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Research article

Topological properties and duals of difference net spaces over directed sets with fixed-point applications

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Abstract: Kızmaz introduced the concept of difference sequence spaces, studied their topological properties and inclusion relations, and computed their duals. In this paper, we generalize difference sequence spaces by introducing difference net spaces over directed sets. We define the norm on these spaces and study their properties like completeness, separability, and inclusion relations. We characterize the dual spaces and establish the isometric isomorphisms $(\ell^\infty(\Delta, D))^* \cong \frac{\mathbb{K}\times ba(D)}{(T(\ell^\infty(\Delta,D)))^\perp}$, $c(\Delta, D)^* \cong \frac{\mathbb{K}\times(\ell^1(D)\oplus\mathbb{K})}{R^\perp}$ and $(c_0(\Delta, D))^* \cong \frac{\mathbb{K}\times\ell^1(D)}{(T(c_0(\Delta,D)))^\perp}$. Lastly, we establish an application to fixed-point theory, demonstrating that iterative nets generated by contraction mappings constitute natural elements of the space $C_0(\Delta, D)$.

Keywords: fixed point; net space; dual; difference sequence space; convergence; Banach space
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1. Introduction

In functional analysis and summability theory, sequence spaces play a fundamental role. Classical sequence spaces such as ℓ_p $(1 \leq p \leq \infty)$, c, and c_0 have numerous applications in analysis, operator theory, and approximation theory, and so they are extensively studied. They have been generalized in many directions using matrix transformations and difference operators.

In 1981, Kızmaz [1] introduced difference sequence spaces, which are constructed by applying the difference operator to classical sequence spaces. The motivation behind this construction is that the behavior of the differences of a sequence often provides more refined information about the sequence than the sequence itself. Consider the sequence $x = (1, 2, 3, 4, 5, \cdots)$, which is not bounded and hence does not belong to l_∞ but $\Delta x = (-1, -1, -1, \cdots) \in l_\infty$.

Let ω denote the space of all real or complex sequences $x = (x_k)_{k=1}^{\infty}$. The forward difference operator Δ is defined on ω by

$$(\Delta x)_k = x_k - x_{k+1}, \qquad k \in \mathbb{N}.$$

Using this operator, Kızmaz defined the following sequence spaces:

$$X(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in X\},$$

where X is any classical sequence space such as ℓ_∞, c, or c_0.

Thus, the Kızmaz sequence spaces are defined as

$$\ell_\infty(\Delta) = \{x \in \omega : (\Delta x_k) \in \ell_\infty\},$$

$$c(\Delta) = \{x \in \omega : (\Delta x_k) \in c\},$$

$$c_0(\Delta) = \{x \in \omega : (\Delta x_k) \in c_0\}.$$

These are the sequence spaces whose difference sequences are in the corresponding classical sequence spaces. These spaces impose conditions on the sequence's first-order differences rather than the sequence itself.

The study of sequence spaces took a new turn with the advent of difference sequence spaces. These spaces offer a helpful framework for examining stability characteristics, smoothness behavior, and convergence patterns of sequences since the difference operator captures the variation between subsequent terms of a sequence.

Difference operators have also been widely used in numerical analysis for the approximation of nonlinear and nonlocal problems. In particular, a high-order compact Crank–Nicolson alternating-direction implicit (CN–ADI) scheme on graded meshes was proposed for three-dimensional nonlinear partial integro-differential equations with multiple weakly singular kernels [2]. Moreover, time–space two-grid methods were analyzed for two-dimensional nonlinear nonlocal mobile/immobile transport models [3]. These studies illustrate the relevance of difference-based techniques in the analysis of stability, convergence, and approximation of nonlinear and nonlocal mathematical models. Motivated by this broader role of difference operators, our work develops an abstract framework for difference operators acting on nets indexed by directed sets. In particular, we introduce and investigate difference bounded, difference convergent, and difference null net spaces and establish their fundamental structural properties. It is crucial to remember that the underlying space X imparts numerous structural characteristics to the spaces $X(\Delta)$. Specifically, when equipped with the standard norm

$$\|x\|_{X(\Delta)} = |x_1| + \|\Delta x\|_X,$$

the space $X(\Delta)$ becomes a Banach space, provided the X itself is a Banach space. So, $\ell_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$ are Banach spaces.

Over the years, many researchers have extended the concept of Kımaz sequence spaces in various directions. They extended Kızmaz's spaces through generalized difference operators and duality [4, 5], and later through generalized weighted means [6, 7], matrix domain constructions [8, 9], weighted difference spaces, statistical-convergence extensions include weighted and fractional-order difference sequences [10, 11] and rough statistical convergence in intuitionistic fuzzy normed spaces [13]. Such generalizations have been studied in connection with summability theory, operator theory, compact operators, and spectral theory.

The way these spaces relate to matrix transformations is another significant feature. Their behavior between difference sequence spaces and classical sequence spaces can be used to characterize a broad class of infinite matrices. The characterization of bounded and compact operators between these spaces has advanced significantly as a result.

Difference sequence spaces have also been used in discrete models and numerical analysis, where discrete derivatives are crucial. The study of such spaces can be seen as a discrete counterpart of function spaces in classical analysis as the difference operator is a discrete analogue of differentiation.

Inspired by these advancements, researchers are still investigating novel structures and characteristics of spaces of different types. New theoretical understandings and applications may result from the study of generalized difference operators, net versions of difference spaces, and ordered versions of such spaces. Although sequences provide a natural and useful framework for studying convergence, they do not characterize the topology of an arbitrary topological space. In particular, in spaces that are not first-countable, sequential convergence may fail to detect all topological closure and continuity properties. Nets overcome this limitation: Every point in the closure of a set is the limit of a net contained in that set, and the directed index set can be chosen naturally from the neighborhood system of the point. Thus, nets provide a complete characterization of topological convergence without requiring first-countability or metrizability. Replacing the natural-number index set by an arbitrary directed set therefore extends the theory of difference sequence spaces to settings in which sequences alone are insufficient. This observation motivates the introduction of difference net spaces and the investigation of their functional-analytic properties, including completeness, duality, separability, inclusion relations, and their applications to iterative processes.

In this paper, we extend these notions into the more general concept of difference net spaces. A net is a generalization of a sequence indexed by a directed set rather than $\mathbb{N}$. Difference net spaces capture the variation of nets and allow for convergence acceleration via iterated difference operators.

This is how the rest of the paper is structured. We provide the fundamental definitions and findings required for the ensuing advancements in Section 2. We prove the completeness of difference net spaces in Section 3. The dual spaces of difference net spaces are covered in Section 4. The separability of difference net spaces is covered in Section 5. Lastly, a fixed-point theory application of difference net spaces is shown in Section 6.

2. Basic definitions

Definition 2.1. *[14] Let D be a non-empty set. A relation $\leq$ defined on D is called a partial order if it satisfies the following properties for all $a, b, c \in D$:*

1. *$a \leq a$ (reflexivity),*
2. *if $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetry),*
3. *if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity).*

In this case, the pair $(D, \leq)$ is called a partially ordered set (or simply a poset).

Definition 2.2. *[14] Let $(D, \leq)$ be a partially ordered set. The set D is said to be directed if, for any two elements $a, b \in D$, there exists an element $c \in D$ such that $a \leq c$ and $b \leq c$. In this situation, c is called a common upper bound of a and b.*

Definition 2.3. *[15] Let $(D, \leq)$ be a directed set, and let X be a topological space. A net in X indexed by D is a mapping $x : D \to X$. For convenience, we denote the value of the mapping at $d \in D$ by x_d instead of $x(d)$, and the net itself is written as $(x_d)_{d\in D}$.*

Definition 2.4. *[15] Let $(x_d)_{d\in D}$ be a net in a topological space X, and let $L \in X$. The net (x_d) is said to converge to L if, for every neighborhood U of L, there exists an index $d_0 \in D$ such that $x_d \in U$ whenever $d \geq d_0$. In this case, we write $x_d \to L$.*

Definition 2.5. *Let $(D, \leq)$ be a directed set. For $\alpha, \beta \in D$, we say that β* covers *α and write $\alpha \lessdot \beta$, if $\alpha \prec \beta$ and there is no $\gamma \in D$ such that $\alpha \prec \gamma \prec \beta$.*

We say that D has the immediate successor property if, for every $\alpha \in D$, there exists at least one $\beta \in D$ such that $\alpha \lessdot \beta$. A mapping $\sigma : D \longrightarrow D$ is called a successor mapping if $\alpha \lessdot \sigma(\alpha)$ for every $\alpha \in D$. Thus, $\alpha \prec \sigma(\alpha)$ and there is no $\gamma \in D$ satisfying $\alpha \prec \gamma \prec \sigma(\alpha)$. If an element has more than one immediate successor, the mapping σ represents a fixed choice of one such successor.

