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Research article

Generalized $(\mu, \wp)$-cotangent fractional operators: Theory, properties, and an extended Gronwall inequality

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Abstract: This paper studies a family of $(\mu, \wp)$-cotangent fractional integral and differential operators of Riemann–Liouville and Caputo type. A central structural observation is made explicit. After the change of variable $x = \wp(z) - \wp(c)$ and an exponential conjugation, the proposed operators reduce to classical Riemann–Liouville/Caputo operators of normalized order $\alpha = \sigma/\mu$, multiplied by an explicit normalization factor. Consequently, the parameters (σ, μ, r) do not represent three independent degrees of freedom in the kernel shape; up to normalization, their effect is described by the normalized order α and the attenuation rate $a = \cot(\pi r/2)/\mu$. We retain σ and μ separately because this notation is compatible with the μ-Gamma calculus and makes the scaling relations and special cases transparent. Within this corrected interpretation, we establish semigroup and composition rules, Riemann–Liouville/Caputo inversion formulas, and an extended Gronwall inequality. The induction step in the Gronwall proof is written with the complete Beta-to-Γ_μ conversion, and the associated Mittag–Leffler series is identified explicitly with a scaled classical one-parameter Mittag–Leffler function. As a concrete application, we prove an existence result and a Gronwall-based uniqueness and continuous-dependence estimate for a nonlinear Cauchy problem governed by the $(\mu, \wp)$-cotangent Caputo derivative. In addition to the classical-limit tests, a nontrivial example with $\mu = 2$, $r = 1/2$, and $\wp(z) = \log z$ illustrates how the exponential attenuation and normalization jointly affect the bound.

Keywords: fractional derivatives; $(\mu, \wp)$-cotangent fractional derivative; $(\mu, \wp)$-cotangent fractional integral; Gronwall inequality
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1. Introduction

The fractional calculus has its history going back to the famous problem raised by L'Hôpital to Leibniz in 1695 on the interpretation of the derivative of order not equal to an integer. It has grown from a purely mathematical concept into a very useful concept for describing complicated processes in different fields of science in over three hundred years. Many outstanding mathematicians have worked on this topic, among whom there are Euler, Lagrange, Laplace, Fourier, Liouville, Riemann, and Grünwald. The 20th century was marked by important advances in the field of fractional calculus that have been made by Weyl, Riesz, and Caputo. All the approaches to this topic proposed by these mathematicians were intended to deal with certain problems arising in physics.

The Caputo derivative with the inclusion of initial conditions became a valuable resource in the field. The field has experienced major advancements over the past decades. The realization that the fractional operators are a better solution to problems in anomalous diffusion, viscoelasticity, and complex systems than the integer-order operators has contributed to a better understanding of the field. The traditional fractional derivatives are the Riemann–Liouville, Caputo, Hadamard, Erdélyi–Kober, Katugampola, and Hilfer derivatives. The study of Riemann–Liouville (RL) fractional derivatives with respect to a function $\wp$ in [1–3] resulted in the definition of the $\wp$-Hilfer operator in [4].

The everlasting quest for fractional operators that are highly adaptable and physically interpretable has led to various remarkable generalizations. The framework of the fractional operators through the cotangent is provided in [5] and has an exponential kernel. This can be advantageous since it can be used when there is a need for a tuneable attenuation effect. Subsequent investigations have involved other generalizations, including μ-Riemann–Liouville (RL) fractional derivatives [6, 7], $(\mu, \wp)$-RL operators, and the $(\mu, \wp)$-Hilfer operator. The contribution made by the investigation by Sadek and Algefary [8] has further boosted the theory of cotangent fractional operators for functions. This has proven yet another potential application of the theory. In contrast, the advanced approach using non-compactness theory analysis has been applied in examining the existence of complicated boundary value problems as in [9].

Related developments include generalized k-fractional integrals and nonlinear (k, Ψ)-Hilfer equations [10, 11]. Non-singular-kernel and generalized fractional operators have also been used in a range of models and fractional differential equations [12–16], while generalized fractional Gronwall inequalities for proportional and ψ-Hilfer-type operators were established in [17, 18].

1.1. Motivation, current research context, and effective parameters

Cotangent-kernel fractional calculus was introduced in [5], where the exponential factor controlled by $\cot(\pi r/2)$ was combined with the usual power-law memory. A subsequent development with respect to another function $\wp$ was given in [8], allowing the memory variable to be expressed in a transformed coordinate. In parallel, k- or μ-scaled Riemann–Liouville-type operators and derivatives with respect to another function have been studied in several settings, including nonlocal boundary-value problems and numerical fractional Cauchy problems [9, 19]. These directions form the immediate background of the present work.

A key point, emphasized in the revision, is that the parameters must not be interpreted as independent sources of generality. Let

$$\alpha = \frac{\sigma}{\mu}, \qquad \theta = \frac{\pi r}{2}, \qquad a = \frac{\cot\theta}{\mu}, \qquad x = \wp(z) - \wp(c).$$

Lemma 1 shows that the proposed integral is an exponentially conjugated classical Riemann–Liouville integral of order α, with the multiplicative normalization $(\mu \sin\theta)^{-\alpha}$. Thus, up to this explicit rescaling, (σ, μ, r) determine the two effective quantities α and a. In particular, varying μ while compensating σ and r so that α and a are unchanged does not create a new kernel shape. The separate symbols σ and μ are nevertheless useful for compatibility with Γ_μ, for tracking the dimensional scaling, and for making contact with the k/μ-fractional literature. Accordingly, the contribution of this paper is positioned as a structural and analytical development of the cotangent calculus rather than as the introduction of three independent fractional parameters.

The principal characteristics of the framework are therefore: (i) Exact reducibility by conjugation to classical fractional calculus, which makes many identities verifiable; (ii) an explicit attenuation factor controlled by r together with a coordinate map $\wp$; and (iii) closed composition and Gronwall estimates that can be transferred to nonlinear fractional equations. The advantage is analytical tractability and coordinate flexibility, not an artificial increase in the number of independent parameters. The nontrivial example in Section 7 and the Cauchy-problem application below provide direct checks of these features.

Fractional differential equations are valuable precisely because the memory and hereditary behavior is captured by the nonlocal kernel. Recent instances of this application include fractional differential equations from chemical graph theory [20], dual-order models of memory for animal learning and numerical sensitivity analysis [21], and nonlinear memory-dependent models of behavioral learning with analytical and numerical validation [22]. These applications provide motivation for the qualitative estimates obtained herein. In the case where a model can be formulated as a Volterra-type integral equation, then a Gronwall inequality is a useful tool to study uniqueness and sensitivity.

1.2. Main contributions and logical relation of the results

The revised contribution is summarized in three points:

- We give the conjugation representation of the $(\mu, \wp)$-cotangent integral and its Riemann–Liouville and Caputo derivatives, and use it to clarify the effective parameter dependence and to establish the main composition and inversion identities.
- We prove an extended Gronwall inequality with a corrected induction calculation, identify the associated series with a scaled classical Mittag–Leffler function, and make the regularity and parameter assumptions explicit.
- We apply the Gronwall inequality to a nonlinear $(\mu, \wp)$-cotangent Caputo Cauchy problem, obtaining existence under an explicit contraction condition and uniqueness/continuous dependence through the new estimate; a genuinely non-classical example verifies the influence of μ, r, and $\wp$.

The results follow in a sequence of dependency chain. Lemma 1 represents the structural link to classical fractional calculus. Lemma 2 provides the power kernel lemma, Lemma 3 provides the differentiation under the fractional integral lemma, and Lemma 4 gives the integer-order integral-differentiation composition lemma. Theorem 1 provides the Riemann-Liouville semigroup and inversion properties, whereas Theorem 2 provides the Caputo inversion property. Then, Theorem 3 relies on positivity and composition of kernels to obtain the Gronwall estimate, which is then used in Theorem 4 for a concrete fractional Cauchy problem.

Section 2 states the function spaces and natural integration domain. Sections 3–5 develop the integral and derivative operators. Section 6 proves the Gronwall inequality and its application to a

Cauchy problem. Section 7 contains four examples. The first three isolate the theorem and its two corollaries in the classical limit, while the fourth tests nontrivial values of μ, r, and $\wp$ and displays the attenuation effect. The later derivation section provides exact checks of the representative formulas.

2. Function spaces and basic definitions

Fix $r_0 \in (0, 1)$. Throughout the analytical results and estimates, we take $r \in [r_0, 1]$. This removes the singular endpoint $r = 0$, where $\sin(\pi r/2) \to 0$ and $\cot(\pi r/2) \to \infty$. The formulas are meaningful for each fixed $r \in (0, 1]$, but no uniform $r \to 0^+$ limit is claimed here.

Let $\wp : [c, d] \to \mathbb{R}$ be continuously differentiable and strictly increasing, with $\wp'(z) > 0$ on $[c, d]$. The kernel is evaluated only on the triangular Volterra domain

$$\Delta_{c,d} = \{(z, \eta) : c \leq \eta < z \leq d\},$$

not on the full square $[c, d] \times [c, d]$. On $\Delta_{c,d}$, the quantity $\wp(z) - \wp(\eta)$ is positive, and hence the real-valued power in

$${}^{r}_{\mu}\Psi_{\wp}^{\frac{\sigma}{\mu}-1}(z, \eta) = e^{-\frac{\cot(\frac{\pi}{2}r)(\wp(z)-\wp(\eta))}{\mu}} (\wp(z) - \wp(\eta))^{\frac{\sigma}{\mu}-1} \tag{2.1}$$

is unambiguous.

2.1. Function spaces and existence conditions

Put $\alpha = \sigma/\mu$, $X = \wp(d) - \wp(c)$, $\theta = \pi r/2$, $a = \cot\theta/\mu$, and

$$\widehat{h}(x) = e^{ax} h\Big(\wp^{-1}(\wp(c) + x)\Big), \qquad 0 \leq x \leq X.$$

For the fractional integral, $h \in L^1([c, d])$ is sufficient because $\alpha > 0$ makes the power singularity locally integrable on $\Delta_{c,d}$ and, for $r \in [r_0, 1]$, the exponential factor is bounded by one.

