pi i}\int_L \frac{\bar z|z|^{\theta}}{1-\bar z|z|^{\theta}\zeta}\frac{d\zeta}{\tau|\tau|^{\theta}-\zeta|\zeta|^{\theta}}\Big]d\sigma_{\tau} \\
&= \frac{1}{(1-\beta)\pi}\int_D |\tau|^{\theta}g(\tau)\Big[\frac{-1}{2\pi i}\int_L \frac{\bar z|z|^{\theta}}{1-\bar z|z|^{\theta}\zeta}\frac{d\zeta}{\zeta-\tau|\tau|^{\theta}}\Big]d\sigma_{\tau} \\
&= \frac{-1}{(1-\beta)\pi}\int_D |\tau|^{\theta}g(\tau)\frac{\bar z|z|^{\theta}d\sigma_{\tau}}{1-\tau|\tau|^{\theta}\bar z|z|^{\theta}\zeta},
\end{aligned} \tag{3.12}$$

combining (3.11) and (3.12), we arrive at (3.9).

(Sufficiency) As

$$\partial_{\bar z}^{\beta}\Big[\frac{1}{2\pi i}\int_L \frac{\gamma(\zeta)d\zeta}{\zeta - \tilde{z}}\Big] = 0,$$

then it follows from Lemma 2.7 that Eq (3.10) satisfies $\partial_{\bar{z}}^{\beta} f(z) = g(z)\ \ (z \in D)$. Now, we verify that Eq (3.10) satisfies the boundary condition.

For $|z| = 1$, we have $\widetilde{z} = z$ and $\frac{1}{\widetilde{\zeta}-\widetilde{z}} + \frac{\widetilde{\bar{z}}}{1-\widetilde{\bar{z}\zeta}} = 0$, which follows that

$$\frac{-1}{(1-\beta)\pi}\int_D |\zeta|^{\theta} g(\zeta)\Big(\frac{1}{\widetilde{\zeta}-\widetilde{z}} + \frac{\widetilde{\bar{z}}}{1-\widetilde{\bar{z}\zeta}}\Big)d\sigma_{\zeta} = 0. \tag{3.13}$$

Furthermore, when $|z| = 1$, we have that

$$\frac{1}{2\pi i}\int_L \gamma(\zeta)\Big(\frac{\zeta}{\zeta-\widetilde{z}} + \frac{\overline{\zeta}}{\overline{\zeta-\widetilde{z}}} - 1\Big)\frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\int_L \gamma(\zeta)\Big(\frac{\zeta}{\zeta-z} + \frac{\overline{\zeta}}{\overline{\zeta-z}} - 1\Big)\frac{d\zeta}{\zeta} = \gamma(z). \tag{3.14}$$

Combining (3.9) and (3.10), we obtain that

$$f(z) = \frac{1}{2\pi i}\int_L \gamma(\zeta)\Big(\frac{1}{\zeta-\widetilde{z}} + \frac{\bar{\bar{\widetilde{z}}}}{1-\bar{\bar{\widetilde{z}}}\zeta}\Big)d\zeta - \frac{1}{(1-\beta)\pi}\int_D |\zeta|^{\theta} g(\zeta)\Big(\frac{1}{\widetilde{\zeta}-\widetilde{z}} + \frac{\bar{\bar{\widetilde{z}}}}{1-\widetilde{\bar{\bar{z}}\zeta}}\Big)d\sigma_{\zeta}. \tag{3.15}$$

Thus, it follows from (3.13)–(3.15) that for $|z| = 1$, we have the following:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_L \gamma(\zeta)\Big(\frac{\zeta}{\zeta-\widetilde{z}} + \frac{\overline{\zeta}}{\overline{\zeta-\widetilde{z}}} - 1\Big)\frac{d\zeta}{\zeta} - \frac{1}{(1-\beta)\pi}\int_D |\zeta|^{\theta} g(\zeta)\Big(\frac{1}{\widetilde{\zeta}-\widetilde{z}} + \frac{\bar{\bar{\widetilde{z}}}}{1-\widetilde{\bar{\bar{z}}\zeta}}\Big)d\sigma_{\zeta} \\ &= \gamma(z), \end{aligned}$$

which means that (3.10) satisfies the boundary condition $f = \gamma$ on L. $\square$

4. Neumann problems

In this section, we discuss the Neumann problems for β-analytic functions on the unit disc D with the boundary L.

Theorem 4.1. *Let $\gamma \in C(\partial D, \mathbb{C})$, $c \in \mathbb{C}$, and $0 \le \beta < 1$; then, the Neumann Problem N1*

$$\partial_{\bar{z}}^{\beta} f(z) = 0\ \ (z \in D), \quad \partial_{\nu} f(\zeta) = \gamma(\zeta)\ \ (\zeta \in L), \quad f(0) = c \tag{4.1}$$

is solvable if and only if

$$\frac{1}{2\pi i}\int_L \gamma(\zeta)\frac{d\zeta}{(1-\bar{\bar{z}}\zeta)\zeta} = 0 \quad (|z| < 1), \tag{4.2}$$

and the solution is given by

$$f(z) = c - \frac{1-\beta}{1+\beta}\frac{1}{2\pi i}\int_L \gamma(\zeta)\ln(1-\overline{\bar{z}\zeta})\frac{d\zeta}{\zeta}, \tag{4.3}$$

where $\partial_{\nu} f = z\partial_z f + \bar{z}\partial_{\bar{z}} f$.