Remark 2.1. *A general directed set need not possess the immediate successor property. In particular, densely ordered directed sets, such as $(\mathbb{R}, \leq)$ and $(\mathbb{Q}, \leq)$, have no immediate successors and therefore do not admit a successor mapping in the sense of Definition 2.5. Moreover, a directed set with a maximal element cannot satisfy the immediate successor property.*

Consequently, all definitions and results in this paper involving σ are restricted to directed sets admitting a specified successor mapping. They are not asserted for arbitrary directed sets. This class remains more general than the natural numbers because it may include partially ordered and branching directed systems, but it does not include densely ordered directed sets.

The term "discrete", when used in this context, refers only to the existence of immediate order successors and should not be confused with discreteness with respect to an independently specified topology.

Example 2.1. *Consider the set $D = \{A \subset \mathbb{N} : A \text{ is finite}\}$, ordered by inclusion; that is, for $A, B \in D$, $A \leq B \iff A \subseteq B$. Then $(D, \leq)$ is a directed set. Define the mapping $\sigma : D \longrightarrow D$ by $\sigma(A) = A \cup \{\min(\mathbb{N} \setminus A)\}$, $A \in D$. Since A is finite, $\mathbb{N} \setminus A \neq \emptyset$, and hence $\min(\mathbb{N} \setminus A)$ exists. Therefore, $A \subsetneq \sigma(A)$. Moreover, there is no $B \in D$ such that $A \subsetneq B \subsetneq \sigma(A)$. Indeed, since $\sigma(A) \setminus A$ contains exactly one element, namely $\min(\mathbb{N} \setminus A)$, there is no set properly lying between A and $\sigma(A)$ with respect to the inclusion order. Thus, $\sigma(A)$ is the immediate successor of A in the poset $(D, \subseteq)$, and σ defines a successor mapping on D. Consequently, for a net $x = (x_A)_{A\in D}$, its difference net is defined by $(\Delta x)_A = x_{\sigma(A)} - x_A$, $A \in D$.*

2.1. Net spaces

From here on, we denote by $(D, \leq)$ a directed set and let $x = (x_\alpha)_{\alpha\in D}$ be a real or complex valued net.

Definition 2.6. *The space of bounded nets indexed by D is denoted by $\ell^\infty(D)$ and is defined by $\ell^\infty(D) = \{x = (x_\alpha)_{\alpha\in D} : \sup_{\alpha\in D} |x_\alpha| < \infty\}$. For $x \in \ell^\infty(D)$, the norm is given by $\|x\|_\infty = \sup_{\alpha\in D} |x_\alpha|$.*

Definition 2.7. *The space $C(D)$ consists of all nets in D that converge in the usual sense. Thus* $C(D) = \{x = (x_\alpha)_{\alpha\in D} : \exists\, L \in \mathbb{R} \text{ such that } x_\alpha \to L\}$.

Definition 2.8. *The space $C_0(D)$ is the collection of all nets that converge to zero, that is,* $C_0(D) = \{x = (x_\alpha)_{\alpha\in D} : x_\alpha \to 0\}$.

It is immediate from the definitions that $C_0(D) \subset C(D) \subset \ell^\infty(D)$.

Definition 2.9. *Let $\sigma : D \to D$ be a successor mapping and $(D, \preceq)$ be a directed set. For a net $x = (x_\alpha)_{\alpha\in D}$, we define the difference operator Δ by $(\Delta x)_\alpha = x_{\sigma(\alpha)} - x_\alpha, \alpha \in D$.*

Throughout the sequel, whenever $\ell^\infty(\Delta, D)$ is equipped with the norm $\|\cdot\|_\Delta$, we assume that D has a least element α_0 and that, for every $\alpha \in D$, there exist non-negative integers m and n such that

$$\sigma^m(\alpha) = \sigma^n(\alpha_0).$$

Definition 2.10. *The difference-bounded net space is defined by*

$$\ell^\infty(\Delta, D) = \left\{ x = (x_\alpha)_{\alpha\in D} : \sup_{\alpha\in D} \left| x_{\sigma(\alpha)} - x_\alpha \right| < \infty \right\}.$$

For $x \in \ell^\infty(\Delta, D)$, we define $\|x\|_\Delta = |x_{\alpha_0}| + \sup_{\alpha\in D} \left| x_{\sigma(\alpha)} - x_\alpha \right|$.

Definition 2.11. *The difference convergent net space is defined as*

$$C(\Delta, D) = \{x = (x_\alpha)_{\alpha\in D} : (x_{\sigma(\alpha)} - x_\alpha) \text{ converges}\}.$$

Definition 2.12. *The difference null net space is given by*

$$C_0(\Delta, D) = \{x = (x_\alpha)_{\alpha\in D} : x_{\sigma(\alpha)} - x_\alpha \to 0\}.$$

Clearly, $C_0(\Delta, D) \subset C(\Delta, D) \subset \ell^\infty(\Delta, D)$.

We now present examples illustrating how the difference operator may transform certain nets that are not bounded or not convergent into nets having better properties.

Example 2.2. *Let $D = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$ be the family of all finite subsets of $\mathbb{N}$, ordered by inclusion:*

$$A \preceq B \iff A \subseteq B.$$

Then $(D, \preceq)$ is a directed set. Indeed, for any $A, B \in D$, the finite set $A \cup B$ is an upper bound of both A and B.

For every $A \in D$, define $m_A = \min(\mathbb{N} \setminus A)$ and $\sigma(A) = A \cup \{m_A\}$. Since A is finite, $\mathbb{N} \setminus A$ is nonempty, and hence m_A is well defined. Clearly,

$$A \subsetneq \sigma(A).$$

Moreover, there is no $B \in D$ such that $A \subsetneq B \subsetneq \sigma(A)$. Indeed,

$$\sigma(A) \setminus A = \{m_A\}$$

contains exactly one element. Thus, if

$$A \subseteq B \subseteq \sigma(A),$$

then either $B = A$ or $B = \sigma(A)$. Therefore,

$$A \lessdot \sigma(A),$$

and σ is a successor mapping on D.

Now define the scalar-valued net $x = (x_A)_{A\in D}$ by

$$x_A = |A|,$$

where $|A|$ denotes the cardinality of A. This net is unbounded. In fact, for every $N \in \mathbb{N}$, there exists a finite set $A \in D$ such that

$$x_A = |A| > N.$$

Consequently,

$$\sup_{A\in D} |x_A| = \infty.$$

On the other hand,

$$(\Delta x)_A = x_{\sigma(A)} - x_A.$$

Since $m_A \notin A$, we have

$$|\sigma(A)| = |A \cup \{m_A\}| = |A| + 1.$$

Therefore,

$$(\Delta x)_A = (|A| + 1) - |A| = 1$$

for every $A \in D$. Hence,

$$\sup_{A\in D} |(\Delta x)_A| = 1 < \infty.$$

Thus, x is an unbounded net whose difference net Δx is bounded.