For the Riemann–Liouville derivative with $i = \lceil \alpha \rceil$, the natural domain is

$$\mathcal{D}^{\sigma}_{RL} = \left\{h \in L^1([c, d]) : I^{i-\alpha}_{0+}\widehat{h} \in AC^i([0, X])\right\}.$$

For the Caputo derivative, the natural domain is

$$\mathcal{D}^{\sigma}_{C} = \left\{h : \widehat{h} \in AC^i([0, X])\right\}.$$

These conditions are the minimal regularity used by the iterated differential operators in the transformed variable. The stronger assumptions $h \in C^i$ and $\wp \in C^i$ with $\wp' > 0$ are convenient sufficient conditions and are invoked in some statements where endpoint traces are needed.

2.2. Special functions

The generalized Gamma function Γ_μ plays a crucial role in our development and is defined for $\mu > 0$ as

$$\Gamma_\mu(z) = \int_0^\infty s^{z-1} e^{-\frac{s^\mu}{\mu}} ds, \quad z \in \mathbb{C}, \quad \mathrm{Re}(z) > 0,$$

with the fundamental properties

$$\Gamma_\mu(z+\mu) = z\Gamma_\mu(z), \quad \Gamma_\mu(\mu) = 1, \quad \Gamma_\mu(z) = \mu^{\frac{z}{\mu}-1}\Gamma\left(\frac{z}{\mu}\right), \quad \Gamma(z) = \lim_{\mu\to 1}\Gamma_\mu(z). \tag{2.2}$$

This generalized Gamma function reduces to the classical Euler Gamma function when $\mu = 1$ and provides the necessary normalization for our fractional operators. Because

$$\Gamma_\mu(z) = \mu^{z/\mu-1}\Gamma(z/\mu),$$

its meromorphic continuation has simple poles precisely at

$$z = 0, -\mu, -2\mu, -3\mu, \ldots$$

and no zeros. This explicit pole set is used below when a denominator involving Γ_μ is discussed.

3. The $(\mu, \wp)$-cotangent fractional integral

Definition 1. *Let $\sigma, \mu \in \mathbb{R}^+, r_0 \leq r \leq 1$, and $h \in L^1([c,d], \mathbb{R})$. The $(\mu, \wp)$-cotangent fractional integral operator (CFIO) of order σ for h is defined by*

$${}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}h(z) = \frac{1}{\sin(\frac{\pi}{2}r)^{\frac{\sigma}{\mu}}\mu\Gamma_\mu(\sigma)}\int_c^z {}^{r}_{\mu}\Psi_{\wp}^{\frac{\sigma}{\mu}-1}(z,\eta)\wp'(\eta)h(\eta)d\eta.$$

This operator represents a significant generalization that unifies several well-known fractional integrals:

- When $\mu = 1$, $\wp(z) = z$, and $r = 1$, we recover the classical Riemann–Liouville fractional integral [4].
- When $\mu = 1$ and $\wp(z) = z$, we obtain the cotangent fractional integral introduced in [5].
- When $\mu = 1$, we recover the $\wp$-cotangent fractional integral developed in [8].
- When $r = 1$, the operator reduces to the $(\mu, \wp)$-Riemann–Liouville fractional integral studied in [9].
- Specific choices of $\wp$ yield other important fractional integrals, up to the standard normalizations used in each convention: $\wp(z) = \log z$ gives a Hadamard-type integral, $\wp(z) = z^\rho$ gives a Katugampola-type integral, and $\wp(z) = \varphi(z)^\rho$ is related to the general Caputo–Katugampola setting discussed in [19].

The parameter r controls the exponential attenuation, while the pair (σ, μ) determines the normalized order $\alpha = \sigma/\mu$ and the associated scaling. The function $\wp$ provides coordinate flexibility through the transformed variable $\wp(z) - \wp(c)$.

4. The $(\mu, \wp)$-RL cotangent fractional derivative

Definition 2. *Let $\sigma, \mu > 0$, $r \in [r_0, 1]$, $i = \lceil\sigma/\mu\rceil$, $\wp \in C^1([c,d])$ be strictly increasing with $\wp' > 0$, and $h \in \mathcal{D}^\sigma_{RL}$. The $(\mu, \wp)$-Riemann–Liouville cotangent fractional derivative (RL-CFDO) of order σ is*

$${}^{R}_{c,\mu}\mathbb{D}^{\sigma,r,\wp}h(z) := \mathbb{D}^{i,r,\wp}\left({}_{c,\mu}\mathbb{I}^{i\mu-\sigma,r,\wp}h\right)(z),$$

where $\mathbb{D}^{i,r,\wp}$ *is the i-fold iterate of*

$$\mathbb{D}^{r,\wp}h(z) = \cos\left(\frac{\pi r}{2}\right)h(z) + \mu\sin\left(\frac{\pi r}{2}\right)\frac{h'(z)}{\wp'(z)}.$$

If $i\mu = \sigma$, ${}_{c,\mu}\mathbb{I}^{0,r,\wp}$ *is understood as the identity.*

Remark 1. *The integer* $i = \lceil \sigma/\mu \rceil$ *is the smallest positive integer such that* $(i-1)\mu < \sigma \leq i\mu$. *Whenever* $i\mu = \sigma$, *the operator* ${}_{c,\mu}\mathbb{I}^{0,r,\wp}$ *is understood as the identity operator.*

Remark 2. *The RL-CFDO can be equivalently expressed as*

$${}^{R}_{c,\mu}\mathbb{D}^{\sigma,r,\wp}h(z) = \frac{\mathbb{D}^{i,r,\wp}}{\sin(\frac{\pi}{2}r)^{\frac{i\mu-\sigma}{\mu}}\mu\Gamma_{\mu}(i\mu-\sigma)}\int_{c}^{z} {}^{r}_{\mu}\Psi_{\wp}^{\frac{i\mu-\sigma}{\mu}-1}(z,\eta)\wp'(\eta)h(\eta)d\eta.$$

4.1. Fundamental properties of RL-CFDO

The following conjugation formula is useful throughout the paper. It reduces the proposed operators to classical fractional operators after a change of variables and an exponential rescaling.

Lemma 1. *(Conjugation representation) Let*

$$\theta = \frac{\pi r}{2}, \qquad a = \frac{\cot\theta}{\mu}, \qquad x = \wp(z) - \wp(c),$$

and define

$$\widehat{h}(x) = e^{ax}h\Big(\wp^{-1}(\wp(c)+x)\Big).$$

For every admissible function h and every $\sigma > 0$, *with* $\alpha = \sigma/\mu$, *we have*

$${}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}h(z) = \frac{e^{-ax}}{(\mu\sin\theta)^{\alpha}}\left(I^{\alpha}_{0+}\widehat{h}\right)(x), \tag{4.1}$$

$$\mathbb{D}^{r,\wp}h(z) = \mu\sin\theta\, e^{-ax}\widehat{h}'(x), \tag{4.2}$$

$${}^{R}_{c,\mu}\mathbb{D}^{\sigma,r,\wp}h(z) = (\mu\sin\theta)^{\alpha}e^{-ax}\left(D^{\alpha}_{0+}\widehat{h}\right)(x), \tag{4.3}$$

$${}^{C}_{c,\mu}\mathbb{D}^{\sigma,r,\wp}h(z) = (\mu\sin\theta)^{\alpha}e^{-ax}\left({}^{C}D^{\alpha}_{0+}\widehat{h}\right)(x). \tag{4.4}$$

Here, I^{α}_{0+}, D^{α}_{0+}, *and* ${}^{C}D^{\alpha}_{0+}$ *denote the classical left-sided Riemann–Liouville integral, Riemann–Liouville derivative, and Caputo derivative with respect to the variable x.*

Proof. Starting from Definition 1, use $y = \wp(\eta) - \wp(c)$. Since $\wp$ is strictly increasing, this change of variables is admissible and gives

$$\wp(z) - \wp(\eta) = x - y, \qquad \wp'(\eta)d\eta = dy.$$

Furthermore, the identity

$$\mu\Gamma_{\mu}(\sigma) = \mu^{\sigma/\mu}\Gamma\left(\frac{\sigma}{\mu}\right)$$

follows from (2.2). Therefore,

$$
\begin{aligned}
{}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}h(z) &= \frac{e^{-ax}}{(\mu\sin\theta)^{\alpha}\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1}e^{ay}h\Big(\wp^{-1}(\wp(c)+y)\Big)\,dy\\
&= \frac{e^{-ax}}{(\mu\sin\theta)^{\alpha}}\left(I_{0+}^{\alpha}\widehat{h}\right)(x),
\end{aligned}
$$

which proves (4.1). Next, differentiating $h(z) = e^{-ax}\widehat{h}(x)$ with respect to z and using $x' = \wp'(z)$ yields

$$
\cos\theta\, h(z) + \mu\sin\theta\frac{h'(z)}{\wp'(z)} = \mu\sin\theta\, e^{-ax}\widehat{h}'(x).
$$

Indeed, $\mu\sin\theta\, a = \cos\theta$. Iterating this identity and combining it with (4.1) gives (4.3) and (4.4). $\square$

Lemma 2. *Let $\sigma, \mu > 0$, $r \in [r_0, 1]$, and let the expressions below be well defined. Then:*

1. *For $\omega > 0$,*

$$
{}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}\left[{}^{r}_{\mu}\Psi_{\wp}^{\frac{\omega}{\mu}-1}(z,c)\right] = \frac{\Gamma_\mu(\omega)}{\sin(\frac{\pi}{2}r)^{\sigma/\mu}\Gamma_\mu(\omega+\sigma)}{}^{r}_{\mu}\Psi_{\wp}^{\frac{\omega+\sigma}{\mu}-1}(z,c).
$$

2. *For the pointwise function identity, if $\omega > \sigma$,*

$$
{}^{R}_{c,\mu}\mathbb{D}^{\sigma,r,\wp}\left[{}^{r}_{\mu}\Psi_{\wp}^{\frac{\omega}{\mu}-1}(z,c)\right] = \frac{\sin(\frac{\pi}{2}r)^{\sigma/\mu}\Gamma_\mu(\omega)}{\Gamma_\mu(\omega-\sigma)}{}^{r}_{\mu}\Psi_{\wp}^{\frac{\omega-\sigma}{\mu}-1}(z,c).
$$

The right-hand side also supplies the meromorphic continuation in ω wherever $\Gamma_\mu(\omega-\sigma)$ is finite; for $\omega \le \sigma$, this continuation should not be confused with an ordinary locally integrable power kernel at $z = c$.