Proof. Obviously, $f(z)$ expressed by (4.3) satisfies $\partial_{\bar{z}}^{\beta} f(z) = 0$ $(z \in D)$. The β-analyticity of f means that $\partial_{\bar{z}} f = \beta\frac{z}{\bar{z}}\partial_z f$. Thus,

$$\partial_{\nu} f = z\partial_z f + \bar{z}\partial_{\bar{z}} f = (1+\beta)z\partial_z f,$$

and

$$\begin{aligned}\partial_{\bar{z}}^{\beta}(z\partial_z f) &= (\partial_{\bar{z}} - \beta\frac{z}{\bar{z}}\partial_z)(z\partial_z f) = z\partial_{\bar{z}}\partial_z f - \beta\frac{z}{\bar{z}}\partial_z(z\partial_z f)\\ &= z\partial_z\Big(\beta\frac{z}{\bar{z}}\partial_z f\Big) - \beta\frac{z}{\bar{z}}\partial_z(z\partial_z f)\\ &= 0.\end{aligned}$$

Therefore, $z\partial_z f$ is also β-analytic and satisfies the boundary condition $\zeta\partial_\zeta f(\zeta) = \frac{\partial_\nu f(\zeta)}{1+\beta}$. As a consequence, Problem N1 is equivalent to the following:

$$\partial_{\bar{z}}^{\beta}(z\partial_z f)(z) = 0 \ \ (z \in D), \quad \zeta\partial_\zeta f(\zeta) = \frac{\gamma(\zeta)}{1+\beta} \doteq \widetilde{\gamma}(\zeta) \ \ (\zeta \in L), \quad f(0) = c. \tag{4.4}$$

By Theorem 3.7, problem

$$\partial_{\bar{z}}^{\beta}(z\partial_z f)(z) = 0, \ \ \zeta\partial_\zeta f(\zeta) = \frac{\gamma(\zeta)}{1+\beta}$$

is solvable if and only if

$$\frac{1}{2\pi i}\int_L \frac{\bar{z}|z|^\theta\widetilde{\gamma}(\zeta)d\zeta}{1-\bar{z}|z|^\theta\zeta} = 0 \quad (|z| < 1), \tag{4.5}$$

and the solution is given by

$$z\partial_z f(z) = \frac{1}{2\pi i}\int_L \frac{\widetilde{\gamma}(\zeta)d\zeta}{\zeta - z|z|^\theta}. \tag{4.6}$$

(4.6) follows that

$$\partial_z f(z) = \frac{1}{z}\frac{1}{2\pi i}\int_L \frac{\widetilde{\gamma}(\zeta)d\zeta}{\zeta - z|z|^\theta}, \tag{4.7}$$

and

$$0 = \frac{1}{2\pi i}\int_L \frac{\widetilde{\gamma}(\zeta)d\zeta}{\zeta}, \tag{4.8}$$

which leads to

$$0 = \frac{1}{z}\frac{1}{2\pi i}\int_L \frac{\widetilde{\gamma}(\zeta)d\zeta}{\zeta}. \tag{4.9}$$

Combining (4.7) and (4.9), we get that

$$\partial_z f(z) = \frac{1}{z}\frac{1}{2\pi i}\int_L \frac{z|z|^\theta\widetilde{\gamma}(\zeta)d\zeta}{(\zeta - z|z|^\theta)\zeta} = \frac{1}{2\pi i}\int_L \frac{|z|^\theta\widetilde{\gamma}(\zeta)}{(\zeta - \widetilde{z})\zeta}d\zeta.$$

Furthermore, since

$$[\ln(\zeta - \widetilde{z})]_z' = \frac{-1}{\zeta - \widetilde{z}}[\partial z(z|z|^\theta)] = \frac{-1}{1-\beta}\frac{|z|^\theta}{\zeta - \widetilde{z}}$$

follows

$$\frac{|z|^\theta}{\zeta - \widetilde{z}} = (\beta - 1)[\ln(\zeta - \widetilde{z})]_z',$$

then

$$\partial_z f(z) = \frac{1}{2\pi i}\int_L (\beta - 1)[\ln(\zeta - \widetilde{z})]_z' \frac{\widetilde{\gamma}(\zeta)d\zeta}{\zeta}.$$

Integrating both sides yields that

$$f(z) = \frac{1}{2\pi i}\int_L (\beta-1)\ln(\zeta-\widetilde{z})\frac{\widetilde{\gamma}(\zeta)d\zeta}{\zeta} + C, \tag{4.10}$$

which follows that

$$f(0) = \frac{1}{2\pi i}\int_L (\beta-1)\ln\zeta\frac{\widetilde{\gamma}(\zeta)d\zeta}{\zeta} + C.$$

As $f(0) = c$, then

$$C = c + \frac{1-\beta}{2\pi i}\int_L \ln\zeta\frac{\widetilde{\gamma}(\zeta)d\zeta}{\zeta}. \tag{4.11}$$

Plugging (4.11) into (4.10), we get that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_L (\beta-1)\ln(\zeta-\widetilde{z})\frac{\widetilde{\gamma}(\zeta)d\zeta}{\zeta} + c + \frac{1-\beta}{2\pi i}\int_L \ln\zeta\frac{\widetilde{\gamma}(\zeta)d\zeta}{\zeta} \\ &= c - \frac{1-\beta}{2\pi i}\int_L \ln(1-\widetilde{\overline{z}\zeta})\frac{\widetilde{\gamma}(\zeta)d\zeta}{\zeta} \\ &= c - \frac{1-\beta}{1+\beta}\frac{1}{2\pi i}\int_L \gamma(\zeta)\ln(1-\widetilde{\overline{z}\zeta})\frac{d\zeta}{\zeta}. \end{aligned}$$

Combining (4.5) and (4.8), we get that

$$\frac{1}{2\pi i}\int_L \frac{\widetilde{\overline{z}}\widetilde{\gamma}(\zeta)d\zeta}{1-\widetilde{\overline{z}}\zeta} + \frac{1}{2\pi i}\int_L \frac{\widetilde{\gamma}(\zeta)d\zeta}{\zeta} = \frac{1}{2\pi i}\int_L \frac{\widetilde{\gamma}(\zeta)d\zeta}{(1-\widetilde{\overline{z}}\zeta)\zeta} = 0,$$

which is equivalent to (4.2).

