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Research article

Non-additive families satisfying a mixed bi-skew higher derivation identity on $\star$-algebras

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Abstract: Let $\mathscr{A}$ be a unital $\star$-algebra over the complex field that possesses a nontrivial projection $\mathscr{P}$ satisfying the faithfulness property $\mathscr{X}\mathscr{A}\mathscr{P} = 0 \Rightarrow \mathscr{X} = 0$ (and the analogous condition for $I - \mathscr{P}$). Suppose a family $\{\Delta_n\}_{n=0}^{\infty}$ of (not necessarily additive) maps $\Delta_n : \mathscr{A} \to \mathscr{A}$ where $\Delta_0 = \mathrm{id}_{\mathscr{A}}$ satisfies the mixed bi-skew Jordan–Lie higher derivation relation

$$\Delta_n([\mathscr{X} \bullet \mathscr{Y}, \mathscr{Z}]_\star) = [\Delta_n(\mathscr{X}) \bullet \mathscr{Y}, \mathscr{Z}]_\star + [\mathscr{X} \bullet \Delta_n(\mathscr{Y}), \mathscr{Z}]_\star + [\mathscr{X} \bullet \mathscr{Y}, \Delta_n(\mathscr{Z})]_\star + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [\Delta_r(\mathscr{X}) \bullet \Delta_s(\mathscr{Y}), \Delta_t(\mathscr{Z})]_\star$$

for all $\mathscr{X}, \mathscr{Y}, \mathscr{Z} \in \mathscr{A}$, and each $n \geq 1$. We prove that all maps Δ_n are additive and preserve the involution. Consequently, the family $\{\Delta_n\}_{n=0}^{\infty}$ forms an additive $\star$-higher derivation with respect to the ternary product $[\cdot \bullet \cdot, \cdot]_\star$. Direct consequences for prime $\star$-algebras, factor von Neumann algebras, and standard operator algebras are presented, and an example is provided to illustrate the theory.

Keywords: bi-skew Jordan triple product; bi-skew Lie triple product; higher derivation; $\star$-derivation; $\star$-algebra
Mathematics Subject Classification: 16W25, 47B47

1. Introduction

Understanding when a non-additive mapping automatically becomes additive is a prominent theme in the theory of functional identities. In the context of associative algebras, much effort has been devoted to maps that preserve Jordan or Lie structures built from the ordinary product. Over the past two decades, the focus has shifted to non-standard products obtained by inserting an involution. Skew and bi-skew Jordan and Lie products have proved to be a highly productive framework for distinctive algebraic features. In particular, bi-skew derivations on these structures were investigated in [1–3], while maps preserving specific triple symmetric products were explored in [4–6]. More recently, substantial progress has been made in characterizing mixed and higher order derivations [7–9] as well as general multiplicative structures on von Neumann algebras [10–12].

For an associative $\star$-algebra $\mathscr{A}$ over $\mathbb{C}$, we shall work with the bi-skew Jordan product $\mathscr{X} \bullet \mathscr{Y} = \mathscr{X}\mathscr{Y}^\star + \mathscr{Y}\mathscr{X}^\star$ and the bi-skew Lie product $[\mathscr{X}, \mathscr{Y}]_\star = \mathscr{X}\mathscr{Y}^\star - \mathscr{Y}\mathscr{X}^\star$. A map $\Phi : \mathscr{A} \to \mathscr{A}$ (without any additivity assumption) is called a bi-skew Jordan triple derivation if

$$\Phi((\mathscr{X} \bullet \mathscr{Y}) \bullet \mathscr{Z}) = \Phi(\mathscr{X}) \bullet \mathscr{Y} \bullet \mathscr{Z} + \mathscr{X} \bullet \Phi(\mathscr{Y}) \bullet \mathscr{Z} + \mathscr{X} \bullet \mathscr{Y} \bullet \Phi(\mathscr{Z}),$$

and a bi-skew Lie triple derivation if

$$\Phi([[\mathscr{X}, \mathscr{Y}]_\star, \mathscr{Z}]_\star) = [[\Phi(\mathscr{X}), \mathscr{Y}]_\star, \mathscr{Z}]_\star + [[\mathscr{X}, \Phi(\mathscr{Y})]_\star, \mathscr{Z}]_\star + [[\mathscr{X}, \mathscr{Y}]_\star, \Phi(\mathscr{Z})]_\star.$$

Investigating such mappings has led to striking additivity results: Many nonlinear maps of this kind turn out to be $\star$-derivations (see [12–14]).