Proof. For part 1, substitute the kernel into Definition 1, and set

$$
s = \frac{\wp(\eta)-\wp(c)}{\wp(z)-\wp(c)}.
$$

The exponential factors combine, while the remaining integral is

$$
B\left(\frac{\sigma}{\mu},\frac{\omega}{\mu}\right) = \frac{\Gamma(\sigma/\mu)\Gamma(\omega/\mu)}{\Gamma((\sigma+\omega)/\mu)} = \mu\,\frac{\Gamma_\mu(\sigma)\Gamma_\mu(\omega)}{\Gamma_\mu(\sigma+\omega)}.
$$

After cancellation with the normalization in ${}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}$, the first identity follows.

For part 2, let $i = \lceil\sigma/\mu\rceil$, and first apply part 1 with order $i\mu-\sigma$:

$$
{}_{c,\mu}\mathbb{I}^{i\mu-\sigma,r,\wp}\left[{}^{r}_{\mu}\Psi_{\wp}^{\frac{\omega}{\mu}-1}(z,c)\right] = \frac{\Gamma_\mu(\omega)}{\sin(\frac{\pi}{2}r)^{i-\sigma/\mu}\Gamma_\mu(\omega+i\mu-\sigma)}{}^{r}_{\mu}\Psi_{\wp}^{\frac{\omega+i\mu-\sigma}{\mu}-1}(z,c). \tag{4.5}
$$

The iterated differentiation omitted in the earlier version is as follows. For every admissible β,

$$
\mathbb{D}^{r,\wp}\left[{}^{r}_{\mu}\Psi_{\wp}^{\frac{\beta}{\mu}-1}(z,c)\right] = \sin(\frac{\pi}{2}r)(\beta-\mu){}^{r}_{\mu}\Psi_{\wp}^{\frac{\beta-\mu}{\mu}-1}(z,c),
$$

and hence, by induction,

$$\mathbb{D}^{i,r,\wp}\left[{}_{\mu}^{r}\Psi_{\wp}^{\frac{\beta}{\mu}-1}(z,c)\right] = \sin(\frac{\pi}{2}r)^{i}\frac{\Gamma_{\mu}(\beta)}{\Gamma_{\mu}(\beta-i\mu)}{}_{\mu}^{r}\Psi_{\wp}^{\frac{\beta-i\mu}{\mu}-1}(z,c). \tag{4.6}$$

Indeed, the recurrence $\Gamma_{\mu}(q+\mu) = q\Gamma_{\mu}(q)$ converts the product $\prod_{j=1}^{i}(\beta - j\mu)$ into the quotient in (4.6). Applying (4.6) to (4.5) with $\beta = \omega + i\mu - \sigma$ gives

$${}_{c,\mu}^{R}\mathbb{D}^{\sigma,r,\wp}\left[{}_{\mu}^{r}\Psi_{\wp}^{\frac{\omega}{\mu}-1}(z,c)\right] = \frac{\sin(\frac{\pi}{2}r)^{\sigma/\mu}\Gamma_{\mu}(\omega)}{\Gamma_{\mu}(\omega-\sigma)}{}_{\mu}^{r}\Psi_{\wp}^{\frac{\omega-\sigma}{\mu}-1}(z,c),$$

as claimed. □

Remark 3. *The phrase "a pole of* Γ_{μ}*" can now be made precise. Since the poles are* $0, -\mu, -2\mu, \ldots$*, a denominator such as* $\Gamma_{\mu}(\omega-\sigma)$ *is singular exactly when* $\omega-\sigma = -k\mu$ *for some* $k \in \mathbb{N}_0$*. Consequently, a formal zero produced by the reciprocal Gamma factor is a consequence of meromorphic continuation; it does not imply that every low-order kernel belongs to the pointwise domain of the Riemann–Liouville derivative. This is why the ordinary function identity above is stated first for* $\omega > \sigma$*.*

Lemma 3. *Let* $\sigma, \mu > 0$*,* $r \in [r_0, 1]$*, and* $\wp \in C^1([c,d])$ *be strictly increasing with* $\wp' > 0$*, and* $h \in L^1([c,d])$*. If* $m \in \mathbb{N}^*$ *and* $m < \sigma/\mu$*, then, almost everywhere on* $(c,d]$*,*

$$\mathbb{D}^{m,r,\wp}\left({}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}h\right)(z) = {}_{c,\mu}\mathbb{I}^{\sigma-m\mu,r,\wp}h(z).$$

Proof. Set $\alpha = \sigma/\mu$ and $b = \mu\sin(\pi r/2)$, and use the transformed function $\widehat{h}$ from Section 2. By Lemma 1,

$${}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}h(z) = b^{-\alpha}e^{-ax}I_{0+}^{\alpha}\widehat{h}(x), \qquad \mathbb{D}^{r,\wp} = b\,e^{-ax}\frac{d}{dx}e^{ax}.$$

Therefore,

$$\mathbb{D}^{m,r,\wp}\left({}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}h\right) = b^{m-\alpha}e^{-ax}\frac{d^m}{dx^m}I_{0+}^{\alpha}\widehat{h} = b^{m-\alpha}e^{-ax}I_{0+}^{\alpha-m}\widehat{h},$$

where the classical identity $d^mI^{\alpha}/dx^m = I^{\alpha-m}$ is valid for $\widehat{h} \in L^1$ because $\alpha > m$. Reversing the conjugation gives the stated formula.

For completeness, this also justifies the direct differentiation used in the earlier proof. In the base case $m = 1$, the assumption $1 < \alpha$ implies that the kernel behaves as $(x-y)^{\alpha-1}$ at $y = x$, so its endpoint value is zero, while its derivative behaves as $(x-y)^{\alpha-2}$ with an exponent greater than -1 and is integrable. The same argument applies at each subsequent step because before the k-th differentiation, one has $\alpha - k + 1 > 1$. The vacuous case $m = 0$ has therefore been removed from the statement. □

Lemma 4. *Let* $i \in \mathbb{N}^*$*,* $\mu > 0$*,* $r \in [r_0, 1]$ *and* $\wp \in C^1([c,d])$ *be strictly increasing with* $\wp' > 0$*, and assume that the transformed function* $\widehat{h}$ *belongs to* $AC^i([0,X])$*. Then*

$${}_{c,\mu}\mathbb{I}^{i\mu,r,\wp}\left(\mathbb{D}^{i,r,\wp}h\right)(z) = h(z) - \sum_{j=1}^{i}\frac{{}_{\mu}^{r}\Psi_{\wp}^{i-j}(z,c)}{(\mu\sin(\pi r/2))^{i-j}(i-j)!}\mathbb{D}^{i-j,r,\wp}h(c).$$

Proof. Let $b = \mu\sin(\pi r/2)$ and $x = \wp(z) - \wp(c)$. By Lemma 1,

$$\mathbb{D}^{i,r,\wp}h(z) = b^i e^{-ax}\widehat{h}^{(i)}(x), \qquad {}_{c,\mu}\mathbb{I}^{i\mu,r,\wp}g(z) = b^{-i}e^{-ax}I^i_{0+}\widehat{g}(x).$$

Thus

$${}_{c,\mu}\mathbb{I}^{i\mu,r,\wp}\left(\mathbb{D}^{i,r,\wp}h\right)(z) = e^{-ax}I^i_{0+}\widehat{h}^{(i)}(x) = e^{-ax}\left[\widehat{h}(x) - \sum_{k=0}^{i-1}\frac{\widehat{h}^{(k)}(0+)}{k!}x^k\right].$$

Since $\mathbb{D}^{k,r,\wp}h(c) = b^k\widehat{h}^{(k)}(0+)$ and ${}^{r}_{\mu}\Psi^k_{\wp}(z,c) = e^{-ax}x^k$, the formula follows after reindexing.

We also record explicitly the correction to the inductive calculation in the earlier version. Lemma 2 with $\sigma = \mu$ and $\omega = \mu(q+1)$ gives

$${}_{c,\mu}\mathbb{I}^{\mu,r,\wp}\left[{}^{r}_{\mu}\Psi^q_{\wp}(z,c)\right] = \frac{1}{\mu\sin(\pi r/2)(q+1)}{}^{r}_{\mu}\Psi^{q+1}_{\wp}(z,c). \tag{4.7}$$

The factor $1/\mu$ was missing in the previous intermediate line. Equation (4.7) produces exactly the factor $(\mu\sin(\pi r/2))^{-(q+1)}$ already present in the final statement, so the final formula of the lemma is unchanged. The present proof is independent of Theorem 1; hence there is no longer any use of the semigroup property before it is established. □

Theorem 1. *Let $\sigma, \delta, \mu > 0$ and $r_0 \le r \le 1$, and let $\wp \in C^1([c,d],\mathbb{R})$ be strictly increasing with $\wp'(z) > 0$. Assume that all the displayed expressions are well defined. Then:*

1. *The semigroup property holds:*

$${}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}\left({}_{c,\mu}\mathbb{I}^{\delta,r,\wp}h\right)(z) = {}_{c,\mu}\mathbb{I}^{\sigma+\delta,r,\wp}h(z) = {}_{c,\mu}\mathbb{I}^{\delta,r,\wp}\left({}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}h\right)(z).$$

2. *For $\delta > \sigma$,*

$${}^{R}_{c,\mu}\mathbb{D}^{\sigma,r,\wp}\left({}_{c,\mu}\mathbb{I}^{\delta,r,\wp}h\right)(z) = {}_{c,\mu}\mathbb{I}^{\delta-\sigma,r,\wp}h(z).$$

3. *Let $\alpha = \sigma/\mu$ and $i = \lceil\alpha\rceil$, and set $x = \wp(z) - \wp(c)$. Let $\widehat{h}$ be defined as in Lemma 1. If the traces $\left(D^{\alpha-k}_{0+}\widehat{h}\right)(0+)$ exist for $k = 1, \ldots, i$, then*

$${}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}\left({}^{R}_{c,\mu}\mathbb{D}^{\sigma,r,\wp}h\right)(z) = h(z) - e^{-\frac{\cot(\pi r/2)}{\mu}x}\sum_{k=1}^{i}\frac{x^{\alpha-k}}{\Gamma(\alpha-k+1)}\left(D^{\alpha-k}_{0+}\widehat{h}\right)(0+). \tag{4.8}$$

Proof. Let $\theta = \pi r/2$ and $a = \cot\theta/\mu$. Use the notation introduced in Lemma 1.