Hence, (4.3) is the solution to Problem N1 if and only if (4.2) holds. □

Theorem 4.2. *Let* $\gamma \in C(\partial D, \mathbb{C})$*,* $c \in \mathbb{C}$*, and* $0 \le \beta < 1$*, and let* $g \in L_1(\overline{D}, \mathbb{C})$*. Then, the inhomogeneous Neumann Problem N2*

$$\partial_{\overline{z}}^{\beta} f(z) = g(z)\ \ (z \in D), \quad \partial_\nu f(\zeta) = \gamma(\zeta)\ \ (\zeta \in L), \quad f(0) = c \tag{4.12}$$

is solvable if and only if

$$\frac{1}{2\pi i}\int_L \frac{\gamma(\zeta) - \overline{\zeta}g(\zeta)}{(1-\overline{z}\zeta)\zeta}d\zeta + \frac{1+\beta}{(1-\beta)^2\pi}\int_D |\tau|^\theta g(\tau)\frac{\widetilde{\overline{z}}d\sigma_\tau}{(1-\widetilde{\overline{z}\tau})^2} = 0 \quad (|z| < 1), \tag{4.13}$$

and the solution is given by

$$f(z) = c - \frac{1}{(1-\beta)\pi}\int_D \frac{\widetilde{z}g(\zeta)}{\zeta(\widetilde{\zeta}-\widetilde{z})}d\sigma_\zeta - \frac{1-\beta}{1+\beta}\frac{1}{2\pi i}\int_L [\gamma(\zeta) - \overline{\zeta}g(\zeta)]\ln(1-\widetilde{\overline{z}\zeta})\frac{d\zeta}{\zeta}, \tag{4.14}$$

where $\partial_\nu f = z\partial_z f + \overline{z}\partial_{\overline{z}} f$*, and* $\theta = \frac{2\beta}{1-\beta}$*.*

Proof. By Lemma 2.7, $\partial_{\bar z}^{\beta} T_\beta g(z) = g(z)$, where

$$T_\beta g(z) = \frac{-1}{(1-\beta)\pi} \int_D \frac{|\zeta|^\theta g(\zeta)}{\zeta|\zeta|^\theta - z|z|^\theta} d\sigma_\zeta.$$

Therefore,

$$\begin{aligned}\partial_{\bar z}^{\beta} f(z) = g(z),\ \ f(0) = c &\Longleftrightarrow \partial_{\bar z}^{\beta}[f(z) - T_\beta g(z)] = 0,\ \ f(0) - T_\beta g(0) = c - T_\beta g(0)\\ &\Longleftrightarrow \partial_{\bar z}^{\beta}\varphi(z) = 0,\ \ \varphi(0) = c - T_\beta g(0),\end{aligned}$$

in which

$$\varphi(z) = f(z) - T_\beta g(z).$$

Furthermore, since

$$\partial_{\bar z}^{\beta} T_\beta g(z) = g(z) \Longleftrightarrow (\partial_{\bar z} - \beta\frac{z}{\bar z}\partial_z)T_\beta g(z) = g(z) \Longleftrightarrow \partial_{\bar z}T_\beta g(z) = g(z) + \beta\frac{z}{\bar z}\partial_z(T_\beta g)(z),$$

then

$$\begin{aligned}\partial_\nu\varphi(\zeta) = \partial_\nu(f - T_\beta g)(\zeta) &= \partial_\nu f(\zeta) - [\zeta\partial_\zeta(T_\beta g)(\zeta) + \bar\zeta\partial_{\bar\zeta}(T_\beta g)(\zeta)]\\ &= \partial_\nu f(\zeta) - [\zeta\partial_\zeta(T_\beta g)(\zeta) + \bar\zeta g(\zeta) + \beta\zeta\partial_\zeta(T_\beta g)(\zeta)]\\ &= \partial_\nu f(\zeta) - (1+\beta)\zeta\partial_\zeta(T_\beta g)(\zeta) - \bar\zeta g(\zeta).\end{aligned}$$

Therefore,

$$\partial_\nu f(\zeta) = \gamma(\zeta) \Longleftrightarrow \partial_\nu\varphi(\zeta) = \gamma(\zeta) - (1+\beta)\zeta\partial_\zeta(T_\beta g)(\zeta) - \bar\zeta g(\zeta).$$

From the above, it follows that Problem N2 is equivalent to the following:

$$\partial_{\bar z}^{\beta}\varphi(z) = 0,\ \ \partial_\nu\varphi(\zeta) = \gamma(\zeta) - (1+\beta)\zeta\partial_\zeta(T_\beta g)(\zeta) - \bar\zeta g(\zeta),\ \ \varphi(0) = c - T_\beta g(0). \tag{4.15}$$

By applying Theorem 4.1, the solution to (4.15) is

$$\varphi(z) = c - T_\beta g(0) - \frac{1-\beta}{1+\beta}\frac{1}{2\pi i}\int_L [\gamma(\zeta) - (1+\beta)\zeta\partial_\zeta(T_\beta g)(\zeta) - \bar\zeta g(\zeta)] \ln(1 - \tilde{z}\bar{\tilde\zeta})\frac{d\zeta}{\zeta}, \tag{4.16}$$

if and only if

$$\frac{1}{2\pi i}\int_L [\gamma(\zeta) - (1+\beta)\zeta\partial_\zeta(T_\beta g)(\zeta) - \bar\zeta g(\zeta)]\frac{d\zeta}{(1 - \bar{\tilde z}\zeta)\zeta} = 0 \quad (|z| < 1). \tag{4.17}$$

As $\partial_\zeta(\zeta|\zeta|^\theta) = (\frac{\theta}{2} + 1)\zeta^{\frac{\theta}{2}}\bar\zeta^{\frac{\theta}{2}} = \frac{|\zeta|^\theta}{1-\beta}$, then

$$\begin{aligned}\partial_\zeta(T_\beta g)(\zeta) &= \partial_\zeta\Big[\frac{-1}{(1-\beta)\pi}\int_D \frac{|\tau|^\theta g(\tau)}{\tau|\tau|^\theta - \zeta|\zeta|^\theta} d\sigma_\tau\Big]\\ &= \frac{-1}{(1-\beta)\pi}\int_D |\tau|^\theta g(\tau)\frac{|\zeta|^\theta/(1-\beta)}{(\tau|\tau|^\theta - \zeta|\zeta|^\theta)^2} d\sigma_\tau\\ &= \frac{-|\zeta|^\theta}{(1-\beta)^2\pi}\int_D \frac{|\tau|^\theta g(\tau)}{(\tau|\tau|^\theta - \zeta|\zeta|^\theta)^2} d\sigma_\tau.\end{aligned} \tag{4.18}$$