Example 2.3. *We now give an example of a net that does not belong to (C, D) or (C_0, D), but whose difference net belongs to both $C(\Delta, D)$ and $C_0(\Delta, D)$.*

Let $D = \mathbb{N} \times \mathbb{N}$ with the order

$$(m, n) \leq (p, q) \quad \Longleftrightarrow \quad m \leq p \text{ and } n \leq q.$$

For $\alpha = (m, n) \in D$, define the successor mapping by

$$\sigma(m, n) = (m + 1, n + 1).$$

Clearly,

$$(m, n) \prec \sigma(m, n),$$

and there is no $(p, q) \in D$ such that

$$(m, n) \prec (p, q) \prec (m + 1, n + 1).$$

Thus, σ is an immediate successor mapping on D.

Define a net $x = (x_{m,n})$ by

$$x_{m,n} = \frac{m}{m + n}.$$

First, we show that the net does not converge. Consider the two cofinal paths. If $m = n$, then

$$x_{n,n} = \frac{n}{2n} = \frac{1}{2}.$$

On the other hand, if $n = 1$ and $m \to \infty$, then

$$x_{m,1} = \frac{m}{m+1} \longrightarrow 1.$$

Since the two limits are different, the net does not converge. Hence,

$$x \notin (C, D).$$

In particular, $x \notin (C_0, D)$.
Now, according to Definition 2.9, the difference net is defined using the successor mapping as

$$(\Delta x)_{m,n} = x_{\sigma(m,n)} - x_{m,n} = x_{m+1,n+1} - x_{m,n}.$$

Therefore,

$$(\Delta x)_{m,n} = \frac{m+1}{m+n+2} - \frac{m}{m+n}.$$

After simplification, we obtain

$$(\Delta x)_{m,n} = \frac{n-m}{(m+n)(m+n+2)}.$$

Taking absolute values gives

$$|(\Delta x)_{m,n}| = \frac{|n-m|}{(m+n)(m+n+2)}.$$

Since

$$|n-m| \leq m+n,$$

we have

$$|(\Delta x)_{m,n}| \leq \frac{1}{m+n+2}.$$

As $(m, n) \to \infty$ in the directed set D, we have

$$m + n \longrightarrow \infty,$$

and consequently

$$(\Delta x)_{m,n} \longrightarrow 0.$$

Hence,

$$\Delta x \in C_0(\Delta, D) \subset C(\Delta, D).$$

Thus, the net x does not belong to (C, D) or (C_0, D), where as its forward difference net belongs to both $C(\Delta, D)$ and $C_0(\Delta, D)$.

The generalized difference operator provides a systematic way of studying nets that may not themselves possess a simple convergence behavior. Given a net $x = (x_\alpha)_{\alpha\in D}$, the operator Δ_D produces the transformed net

$$\Delta_D x = \big((\Delta_D x)_\alpha\big)_{\alpha\in D}.$$

Although the original net may be divergent or irregular, its generalized differences may satisfy a useful regularity condition. In particular, the transformed net may be bounded, convergent, or convergent to zero. These properties determine whether the original net belongs to the corresponding generalized difference space. This conceptual process is illustrated in Figure 1.

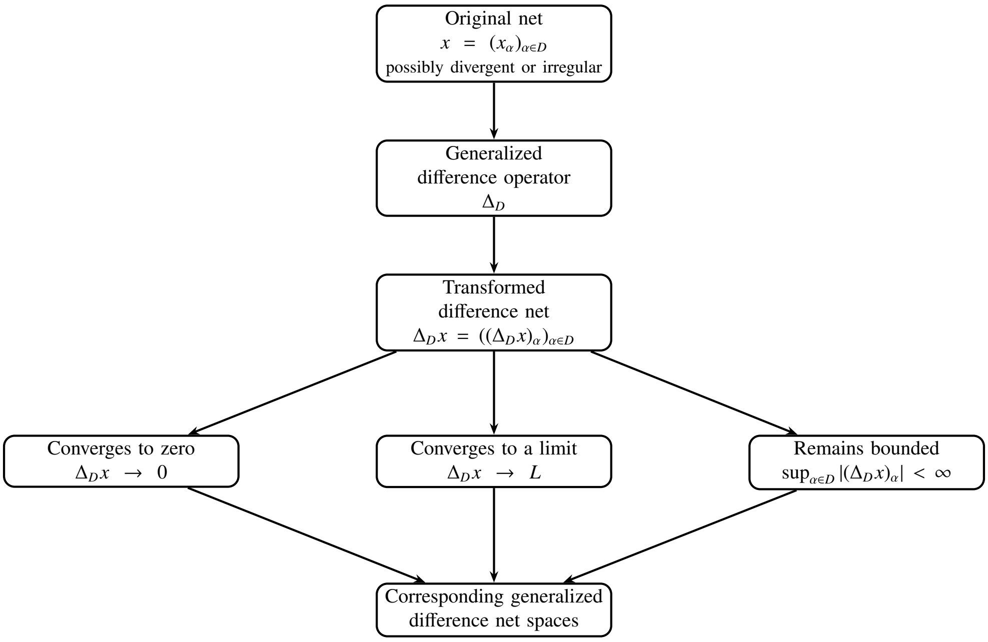

Figure 1. Conceptual role of the generalized difference operator in transforming a potentially irregular net into a difference net whose regularity determines membership in the associated spaces.

3. Main results

In this section, we prove the completeness of the spaces $\ell^\infty(\Delta, D)$, $C(\Delta, D)$, and $C_0(\Delta, D)$.

Theorem 3.1. *Let $(D, \leq)$ be a directed set with a least element α_0, and let $\sigma : D \to D$ be a successor mapping. Assume that, for every $\alpha \in D$, there exist non-negative integers m and n such that $\sigma^m(\alpha) = \sigma^n(\alpha_0)$. Let X be a Banach space of scalar-valued nets on D, and suppose that the difference operator $\Delta x = (x_{\sigma(\alpha)} - x_\alpha)_{\alpha\in D}$ is defined on X. Set $X(\Delta, D) = \{x \in X : \Delta x \in X\}$. For $x \in X(\Delta, D)$, define $\|x\|_\Delta = |x_{\alpha_0}| + \|\Delta x\|_X$. Assume, in addition, that the range of the mapping $T : X(\Delta, D) \longrightarrow \mathbb{K} \times X$ defined by $T(x) = (x_{\alpha_0}, \Delta x)$, is closed in $\mathbb{K} \times X$. Then $(X(\Delta, D), \|\cdot\|_\Delta)$ is a Banach space.*

Proof. First, we show that $\|\cdot\|_\Delta$ is a norm. Clearly, $\|x\|_\Delta \geq 0$ and the homogeneity and triangle inequality follow directly from the corresponding properties of the absolute value and the norm on X.

Suppose that

$$\|x\|_\Delta = 0.$$

Then

$$x_{\alpha_0} = 0 \textit{ and } \Delta x = 0.$$

Thus

$$x_{\sigma(\alpha)} = x_\alpha \qquad \text{for every } \alpha \in D.$$

Hence

$$x_{\sigma^k(\alpha)} = x_\alpha \qquad \text{for every } k \geq 0.$$

Given $\alpha \in D$, by assumption, there exist non-negative integers m and n such that

$$\sigma^m(\alpha) = \sigma^n(\alpha_0).$$

Therefore,

$$x_\alpha = x_{\sigma^m(\alpha)} = x_{\sigma^n(\alpha_0)} = x_{\alpha_0} = 0.$$

Thus $x = 0$, and so $\|\cdot\|_\Delta$ is a norm. Now define $T : X(\Delta, D) \longrightarrow \mathbb{K} \times X$ by

$$T(x) = (x_{\alpha_0}, \Delta x).$$

By the definition of $\|\cdot\|_\Delta$,

$$\|x\|_\Delta = |x_{\alpha_0}| + \|\Delta x\|_X = \|T(x)\|_{\mathbb{K}\times X},$$

where $\mathbb{K} \times X$ is equipped with the norm

$$\|(a, y)\| = |a| + \|y\|_X.$$

Hence T is an isometry from $(X(\Delta, D), \|\cdot\|_\Delta)$ onto $T(X(\Delta, D))$.