More recently, mixed products that intertwine the bi-skew Jordan and bi-skew Lie products have attracted attention and were explored in [15–18]. The higher analog $\{\Delta_n\}_{n\geq 0}$ satisfying an n-term functional equation on the mixed product arise naturally. In the present work, we study one such family. Precisely, we consider a sequence $\Delta_n : \mathscr{A} \to \mathscr{A}$ (with $\Delta_0 = \mathrm{id}_{\mathscr{A}}$) that fulfills the following relation for every $\mathscr{X}, \mathscr{Y}, \mathscr{Z} \in \mathscr{A}$ and every integer $n \geq 1$:

$$\begin{aligned}\Delta_n([\mathscr{X} \bullet \mathscr{Y},\, \mathscr{Z}]_\star) = {} & [\Delta_n(\mathscr{X}) \bullet \mathscr{Y},\, \mathscr{Z}]_\star + [\mathscr{X} \bullet \Delta_n(\mathscr{Y}),\, \mathscr{Z}]_\star + [\mathscr{X} \bullet \mathscr{Y},\, \Delta_n(\mathscr{Z})]_\star \\ & + \sum_{\substack{r+s+t=n \\ 0\leq r,s,t\leq n-1}} [\Delta_r(\mathscr{X}) \bullet \Delta_s(\mathscr{Y}),\, \Delta_t(\mathscr{Z})]_\star. \end{aligned} \tag{1.1}$$

The case $n = 1$ (where the tail sum is empty) was treated in [15], where it was shown that a map satisfying the corresponding ternary mixed identity is an additive $\star$-derivation. Our purpose is to establish the analogous statement for every n. The main result reads as follows.

Theorem 1.1. *Let $\mathscr{A}$ be a unital $\star$-algebra with identity I. Assume that $\mathscr{A}$ contains a nontrivial projection $\mathscr{P}$ such that*

$$\mathscr{X}\mathscr{A}\mathscr{P} = 0 \implies \mathscr{X} = 0, \qquad \mathscr{X}\mathscr{A}(I - \mathscr{P}) = 0 \implies \mathscr{X} = 0. \tag{1.2}$$

If a family $\{\Delta_n\}_{n=0}^\infty$ of (not necessarily additive) maps $\Delta_n : \mathscr{A} \to \mathscr{A}$ with $\Delta_0 = \mathrm{id}_{\mathscr{A}}$ satisfies (1.1) for all $\mathscr{X}, \mathscr{Y}, \mathscr{Z} \in \mathscr{A}$, and every $n \geq 1$, then every Δ_n is additive and satisfies $\Delta_n(\mathscr{X}^\star) = \Delta_n(\mathscr{X})^\star$ for all $\mathscr{X} \in \mathscr{A}$. Hence, $\{\Delta_n\}_{n=0}^\infty$ is an additive $\star$-higher derivation with respect to the ternary product $[\cdot\bullet\cdot, \cdot]_\star$.

The proof is organized by induction on n; the base step $n = 1$ is furnished by [15, Theorem 2.1]. After fixing some notation in Section 2, we carry out the induction in Section 3. The method relies on a Peirce decomposition with respect to the projection $\mathscr{P}$ and on a detailed analysis of how the family acts on Hermitian and skew-Hermitian elements. An example and direct applications are given in Section 4.