Thus,

$$\begin{aligned}\frac{1}{2\pi i}\int_L \zeta\partial_\zeta(T_\beta g)(\zeta)\frac{d\zeta}{(1-\overline{\tilde{z}}\zeta)\zeta} &= \frac{1}{2\pi i}\int_L\Big[\frac{-|\zeta|^\theta}{(1-\beta)^2\pi}\int_D\frac{|\tau|^\theta g(\tau)}{(\tau|\tau|^\theta-\zeta|\zeta|^\theta)^2}d\sigma_\tau\Big]\frac{d\zeta}{1-\overline{\tilde{z}}\zeta}\\ &=\frac{-1}{(1-\beta)^2\pi}\int_D|\tau|^\theta g(\tau)\Big[\frac{1}{2\pi i}\int_L\frac{|\zeta|^\theta}{(\tau|\tau|^\theta-\zeta|\zeta|^\theta)^2}\frac{d\zeta}{1-\overline{\tilde{z}}\zeta}\Big]d\sigma_\tau\\ &= \frac{-1}{(1-\beta)^2\pi}\int_D|\tau|^\theta g(\tau)\Big[\frac{1}{2\pi i}\int_L\frac{\frac{1}{1-\overline{\tilde{z}}\zeta}}{(\zeta-\tau|\tau|^\theta)^2}d\zeta\Big]d\sigma_\tau\\ &= \frac{-1}{(1-\beta)^2\pi}\int_D|\tau|^\theta g(\tau)\frac{\overline{\tilde{z}}d\sigma_\tau}{(1-\overline{\tilde{z}}\tau|\tau|^\theta)^2}.\end{aligned} \tag{4.19}$$

From (4.17) and (4.19), we derive that the solvability condition of Problem N2 is (4.13).

In addition, by (4.18), we obtain that

$$\begin{aligned}\frac{1}{2\pi i}\int_L\zeta\partial_\zeta(T_\beta g)(\zeta)\ln(1-\overline{\tilde{z}}\overline{\zeta})\frac{d\zeta}{\zeta} &= \frac{1}{2\pi i}\int_L\Big[\frac{-|\zeta|^\theta}{(1-\beta)^2\pi}\int_D\frac{|\tau|^\theta g(\tau)}{(\tau|\tau|^\theta-\zeta|\zeta|^\theta)^2}d\sigma_\tau\Big]\ln(1-\overline{\tilde{z}}\overline{\zeta})d\zeta\\ &= \frac{-1}{(1-\beta)^2\pi}\int_D|\tau|^\theta g(\tau)\Big[\frac{1}{2\pi i}\int_L\frac{|\zeta|^\theta\ln(1-\overline{\tilde{z}}\overline{\zeta})}{(\tau|\tau|^\theta-\zeta|\zeta|^\theta)^2}d\zeta\Big]d\sigma_\tau\\ &= \frac{-1}{(1-\beta)^2\pi}\int_D|\tau|^\theta g(\tau)\Big[\frac{1}{2\pi i}\int_L\frac{\ln(1-\overline{\tilde{z}}\overline{\zeta})}{(1-\tau|\tau|^\theta\overline{\zeta})^2}d\overline{\zeta}\Big]d\sigma_\tau\\ &= 0.\end{aligned} \tag{4.20}$$

Combining (4.16) and (4.20), we obtain that

$$\varphi(z) = c - T_\beta g(0) - \frac{1-\beta}{1+\beta}\frac{1}{2\pi i}\int_L[\gamma(\zeta)-\overline{\zeta}g(\zeta)]\ln(1-\overline{\tilde{z}}\overline{\zeta})\frac{d\zeta}{\zeta}, \tag{4.21}$$

in which

$$T_\beta g(0) = \frac{-1}{(1-\beta)\pi}\int_D\frac{g(\zeta)}{\zeta}d\sigma_\zeta. \tag{4.22}$$

From (4.21) and (4.22), we obtain that

$$\begin{aligned}f(z) &= \varphi(z) + T_\beta g(z)\\ &= c - T_\beta g(0) + T_\beta g(z) - \frac{1-\beta}{1+\beta}\frac{1}{2\pi i}\int_L[\gamma(\zeta)-\overline{\zeta}g(\zeta)]\ln(1-\overline{\tilde{z}}\overline{\zeta})\frac{d\zeta}{\zeta}\\ &= c - \frac{-1}{(1-\beta)\pi}\int_D\frac{g(\zeta)}{\zeta}d\sigma_\zeta + \frac{-1}{(1-\beta)\pi}\int_D\frac{|\zeta|^\theta g(\zeta)}{\zeta|\zeta|^\theta - z|z|^\theta}d\sigma_\zeta\\ &\quad - \frac{1-\beta}{1+\beta}\frac{1}{2\pi i}\int_L[\gamma(\zeta)-\overline{\zeta}g(\zeta)]\ln(1-\overline{\tilde{z}}\overline{\zeta})\frac{d\zeta}{\zeta}\\ &= c - \frac{1}{(1-\beta)\pi}\int_D\frac{z|z|^\theta g(\zeta)}{\zeta(\zeta|\zeta|^\theta - z|z|^\theta)}d\sigma_\zeta - \frac{1-\beta}{1+\beta}\frac{1}{2\pi i}\int_L[\gamma(\zeta)-\overline{\zeta}g(\zeta)]\ln(1-\overline{\tilde{z}}\overline{\zeta})\frac{d\zeta}{\zeta}.\end{aligned}$$

Therefore, (4.14) is the solution to Problem N2 if and only if (4.13) holds. □

Theorem 4.3. *Let $\gamma \in C(\partial D, \mathbb{C})$, $c \in \mathbb{C}$, and $0 \le \beta < 1$, and let $g \in L_1(\overline{D}, \mathbb{C})$. Then, the inhomogeneous half-Neumann Problem N3*

$$\partial_{\bar{z}}^{\beta} f(z) = g(z) \ \ (z \in D), \quad \zeta \partial_{\zeta} f(\zeta) = \gamma(\zeta) \ \ (\zeta \in L), \quad f(0) = c$$

is solvable if and only if

$$\frac{1}{2\pi i} \int_L \frac{\gamma(\zeta)}{(1 - \bar{z}\zeta)\zeta} d\zeta + \frac{1}{(1-\beta)^2 \pi} \int_D |\tau|^{\theta} g(\tau) \frac{\widetilde{\bar{z}} d\sigma_{\tau}}{(1 - \widetilde{\bar{z}}\widetilde{\tau})^2} = 0 \quad (|z| < 1),$$

and the solution is given by

$$f(z) = c - \frac{1}{(1-\beta)\pi} \int_D \frac{\widetilde{z} g(\zeta)}{\zeta(\widetilde{\zeta} - \widetilde{z})} d\sigma_{\zeta} - \frac{1-\beta}{2\pi i} \int_L \gamma(\zeta) \ln(1 - \widetilde{\bar{z}}\widetilde{\zeta}) \frac{d\zeta}{\zeta},$$

where $\theta = \frac{2\beta}{1-\beta}$.