For the first assertion, relation (4.1) and the classical semigroup property give

$$\begin{aligned}{}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}\left({}_{c,\mu}\mathbb{I}^{\delta,r,\wp}h\right)(z) &= \frac{e^{-ax}}{(\mu\sin\theta)^{(\sigma+\delta)/\mu}}I^{\sigma/\mu}_{0+}I^{\delta/\mu}_{0+}\widehat{h}(x)\\ &= \frac{e^{-ax}}{(\mu\sin\theta)^{(\sigma+\delta)/\mu}}I^{(\sigma+\delta)/\mu}_{0+}\widehat{h}(x)\\ &= {}_{c,\mu}\mathbb{I}^{\sigma+\delta,r,\wp}h(z).\end{aligned}$$

The reverse order follows in exactly the same way. Equivalently, this calculation can be verified directly by changing the order of integration and using the Beta identity

$$\int_0^1(1-s)^{\frac{\sigma}{\mu}-1}s^{\frac{\delta}{\mu}-1}ds = \frac{\Gamma(\sigma/\mu)\Gamma(\delta/\mu)}{\Gamma((\sigma+\delta)/\mu)}.$$

For the second assertion, Eqs (4.1) and (4.3) yield

$$\begin{aligned}
{}^{R}_{c,\mu}\mathbb{D}^{\sigma,r,\wp}\Big({}_{c,\mu}\mathbb{I}^{\delta,r,\wp}h\Big)(z) &= (\mu\sin\theta)^{\sigma/\mu}e^{-ax}D_{0+}^{\sigma/\mu}\left[(\mu\sin\theta)^{-\delta/\mu}I_{0+}^{\delta/\mu}\widehat{h}\right](x)\\
&= \frac{e^{-ax}}{(\mu\sin\theta)^{(\delta-\sigma)/\mu}}I_{0+}^{(\delta-\sigma)/\mu}\widehat{h}(x)\\
&= {}_{c,\mu}\mathbb{I}^{\delta-\sigma,r,\wp}h(z).
\end{aligned}$$

Finally, by applying (4.1), (4.3), and the classical Riemann–Liouville composition rule, we obtain

$$\begin{aligned}
{}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}\Big({}^{R}_{c,\mu}\mathbb{D}^{\sigma,r,\wp}h\Big)(z) &= e^{-ax}I_{0+}^{\alpha}D_{0+}^{\alpha}\widehat{h}(x)\\
&= e^{-ax}\left[\widehat{h}(x) - \sum_{k=1}^{i}\frac{x^{\alpha-k}}{\Gamma(\alpha-k+1)}\Big(D_{0+}^{\alpha-k}\widehat{h}\Big)(0+)\right].
\end{aligned}$$

Since $e^{-ax}\widehat{h}(x) = h(z)$, formula (4.8) follows. □

5. The $(\mu,\wp)$-Caputo cotangent fractional derivative

Definition 3. *Let $\sigma,\mu > 0$, $r \in [r_0, 1]$, $i = \lceil\sigma/\mu\rceil$, $\wp \in C^1([c,d])$ be strictly increasing with $\wp' > 0$, and $h \in \mathcal{D}_C^{\sigma}$. The $(\mu,\wp)$-Caputo cotangent fractional derivative (C-CFDO) of order σ is*

$$ {}^{C}_{c,\mu}\mathbb{D}^{\sigma,r,\wp}h(z) := {}_{c,\mu}\mathbb{I}^{i\mu-\sigma,r,\wp}\Big(\mathbb{D}^{i,r,\wp}h\Big)(z), \tag{5.1}$$

with ${}_{c,\mu}\mathbb{I}^{0,r,\wp}$ interpreted as the identity when $i\mu = \sigma$.

Remark 4. *The C-CFDO can be equivalently expressed as*

$$ {}^{C}_{c,\mu}\mathbb{D}^{\sigma,r,\wp}h(z) = \frac{1}{\sin(\frac{\pi}{2}r)^{\frac{i\mu-\sigma}{\mu}}\mu\Gamma_\mu(i\mu-\sigma)}\int_c^z {}^{r}_{\mu}\Psi_{\wp}^{\frac{i\mu-\sigma}{\mu}-1}(z,\eta)\wp'(\eta)\mathbb{D}^{i,r,\wp}h(\eta)d\eta.$$

5.1. Fundamental properties of C-CFDO

Lemma 5. *Let $\sigma,\mu > 0$, $r_0 \le r \le 1$, and $i = \lceil\sigma/\mu\rceil$. For $\delta > i\mu$, we have*

$$ {}^{C}_{c,\mu}\mathbb{D}^{\sigma,r,\wp}\left[{}^{r}_{\mu}\Psi_{\wp}^{\frac{\delta}{\mu}-1}(z,c)\right] = \frac{\sin(\frac{\pi}{2}r)^{\frac{\sigma}{\mu}}\Gamma_\mu(\delta)}{\Gamma_\mu(\delta-\sigma)}{}^{r}_{\mu}\Psi_{\wp}^{\frac{\delta-\sigma}{\mu}-1}(z,c).$$

Moreover, for $j = 0, 1, \ldots, i-1$,

$$ {}^{C}_{c,\mu}\mathbb{D}^{\sigma,r,\wp}\left[{}^{r}_{\mu}\Psi_{\wp}^{j}(z,c)\right] = 0.$$

Proof. Let $\theta = \pi r/2$, $a = \cot\theta/\mu$, $x = \wp(z) - \wp(c)$, and $\alpha = \sigma/\mu$. For

$$ h(z) = {}^{r}_{\mu}\Psi_{\wp}^{\frac{\delta}{\mu}-1}(z,c) = e^{-ax}x^{\frac{\delta}{\mu}-1},$$

the transformed function in Lemma 1 is $\widehat{h}(x) = x^{\delta/\mu-1}$. Since $\delta > i\mu$, the classical Caputo power rule is applicable and gives

$$ {}^{C}D_{0+}^{\alpha}x^{\frac{\delta}{\mu}-1} = \frac{\Gamma(\delta/\mu)}{\Gamma((\delta-\sigma)/\mu)}x^{\frac{\delta-\sigma}{\mu}-1}.$$

Using (4.4) and $\Gamma_\mu(s) = \mu^{s/\mu-1}\Gamma(s/\mu)$, we obtain

$$\begin{aligned}{}^{C}_{c,\mu}\mathbb{D}^{\sigma,r,\wp}\left[{}^{r}_{\mu}\Psi_{\wp}^{\frac{\delta}{\mu}-1}(z,c)\right] &= (\mu\sin\theta)^{\alpha}e^{-ax}\frac{\Gamma(\delta/\mu)}{\Gamma((\delta-\sigma)/\mu)}x^{\frac{\delta-\sigma}{\mu}-1}\\ &= \frac{\sin(\theta)^{\sigma/\mu}\Gamma_\mu(\delta)}{\Gamma_\mu(\delta-\sigma)}{}^{r}_{\mu}\Psi_{\wp}^{\frac{\delta-\sigma}{\mu}-1}(z,c).\end{aligned}$$

For the second relation, the transformed function associated with ${}^{r}_{\mu}\Psi^{j}_{\wp}(z,c) = e^{-ax}x^{j}$ is x^j. Its classical Caputo derivative of order α vanishes for $j = 0,\ldots,i-1$. Equation (4.4) completes the proof. □

Theorem 2. *Let $\sigma,\mu > 0$, $r_0 \le r \le 1$, and $i = \lceil\sigma/\mu\rceil$, and let $\wp \in C^i([c,d],\mathbb{R})$ satisfy $\wp'(z) > 0$. For every admissible function h, we have*

1.

$$ {}^{C}_{c,\mu}\mathbb{D}^{\sigma,r,\wp}\left({}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}h\right)(z) = h(z);$$

2.

$$ {}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}\left({}^{C}_{c,\mu}\mathbb{D}^{\sigma,r,\wp}h\right)(z) = h(z) - \sum_{k=0}^{i-1}\frac{{}^{r}_{\mu}\Psi^{k}_{\wp}(z,c)}{(\mu\sin(\pi r/2))^k k!}\mathbb{D}^{k,r,\wp}h(c). \tag{5.2}$$

Proof. Let $\alpha = \sigma/\mu$, $\theta = \pi r/2$, and $a = \cot\theta/\mu$. Set $x = \wp(z) - \wp(c)$, and let $\widehat{h}$ be given by Lemma 1.

For the first relation, Eqs (4.1) and (4.4), together with the classical identity ${}^{C}D^{\alpha}_{0+}I^{\alpha}_{0+}\widehat{h} = \widehat{h}$, yield

$$\begin{aligned}{}^{C}_{c,\mu}\mathbb{D}^{\sigma,r,\wp}\left({}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}h\right)(z) &= e^{-ax}{}^{C}D^{\alpha}_{0+}I^{\alpha}_{0+}\widehat{h}(x)\\ &= e^{-ax}\widehat{h}(x) = h(z).\end{aligned}$$

For the second relation, the classical Caputo composition rule gives

$$I^{\alpha}_{0+}{}^{C}D^{\alpha}_{0+}\widehat{h}(x) = \widehat{h}(x) - \sum_{k=0}^{i-1}\frac{\widehat{h}^{(k)}(0+)}{k!}x^k.$$

Therefore,

$$ {}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}\left({}^{C}_{c,\mu}\mathbb{D}^{\sigma,r,\wp}h\right)(z) = h(z) - e^{-ax}\sum_{k=0}^{i-1}\frac{\widehat{h}^{(k)}(0+)}{k!}x^k.$$

By iterating (4.2), we have

$$\mathbb{D}^{k,r,\wp}h(c) = (\mu\sin\theta)^k\widehat{h}^{(k)}(0+).$$

Since ${}^{r}_{\mu}\Psi^{k}_{\wp}(z,c) = e^{-ax}x^k$, formula (5.2) follows. □

6. Extended Gronwall inequality

We now derive the estimate that will be used in the Cauchy-problem application. The assumptions are stated with $\wp' > 0$ consistently with the Volterra change of variables.

Theorem 3. *(Extended $(\mu,\wp)$-cotangent fractional Gronwall inequality) Let $\sigma,\mu > 0$ and $r \in [r_0, 1]$, and let $\wp \in C^1([c,d])$ be strictly increasing with $\wp'(z) > 0$. Assume*

(H1) u and v are non-negative and integrable on $[c, d]$*;*
(H2) w is non-negative, continuous, and non-decreasing on $[c, d]$.