2. Preliminaries and notation

Throughout this paper, $\mathscr{A}$ denotes an associative unital $\star$-algebra over $\mathbb{C}$ with the identity I. The involution on $\mathscr{A}$ is denoted by $\star$, and its Hermitian and skew-Hermitian parts are defined by

$$\mathscr{A}_h = \{\mathscr{X} \in \mathscr{A} : \mathscr{X}^\star = \mathscr{X}\}, \qquad \mathscr{A}_s = \{\mathscr{X} \in \mathscr{A} : \mathscr{X}^\star = -\mathscr{X}\}.$$

We fix a nontrivial projection $\mathscr{P} \in \mathscr{A}$ and set $\mathscr{Q} = I - \mathscr{P}$. Condition (1.2) is assumed throughout; it is a primeness-type requirement that forces various operators determined by the Peirce components to vanish. The Peirce decomposition of $\mathscr{A}$ relative to $\mathscr{P}$ is given by

$$\mathscr{A} = \bigoplus_{i,j=1}^{2} \mathscr{A}_{ij}, \qquad \mathscr{A}_{ij} = \mathscr{P}_i \mathscr{A} \mathscr{P}_j \ \ (\mathscr{P}_1 = \mathscr{P},\ \mathscr{P}_2 = \mathscr{Q}).$$

For the skew-Hermitian part, we also define

$$\mathscr{A}_{s,12} = \mathscr{P}\mathscr{A}_s\mathscr{Q} + \mathscr{Q}\mathscr{A}_s\mathscr{P}, \qquad \mathscr{A}_{s,ii} = \mathscr{P}_i\mathscr{A}_s\mathscr{P}_i \ \ (i = 1, 2).$$

Every $\mathscr{S} \in \mathscr{A}_s$ can be uniquely expanded as $\mathscr{S} = \mathscr{S}_{11} + \mathscr{S}_{12} + \mathscr{S}_{22}$ with $\mathscr{S}_{12} \in \mathscr{A}_{s,12}$ and $\mathscr{S}_{ii} \in \mathscr{A}_{s,ii}$.

We consider a family $\Delta = \{\Delta_n\}_{n=0}^{\infty}$ of maps $\Delta_n : \mathscr{A} \to \mathscr{A}$ with $\Delta_0 = \mathrm{id}_{\mathscr{A}}$. For the remainder of the paper, we assume that Δ satisfies Identity (1.1) for all $\mathscr{X}, \mathscr{Y}, \mathscr{Z} \in \mathscr{A}$ and each $n \geq 1$. A family $\{\Delta_n\}_{n=0}^{\infty}$ of additive maps on $\mathscr{A}$ is called an additive $\star$-higher derivation if $\Delta_0 = \mathrm{id}_{\mathscr{A}}$ and, for every $n \geq 1$, we have

$$\Delta_n(\mathscr{X}\mathscr{Y}) = \sum_{k=0}^{n} \Delta_k(\mathscr{X})\Delta_{n-k}(\mathscr{Y}), \qquad \Delta_n(\mathscr{X}^\star) = \Delta_n(\mathscr{X})^\star \qquad \text{for all } \mathscr{X}, \mathscr{Y} \in \mathscr{A}.$$

3. Proof of Theorem 1.1

We establish Theorem 1.1 by induction on n. The base case $n = 1$ is provided by [15, Theorem 2.1], which ensures that Δ_1 is an additive $\star$-derivation; in particular, Δ_1 is additive, preserves the involution, is complex linear, and satisfies the higher derivation rule $\Delta_1(\mathscr{X}\mathscr{Y}) = \Delta_1(\mathscr{X})\mathscr{Y} + \mathscr{X}\Delta_1(\mathscr{Y})$ (the case $n = 1$ in the definition above). For the inductive step, we fix an integer $n \geq 2$ and assume that for all $1 \leq k < n$, the maps Δ_k are additive and satisfy $\Delta_k(\mathscr{X}^\star) = \Delta_k(\mathscr{X})^\star$, $\Delta_k(i\mathscr{X}) = i\Delta_k(\mathscr{X})$, and the higher derivation rule

$$\Delta_k(\mathscr{X}\mathscr{Y}) = \sum_{i=0}^{k} \Delta_i(\mathscr{X})\Delta_{k-i}(\mathscr{Y}), \qquad \text{for all } \mathscr{X}, \mathscr{Y} \in \mathscr{A}.$$

We summarize the immediate consequences of this induction hypothesis in the following lemma.