Proof. As

$$\partial_{\bar{z}}^{\beta} f(z) = g(z) \Longleftrightarrow (\partial_{\bar{z}} - \beta \frac{z}{\bar{z}} \partial_z) f(z) = g(z) \Longleftrightarrow \bar{z}\partial_{\bar{z}} f(z) = \bar{z} g(z) + \beta z \partial_z f(z).$$

Therefore,

$$\zeta \partial_{\zeta} f(\zeta) = \gamma(\zeta) \Longleftrightarrow \partial_{\nu} f(\zeta) = \zeta \partial_{\zeta} f + \bar{\zeta} \partial_{\bar{\zeta}} f = (1+\beta)\gamma(\zeta) + \bar{\zeta} g(\zeta).$$

Applying Theorem 4.2 and substituting $(1+\beta)\gamma + \bar{\zeta} g$ for γ yields the desired conclusion. □

Analogously to Theorem 4.3, we obtain the following result.

Theorem 4.4. *Let $\gamma \in C(\partial D, \mathbb{C})$, $c \in \mathbb{C}$, and $0 \le \beta < 1$, and let $g \in L_1(\overline{D}, \mathbb{C})$. Then, the inhomogeneous half-Neumann Problem N4*

$$\partial_{\bar{z}}^{\beta} f(z) = g(z) \ \ (z \in D), \quad \bar{\zeta} \partial_{\bar{\zeta}} f(\zeta) = \gamma(\zeta) \ \ (\zeta \in L), \quad f(0) = c$$

is solvable if and only if

$$\frac{1}{2\pi i} \int_L \frac{1}{\beta} \frac{\gamma(\zeta) - \bar{\zeta} g(\zeta)}{(1 - \bar{z}\zeta)\zeta} d\zeta + \frac{1}{(1-\beta)^2 \pi} \int_D |\tau|^{\theta} g(\tau) \frac{\widetilde{\bar{z}} d\sigma_{\tau}}{(1 - \widetilde{\bar{z}}\widetilde{\tau})^2} = 0 \quad (|z| < 1),$$

and the solution is given by

$$f(z) = c - \frac{1}{(1-\beta)\pi} \int_D \frac{\widetilde{z} g(\zeta)}{\zeta(\widetilde{\zeta} - \widetilde{z})} d\sigma_{\zeta} - \frac{1-\beta}{\beta} \frac{1}{2\pi i} \int_L [\gamma(\zeta) - \bar{\zeta} g(\zeta)] \ln(1 - \widetilde{\bar{z}}\widetilde{\zeta}) \frac{d\zeta}{\zeta},$$

where $\theta = \frac{2\beta}{1-\beta}$.

Remark 4.5. *Let $\beta = 0$ in Theorems 4.1–4.4; we can get the corresponding results of Neumann and half-Neumann problems for analytic functions, and the case $\beta = 0$ yields simpler solutions and solvability conditions compared with $\beta > 0$.*

5. Rubin problems

In this section, we discuss the Rubin problems for β-analytic functions on the unit disc D with the boundary L.

Theorem 5.1. *Let $\gamma \in C(\partial D, \mathbb{C})$ and $0 \le \beta < 1$; then, the Rubin Problem R1*

$$\partial_{\bar{z}}^{\beta} f(z) = 0 \ \ (z \in D), \quad f(\zeta) + \partial_{\nu} f(\zeta) = \gamma(\zeta) \ \ (\zeta \in L) \tag{5.1}$$

is solvable if and only if

$$\frac{1}{2\pi i}\int_L \gamma(\zeta)\frac{\widetilde{\bar{z}}d\zeta}{1-\widetilde{\bar{z}}\zeta} = 0 \quad (|z| < 1), \tag{5.2}$$

and the solution is given by

$$f(z) = \frac{1}{2\pi i}\int_L \gamma(\zeta)\Big[\sum_{n=0}^{\infty}\frac{1}{1+(\theta+1)n}\frac{\widetilde{z}^n}{\zeta^{n+1}}\Big]d\zeta, \tag{5.3}$$

where $\partial_\nu f = z\partial_z f + \bar{z}\partial_{\bar{z}} f$ and $\theta = \frac{2\beta}{1-\beta}$.

Proof. As $\partial_{\bar{z}}^{\beta} f(z) = 0$, then $\partial_{\bar{z}} f = \beta\frac{z}{\bar{z}}\partial_z f$. Therefore,

$$\begin{cases} \partial_\nu f = z\partial_z f + \bar{z}\partial_{\bar{z}} f = (1+\beta)z\partial_z f, \\ \partial_{\bar{z}}^{\beta}(z\partial_z f) = (\partial_{\bar{z}} - \beta\frac{z}{\bar{z}}\partial_z)(z\partial_z f) = z\partial_{\bar{z}}\partial_z f - \beta\frac{z}{\bar{z}}\partial_z(z\partial_z f) = z\partial_z\Big(\partial_{\bar{z}} f - \beta\frac{z}{\bar{z}}\partial_z f\Big) = 0. \end{cases}$$

Then, it follows that

$$\begin{cases} f(\zeta) + (1+\beta)\zeta\partial_\zeta f(\zeta) = \gamma(\zeta), \\ \partial_{\bar{z}}^{\beta}[f(z) + (1+\beta)z\partial_z f] = 0. \end{cases} \tag{5.4}$$

By Theorem 3.7,

$$f(z) + (1+\beta)z\partial_z f(z) = \frac{1}{2\pi i}\int_L \frac{\gamma(\zeta)d\zeta}{\zeta - z|z|^\theta} \tag{5.5}$$

is the solution to (5.4) if and only if (5.2) holds.