If

$$u(z) \le v(z) + \frac{\Gamma_\mu(\sigma)}{\mu} w(z)_{c,\mu}\mathbb{I}^{\sigma,r,\wp} u(z), \tag{6.1}$$

then, for $z \in [c, d]$*,*

$$u(z) \le v(z) + \int_c^z \left[\sum_{n=1}^{\infty} \frac{[\Gamma_\mu(\sigma) w(z)]^n}{\sin(\frac{\pi}{2} r)^{n\sigma/\mu} \mu^{n+1} \Gamma_\mu(n\sigma)} {}^{r}_{\mu}\Psi_{\wp}^{\frac{n\sigma}{\mu}-1}(z, \eta)\, \wp'(\eta) v(\eta) \right] d\eta. \tag{6.2}$$

Proof. Define the positive Volterra operator

$$(\Re u)(z) = \frac{w(z)}{\sin(\frac{\pi}{2} r)^{\sigma/\mu} \mu^2} \int_c^z {}^{r}_{\mu}\Psi_{\wp}^{\frac{\sigma}{\mu}-1}(z, \eta) \wp'(\eta) u(\eta) d\eta. \tag{6.3}$$

Then (6.1) is $u \le v + \Re u$, and the iteration gives

$$u \le \sum_{j=0}^{n-1} \Re^j v + \Re^n u.$$

We claim that

$$\Re^n u(z) \le \int_c^z \frac{[\Gamma_\mu(\sigma) w(z)]^n}{\sin(\frac{\pi}{2} r)^{n\sigma/\mu} \mu^{n+1} \Gamma_\mu(n\sigma)} {}^{r}_{\mu}\Psi_{\wp}^{\frac{n\sigma}{\mu}-1}(z, \xi) \wp'(\xi) u(\xi) d\xi. \tag{6.4}$$

The case $n = 1$ is exactly (6.3). Suppose (6.4) holds for $n = m$. Substitution into $\Re(\Re^m u)$, use of $w(\eta) \le w(z)$, and Fubini's theorem give

$$\Re^{m+1} u(z) \le \frac{\Gamma_\mu(\sigma)^m w(z)^{m+1}}{\sin(\frac{\pi}{2} r)^{(m+1)\sigma/\mu} \mu^{m+3} \Gamma_\mu(m\sigma)} \times \int_c^z \wp'(\xi) u(\xi) \left[\int_\xi^z {}^{r}_{\mu}\Psi_{\wp}^{\frac{\sigma}{\mu}-1}(z, \eta) {}^{r}_{\mu}\Psi_{\wp}^{\frac{m\sigma}{\mu}-1}(\eta, \xi) \wp'(\eta) d\eta \right] d\xi.$$

This is the corrected prefactor. It contains $\Gamma_\mu(\sigma)^m$, not $\Gamma_\mu(\sigma)^{m+1}$, before the Beta integral, and there is no extraneous factor $1/\mu$ in the change of variables. Indeed, with

$$t = \frac{\wp(\eta) - \wp(\xi)}{\wp(z) - \wp(\xi)}, \qquad \wp'(\eta) d\eta = (\wp(z) - \wp(\xi)) dt,$$

the inner integral becomes

$${}^{r}_{\mu}\Psi_{\wp}^{\frac{(m+1)\sigma}{\mu}-1}(z, \xi) B\left(\frac{\sigma}{\mu}, \frac{m\sigma}{\mu}\right).$$

The requested classical-to-μ conversion is

$$\begin{aligned} B\left(\frac{\sigma}{\mu}, \frac{m\sigma}{\mu}\right) &= \frac{\Gamma(\sigma/\mu)\Gamma(m\sigma/\mu)}{\Gamma((m+1)\sigma/\mu)} \\ &= \mu \frac{\Gamma_\mu(\sigma)\Gamma_\mu(m\sigma)}{\Gamma_\mu((m+1)\sigma)}. \end{aligned}$$

After cancellation, we obtain exactly (6.4) with $n = m + 1$.

Since w is continuous on the compact interval, $0 \le w(z) \le w^*$. Moreover, $\wp' \in C([c,d])$, so $M_\wp := \|\wp'\|_{\infty,[c,d]} < \infty$. Therefore, for all sufficiently large n,

$$\Re^n u(z) \le \frac{[w^*\Gamma_\mu(\sigma)]^n[\wp(d)-\wp(c)]^{n\sigma/\mu-1}}{\sin(\frac{\pi}{2}r)^{n\sigma/\mu}\mu^{n+1}\Gamma_\mu(n\sigma)} M_\wp \|u\|_{L^1([c,d])}.$$

Indeed, the factor $M_\wp$ comes from the estimate $\int_c^z \wp'(\xi)u(\xi)\,d\xi \le M_\wp\|u\|_{L^1([c,d])}$. Using $\Gamma_\mu(n\sigma) = \mu^{n\sigma/\mu-1}\Gamma(n\sigma/\mu)$ and Stirling's formula shows that the right-hand side tends to zero. Letting $n \to \infty$ in the iterated inequality yields (6.2). □

Corollary 1. *Under the assumptions of Theorem 3, let $w(z) \equiv M \ge 0$. If*

$$u(z) \le v(z) + \frac{M\Gamma_\mu(\sigma)}{\mu}\, {}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}u(z),$$

then

$$u(z) \le v(z) + \int_c^z \left[\sum_{n=1}^{\infty} \frac{[M\Gamma_\mu(\sigma)]^n}{\sin(\frac{\pi}{2}r)^{n\sigma/\mu}\mu^{n+1}\Gamma_\mu(n\sigma)}\, {}^{r}_{\mu}\Psi_\wp^{\frac{n\sigma}{\mu}-1}(z,\eta)\wp'(\eta)v(\eta)\right] d\eta.$$

Corollary 2. *Assume the hypotheses of Theorem 3, and suppose in addition that v is non-decreasing. Then*

$$u(z) \le v(z)E_{\mu,\sigma}\left(\frac{\Gamma_\mu(\sigma)w(z)}{\sin(\frac{\pi}{2}r)^{\sigma/\mu}\mu}[\wp(z)-\wp(c)]^{\sigma/\mu}\right), \tag{6.5}$$

where

$$E_{\mu,\sigma}(q) = \sum_{n=0}^{\infty} \frac{q^n}{\Gamma_\mu(n\sigma+\mu)}. \tag{6.6}$$

Proof. Because $v(\eta) \le v(z)$ and the exponential in the kernel is at most one, Theorem 3 gives

$$u(z) \le v(z)\sum_{n=0}^{\infty} \frac{1}{\Gamma_\mu(n\sigma+\mu)}\left[\frac{\Gamma_\mu(\sigma)w(z)}{\sin(\frac{\pi}{2}r)^{\sigma/\mu}\mu}[\wp(z)-\wp(c)]^{\sigma/\mu}\right]^n,$$

which is (6.5). □

Remark 5. *The series in* (6.6) *is not a new special function. If $\alpha = \sigma/\mu$, then*

$$\Gamma_\mu(n\sigma+\mu) = \mu^{n\alpha}\Gamma(n\alpha+1),$$

and consequently

$$E_{\mu,\sigma}(q) = E_\alpha\left(\frac{q}{\mu^\alpha}\right), \tag{6.7}$$

where E_α is the classical one-parameter Mittag–Leffler function. Since E_α is an entire function for every $\alpha > 0$, the series (6.6) *converges for every complex, and hence every real, q.*

6.1. Application to a nonlinear cotangent-Caputo Cauchy problem

Consider, for $0 < \alpha = \sigma/\mu \leq 1$,

$$ {}^{C}_{c,\mu}\mathbb{D}^{\sigma,r,\wp}x(z) = F(z, x(z)), \qquad x(c) = x_c. \tag{6.8} $$

By Theorem 2, (6.8) is equivalent to

$$ x(z) = {}^{r}_{\mu}\Psi^{0}_{\wp}(z, c)x_c + {}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}F(\cdot, x(\cdot))(z). \tag{6.9} $$

Theorem 4. *Assume that $F : [c, d]\times\mathbb{R} \to \mathbb{R}$ is continuous and globally Lipschitz in its second variable,*

$$ |F(z, x) - F(z, y)| \leq L|x - y|. $$

If

$$ Q := \frac{L[\wp(d) - \wp(c)]^{\alpha}}{(\mu \sin(\pi r/2))^{\alpha}\Gamma(\alpha + 1)} < 1, \tag{6.10} $$

then (6.8) *has a unique solution in* $C([c, d])$. *Moreover, if x and y are any two solutions corresponding to initial data x_c and y_c, then*

$$ |x(z) - y(z)| \leq |x_c - y_c|\, E_{\alpha}\left(\frac{L[\wp(z) - \wp(c)]^{\alpha}}{(\mu \sin(\pi r/2))^{\alpha}}\right). \tag{6.11} $$

In particular, solutions with the same initial value coincide.

Proof. Define $\mathcal{T}$ by the right-hand side of (6.9). Using the kernel bound $e^{-a(\wp(z)-\wp(\eta))} \leq 1$ and the substitution $s = \wp(z) - \wp(\eta)$ gives

$$ \|\mathcal{T}x - \mathcal{T}y\|_{\infty} \leq Q\|x - y\|_{\infty}. $$

Hence, Banach's fixed-point theorem proves existence and uniqueness under (6.10).

To show explicitly how Theorem 3 enters, let $u(z) = |x(z)-y(z)|$ and $\delta = |x_c-y_c|$. Since ${}^{r}_{\mu}\Psi^{0}_{\wp}(z, c) \leq 1$,

$$ u(z) \leq \delta + L\,{}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}u(z) = \delta + \frac{\Gamma_{\mu}(\sigma)}{\mu}\left(\frac{\mu L}{\Gamma_{\mu}(\sigma)}\right){}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}u(z). $$

Apply Corollary 2 with the constant $w = \mu L/\Gamma_{\mu}(\sigma)$, and use (6.7). This yields (6.11). If $x_c = y_c$, then $\delta = 0$ and $u \equiv 0$, which is the Gronwall uniqueness argument. □

7. Illustrative examples

The first three examples deliberately use the classical limit to isolate, respectively, Theorem 3, Corollaries 1 and 2. Their aim is to verify the hypotheses and distinguish the assumed integral inequality from the final majorant. Example 4 then leaves the classical limit and tests $\mu \neq 1$, $r < 1$, and a nonlinear coordinate map $\wp$.