Lemma 3.1. *Under the induction hypothesis above, for every* $1 \leq k < n$*, we have*

$$\Delta_k(0) = 0, \quad \Delta_k(I) = 0, \quad \Delta_k(iI) = 0, \quad \Delta_k(\mathscr{X}^\star) = \Delta_k(\mathscr{X})^\star, \quad \Delta_k(i\mathscr{X}) = i\Delta_k(\mathscr{X}).$$

Proof. The properties $\Delta_k(\mathscr{X}^\star) = \Delta_k(\mathscr{X})^\star$ and $\Delta_k(i\mathscr{X}) = i\Delta_k(\mathscr{X})$ are part of the hypothesis, and additivity gives $\Delta_k(0) = 0$. To see $\Delta_k(I) = 0$, apply the higher derivation rule with $\mathscr{X} = \mathscr{Y} = I$:

$$\Delta_k(I) = \Delta_k(I \cdot I) = \sum_{i=0}^{k} \Delta_i(I)\Delta_{k-i}(I) = 2\Delta_k(I) + \sum_{i=1}^{k-1} \Delta_i(I)\Delta_{k-i}(I).$$

For $k = 1$, the sum is empty and we already know $\Delta_1(I) = 0$ (base case). For $k > 1$, we may assume inductively that $\Delta_i(I) = 0$ for all $1 \leq i < k$. Hence, the sum vanishes and we obtain $\Delta_k(I) = 2\Delta_k(I)$, so $\Delta_k(I) = 0$. Finally, $\Delta_k(iI) = i\Delta_k(I) = 0$. □

We assume this induction hypothesis throughout the remainder of this section and will use the facts from Lemma 3.1 without explicit reference. To prove the main result for Δ_n, we divide the argument into a sequence of lemmas, beginning with the behavior of the map at zero.

Lemma 3.2. $\Delta_n(0) = 0$.

Proof. Put $\mathscr{X} = \mathscr{Y} = \mathscr{Z} = 0$ in (1.1). Because $\Delta_k(0) = 0$ for all $k < n$ (by Lemma 3.1), the sum over r, s, t vanishes and the first three terms are also zero. Hence, we have

$$\begin{aligned}\Delta_n(0) &= \Delta_n([0 \bullet 0, 0]_\star) \\ &= [\Delta_n(0) \bullet 0, 0]_\star + [0 \bullet \Delta_n(0), 0]_\star + [0 \bullet 0, \Delta_n(0)]_\star \\ &\quad + \sum_{\substack{r+s+t=n \\ 0 \leq r,s,t \leq n-1}} [\Delta_r(0) \bullet \Delta_s(0),\ \Delta_t(0)]_\star \\ &= 0.\end{aligned}$$

Thus, $\Delta_n(0) = 0$ for each $n \in \mathbb{N}$. □

Lemma 3.3. *For every $\mathscr{S} \in \mathscr{A}_s$, we have $\Delta_n(\mathscr{S})^\star = -\Delta_n(\mathscr{S})$.*

Proof. Any $\mathscr{S} \in \mathscr{A}_s$ satisfies $\mathscr{S} = [-\frac{iI}{2} \bullet \mathscr{S}, \frac{iI}{2}]_\star$. Applying Δ_n and using (1.1) gives

$$\begin{aligned}\Delta_n(\mathscr{S}) = {} & [\Delta_n(-\tfrac{iI}{2}) \bullet \mathscr{S}, \tfrac{iI}{2}]_\star + [-\tfrac{iI}{2} \bullet \Delta_n(\mathscr{S}), \tfrac{iI}{2}]_\star + [-\tfrac{iI}{2} \bullet \mathscr{S}, \Delta_n(\tfrac{iI}{2})]_\star \\ & + \sum_{\substack{r+s+t=n \\ 0 \leq r,s,t \leq n-1}} [\Delta_r(-\tfrac{iI}{2}) \bullet \Delta_s(\mathscr{S}), \Delta_t(\tfrac{iI}{2})]_\star.\end{aligned} \tag{3.1}$$