Since $\mathbb{K}$ and X are Banach spaces, $\mathbb{K} \times X$ is a Banach space. By assumption, $T(X(\Delta, D))$ is a closed subspace of $\mathbb{K} \times X$ and is therefore complete. Since T is an isometry, it follows that

$$(X(\Delta, D), \|\cdot\|_\Delta)$$

is complete. Hence it is a Banach space. □

Corollary 3.1. *Under the assumptions of Theorem 3.1, each of the spaces* $\ell^\infty(\Delta, D)$, $C(\Delta, D)$, *and* $C_0(\Delta, D)$ *is a Banach space with respect to the norm*

$$\|x\|_\Delta = |x_{\alpha_0}| + \sup_{\alpha\in D} \left| x_{\sigma(\alpha)} - x_\alpha \right|.$$

Proof. The spaces $\ell^\infty(D)$, $C(D)$, and $C_0(D)$ are Banach spaces under the supremum norm. Applying Theorem 3.1 to each of these spaces, with the difference operator defined by

$$(\Delta x)_\alpha = x_{\sigma(\alpha)} - x_\alpha,$$

gives the completeness of $\ell^\infty(\Delta, D)$, $C(\Delta, D)$, and $C_0(\Delta, D)$, respectively, with respect to the norm

$$\|x\|_\Delta = |x_{\alpha_0}| + \|\Delta x\|_\infty = |x_{\alpha_0}| + \sup_{\alpha\in D} \left| x_{\sigma(\alpha)} - x_\alpha \right|.$$

Hence, each of the spaces $\ell^\infty(\Delta, D)$, $C(\Delta, D)$, and $C_0(\Delta, D)$ is a Banach space. □

4. Duals of difference net spaces

In this section, we present the duals of difference net spaces.

Theorem 4.1. *Let $(D, \preceq)$ be a directed set with a least element α_0, and let $\sigma : D \to D$ be a successor mapping. Assume that, for every $\alpha \in D$, there exist non-negative integers m and n such that $\sigma^m(\alpha) = \sigma^n(\alpha_0)$. Let $\ell^\infty(\Delta, D) = \left\{x = (x_\alpha)_{\alpha\in D} : \sup_{\alpha\in D} \left|x_{\sigma(\alpha)} - x_\alpha\right| < \infty\right\}$, equipped with the norm*

$$\|x\|_\Delta = |x_{\alpha_0}| + \sup_{\alpha\in D} \left|x_{\sigma(\alpha)} - x_\alpha\right|.$$

Define $T : \ell^\infty(\Delta, D) \longrightarrow \mathbb{K} \times \ell^\infty(D)$ by $T(x) = (x_{\alpha_0}, \Delta x)$. Then T is an isometric linear embedding. Consequently,

$$(\ell^\infty(\Delta, D))^* \cong T(\ell^\infty(\Delta, D))^*.$$

If, in addition, $T(\ell^\infty(\Delta, D))$ is a closed subspace of $\mathbb{K} \times \ell^\infty(D)$, then

$$(\ell^\infty(\Delta, D))^* \cong \frac{\mathbb{K} \times ba(D)}{(T(\ell^\infty(\Delta, D)))^\perp} \text{ as Banach spaces.}$$

Proof. Define $T : \ell^\infty(\Delta, D) \longrightarrow \mathbb{K} \times \ell^\infty(D)$ by

$$T(x) = (x_{\alpha_0}, \Delta x).$$

The map T is clearly linear. Moreover, for every $x \in \ell^\infty(\Delta, D)$,

$$\begin{aligned}\|T(x)\|_{\mathbb{K}\times\ell^\infty(D)} &= |x_{\alpha_0}| + \|\Delta x\|_\infty \\ &= |x_{\alpha_0}| + \sup_{\alpha\in D} \left|x_{\sigma(\alpha)} - x_\alpha\right| \\ &= \|x\|_\Delta.\end{aligned}$$

Thus, T is an isometry.
We next show that T is injective. Suppose that

$$T(x) = (0, 0).$$

Then

$$x_{\alpha_0} = 0 \text{ and } \Delta x = 0.$$

Hence

$$x_{\sigma(\alpha)} = x_\alpha \qquad \text{for every } \alpha \in D.$$

Therefore,

$$x_{\sigma^k(\alpha)} = x_\alpha \qquad \text{for every } k \geq 0.$$

Fix $\alpha \in D$. By assumption, there exist non-negative integers m and n such that

$$\sigma^m(\alpha) = \sigma^n(\alpha_0).$$

It follows that

$$x_\alpha = x_{\sigma^m(\alpha)} = x_{\sigma^n(\alpha_0)} = x_{\alpha_0} = 0.$$

Thus $x = 0$, and hence T is injective.
Since T is an isometry, $\ell^\infty(\Delta, D)$ is isometrically isomorphic to its image

$$R := T(\ell^\infty(\Delta, D)).$$

Therefore,

$$(\ell^\infty(\Delta, D))^* \cong R^* \text{ isometrically.}$$

Suppose now that R is closed in $\mathbb{K} \times \ell^\infty(D)$. Since $\mathbb{K} \times \ell^\infty(D)$ is a Banach space, R is also a Banach space. By the Hahn–Banach theorem, every continuous linear functional on R extends to a continuous linear functional on $\mathbb{K} \times \ell^\infty(D)$. Hence

$$R^* \cong \frac{(\mathbb{K} \times \ell^\infty(D))^*}{R^\perp}.$$

Since

$$(\mathbb{K} \times \ell^\infty(D))^* \cong \mathbb{K} \times ba(D),$$

we obtain

$$(\ell^\infty(\Delta, D))^* \cong \frac{\mathbb{K} \times ba(D)}{R^\perp}.$$

This proves the result. □

Theorem 4.2. *Let $(D, \preceq)$ be a directed set with a least element α_0, and let $\sigma : D \to D$ be a successor mapping. Assume that, for every $\alpha \in D$, there exist non-negative integers m and n such that*

$$\sigma^m(\alpha) = \sigma^n(\alpha_0).$$

Define $c(\Delta, D) = \{x \in \ell^\infty(\Delta, D) : \lim_\alpha (\Delta x)_\alpha \text{ exists}\}$, where $(\Delta x)_\alpha = x_{\sigma(\alpha)} - x_\alpha$. Equip $c(\Delta, D)$ with the norm

$$\|x\|_\Delta = |x_{\alpha_0}| + \sup_{\alpha \in D} |x_{\sigma(\alpha)} - x_\alpha|.$$

Then the mapping $T : c(\Delta, D) \longrightarrow \mathbb{K} \times c(D)$ defined by

$$T(x) = (x_{\alpha_0}, \Delta x)$$

is an isometric linear embedding. Consequently,

$$c(\Delta, D)^* \cong T(c(\Delta, D))^* \text{ isometrically.}$$

If, in addition, $T(c(\Delta, D))$ is a closed subspace of $\mathbb{K} \times c(D)$, then

$$c(\Delta, D)^* \cong \frac{\mathbb{K} \times c(D)^*}{T(c(\Delta, D))^\perp}.$$

Since $c(D)^* \cong \ell^1(D) \oplus \mathbb{K}$, the dual can be represented as a quotient of $\mathbb{K} \times (\ell^1(D) \oplus \mathbb{K})$.

Proof. Define
$$T : c(\Delta, D) \longrightarrow \mathbb{K} \times c(D)$$
by
$$T(x) = (x_{\alpha_0}, \Delta x).$$
The map T is linear. Moreover, for every $x \in c(\Delta, D)$,
$$\begin{aligned}\|T(x)\| &= |x_{\alpha_0}| + \|\Delta x\|_\infty \\ &= |x_{\alpha_0}| + \sup_{\alpha \in D} |x_{\sigma(\alpha)} - x_\alpha| \\ &= \|x\|_\Delta.\end{aligned}$$
Thus, T is an isometry.