In the first three examples, we take

$$ c = 0, \qquad d = 1, \qquad \mu = 1, \qquad r = 1, \qquad \wp(z) = z, \qquad \sigma = \frac{1}{2}. $$

Consequently, the $(\mu, \wp)$-cotangent fractional integral reduces to the classical Riemann–Liouville integral $I_{0+}^{1/2}$. For

$$u(z) = 1 + z,$$

we have

$$I_{0+}^{1/2} u(z) = \frac{z^{1/2}}{\Gamma(3/2)} + \frac{z^{3/2}}{\Gamma(5/2)} = \frac{2}{\sqrt{\pi}} z^{1/2} + \frac{4}{3\sqrt{\pi}} z^{3/2}. \tag{7.1}$$

Example 1. *(Illustration of Theorem 3) Let*

$$v(z) = 1, \qquad w(z) = \frac{1+z}{2}, \qquad u(z) = 1 + z.$$

The functions u and v are non-negative and integrable, whereas w is continuous, non-negative, and non-decreasing on $[0, 1]$. *By* (7.1)*,*

$$\begin{aligned} v(z) + \Gamma\left(\frac{1}{2}\right) w(z) I_{0+}^{1/2} u(z) &= 1 + \frac{1+z}{2}\left(2z^{1/2} + \frac{4}{3} z^{3/2}\right) \\ &\geq 1 + z = u(z), \qquad 0 \leq z \leq 1. \end{aligned}$$

Thus, all hypotheses of Theorem 3 are fulfilled. Since v is constant, the series in the theorem can be written in terms of the classical Mittag–Leffler function $E_{1/2}$*:*

$$u(z) \leq E_{1/2}\left(\sqrt{\pi}\, w(z) z^{1/2} \right). \tag{7.2}$$

Figure 1 compares u, the right-hand side of the assumed integral inequality, and the upper bound in (7.2)*.*

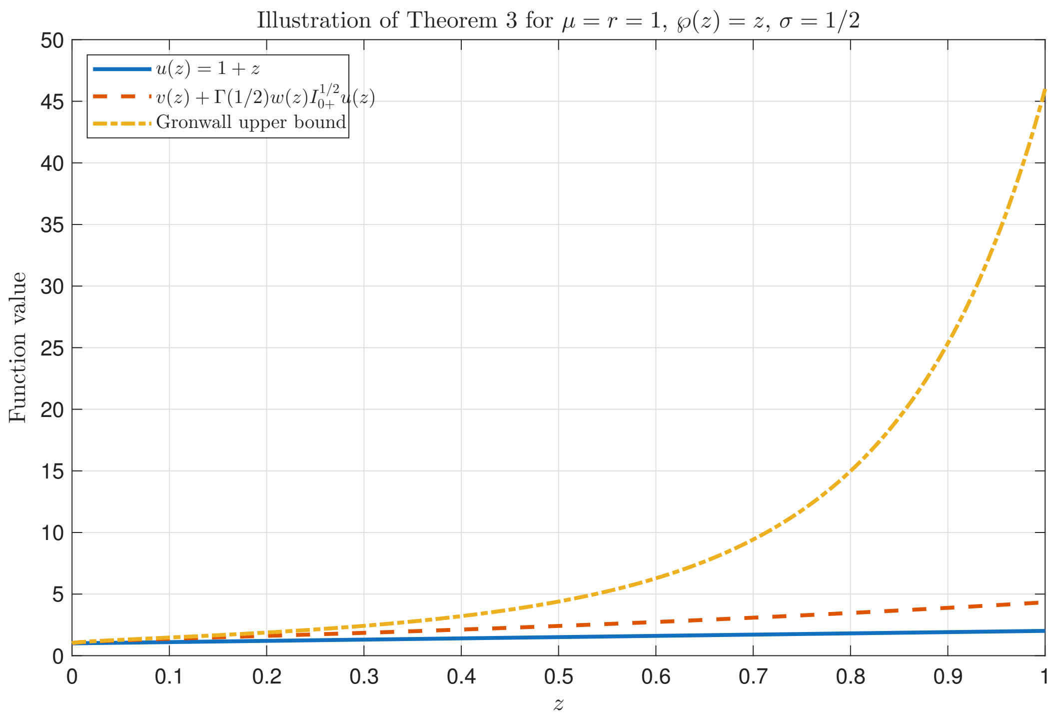

Figure 1. Graphical illustration of Theorem 3 for $\mu = r = 1$, $\wp(z) = z$, and $\sigma = 1/2$. The assumed integral inequality and the Gronwall upper bound are both satisfied on $[0, 1]$.

Example 2. *(Illustration of Corollary 1) Take*

$$v(z) = 1, \qquad w(z) \equiv M = \frac{1}{2}, \qquad u(z) = 1 + z.$$

Using (7.1)*, we obtain*

$$\begin{aligned} v(z) + M\Gamma\left(\frac{1}{2}\right) I_{0+}^{1/2} u(z) &= 1 + z^{1/2} + \frac{2}{3} z^{3/2} \\ &\geq 1 + z = u(z), \qquad 0 \leq z \leq 1. \end{aligned}$$

Corollary 1 therefore gives

$$u(z) \leq E_{1/2}\left(\frac{\sqrt{\pi}}{2} z^{1/2}\right). \tag{7.3}$$

For instance, at $z = 1$, the left-hand side is $u(1) = 2$, whereas the upper bound in (7.3) *is approximately* 3.925771.

Example 3. *(Illustration of Corollary 2) Let*

$$v(z) = 1 + \frac{z}{2}, \qquad w(z) \equiv \frac{1}{4}, \qquad u(z) = 1 + z.$$

The function v is non-decreasing. Moreover,

$$\begin{aligned} v(z) + \frac{1}{4}\Gamma\left(\frac{1}{2}\right) I_{0+}^{1/2} u(z) &= 1 + \frac{z}{2} + \frac{z^{1/2}}{2} + \frac{z^{3/2}}{3} \\ &\geq 1 + z = u(z), \qquad 0 \leq z \leq 1. \end{aligned}$$

Hence, Corollary 2 yields the explicit estimate

$$u(z) \leq \left(1 + \frac{z}{2}\right) E_{1/2}\left(\frac{\sqrt{\pi}}{4} z^{1/2}\right). \tag{7.4}$$

At $z = 1$, inequality (7.4) *gives $u(1) = 2 \leq 2.681766$.*

Example 4. *(Nontrivial parameters and attenuation) Let*

$$c = 1, \quad d = e, \quad \mu = 2, \quad \sigma = 1, \quad \alpha = \frac{1}{2}, \quad \wp(z) = \log z, \quad u(z) = v(z) = 1, \quad w(z) \equiv M = \frac{1}{5}.$$

Choose first $r = 1/2$. Then $\theta = \pi/4$, $\sin\theta = \sqrt{2}/2$, and $a = \cot\theta/\mu = 1/2$. Since the memory term is non-negative, condition (6.1) *is automatically satisfied:*

$$1 = u(z) \leq 1 + \frac{\Gamma_2(1)}{2} M_{c,\mu} \mathbb{I}^{1,r,\wp} 1.$$

For $x = \log z$, the exact series bound from Theorem 3 can be evaluated without dropping the exponential factor. Define

$$J_{n,r}(x) = \int_0^x e^{-a_r t} t^{n\alpha - 1} dt, \qquad a_r = \frac{\cot(\pi r/2)}{\mu}.$$

For $r < 1$,

$$J_{n,r}(x) = a_r^{-n\alpha}\,\gamma(n\alpha, a_r x),$$

where γ is the lower incomplete Gamma function, while for $r = 1$, the limit is $J_{n,1}(x) = x^{n\alpha}/(n\alpha)$. Hence,

$$B_r(z) = 1 + \sum_{n=1}^{\infty} \frac{[\Gamma_2(1)M]^n}{\sin(\pi r/2)^{n/2} 2^{n+1}\Gamma_2(n)} J_{n,r}(\log z) \tag{7.5}$$

is an explicit Gronwall majorant for $u(z) = 1$.

Figure 2 compares (7.5) for $r = 1/2$ and $r = 1$. The $r = 1/2$ kernel contains the attenuation factor $e^{-t/2}$, whereas $r = 1$ has no exponential attenuation. The plot also illustrates that attenuation is not a standalone phenomenon since the normalizing function $\sin(\pi r/2)^{-n\alpha}$ varies in parallel, and, therefore, the majorant may not be monotonically decreasing as r becomes smaller. It is for this reason that the phenomenon needs to be considered based on the whole normalized kernel.

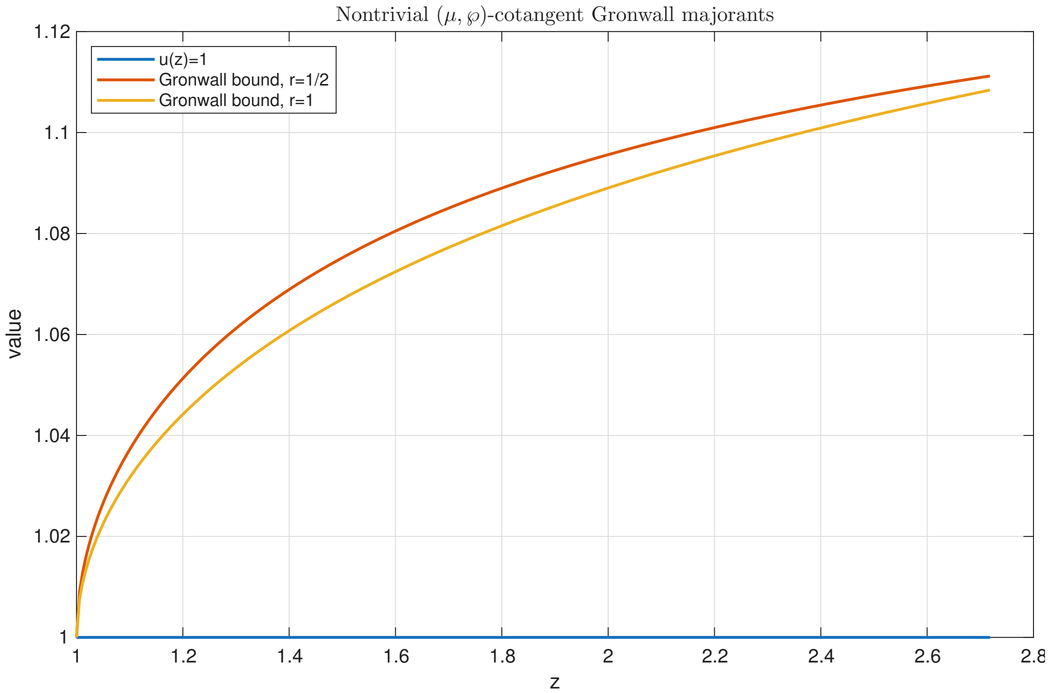

Figure 2. Exact series majorant (7.5) for $\mu = 2$, $\sigma = 1$, $\wp(z) = \log z$, and $M = 1/5$, comparing the attenuated case $r = 1/2$ with the unattenuated case $r = 1$.