In the sum, $r + s + t = n$ with $0 \leq r, s, t \leq n - 1$. If $r \geq 1$, then $\Delta_r(-\frac{iI}{2}) = 0$ by Lemma 3.1; similarly, if $t \geq 1$, then $\Delta_t(\frac{iI}{2}) = 0$. Because $s \leq n - 1$, the constraint $r + s + t = n$ forces at least one of r, t to be ≥ 1 (otherwise, $r = t = 0$ would imply $s = n$). Hence, every term in the sum contains a factor that vanishes, and the whole sum is zero.
Expanding the three remaining brackets using $\mathscr{S}^\star = -\mathscr{S}$ yields

$$\Delta_n(\mathscr{S}) = i(\Delta_n(-\tfrac{iI}{2}) - \Delta_n(\tfrac{iI}{2}))\mathscr{S} + i\mathscr{S}(\Delta_n(\tfrac{iI}{2})^\star - \Delta_n(-\tfrac{iI}{2})^\star) + \tfrac{1}{2}(\Delta_n(\mathscr{S}) - \Delta_n(\mathscr{S})^\star).$$

Rearranging, we have

$$\tfrac{1}{2}(\Delta_n(\mathscr{S}) + \Delta_n(\mathscr{S})^\star) = i(\Delta_n(-\tfrac{iI}{2}) - \Delta_n(\tfrac{iI}{2}))\mathscr{S} + i\mathscr{S}(\Delta_n(\tfrac{iI}{2})^\star - \Delta_n(-\tfrac{iI}{2})^\star). \tag{3.2}$$

The left-hand side is Hermitian. Taking the adjoint of the right-hand side and using $\mathscr{S}^\star = -\mathscr{S}$ shows that it is skew-Hermitian. Consequently, both sides vanish, giving $\Delta_n(\mathscr{S}) + \Delta_n(\mathscr{S})^\star = 0$, i.e. $\Delta_n(\mathscr{S})^\star = -\Delta_n(\mathscr{S})$. □

Lemma 3.4. *For any $\mathscr{X}_{11} \in \mathscr{A}_{s,11}$, $\mathscr{Y}_{12} \in \mathscr{A}_{s,12}$, and $\mathscr{Z}_{22} \in \mathscr{A}_{s,22}$, we have*

$$\begin{aligned}\Delta_n(\mathscr{X}_{11} + \mathscr{Y}_{12}) &= \Delta_n(\mathscr{X}_{11}) + \Delta_n(\mathscr{Y}_{12}), \\ \Delta_n(\mathscr{Y}_{12} + \mathscr{Z}_{22}) &= \Delta_n(\mathscr{Y}_{12}) + \Delta_n(\mathscr{Z}_{22}).\end{aligned}$$

Proof. We prove the first equality; the second is analogous. In view of Lemma 3.1, for $k < n$, we have

$$\Delta_k(\mathscr{X}_{11} + \mathscr{Y}_{12}) = \Delta_k(\mathscr{X}_{11}) + \Delta_k(\mathscr{Y}_{12}).$$

Set $\mathscr{S} = \Delta_n(\mathscr{X}_{11} + \mathscr{Y}_{12}) - \Delta_n(\mathscr{X}_{11}) - \Delta_n(\mathscr{Y}_{12})$. By Lemma 3.3, for every $\mathscr{S} \in \mathscr{A}_s$, we have $\mathscr{S}^\star = -\mathscr{S}$. Since

$$[\mathscr{P} \bullet (\mathscr{X}_{11} + \mathscr{Y}_{12}), \mathscr{Q}]_\star = [\mathscr{P} \bullet \mathscr{X}_{11}, \mathscr{Q}]_\star + [\mathscr{P} \bullet \mathscr{Y}_{12}, \mathscr{Q}]_\star .$$