We next show that T is injective. Suppose that
$$T(x) = (0, 0).$$
Then
$$x_{\alpha_0} = 0$$
and
$$x_{\sigma(\alpha)} = x_\alpha \qquad \text{for every } \alpha \in D.$$
Hence,
$$x_{\sigma^k(\alpha)} = x_\alpha \qquad \text{for every } k \geq 0.$$
For any $\alpha \in D$, by the standing assumption, there exist non-negative integers m and n such that
$$\sigma^m(\alpha) = \sigma^n(\alpha_0).$$
Therefore,
$$x_\alpha = x_{\sigma^m(\alpha)} = x_{\sigma^n(\alpha_0)} = x_{\alpha_0} = 0.$$
Thus $x = 0$, and T is injective.

Since T is an isometry, $c(\Delta, D)$ is isometrically isomorphic to its range
$$R = T(c(\Delta, D)).$$
Therefore,
$$c(\Delta, D)^* \cong R^*$$
isometrically.

If R is closed in $\mathbb{K} \times c(D)$, then R is a Banach space. By the Hahn–Banach theorem,
$$R^* \cong \frac{(\mathbb{K} \times c(D))^*}{R^\perp}.$$
Since
$$(\mathbb{K} \times c(D))^* \cong \mathbb{K} \times c(D)^*$$

and

$$c(D)^* \cong \ell^1(D) \oplus \mathbb{K},$$

we obtain

$$c(\Delta, D)^* \cong \frac{\mathbb{K} \times (\ell^1(D) \oplus \mathbb{K})}{R^\perp}.$$

This completes the proof. □

Theorem 4.3. *Let $(D, \leq)$ be a directed set with a least element α_0, and let $\sigma : D \to D$ be a successor mapping. Assume that, for every $\alpha \in D$, there exist non-negative integers m and n such that*

$$\sigma^m(\alpha) = \sigma^n(\alpha_0).$$

Let

$$c_0(\Delta, D) = \{x \in \ell^\infty(\Delta, D) : \Delta x \in c_0(D)\},$$

where

$$(\Delta x)_\alpha = x_{\sigma(\alpha)} - x_\alpha.$$

Equip $c_0(\Delta, D)$ with the norm

$$\|x\|_\Delta = |x_{\alpha_0}| + \sup_{\alpha \in D} |x_{\sigma(\alpha)} - x_\alpha|.$$

Define

$$T : c_0(\Delta, D) \longrightarrow \mathbb{K} \times c_0(D)$$

by

$$T(x) = (x_{\alpha_0}, \Delta x).$$

Then T is a linear isometric embedding. Consequently,

$$(c_0(\Delta, D))^* \cong T(c_0(\Delta, D))^*$$

isometrically.

If, in addition, $T(c_0(\Delta, D))$ is a closed subspace of $\mathbb{K} \times c_0(D)$, then

$$(c_0(\Delta, D))^* \cong \frac{\mathbb{K} \times \ell^1(D)}{(T(c_0(\Delta, D)))^\perp}$$

as Banach spaces.

Proof. Define

$$T : c_0(\Delta, D) \longrightarrow \mathbb{K} \times c_0(D)$$

by

$$T(x) = (x_{\alpha_0}, \Delta x).$$

The map T is linear. Moreover, for every $x \in c_0(\Delta, D)$,

$$\begin{aligned} \|T(x)\| &= |x_{\alpha_0}| + \|\Delta x\|_\infty \\ &= |x_{\alpha_0}| + \sup_{\alpha \in D} |x_{\sigma(\alpha)} - x_\alpha| \\ &= \|x\|_\Delta. \end{aligned}$$

Hence, T is an isometry.

We now show that T is injective. Suppose that

$$T(x) = (0, 0).$$

Then

$$x_{\alpha_0} = 0$$

and

$$x_{\sigma(\alpha)} - x_\alpha = 0 \qquad \text{for every } \alpha \in D.$$

Thus

$$x_{\sigma(\alpha)} = x_\alpha \qquad \text{for every } \alpha \in D.$$

Consequently,

$$x_{\sigma^k(\alpha)} = x_\alpha \qquad \text{for every } k \geq 0.$$

Fix $\alpha \in D$. By the standing assumption, there exist non-negative integers m and n such that

$$\sigma^m(\alpha) = \sigma^n(\alpha_0).$$

Therefore,

$$x_\alpha = x_{\sigma^m(\alpha)} = x_{\sigma^n(\alpha_0)} = x_{\alpha_0} = 0.$$

Since α is arbitrary, $x = 0$. Hence T is injective.

Thus, T is an isometric isomorphism from $c_0(\Delta, D)$ onto its range

$$R = T(c_0(\Delta, D)).$$

It follows that

$$(c_0(\Delta, D))^* \cong R^*$$

isometrically.

If R is closed in $\mathbb{K} \times c_0(D)$, then R is a Banach space. By the Hahn–Banach theorem,

$$R^* \cong \frac{(\mathbb{K} \times c_0(D))^*}{R^\perp}.$$

Since

$$c_0(D)^* \cong \ell^1(D),$$

we have

$$(\mathbb{K} \times c_0(D))^* \cong \mathbb{K} \times \ell^1(D).$$

Consequently,

$$(c_0(\Delta, D))^* \cong \frac{\mathbb{K} \times \ell^1(D)}{R^\perp}.$$

This completes the proof. □

5. Separability of difference net spaces

In this section, we will see whether the difference net spaces are separable or not.

Theorem 5.1. *Let $(D, \preceq)$ be an infinite directed set with a least element α_0, and let $\sigma : D \to D$ be a successor mapping. Assume that, for every $\alpha \in D$, there exist non-negative integers m and n such that*

$$\sigma^m(\alpha) = \sigma^n(\alpha_0).$$

Then the difference bounded net space

$$\ell^\infty(\Delta, D) = \left\{ x = (x_\alpha)_{\alpha\in D} : \sup_{\alpha\in D} |x_{\sigma(\alpha)} - x_\alpha| < \infty \right\}$$

is non-separable with respect to the norm

$$||x||_\Delta = |x_{\alpha_0}| + \sup_{\alpha\in D} |x_{\sigma(\alpha)} - x_\alpha|.$$

Proof. For each bounded scalar net $y = (y_\alpha)_{\alpha\in D} \in \ell^\infty(D)$, define $x = T(y)$ by

$$x_{\alpha_0} = 0$$

and, along the successor orbits, define x recursively by

$$x_{\sigma(\alpha)} = x_\alpha + y_\alpha.$$

Under the standing assumptions on D and σ, this construction gives a well-defined element $x \in \ell^\infty(\Delta, D)$ satisfying

$$\Delta x = y.$$

Consequently,

$$||x||_\Delta = |x_{\alpha_0}| + ||\Delta x||_\infty = ||y||_\infty.$$

Thus, T is an isometric embedding of $\ell^\infty(D)$ into $\ell^\infty(\Delta, D)$.

Since D is infinite, $\ell^\infty(D)$ is non-separable with respect to the supremum norm. Hence $\ell^\infty(\Delta, D)$ contains an isometric copy of a non-separable Banach space. Therefore, $\ell^\infty(\Delta, D)$ itself cannot be separable.

Thus

$$\ell^\infty(\Delta, D)$$

is non-separable with respect to the norm $|| \cdot ||_\Delta$. □

Theorem 5.2. *Let $(D, \preceq)$ be a countable directed set with a least element α_0, and let $\sigma : D \to D$ be a successor mapping. Assume that, for every $\alpha \in D$, there exist non-negative integers m and n such that*

$$\sigma^m(\alpha) = \sigma^n(\alpha_0).$$

Let

$$c(\Delta, D) = \left\{ x \in \ell^\infty(\Delta, D) : \lim_\alpha (\Delta x)_\alpha \text{ exists} \right\},$$

where

$$(\Delta x)_\alpha = x_{\sigma(\alpha)} - x_\alpha.$$

Then $c(\Delta, D)$ *is separable with respect to the norm*

$$\|x\|_\Delta = |x_{\alpha_0}| + \sup_{\alpha \in D} |x_{\sigma(\alpha)} - x_\alpha|.$$

Proof. Define

$$T : c(\Delta, D) \longrightarrow \mathbb{K} \times c(D)$$

by

$$T(x) = (x_{\alpha_0}, \Delta x).$$

By the definition of the norm,

$$\|x\|_\Delta = |x_{\alpha_0}| + \|\Delta x\|_\infty.$$

Thus, T is an isometry onto its range.