Table 1. Representative numerical values for the three illustrative examples.

z	$u(z)$	RHS in Example 1	Bound in Example 1	Bound in Example 2	Bound in Example 3
0.00	1.000000	1.000000	1.000000	1.000000	1.000000
0.25	1.250000	1.729167	2.129057	1.787844	1.472238
0.50	1.500000	2.414214	4.394504	2.405847	1.850980
0.75	1.750000	3.273317	11.798055	3.103976	2.250592
1.00	2.000000	4.333333	45.999326	3.925771	2.681766

8. Expanded derivations and exact verification of the examples

For the sake of completeness, this section describes the direct computations of the basic composition rules and the first three illustrative examples. The computations are meant to complement the conjugation argument of Lemma 1. They are presented for completeness of the paper and for illustration of the interplay of the exponential factor, rescaling due to $\wp$, and μ-Gamma normalization.

8.1. Direct kernel calculation for the semigroup property

Set

$$\theta = \frac{\pi r}{2}, \qquad a = \frac{\cot\theta}{\mu},$$

and introduce the kernel

$$\mathcal{K}_\lambda(z,\eta) = e^{-a(\wp(z)-\wp(\eta))}(\wp(z)-\wp(\eta))^{\frac{\lambda}{\mu}-1}, \qquad c \le \eta < z \le d. \tag{8.1}$$

From Definition 1, we get

$${}_{c,\mu}\mathbb{I}^{\lambda,r,\wp}h(z) = \frac{1}{(\sin\theta)^{\lambda/\mu}\mu\Gamma_\mu(\lambda)}\int_c^z \mathcal{K}_\lambda(z,\eta)\wp'(\eta)h(\eta)\,d\eta. \tag{8.2}$$

Let $\sigma, \delta > 0$. Substituting (8.2) twice and using Fubini's theorem gives

$${}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}\Big({}_{c,\mu}\mathbb{I}^{\delta,r,\wp}h\Big)(z) = \frac{1}{(\sin\theta)^{(\sigma+\delta)/\mu}\mu^2\Gamma_\mu(\sigma)\Gamma_\mu(\delta)} \times \int_c^z\left[\int_\xi^z \mathcal{K}_\sigma(z,\eta)\mathcal{K}_\delta(\eta,\xi)\wp'(\eta)\,d\eta\right]\wp'(\xi)h(\xi)\,d\xi. \tag{8.3}$$

The exponential factors in the inner integral combine into

$$e^{-a(\wp(z)-\wp(\eta))}e^{-a(\wp(\eta)-\wp(\xi))} = e^{-a(\wp(z)-\wp(\xi))}.$$

Next, put

$$s = \frac{\wp(\eta)-\wp(\xi)}{\wp(z)-\wp(\xi)}.$$

Since $\wp'(z) > 0$, the change of variables is allowed. As η moves from ξ to z, s changes from 0 to 1. Thus

$$\wp'(\eta)d\eta = (\wp(z)-\wp(\xi))ds.$$

Therefore,

$$\begin{aligned}
&\int_\xi^z \mathcal{K}_\sigma(z,\eta)\mathcal{K}_\delta(\eta,\xi)\wp'(\eta)\,d\eta \\
&= e^{-a(\wp(z)-\wp(\xi))}(\wp(z)-\wp(\xi))^{\frac{\sigma+\delta}{\mu}-1}\int_0^1 (1-s)^{\frac{\sigma}{\mu}-1}s^{\frac{\delta}{\mu}-1}\,ds \\
&= e^{-a(\wp(z)-\wp(\xi))}(\wp(z)-\wp(\xi))^{\frac{\sigma+\delta}{\mu}-1}B\Big(\frac{\sigma}{\mu},\frac{\delta}{\mu}\Big).
\end{aligned} \tag{8.4}$$

Using

$$B\left(\frac{\sigma}{\mu}, \frac{\delta}{\mu}\right) = \frac{\Gamma(\sigma/\mu)\Gamma(\delta/\mu)}{\Gamma((\sigma+\delta)/\mu)} = \mu \frac{\Gamma_\mu(\sigma)\Gamma_\mu(\delta)}{\Gamma_\mu(\sigma+\delta)}, \tag{8.5}$$

we see that all intermediate constants cancel exactly. Substitution of (8.4) into (8.3) yields

$${}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}\left({}_{c,\mu}\mathbb{I}^{\delta,r,\wp}h\right)(z) = {}_{c,\mu}\mathbb{I}^{\sigma+\delta,r,\wp}h(z).$$

The same process with σ and δ switched shows commutativity. Hence, the computation of the direct kernel matches the proof using conjugation as stated in Theorem 1.

8.2. *First-order cancellation and its iterated form*

The case $\lambda = \mu$ is particularly informative. Since $\Gamma_\mu(\mu) = 1$, formula (8.2) becomes

$${}_{c,\mu}\mathbb{I}^{\mu,r,\wp}g(z) = \frac{1}{\mu \sin\theta} \int_c^z e^{-a(\wp(z)-\wp(\eta))}\wp'(\eta)g(\eta)\, d\eta. \tag{8.6}$$

Apply (8.6) to

$$\mathbb{D}^{r,\wp}h(\eta) = \cos\theta\, h(\eta) + \mu \sin\theta \frac{h'(\eta)}{\wp'(\eta)}.$$

We obtain

$${}_{c,\mu}\mathbb{I}^{\mu,r,\wp}(\mathbb{D}^{r,\wp}h)(z) = \frac{\cos\theta}{\mu\sin\theta}\int_c^z e^{-a(\wp(z)-\wp(\eta))}\wp'(\eta)h(\eta)\, d\eta + \int_c^z e^{-a(\wp(z)-\wp(\eta))}h'(\eta)\, d\eta. \tag{8.7}$$

An integration by parts in the last integral gives

$$\int_c^z e^{-a(\wp(z)-\wp(\eta))}h'(\eta)\, d\eta = h(z) - e^{-a(\wp(z)-\wp(c))}h(c) - a\int_c^z e^{-a(\wp(z)-\wp(\eta))}\wp'(\eta)h(\eta)\, d\eta. \tag{8.8}$$

Because $a = \cos\theta/(\mu\sin\theta)$, the integral terms in (8.7) and (8.8) cancel. Consequently,

$${}_{c,\mu}\mathbb{I}^{\mu,r,\wp}(\mathbb{D}^{r,\wp}h)(z) = h(z) - e^{-a(\wp(z)-\wp(c))}h(c). \tag{8.9}$$

The identity (8.9) describes the boundary term that remains when the first-order derivative is integrated. It also explains the structure of the correction polynomial in the Caputo composition formula.

Iterating (8.9) gives a useful integer-multiple identity. For every integer $m \geq 1$ and every sufficiently smooth function h,

$${}_{c,\mu}\mathbb{I}^{m\mu,r,\wp}(\mathbb{D}^{m,r,\wp}h)(z) = h(z) - \sum_{k=0}^{m-1} \frac{{}^{r}_{\mu}\Psi^{k}_{\wp}(z,c)}{(\mu\sin\theta)^k k!}\mathbb{D}^{k,r,\wp}h(c). \tag{8.10}$$

Indeed, the case $m = 1$ is exactly (8.9). Assume that (8.10) holds for a fixed integer m. Apply it to $\mathbb{D}^{r,\wp}h$, then integrate once more with ${}_{c,\mu}\mathbb{I}^{\mu,r,\wp}$. The semigroup law and Lemma 2 shift the power of ${}^{r}_{\mu}\Psi^{k}_{\wp}$ by one, while (8.9) supplies the missing boundary term corresponding to $k = 0$. This proves the formula for $m + 1$.

Formula (8.10) is consistent with Theorem 2. In fact, if $i = \lceil \sigma/\mu \rceil$, then

$$\begin{aligned} {}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}\left({}^{C}_{c,\mu}\mathbb{D}^{\sigma,r,\wp}h\right) &= {}_{c,\mu}\mathbb{I}^{\sigma,r,\wp}\left({}_{c,\mu}\mathbb{I}^{i\mu-\sigma,r,\wp}\mathbb{D}^{i,r,\wp}h\right) \\ &= {}_{c,\mu}\mathbb{I}^{i\mu,r,\wp}\left(\mathbb{D}^{i,r,\wp}h\right), \end{aligned}$$

and the right-hand side is evaluated by (8.10). This provides a direct derivation of (5.2) that is independent of the conjugation representation.

8.3. Interpretation of the transformed trace terms

The transformed traces in (4.8) are not arbitrary additions. They are inherited from the classical Riemann–Liouville composition rule after the change of variables $x = \wp(z) - \wp(c)$ and the exponential rescaling

$$\widehat{h}(x) = e^{ax}h\Big(\wp^{-1}(\wp(c) + x)\Big).$$

More precisely, if $\alpha = \sigma/\mu$ and $i = \lceil\alpha\rceil$, then

$$I^{\alpha}_{0+}D^{\alpha}_{0+}\widehat{h}(x) = \widehat{h}(x) - \sum_{k=1}^{i} \frac{x^{\alpha-k}}{\Gamma(\alpha - k + 1)} \left(D^{\alpha-k}_{0+}\widehat{h}\right)(0+). \tag{8.11}$$

Multiplication by e^{-ax} immediately yields (4.8). For integer values $\alpha = m$, the terms in (8.11) reduce to the usual endpoint data for an mth-order differential equation. For non-integer α, they are fractional traces. This distinction is important: The Riemann–Liouville and Caputo operators generally have different natural boundary data even though they are built from the same kernel.