Applying Δ_n to both sides, expand each term via (1.1), we see that

$$\begin{aligned}
\Delta_n([\mathscr{P} \bullet (\mathscr{X}_{11} + \mathscr{Y}_{12}), \mathscr{Q}]_\star) &= \Delta_n([\mathscr{P} \bullet \mathscr{X}_{11}, \mathscr{Q}]_\star) + \Delta_n([\mathscr{P} \bullet \mathscr{Y}_{12}, \mathscr{Q}]_\star) \\
&= [\Delta_n(\mathscr{P}) \bullet \mathscr{X}_{11}, \mathscr{Q}]_\star + [\mathscr{P} \bullet \Delta_n(\mathscr{X}_{11}), \mathscr{Q}]_\star \\
&\quad + [\mathscr{P} \bullet \mathscr{X}_{11}, \Delta_n(\mathscr{Q})]_\star \\
&\quad + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [\Delta_r(\mathscr{P}) \bullet \Delta_s(\mathscr{X}_{11}), \Delta_t(\mathscr{Q})]_\star \\
&\quad + [\Delta_n(\mathscr{P}) \bullet \mathscr{Y}_{12}, \mathscr{Q}]_\star + [\mathscr{P} \bullet \Delta_n(\mathscr{Y}_{12}), \mathscr{Q}]_\star + [\mathscr{P} \bullet \mathscr{Y}_{12}, \Delta_n(\mathscr{Q})]_\star \\
&\quad + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [\Delta_r(\mathscr{P}) \bullet \Delta_s(\mathscr{Y}_{12}), \Delta_t(\mathscr{Q})]_\star \\
&= [\Delta_n(\mathscr{P}) \bullet (\mathscr{X}_{11} + \mathscr{Y}_{12}), \mathscr{Q}]_\star + [\mathscr{P} \bullet (\Delta_n(\mathscr{X}_{11}) + \Delta_n(\mathscr{Y}_{12})), \mathscr{Q}]_\star \\
&\quad + [\mathscr{P} \bullet (\mathscr{X}_{11} + \mathscr{Y}_{12}), \Delta_n(\mathscr{Q})]_\star \\
&\quad + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [\Delta_r(\mathscr{P}) \bullet \Delta_s(\mathscr{X}_{11} + \mathscr{Y}_{12}), \Delta_t(\mathscr{Q})]_\star.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Delta_n([\mathscr{P} \bullet (\mathscr{X}_{11} + \mathscr{Y}_{12}), \mathscr{Q}]_\star) &= [\Delta_n(\mathscr{P}) \bullet (\mathscr{X}_{11} + \mathscr{Y}_{12}), \mathscr{Q}]_\star + [\mathscr{P} \bullet \Delta_n(\mathscr{X}_{11} + \mathscr{Y}_{12}), \mathscr{Q}]_\star \\
&\quad + [\mathscr{P} \bullet (\mathscr{X}_{11} + \mathscr{Y}_{12}), \Delta_n(\mathscr{Q})]_\star \\
&\quad + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [\Delta_r(\mathscr{P}) \bullet \Delta_s(\mathscr{X}_{11} + \mathscr{Y}_{12}), \Delta_t(\mathscr{Q})]_\star.
\end{aligned}$$

A direct comparison of the last two resulting expressions gives $[\mathscr{P} \bullet \mathscr{S}, \mathscr{Q}]_\star = 0$, which gives $\mathscr{S}_{12} = \mathscr{S}_{21} = 0$. Next, using the same technique with the products $[\mathscr{P} \bullet \mathscr{Y}_{12}, \mathscr{P}]_\star$, we see that