Since D is countable, the space $c(D)$ of convergent scalar nets can be identified, after fixing an enumeration of D with the corresponding space of convergent sequences. In particular, $c(D)$ is separable with respect to the supremum norm. Since $\mathbb{K}$ is separable, the product space

$$\mathbb{K} \times c(D)$$

is separable.

The range

$$T(c(\Delta, D))$$

is a subspace of $\mathbb{K} \times c(D)$. Under the standing assumptions, the anchor value x_{α_0} together with the difference net Δx determines x uniquely. Hence T is an isometric identification of $c(\Delta, D)$ with its range.

Therefore $c(\Delta, D)$ is isometric to a subspace of the separable space $\mathbb{K} \times c(D)$. Since every subspace of a separable metric space is separable, it follows that

$$c(\Delta, D)$$

is separable with respect to $\|\cdot\|_\Delta$. □

Theorem 5.3. *Let* $(D, \preceq)$ *be a countable directed set with a least element* α_0*, and let* $\sigma : D \to D$ *be a successor mapping. Assume that, for every* $\alpha \in D$*, there exist non-negative integers* m *and* n *such that*

$$\sigma^m(\alpha) = \sigma^n(\alpha_0).$$

Let

$$c_0(\Delta, D) = \{x \in \ell^\infty(\Delta, D) : \Delta x \in c_0(D)\},$$

where

$$(\Delta x)_\alpha = x_{\sigma(\alpha)} - x_\alpha.$$

Then $c_0(\Delta, D)$ *is separable with respect to the norm*

$$\|x\|_\Delta = |x_{\alpha_0}| + \sup_{\alpha \in D} |x_{\sigma(\alpha)} - x_\alpha|.$$

Proof. Define

$$T : c_0(\Delta, D) \longrightarrow \mathbb{K} \times c_0(D)$$

by

$$T(x) = (x_{\alpha_0}, \Delta x).$$

By the definition of the norm,

$$\|T(x)\| = |x_{\alpha_0}| + \|\Delta x\|_\infty = \|x\|_\Delta.$$

Hence, T is an isometry.

We next show that T is injective. Suppose that

$$T(x) = (0, 0).$$

Then

$$x_{\alpha_0} = 0$$

and

$$x_{\sigma(\alpha)} = x_\alpha \qquad \text{for every } \alpha \in D.$$

Thus

$$x_{\sigma^k(\alpha)} = x_\alpha \qquad \text{for every } k \geq 0.$$

By the standing assumption, for every $\alpha \in D$, there exist non-negative integers m and n such that

$$\sigma^m(\alpha) = \sigma^n(\alpha_0).$$

Consequently,

$$x_\alpha = x_{\sigma^m(\alpha)} = x_{\sigma^n(\alpha_0)} = x_{\alpha_0} = 0.$$

Therefore, $x = 0$, and T is injective.

Since D is countable, $c_0(D)$ is a separable Banach space with respect to the supremum norm. Since $\mathbb{K}$ is also separable, the product space

$$\mathbb{K} \times c_0(D)$$

is separable.

The range

$$T(c_0(\Delta, D))$$

is a subspace of $\mathbb{K} \times c_0(D)$. Since T is an isometry, $c_0(\Delta, D)$ is isometric to this range. Therefore, $c_0(\Delta, D)$ is separable with respect to the norm $\|\cdot\|_\Delta$. □

For clarity, Table 1 summarizes the relationship between the classical Kızmaz sequence spaces and their corresponding net-based generalizations. The classical construction is recovered by taking the directed index set to be $\mathbb{N}$ equipped with its usual order. Therefore, the proposed framework extends the classical sequential setting to more general directed index sets.

Table 1. Comparison between the classical Kızmaz spaces and their net-based generalizations.

Feature	**Classical Kızmaz spaces**	**Proposed net spaces**
Basic object	A sequence $(x_n)_{n\in\mathbb{N}}$	A net $(x_\alpha)_{\alpha\in D}$
Index set	The set $\mathbb{N}$ with its usual order	A general directed set D
Index structure	Countable and linearly ordered	Directed and possibly uncountable
Convergence	Sequential convergence	Net convergence
Difference transformation	The classical difference operator $\Delta x_n = x_n - x_{n-1}$	The corresponding net-difference operator Δx_α, as defined in this paper
Null difference space	$c_0(\Delta)$	$c_0(\Delta, D)$
Convergent difference space	$c(\Delta)$	$c(\Delta, D)$
Bounded difference space	$\ell_\infty(\Delta)$	$\ell_\infty(\Delta, D)$
Defining condition	The relevant condition is imposed on the transformed sequence (Δx_n)	The relevant condition is imposed on the transformed net (Δx_α)
Generalization relationship	Represents the classical sequential case	Reduces to the classical setting when $D = \mathbb{N}$
Scope	Applicable to sequential structures	Applicable to general directed and topological structures

6. Application to fixed point theory

The principal advantage of this framework is that it treats the successive differences of an iterative net as elements of a normed space. Consequently, the convergence and stability of an iterative procedure can be studied through the difference net

$$(\Delta x)_\alpha = x_{\sigma(\alpha)} - x_\alpha.$$

In particular, the quantity

$$\|\Delta x_\alpha\| = \|x_{\sigma(\alpha)} - x_\alpha\|$$

is directly computable from two successive iterates. It can therefore serve as a residual measuring how close the current iterate is to being a fixed point. This is useful because the fixed point itself is usually unknown. For example, if $T : X \to X$ is a contraction with constant $0 < k < 1$ and

$$x_{\sigma(\alpha)} = T(x_\alpha),$$

then

$$\|x_{\sigma^2(\alpha)} - x_{\sigma(\alpha)}\| = \|T(x_{\sigma(\alpha)}) - T(x_\alpha)\| \le k\|x_{\sigma(\alpha)} - x_\alpha\|.$$

By induction,

$$\|x_{\sigma^{n+1}(\alpha)} - x_{\sigma^n(\alpha)}\| \le k^n \|x_{\sigma(\alpha)} - x_\alpha\|.$$

Thus, the associated difference net tends to zero along the successive orbit. This shows that the iteration is asymptotically regular. Moreover, if p denotes the unique fixed point of T, then

$$\begin{aligned}\|x_\alpha - p\| &\le \|x_\alpha - x_{\sigma(\alpha)}\| + \|x_{\sigma(\alpha)} - p\| \\ &= \|x_\alpha - x_{\sigma(\alpha)}\| + \|T(x_\alpha) - T(p)\| \\ &\le \|x_\alpha - x_{\sigma(\alpha)}\| + k\|x_\alpha - p\|.\end{aligned}$$

Therefore,

$$\|x_\alpha - p\| \le \frac{1}{1-k}\|x_{\sigma(\alpha)} - x_\alpha\| = \frac{1}{1-k}\|(\Delta x)_\alpha\|.$$

This is an a posteriori error estimate: It bounds the unknown error $\|x_\alpha - p\|$ in terms of the computable difference between two successive iterates. In particular, the condition

$$\|x_{\sigma(\alpha)} - x_\alpha\| < (1-k)\varepsilon$$

guarantees that

$$\|x_\alpha - p\| < \varepsilon.$$

Hence, the difference net provides a practical stopping criterion for fixed point algorithms. The merit of the paper is therefore not merely that a contraction iteration converges, which is already known from the Banach contraction principle. Rather, the spaces introduced in the paper provide a functional-analytic setting in which

1. successive-iterate differences are treated as elements of normed or Banach net spaces;
2. asymptotic regularity can be expressed as membership in $C_0(\Delta, D)$;
3. perturbations and stability of iterative nets can be measured using the difference norm; and
4. computable residuals yield quantitative error estimates and stopping criteria.