It follows from (5.5) that

$$f(z) + (1+\beta)z\partial_z f(z) = \frac{1}{2\pi i}\int_L \gamma(\zeta)\sum_{n=0}^{\infty}\frac{z^n|z|^{n\theta}}{\zeta^{n+1}}d\zeta = \sum_{n=0}^{\infty}\Big[\frac{1}{2\pi i}\int_L \frac{\gamma(\zeta)}{\zeta^{n+1}}d\zeta\Big]z^n|z|^{n\theta}. \tag{5.6}$$

In addition, by Theorem 3.2, we obtain that $f(z)$ can be expanded into the following:

$$f(z) = \sum_{n=0}^{\infty} c_n\widetilde{z}^n = \sum_{n=0}^{\infty} c_n z^n|z|^{n\theta}.$$

Thus,

$$\begin{aligned} f(z) + (1+\beta)z\partial_z f(z) &= \sum_{n=0}^{\infty} c_n z^n|z|^{n\theta} + (1+\beta)z\sum_{n=0}^{\infty} c_n \cdot n(z|z|^\theta)^{n-1}\frac{1}{1-\beta}|z|^\theta \\ &= \sum_{n=0}^{\infty}[1+(\theta+1)n]c_n z^n|z|^{n\theta}. \end{aligned} \tag{5.7}$$

Comparing (5.6) with (5.7), we obtain that

$$c_n = \frac{1}{1+(\theta+1)n}\frac{1}{2\pi i}\int_L \frac{\gamma(\zeta)}{\zeta^{n+1}}d\zeta \quad (n = 0, 1, 2\cdots).$$

Therefore,

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} c_n \widetilde{z}^n = \sum_{n=0}^{\infty} \frac{1}{1+(\theta+1)n}\frac{1}{2\pi i}\int_L \frac{\gamma(\zeta)}{\zeta^{n+1}}d\zeta \widetilde{z}^n \\ &= \frac{1}{2\pi i}\int_L \gamma(\zeta)\Big[\sum_{n=0}^{\infty}\frac{1}{1+(\theta+1)n}\frac{\widetilde{z}^n}{\zeta^{n+1}}\Big]d\zeta. \end{aligned}$$

Consequently, (5.3) is the solution to (5.1) if and only if (5.2) holds. □

Corollary 5.2. *Let $\gamma \in C(\partial D, \mathbb{C})$; then, the Rubin Problem R2*

$$\partial_{\bar z} f(z) = 0 \ \ (z \in D), \quad f(\zeta) + \partial_\nu f(\zeta) = \gamma(\zeta) \ \ (\zeta \in L)$$

is solvable if and only if

$$\frac{1}{2\pi i}\int_L \gamma(\zeta)\frac{\bar z d\zeta}{1-\bar z\zeta} = 0 \quad (|z| < 1),$$

and the solution is given by

$$f(z) = \frac{-1}{\bar z}\frac{1}{2\pi i}\int_L \gamma(\zeta)\ln(1-z\bar\zeta)d\zeta,$$

where $\partial_\nu f = z\partial_z f + \bar z\partial_{\bar z} f$.

Proof. Let $\beta = 0$ in Theorem 5.1, and thus $\theta = 0$ and $\widetilde{z} = z$. Then, (5.3) reduces to the following:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_L \gamma(\zeta)\Big[\sum_{n=0}^{\infty}\frac{1}{1+n}\frac{z^n}{\zeta^{n+1}}\Big]d\zeta \\ &= \frac{1}{2\pi i}\int_L \gamma(\zeta)\Big[\frac{-1}{\bar z}\sum_{n=0}^{\infty}\frac{-1}{1+n}(z\bar\zeta)^{n+1}\Big]d\zeta \\ &= \frac{-1}{\bar z}\frac{1}{2\pi i}\int_L \gamma(\zeta)\ln(1-z\bar\zeta)d\zeta. \end{aligned}$$

□

Theorem 5.3. *Let $\gamma \in C(\partial D, \mathbb{C})$ and $0 \le \beta < 1$, and let $g \in L_1(\overline{D}, \mathbb{C})$. Then, the Rubin Problem R3*

$$\partial_{\bar z}^{\beta} f(z) = g(z) \ \ (z \in D), \quad f(\zeta) + \partial_\nu f(\zeta) = \gamma(\zeta) \ \ (\zeta \in L) \tag{5.8}$$

is solvable if and only if

$$\frac{1}{2\pi i}\int_L \frac{\gamma(\zeta) - \bar\zeta g(\zeta)}{1-\bar{\widetilde{z}}\zeta}d\zeta = \frac{1}{(1-\beta)\pi}\int_D \frac{|\tau|^\theta g(\tau)}{1-\bar{\widetilde{z}}\widetilde{\tau}}\Big(1-\frac{1+\beta}{1-\beta}\frac{1}{1-\bar{\widetilde{z}}\widetilde{\tau}}\Big)d\sigma_\tau \quad (|z| < 1), \tag{5.9}$$

and the solution is given by

$$f(z) = \frac{-1}{(1-\beta)\pi}\int_G \frac{|\zeta|^\theta g(\zeta)}{\widetilde{\zeta}-\widetilde{z}}d\sigma_\zeta + \frac{1}{2\pi i}\int_L [\gamma(\zeta) - \bar\zeta g(\zeta)]\Big[\sum_{n=0}^{\infty}\frac{1}{1+(\theta+1)n}\frac{\widetilde{z}^n}{\zeta^{n+1}}\Big]d\zeta, \tag{5.10}$$

where $\partial_\nu f = z\partial_z f + \bar z\partial_{\bar z} f$, and $\theta = \frac{2\beta}{1-\beta}$.

Proof. By Lemma 2.7, $\partial_{\bar z}^{\beta} T_{\beta} g(z) = g(z)$, where

$$T_{\beta} g(z) = \frac{-1}{(1-\beta)\pi} \int_D \frac{|\zeta|^{\theta} g(\zeta)}{\zeta|\zeta|^{\theta} - z|z|^{\theta}} d\sigma_{\zeta}.$$