$$\begin{aligned}
&[\Delta_n(\mathscr{P}) \bullet (\mathscr{X}_{11} + \mathscr{Y}_{12}), \mathscr{P}]_\star + [\mathscr{P} \bullet \Delta_n(\mathscr{X}_{11} + \mathscr{Y}_{12}), \mathscr{P}]_\star + [\mathscr{P} \bullet (\mathscr{X}_{11} + \mathscr{Y}_{12}), \Delta_n(\mathscr{P})]_\star \\
&\qquad + \sum_{r+s+t=n} [\Delta_r(\mathscr{P}) \bullet \Delta_s(\mathscr{X}_{11} + \mathscr{Y}_{12}), \Delta_t(\mathscr{Q})]_\star \\
&\quad = \Delta_n([\mathscr{P} \bullet (\mathscr{X}_{11} + \mathscr{Y}_{12}), \mathscr{P}]_\star) \\
&\quad = \Delta_n([\mathscr{P} \bullet \mathscr{X}_{11}, \mathscr{P}]_\star) + \Delta_n([\mathscr{P} \bullet \mathscr{Y}_{12}, \mathscr{P}]_\star) \\
&\quad = [\Delta_n(\mathscr{P}) \bullet \mathscr{X}_{11}, \mathscr{P}]_\star + [\mathscr{P} \bullet \Delta_n(\mathscr{X}_{11}), \mathscr{P}]_\star + [\mathscr{P} \bullet \mathscr{X}_{11}, \Delta_n(\mathscr{P})]_\star \\
&\qquad + \sum_{r+s+t=n} [\Delta_r(\mathscr{P}) \bullet \Delta_s(\mathscr{X}_{11}), \Delta_t(\mathscr{Q})]_\star
\end{aligned}$$

$$
\begin{aligned}
&+ [\Delta_n(\mathscr{P}) \bullet \mathscr{Y}_{12}, \mathscr{P}]_\star + [\mathscr{P} \bullet \Delta_n(\mathscr{Y}_{12}), \mathscr{P}]_\star + [\mathscr{P} \bullet \mathscr{Y}_{12}, \Delta_n(\mathscr{P})]_\star \\
&+ \sum_{r+s+t=n} [\Delta_r(\mathscr{P}) \bullet \Delta_s(\mathscr{Y}_{12}), \Delta_t(\mathscr{Q})]_\star \\
= {}& [\Delta_n(\mathscr{P}) \bullet (\mathscr{X}_{11} + \mathscr{Y}_{12}), \mathscr{P}]_\star + [\mathscr{P} \bullet (\Delta_n(\mathscr{X}_{11}) + \Delta_n(\mathscr{Y}_{12})), \mathscr{P}]_\star \\
&+ [\mathscr{P} \bullet (\mathscr{X}_{11} + \mathscr{Y}_{12}), \Delta_n(\mathscr{P})]_\star \\
&+ \sum_{r+s+t=n} [\Delta_r(\mathscr{P}) \bullet \Delta_s(\mathscr{X}_{11} + \mathscr{Y}_{12}), \Delta_t(\mathscr{Q})]_\star .
\end{aligned}
$$

The last expression gives $[\mathscr{P} \bullet \mathscr{S}, \mathscr{P}]_\star = 0$, which yields $\mathscr{S}_{11} = 0$. In a similar manner, using the same technique with the product $[\mathscr{Q} \bullet \mathscr{Y}_{12}, \mathscr{Q}]_\star$ shows that $\mathscr{S}_{22} = 0$. Hence, $\mathscr{S} = 0$, i.e., $\Delta_n(\mathscr{X}_{11} + \mathscr{Y}_{12}) = \Delta_n(\mathscr{X}_{11}) + \Delta_n(\mathscr{Y}_{12})$. □

Lemma 3.5. *For any* $\mathscr{X}_{11} \in \mathscr{A}_{s,11}$, $\mathscr{Y}_{12} \in \mathscr{A}_{s,12}$, $\mathscr{Z}_{22} \in \mathscr{A}_{s,22}$,

$$\Delta_n(\mathscr{X}_{11} + \mathscr{Y}_{12} + \mathscr{Z}_{22}) = \Delta_n(\mathscr{X}_{11}) + \Delta_n(\mathscr{Y}_{12}) + \Delta_n(\mathscr{Z}_{22}).$$

Proof. In view of Lemma 3.1, for $k < n$, we have

$$\Delta_k(\mathscr{X}_{11} + \mathscr{Y}_{12} + \mathscr{Z}_{22}) = \Delta_k(\mathscr{X}_{11}) + \Delta_k(\mathscr{Y}_{12}) + \Delta_k(\mathscr{Z}_{22}).$$