Theorem 6.1. *Let $(D, \preceq)$ be a directed set, let $\sigma : D \to D$ be a successor mapping, and let $(X, \|\cdot\|)$ be a Banach space. Suppose that $T : X \to X$ is a contraction with constant $k \in (0, 1)$.*

Let $D_0 \subseteq D$, and suppose that there exists a mapping

$$\nu : D \to \mathbb{N}_0$$

such that the following conditions hold:

1. *for every $\alpha \in D$, there exists $\gamma \in D_0$ such that*

$$\alpha = \sigma^{\nu(\alpha)}(\gamma);$$

2. *$\nu(\alpha) \to \infty$ as $\alpha \to \infty$ in D;*
3. *the initial successor differences are uniformly bounded:*

$$M := \sup_{\gamma \in D_0} \|x_{\sigma(\gamma)} - x_\gamma\| < \infty.$$

Assume that the net $x = (x_\alpha)_{\alpha\in D}$ satisfies

$$x_{\sigma(\alpha)} = T(x_\alpha), \qquad \alpha \in D.$$

Then

$$\|(\Delta x)_\alpha\| \le k^{\nu(\alpha)} M, \qquad \alpha \in D.$$

Consequently,

$$(\Delta x)_\alpha = x_{\sigma(\alpha)} - x_\alpha \longrightarrow 0,$$

and hence

$$x \in C_0(\Delta, D; X).$$

Moreover,

$$\sup_{\alpha\in D} \|x_{\sigma(\alpha)} - x_\alpha\| \le M.$$

If D has a least element α_0, then

$$\|x\|_\Delta = \|x_{\alpha_0}\| + \sup_{\alpha\in D} \|x_{\sigma(\alpha)} - x_\alpha\| \le \|x_{\alpha_0}\| + M.$$

Proof. Let $\alpha \in D$. By assumption, there exists $\gamma \in D_0$ such that

$$\alpha = \sigma^{\nu(\alpha)}(\gamma).$$

The contraction property and the iterative identity give

$$\begin{aligned}\|(\Delta x)_{\sigma(\beta)}\| &= \|x_{\sigma^2(\beta)} - x_{\sigma(\beta)}\| \\ &= \|T(x_{\sigma(\beta)}) - T(x_\beta)\| \\ &\le k\|x_{\sigma(\beta)} - x_\beta\| \\ &= k\|(\Delta x)_\beta\|.\end{aligned}$$

Applying this inequality successively along the successor orbit of γ yields

$$\begin{aligned}\|(\Delta x)_\alpha\| &= \|(\Delta x)_{\sigma^{\nu(\alpha)}(\gamma)}\| \\ &\le k^{\nu(\alpha)}\|(\Delta x)_\gamma\| \\ &\le k^{\nu(\alpha)} M.\end{aligned}$$

Since $\nu(\alpha) \to \infty$ and $0 < k < 1$, it follows that

$$k^{\nu(\alpha)} M \longrightarrow 0.$$

Therefore

$$\|(\Delta x)_\alpha\| \longrightarrow 0,$$

so

$$x \in C_0(\Delta, D; X).$$

Finally, $k^{\nu(\alpha)} \le 1$ gives

$$\sup_{\alpha\in D} \|(\Delta x)_\alpha\| \le M.$$

If D has a least element α_0, the asserted estimate for the anchored difference norm follows immediately. □

Illustrative examples

Example 6.1. *Let*

$$D = \mathcal{F}(\mathbb{N})$$

be the family of all finite subsets of $\mathbb{N} = \{1, 2, \ldots\}$*, ordered by inclusion:*

$$A \le B \quad \Longleftrightarrow \quad A \subseteq B.$$

Then D *is a directed set with least element* $\alpha_0 = \emptyset$*. Indeed, for* $A, B \in D$*, the finite set* $A \cup B$ *is an upper bound of both* A *and* B*. Notice that* D *is not linearly ordered. Define the successor mapping* $\sigma : D \to D$ *by*

$$\sigma(A) = A \cup \{\min(\mathbb{N} \setminus A)\}.$$

Since A *is finite,* $\mathbb{N} \setminus A$ *is nonempty, so* $\sigma(A)$ *is well defined. Moreover,*

$$|\sigma(A)| = |A| + 1.$$

Consider the Banach space $X = \mathbb{R}$ *with its usual norm and the contraction* $T : X \to X$ *defined by*

$$T(t) = \frac{t}{2}.$$

Define an iterative net $x = (x_A)_{A \in D}$ *by*

$$x_A = 2^{-|A|}, \qquad A \in D.$$

Using $|\sigma(A)| = |A| + 1$*, we obtain*

$$\begin{aligned} x_{\sigma(A)} &= 2^{-|\sigma(A)|} \\ &= 2^{-(|A|+1)} \\ &= \frac{1}{2} 2^{-|A|} \\ &= T(x_A). \end{aligned}$$

Thus, x *satisfies the contraction iteration*

$$x_{\sigma(A)} = T(x_A), \qquad A \in D.$$

The associated successor difference net is

$$\begin{aligned} (\Delta x)_A &= x_{\sigma(A)} - x_A \\ &= 2^{-(|A|+1)} - 2^{-|A|} \\ &= -2^{-(|A|+1)}. \end{aligned}$$

Therefore,

$$|(\Delta x)_A| = 2^{-(|A|+1)}.$$

We now show that $\Delta x \to 0$ *as a net on* D*. Let* $\varepsilon > 0$*. Choose* $N \in \mathbb{N}$ *such that*

$$2^{-(N+1)} < \varepsilon,$$

and choose $A_0 \in D$ with $|A_0| = N$. Whenever $A \geq A_0$, we have $A_0 \subseteq A$, and hence

$$|A| \geq |A_0| = N.$$

It follows that

$$|(\Delta x)_A| = 2^{-(|A|+1)} \leq 2^{-(N+1)} < \varepsilon.$$

Consequently,

$$(\Delta x)_A \longrightarrow 0 \qquad (A \in D),$$

and therefore

$$x \in C_0(\Delta, D; \mathbb{R}).$$

Finally,

$$x_{\varnothing} = 1$$

and

$$\sup_{A \in D} |x_{\sigma(A)} - x_A| = \sup_{A \in D} 2^{-(|A|+1)} = \frac{1}{2}.$$

Hence the anchored difference norm of x is

$$\begin{aligned} \|x\|_{\Delta} &= |x_{\varnothing}| + \sup_{A \in D} |x_{\sigma(A)} - x_A| \\ &= 1 + \frac{1}{2} = \frac{3}{2}. \end{aligned}$$

Thus, the contraction mapping $T(t) = t/2$ generates an element of the difference-null net space on a genuinely nonlinearly ordered directed set. This example shows that the sequence generated by a contraction mapping naturally produces an element of the difference net space $C_0(\Delta, D)$. Figure 2 illustrates that both the iterative sequence $x_n = 2^{-(n-1)}$ and its difference $\Delta x_n = x_{n+1} - x_n = -2^{-n}$ converge to zero, confirming that the generated sequence belongs to $C_0(\Delta, D)$.