Therefore,

$$\begin{aligned} \partial_{\bar z}^{\beta} f(z) = g(z) &\Longleftrightarrow \partial_{\bar z}^{\beta}[f(z) - T_{\beta} g(z)] = 0 \\ &\Longleftrightarrow \partial_{\bar z}^{\beta} \varphi(z) = 0, \end{aligned}$$

in which

$$\varphi(z) = f(z) - T_{\beta} g(z). \tag{5.11}$$

Furthermore, since

$$\partial_{\bar z}^{\beta} T_{\beta} g(z) = g(z) \Longleftrightarrow (\partial_{\bar z} - \beta \frac{z}{\bar z} \partial_z) T_{\beta} g(z) = g(z) \Longleftrightarrow \partial_{\bar z} T_{\beta} g(z) = g(z) + \beta \frac{z}{\bar z} \partial_z (T_{\beta} g)(z),$$

then

$$\begin{aligned} \varphi(\zeta) + \partial_{\nu} \varphi(\zeta) &= f(\zeta) - T_{\beta} g(\zeta) + \partial_{\nu}(f - T_{\beta} g)(\zeta) \\ &= [f(\zeta) + \partial_{\nu} f(\zeta)] - T_{\beta} g(\zeta) - [\zeta \partial_{\zeta}(T_{\beta} g)(\zeta) + \bar\zeta \partial_{\bar\zeta}(T_{\beta} g)(\zeta)] \\ &= [f(\zeta) + \partial_{\nu} f(\zeta)] - T_{\beta} g(\zeta) - [\zeta \partial_{\zeta}(T_{\beta} g)(\zeta) + \bar\zeta g(\zeta) + \beta \zeta \partial_{\zeta}(T_{\beta} g)(\zeta)] \\ &= [f(\zeta) + \partial_{\nu} f(\zeta)] - T_{\beta} g(\zeta) - (1+\beta) \zeta \partial_{\zeta}(T_{\beta} g)(\zeta) - \bar\zeta g(\zeta). \end{aligned}$$

Therefore,

$$f(\zeta) + \partial_{\nu} f(\zeta) = \gamma(\zeta) \Longleftrightarrow \varphi(\zeta) + \partial_{\nu} \varphi(\zeta) = \gamma(\zeta) - T_{\beta} g(\zeta) - (1+\beta) \zeta \partial_{\zeta}(T_{\beta} g)(\zeta) - \bar\zeta g(\zeta).$$

From the above, it follows that Problem R3 is equivalent to the following:

$$\partial_{\bar z}^{\beta} \varphi(z) = 0, \quad \varphi(\zeta) + \partial_{\nu} \varphi(\zeta) = \gamma(\zeta) - T_{\beta} g(\zeta) - (1+\beta) \zeta \partial_{\zeta}(T_{\beta} g)(\zeta) - \bar\zeta g(\zeta). \tag{5.12}$$

By Theorem 5.1, (5.12) is solvable if and only if

$$\frac{1}{2\pi i} \int_L [\gamma(\zeta) - T_{\beta} g(\zeta) - (1+\beta) \zeta \partial_{\zeta}(T_{\beta} g)(\zeta) - \bar\zeta g(\zeta)] \frac{d\zeta}{1 - \bar z \zeta} = 0 \quad (|z| < 1), \tag{5.13}$$

and the solution is given by

$$\varphi(z) = \frac{1}{2\pi i} \int_L [\gamma(\zeta) - T_{\beta} g(\zeta) - (1+\beta) \zeta \partial_{\zeta}(T_{\beta} g)(\zeta) - \bar\zeta g(\zeta)] \Big[\sum_{n=0}^{\infty} \frac{1}{1 + (\theta+1) n} \frac{\widetilde{z}^n}{\zeta^{n+1}} \Big] d\zeta. \tag{5.14}$$

From (4.18), we get that

$$\begin{aligned} \frac{1}{2\pi i} \int_L \zeta \partial_{\zeta}(T_{\beta} g)(\zeta) \frac{d\zeta}{1 - \widetilde{\bar z} \zeta} &= \frac{1}{2\pi i} \int_L \frac{\zeta}{1 - \widetilde{\bar z} \zeta} \Big[\frac{-|\zeta|^{\theta}}{(1-\beta)^2 \pi} \int_D \frac{|\tau|^{\theta} g(\tau)}{(\widetilde{\tau} - \widetilde{\zeta})^2} d\sigma_{\tau} \Big] d\zeta \\ &= \frac{-1}{(1-\beta)^2 \pi} \int_D |\tau|^{\theta} g(\tau) \Big[\frac{1}{2\pi i} \int_L \frac{\zeta}{1 - \widetilde{\bar z} \zeta} \frac{|\zeta|^{\theta}}{(\widetilde{\tau} - \widetilde{\zeta})^2} d\zeta \Big] d\sigma_{\tau} \\ &= \frac{-1}{(1-\beta)^2 \pi} \int_D |\tau|^{\theta} g(\tau) \Big(\frac{\zeta}{1 - \widetilde{\bar z} \zeta} \Big)'_{\zeta} \Big|_{\zeta = \widetilde{\tau}} d\sigma_{\tau} \\ &= \frac{-1}{(1-\beta)^2 \pi} \int_D \frac{|\tau|^{\theta} g(\tau)}{(1 - \widetilde{\bar z} \widetilde{\tau})^2} d\sigma_{\tau}. \end{aligned} \tag{5.15}$$

Furthermore,

$$
\begin{aligned}
\frac{1}{2\pi i}\int_L (T_\beta g)(\zeta)\frac{d\zeta}{1-\overline{\widetilde{z}}\zeta} &= \frac{1}{2\pi i}\int_L \Big[\frac{-1}{(1-\beta)\pi}\int_D \frac{|\tau|^\theta g(\tau)}{\widetilde{\tau}-\widetilde{\zeta}}d\sigma_\tau\Big]\frac{d\zeta}{1-\overline{\widetilde{z}}\zeta} \\
&= \frac{-1}{(1-\beta)\pi}\int_D |\tau|^\theta g(\tau)\Big[\frac{1}{2\pi i}\int_L \frac{1}{\widetilde{\tau}-\widetilde{\zeta}}\frac{d\zeta}{1-\overline{\widetilde{z}}\zeta}\Big]d\sigma_\tau \\
&= \frac{1}{(1-\beta)\pi}\int_D \frac{|\tau|^\theta g(\tau)}{1-\overline{\widetilde{z}}\widetilde{\tau}}d\sigma_\tau.
\end{aligned} \tag{5.16}
$$

Substituting (5.15) and (5.16) into (5.13) yields (5.9).