8.4. Exact verification of the first three examples

The inequalities in Section 7 can be checked without numerical approximation. Put $s = \sqrt{z}$, so that $0 \leq s \leq 1$. In Example 1, the difference between the right-hand side of the assumed inequality and $u(z)$ is

$$1 + \frac{1+z}{2}\left(2z^{1/2} + \frac{4}{3}z^{3/2}\right) - (1+z) = s\left(1 - s + \frac{5}{3}s^2 + \frac{2}{3}s^4\right) \geq 0.$$

Every term inside the last parentheses is non-negative on $[0, 1]$, except that the first two have already been combined as $1 - s \geq 0$. Hence, the integral inequality required by Theorem 3 holds pointwise on the whole interval.

For Example 2, the corresponding difference is

$$1 + z^{1/2} + \frac{2}{3}z^{3/2} - (1+z) = s\left(1 - s + \frac{2}{3}s^2\right) \geq 0.$$

Thus Corollary 1 is applicable with $M = 1/2$.

Finally, in Example 3, the difference becomes

$$1 + \frac{z}{2} + \frac{z^{1/2}}{2} + \frac{z^{3/2}}{3} - (1+z) = \frac{s}{2}\left(1 - s + \frac{2}{3}s^2\right) \geq 0.$$

Moreover, $v(z) = 1 + z/2$ is non-decreasing. Therefore, the Mittag–Leffler estimate of Corollary 2 applies. When $\mu = 1$ and $\sigma = 1/2$, the generalized function in (6.6) reduces to

$$E_{1,1/2}(q) = \sum_{n=0}^{\infty} \frac{q^n}{\Gamma(n/2 + 1)} = E_{1/2}(q),$$

which is the classical one-parameter Mittag–Leffler function used in Section 7. The numerical values in Table 1 are therefore evaluations of rigorous analytical bounds rather than empirical curve fits.

8.5. Interpretation and use of the resulting bounds

The first three examples clarify the logical role of the estimates, while Example 4 tests the genuinely non-classical parameter regime. The right-hand side in the assumed integral inequality is not the same object as the final Gronwall upper bound. The assumed inequality is the starting point. It describes a function u whose value is controlled by a forcing term v and a fractional-memory contribution. The bound supplied by Theorem 3 is obtained after repeatedly substituting this memory term into itself. Consequently, it is expected to be larger than the initial right-hand side, especially near the right endpoint of the interval. This phenomenon is visible in Figure 1. The theorem provides a guaranteed majorant, not an approximation formula for u.

The distinction is relevant in applications to fractional differential equations. Suppose that two candidate solutions x_1 and x_2 satisfy an estimate of the form

$$|x_1(z) - x_2(z)| \le v(z) + \frac{\Gamma_\mu(\sigma)}{\mu} w(z)_{c,\mu}\mathbb{I}^{\sigma,r,\wp}|x_1 - x_2|(z).$$

Taking $u(z) = |x_1(z) - x_2(z)|$, Theorem 3 yields an explicit upper bound for the discrepancy between the solutions. If the forcing term vanishes, $v \equiv 0$, the series estimate implies $u \equiv 0$ under the hypotheses of the theorem, which is the usual route to a uniqueness argument. If v is small but nonzero, the same estimate provides a continuous-dependence or stability bound. In this sense, the generalized Gronwall inequality can be used as an analytical tool before a numerical discretization is introduced.

The estimate depends on the transformed distance $\wp(z) - \wp(c)$, the normalized order $\alpha = \sigma/\mu$, the attenuation rate $a = \cot(\pi r/2)/\mu$, and the explicit normalization. As stressed in Lemma 1, σ, μ, and r should not be counted as three independent parameters governing the kernel shape. Example 4 retains the full normalization and exponential weight, thereby showing their combined influence on the final majorant.

Corollary 1 isolates the frequent situation in which the coefficient multiplying the memory integral is constant. Corollary 2 imposes the additional monotonicity of v so that $v(\eta)$ can be replaced by $v(z)$ inside the integral. This step is what converts the series of integral terms into a closed Mittag–Leffler expression. Thus, the corollaries are not redundant restatements of Theorem 3: Each one trades a stronger assumption for a simpler and more directly usable estimate.

The tabulated values were computed from the displayed analytical formulas at representative points of $[0, 1]$. They are included to facilitate independent verification of the plot and to distinguish the exact inequalities from purely qualitative illustrations. Additional grid points can be generated directly from the same expressions without changing any assumption or conclusion.

9. Conclusions

This paper has developed the $(\mu, \wp)$-cotangent fractional integral together with its Riemann–Liouville and Caputo derivatives, while clarifying an important structural limitation of the parameterization. Through exponential conjugation and the coordinate $x = \wp(z) - \wp(c)$, the operators reduce to classical fractional operators of order $\alpha = \sigma/\mu$ with an explicit scaling and attenuation. Thus the main value of the framework lies in its transparent transformed structure, coordinate flexibility, and tractable composition rules, rather than in treating σ, μ, and r as three independent parameters governing the kernel shape.

Apart from that, the nonlinear $(\mu, \wp)$-cotangent Caputo Cauchy problem has also been considered. Existence is proven using an explicit contractive condition, while the uniqueness and continuous dependence of the solution upon the initial data are obtained via Gronwall estimate. The first three examples individually show how the theorem and the corollaries can be verified in the classical limit case; however, the fourth example with $\mu = 2$, $r = 1/2$ and $\wp(z) = \log z$ gives an illustration to the complete normalized exponential kernel with competing attenuation and normalization in the bound.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest.

References

1. J. V. C. Sousa, E. C. D. Oliveira, On the ψ-Hilfer fractional derivative, *Commun. Nonlinear Sci.*, **60** (2018), 72–91. https://doi.org/10.1016/j.cnsns.2018.01.005
2. F. Norouzi, G. M. N'Guérékata, A study of ψ-Hilfer fractional differential system with application in financial crisis, *Chaos Soliton. Fract.*, **6** (2021), 100056. https://doi.org/10.1016/j.csfx.2021.100056
3. M. Samraiz, Z. Perveen, G. Rahman, K. S. Nisar, D. Kumar, On the (k, s)-Hilfer–Prabhakar fractional derivative with applications to mathematical physics, *Front. Phys.*, **8** (2020), 309. https://doi.org/10.3389/fphy.2020.00309
4. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science, 2006, 2453–2461.
5. L. Sadek, A cotangent fractional derivative with the application, *Fractal Fract.*, **7** (2023), 444. https://doi.org/10.3390/fractalfract7060444
6. S. Mubeen, G. M. Habibullah, k-fractional integrals and applications, *Int. J. Contemp. Math. Sci.*, **7** (2012), 89–94.
7. G. A. Dorrego, An alternative definition for the k-Riemann–Liouville fractional derivative. *Appl. Math. Sci.*, **9** (2015), 481–491. https://doi.org/10.12988/ams.2015.411893
8. L. Sadek, A. Algefary, The cotangent derivative with respect to another function: Theory, methods and applications, *Fractal Fract.*, **9** (2025), 690. https://doi.org/10.3390/fractalfract9110690
9. A. Aphithana, W. Sudsutad, J. Kongson, C. Thaiprayoon, Measure of non-compactness for nonlocal boundary value problems via (k, ψ)-Riemann–Liouville derivative on unbounded domain, *AIMS Math.*, **8** (2023), 20018–20047. https://doi.org/10.3934/math.20231020

10. Y. C. Kwun, G. Farid, W. Nazeer, S. Ullah, S. M. Kang, Generalized Riemann–Liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of Hadamard inequalities, *IEEE Access*, **6** (2018), 64946–64953. https://doi.org/10.1109/ACCESS.2018.2878266
11. K. D. Kucche, A. D. Mali, On the nonlinear (k, Ψ)-Hilfer fractional differential equations, *Chaos Soliton. Fract.*, **152** (2021), 111335. https://doi.org/10.1016/j.chaos.2021.111335
12. A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model, *Therm. Sci.*, **20** (2016), 763–769. https://doi.org/10.2298/TSCI160111018A
13. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Prog. Fract. Differ. Appl.*, **1** (2015), 73–85. http://dx.doi.org/10.12785/pfda/010201
14. X. J. Yang, J. A. T. Machado, D. Baleanu, Anomalous diffusion models with general fractional derivatives within the kernels of the extended Mittag-Leffler type functions, *Rom. Rep. Phys.*, **69** (2017), 115.
15. N. H. Tuan, D. Baleanu, T. N. Thach, D. O'Regan, N. H. Can, Final value problem for nonlinear time fractional reaction-diffusion equation with discrete data, *J. Comput. Appl. Math.*, **376** (2020), 112883. https://doi.org/10.1016/j.cam.2020.112883
16. J. F. G. Aguilar, L. Torres, H. Y. Martínez, D. Baleanu, J. M. Reyes, I. O. Sosa, Fractional Liénard type model of a pipeline within the fractional derivative without singular kernel, *Adv. Differ. Equ.*, **2016** (2016), 173. https://doi.org/10.1186/s13662-016-0908-1
17. J. Alzabut, T. Abdeljawad, F. Jarad, W. Sudsutad, A Gronwall inequality via the generalized proportional fractional derivative with applications, *J. Inequal. Appl.*, **2019** (2019), 101. https://doi.org/10.1186/s13660-019-2052-4
18. W. Sudsutad, C. Thaiprayoon, B. Khaminsou, J. Kongson, J. Alzabut, A Gronwall inequality and its applications to the Cauchy-type problem under ψ-Hilfer proportional fractional operators, *J. Inequal. Appl.*, **2023** (2023), 20. https://doi.org/10.1186/s13660-023-02929-x
19. L. Sadek, S. A. Idris, F. Jarad, The general Caputo–Katugampola fractional derivative and numerical approach for solving fractional differential equations, *Alex. Eng. J.*, **121** (2025), 539–557. https://doi.org/10.1016/j.aej.2025.02.065
20. A. Turab, N. Rosli, Study of fractional differential equations emerging in the theory of chemical graphs: A robust approach, *Mathematics*, **10** (2022), 4222. https://doi.org/10.3390/math10224222
21. A. Turab, J. A. N. Selva, W. Ali, A. Montoyo, J. J. Tiang, Computational and parameter-sensitivity analysis of dual-order memory-driven fractional differential equations with an application to animal learning, *Fractal Fract.*, **9** (2025), 664. https://doi.org/10.3390/fractalfract9100664
22. A. Turab, J. A. N. Selva, W. Ali, A. Montoyo, J. J. Tiang, Analytical and numerical analysis of a memory-dependent fractional model for behavioral learning dynamics, *Fractal Fract.*, **9** (2025), 710. https://doi.org/10.3390/fractalfract9110710

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