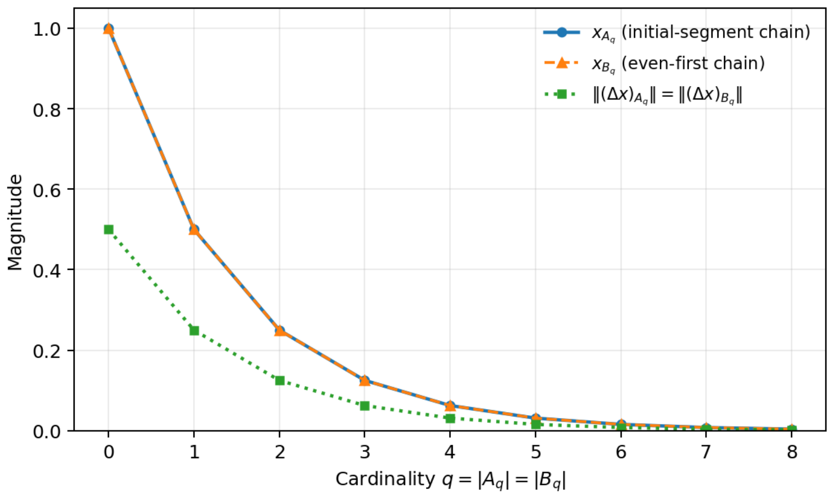

Figure 2. Iterative sequence $x_n = \frac{1}{2n-1}$ and its difference Δx_n.

Example 6.2. *Let*

$$D = \mathbb{N} \times \mathbb{N}$$

be equipped with the product order

$$(m_1, n_1) \leq (m_2, n_2) \quad \Longleftrightarrow \quad m_1 \leq m_2 \text{ and } n_1 \leq n_2.$$

Then D is a directed set with least element $(1,1)$. Define the successor mapping $\sigma : D \to D$ by

$$\sigma(m,n) = \begin{cases} (m+1, n), & m < n, \\ (m, n+1), & m > n, \\ (m+1, n+1), & m = n. \end{cases}$$

Thus, when $m \neq n$, the smaller coordinate is increased by one. Every successor orbit eventually reaches the diagonal, after which it remains on the diagonal. Define the scalar-valued net $x = (x_{(m,n)})_{(m,n)\in D}$ by

$$x_{(m,n)} = \frac{m}{m+n}.$$

The net x does not converge. Indeed, let $(M,N) \in D$. By taking $m \geq M$ sufficiently large while keeping $n = N$, the value

$$x_{(m,N)} = \frac{m}{m+N}$$

can be made arbitrarily close to 1. On the other hand, by taking $n \geq N$ sufficiently large while keeping $m = M$, the value

$$x_{(M,n)} = \frac{M}{M+n}$$

can be made arbitrarily close to 0. Both types of indices lie above (M,N), no tail of the net can converge to a single scalar. We now calculate its successor difference. If $m < n$, then

$$\begin{aligned} (\Delta x)_{(m,n)} &= x_{(m+1,n)} - x_{(m,n)} \\ &= \frac{m+1}{m+n+1} - \frac{m}{m+n} \\ &= \frac{n}{(m+n)(m+n+1)}. \end{aligned}$$

If $m > n$, then

$$\begin{aligned} (\Delta x)_{(m,n)} &= x_{(m,n+1)} - x_{(m,n)} \\ &= \frac{m}{m+n+1} - \frac{m}{m+n} \\ &= -\frac{m}{(m+n)(m+n+1)}. \end{aligned}$$

Finally, if $m = n$, then

$$(\Delta x)_{(m,m)} = x_{(m+1,m+1)} - x_{(m,m)} = 0.$$

Therefore

$$|(\Delta x)_{(m,n)}| \leq \frac{1}{m+n+1}.$$

Let $\varepsilon > 0$. Choose $N_0 \in \mathbb{N}$ such that

$$\frac{1}{2N_0 + 1} < \varepsilon.$$

Whenever

$$(m, n) \geq (N_0, N_0),$$

we have $m + n + 1 \geq 2N_0 + 1$, and consequently

$$|(\Delta x)_{(m,n)}| \leq \frac{1}{m + n + 1} \leq \frac{1}{2N_0 + 1} < \varepsilon.$$

It follows that

$$(\Delta x)_{(m,n)} \longrightarrow 0$$

in the product-directed set. Hence

$$x \in C_0(\Delta, D),$$

although the original net x does not converge.

Figure 3 shows that the net $x_{(m,n)} = \frac{m}{m+n}$ approaches different limits along two directed paths, and therefore does not converge, whereas its difference $\Delta x_{(m,n)}$ converges to zero.

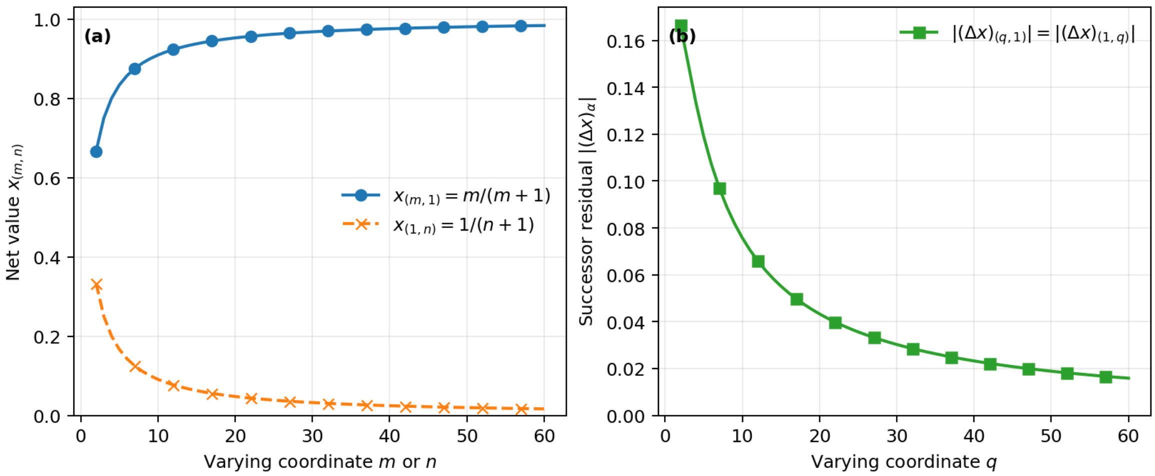

Figure 3. Two directed paths of the net $x_{(m,n)} = \frac{m}{m+n}$ showing non-convergence.

7. Conclusions

In this work, we introduced and studied a new class of spaces, called difference net spaces, which extend the classical framework of difference sequence spaces to nets indexed by directed sets. By introducing a successor mapping on a directed set, we defined a natural difference operator for nets. We used it to construct the spaces $\ell_\infty(\Delta, D)$, $C(\Delta, D)$, and $C_0(\Delta, D)$. These spaces capture the asymptotic behavior of nets through their discrete variation and provide a broader functional-analytic setting for studying generalized convergence phenomena.

Several fundamental structural properties of these spaces were established. In particular, it was shown that $\ell_\infty(\Delta, D)$, $C(\Delta, D)$, and $C_0(\Delta, D)$ are Banach spaces under the norm induced by the supremum of the net together with the supremum of its associated difference net. We also characterized the continuous duals of these spaces and investigated their separability properties. The results indicate that while $\ell_\infty(\Delta, D)$ inherits the non-separability typical of bounded spaces over infinite index sets, the spaces $C(\Delta, D)$ and $C_0(\Delta, D)$ possess separable structures under the same norm.

The examples presented in the paper demonstrate that the difference operator can significantly improve the asymptotic behavior of nets: Nets that are neither bounded nor convergent may still produce bounded or convergent difference nets. The observation underscores the application of the suggested spaces in studying the stability and variation of the generalized index settings. To give an example of this view, we briefly discussed the role of the net spaces of iterations as they occur in fixed point theory and how nets of contraction mappings are naturally in $C_0(\Delta, D)$ as well.

The framework developed here opens several directions for further investigation. Future work may include the study of higher-order difference net spaces, weighted or statistical variants of these spaces, and matrix transformations between difference net spaces. In addition, potential applications may arise in operator theory, summability theory, and the analysis of convergence and stability of iterative algorithms defined over more general index structures.

Author contributions

Adil Hussain Dar: Investigation, software, methodology, writing-original draft preparation, writing-review and editing; Lateef Ahmad Wani: Validation, formal analysis, project administration, writing-review and editing; Mudasir Younis: Conceptualization, methodology, writing-original draft preparation; Saiful R. Mondal: Validation, formal analysis, project administration, writing-review and editing. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

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