In addition, as

$$
\begin{aligned}
\frac{1}{2\pi i}\int_L T_\beta g(\zeta)\frac{d\zeta}{\zeta^{n+1}} &= \frac{1}{2\pi i}\int_L \Big[\frac{-1}{(1-\beta)\pi}\int_D \frac{|\tau|^\theta g(\tau)}{\widetilde{\tau}-\widetilde{\zeta}}d\sigma_\tau\Big]\frac{d\zeta}{\zeta^{n+1}} \\
&= \frac{-1}{(1-\beta)\pi}\int_D |\tau|^\theta g(\tau)\Big[\frac{1}{2\pi i}\int_L \frac{1}{\widetilde{\tau}-\widetilde{\zeta}}\frac{d\zeta}{\zeta^{n+1}}\Big]d\sigma_\tau \\
&= \frac{-1}{(1-\beta)\pi}\int_D |\tau|^\theta g(\tau)\Big[\overline{\frac{-1}{2\pi i}\int_L \frac{1}{\overline{\overline{\widetilde{\tau}}}-\overline{\zeta}}\frac{1}{\overline{\zeta}^{n+1}}\frac{-d\zeta}{\zeta^2}}\Big]d\sigma_\tau \\
&= \frac{-1}{(1-\beta)\pi}\int_D |\tau|^\theta g(\tau)\Big[\overline{\frac{1}{2\pi i}\int_L \frac{\zeta^n d\zeta}{\overline{\overline{\widetilde{\tau}}}\zeta-1}}\Big]d\sigma_\tau \\
&= 0,
\end{aligned} \tag{5.17}
$$

and by (4.18), we have that

$$
\begin{aligned}
\frac{1}{2\pi i}\int_L \zeta\partial_\zeta(T_\beta g)(\zeta)\frac{d\zeta}{\zeta^{n+1}} &= \frac{1}{2\pi i}\int_L \zeta\Big[\frac{-|\zeta|^\theta}{(1-\beta)^2\pi}\int_D \frac{|\tau|^\theta g(\tau)}{(\widetilde{\tau}-\widetilde{\zeta})^2}d\sigma_\tau\Big]\frac{d\zeta}{\zeta^{n+1}} \\
&= \frac{-1}{(1-\beta)^2\pi}\int_D |\tau|^\theta g(\tau)\Big[\frac{1}{2\pi i}\int_L \frac{1}{(\widetilde{\tau}-\widetilde{\zeta})^2}\frac{d\zeta}{\zeta^n}\Big]d\sigma_\tau \\
&= \frac{-1}{(1-\beta)\pi}\int_D |\tau|^\theta g(\tau)\Big[\overline{\frac{-1}{2\pi i}\int_L \frac{1}{(\overline{\overline{\widetilde{\tau}}}-\overline{\zeta})^2}\frac{1}{\overline{\zeta}^n}\frac{-d\zeta}{\zeta^2}}\Big]d\sigma_\tau \\
&= \frac{-1}{(1-\beta)\pi}\int_D |\tau|^\theta g(\tau)\Big[\overline{\frac{1}{2\pi i}\int_L \frac{\zeta^n d\zeta}{(\overline{\overline{\widetilde{\tau}}}\zeta-1)^2}}\Big]d\sigma_\tau \\
&= 0.
\end{aligned} \tag{5.18}
$$

Substituting (5.17) and (5.18) into (5.14) yields that

$$
\varphi(z) = \frac{1}{2\pi i}\int_L [\gamma(\zeta)-\overline{\zeta}g(\zeta)]\Big[\sum_{n=0}^{\infty}\frac{1}{1+(\theta+1)n}\frac{\widetilde{z}^n}{\zeta^{n+1}}\Big]d\zeta. \tag{5.19}
$$

By combining (5.19) and (5.11), we arrive at the solution (5.10). □

Analogously to Corollary 5.2, we apply Theorem 5.3 to obtain the following result.

Corollary 5.4. *Let $\gamma \in C(\partial D, \mathbb{C})$, and let $g \in L_1(\overline{D}, \mathbb{C})$. Then, the Rubin Problem R4*

$$\partial_{\bar{z}} f(z) = g(z) \ \ (z \in D), \quad f(\zeta) + \partial_\nu f(\zeta) = \gamma(\zeta) \ \ (\zeta \in L)$$

is solvable if and only if

$$\frac{1}{2\pi i} \int_L \frac{\gamma(\zeta) - \overline{\zeta} g(\zeta)}{1 - \bar{z}\zeta} d\zeta = \frac{1}{\pi} \int_D \frac{g(\tau)}{1 - \bar{z}\tau} (1 - \frac{1}{1 - \bar{z}\tau}) d\sigma_\tau \quad (|z| < 1),$$

and the solution is given by

$$f(z) = \frac{-1}{\pi} \int_G \frac{g(\zeta)}{\zeta - z} d\sigma_\zeta + \frac{-1}{\bar{z}} \frac{1}{2\pi i} \int_L [\gamma(\zeta) - \overline{\zeta} g(\zeta)] \ln(1 - z\overline{\zeta}) d\zeta,$$

where $\partial_\nu f = z \partial_z f + \bar{z} \partial_{\bar{z}} f$.

Remark 5.5. *Comparing Corollary 5.2 with Theorem 5.1, as well as Corollary 5.4 with Theorem 5.3, we can observe that the solutions and solvability conditions for $\beta = 0$ are simpler than those for $\beta > 0$. In particular, by virtue of the Taylor expansion of the logarithmic function, the solution for $\beta = 0$ can be freed from the representation involving complicated sums, which yields a much simpler expression for the solution of the problem.*

6. Conclusions

We investigated the fundamental properties of β-analytic functions. On this basis and applying the properties of the Poisson kernel, we discussed the Dirichlet boundary value problems for β-analytic functions, and derived the solutions and solvability conditions for these problems. We further studied the Neumann and half-Neumann boundary value problems associated with β-analytic functions. Finally, we addressed the Rubin boundary value problem of β-analytic functions, deduced all admissible solutions, and characterized their solvability conditions. The derived conclusions generalize the existing relevant results for analytic functions, help advance broader applications of β-analytic functions, and lay a solid theoretical foundation for the investigation of other classes of boundary value problems concerning β-analytic functions. For instance, we may investigate high-order boundary value problems for β-analytic functions and related partial differential equations or explore related problems in multi-dimensional complex domains.

Author contributions

Chaojun Wang: Conceptualization, project administration, writing original draft, writing-review and editing. Yanyan Cui: Investigation, writing original draft, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that they have no conflict of interests.

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