Set $\mathscr{S} = \Delta_n(\mathscr{X}_{11} + \mathscr{Y}_{12} + \mathscr{Z}_{22}) - \Delta_n(\mathscr{X}_{11}) - \Delta_n(\mathscr{Y}_{12}) - \Delta_n(\mathscr{Z}_{22})$. Using the fact that $[\mathscr{P} \bullet \mathscr{Z}_{22}, \mathscr{Q}]_\star = 0$, and Lemma 3.4 one expands $\Delta_n([\mathscr{P} \bullet (\mathscr{X}_{11} + \mathscr{Y}_{12} + \mathscr{Z}_{22}), \mathscr{Q}]_\star)$ in two ways

$$
\begin{aligned}
\Delta_n([\mathscr{P} \bullet (\mathscr{X}_{11} + \mathscr{Y}_{12} + \mathscr{Z}_{22}), \mathscr{Q}]_\star) = {}& \Delta_n([\mathscr{P} \bullet (\mathscr{X}_{11} + \mathscr{Y}_{12}), \mathscr{Q}]_\star) + \Delta_n([\mathscr{P} \bullet \mathscr{Z}_{22}, \mathscr{Q}]_\star) \\
= {}& [\Delta_n(\mathscr{P}) \bullet (\mathscr{X}_{11} + \mathscr{Y}_{12}), \mathscr{Q}]_\star + [\mathscr{P} \bullet \Delta_n(\mathscr{X}_{11} + \mathscr{Y}_{12}), \mathscr{Q}]_\star \\
&+ [\mathscr{P} \bullet (\mathscr{X}_{11} + \mathscr{Y}_{12}), \Delta_n(\mathscr{Q})]_\star \\
&+ \sum_{\substack{r+s+t=n \\ 0 \le r,s,t \le n-1}} [\Delta_r(\mathscr{P}) \bullet \Delta_s(\mathscr{X}_{11} + \mathscr{Y}_{12}), \Delta_t(\mathscr{Q})]_\star \\
&+ [\Delta_n(\mathscr{P}) \bullet \mathscr{Z}_{22}, \mathscr{Q}]_\star + [\mathscr{P} \bullet \Delta_n(\mathscr{Z}_{22}), \mathscr{Q}]_\star \\
&+ [\mathscr{P} \bullet \mathscr{Z}_{22}, \Delta_n(\mathscr{Q})]_\star + \sum_{\substack{r+s+t=n \\ 0 \le r,s,t \le n-1}} [\Delta_r(\mathscr{P}) \bullet \Delta_s(\mathscr{Z}_{22}), \Delta_t(\mathscr{Q})]_\star \\
= {}& [\Delta_n(\mathscr{P}) \bullet (\mathscr{X}_{11} + \mathscr{Y}_{12}), \mathscr{Q}]_\star + [\mathscr{P} \bullet \Delta_n(\mathscr{X}_{11}) + \Delta_n(\mathscr{Y}_{12}), \mathscr{Q}]_\star \\
&+ [\mathscr{P} \bullet (\mathscr{X}_{11} + \mathscr{Y}_{12}), \Delta_n(\mathscr{Q})]_\star \\
&+ \sum_{\substack{r+s+t=n \\ 0 \le r,s,t \le n-1}} [\Delta_r(\mathscr{P}) \bullet \Delta_s(\mathscr{X}_{11} + \mathscr{Y}_{12}), \Delta_t(\mathscr{Q})]_\star \\
&+ [\Delta_n(\mathscr{P}) \bullet \mathscr{Z}_{22}, \mathscr{Q}]_\star + [\mathscr{P} \bullet \Delta_n(\mathscr{Z}_{22}), \mathscr{Q}]_\star \\
&+ [\mathscr{P} \bullet \mathscr{Z}_{22}, \Delta_n(\mathscr{Q})]_\star + \sum_{\substack{r+s+t=n \\ 0 \le r,s,t \le n-1}} [\Delta_r(\mathscr{P}) \bullet \Delta_s(\mathscr{Z}_{22}), \Delta_t(\mathscr{Q})]_\star \\
= {}& [\Delta_n(\mathscr{P}) \bullet (\mathscr{X}_{11} + \mathscr{Y}_{12} + \mathscr{Z}_{22}), \mathscr{Q}]_\star \\
&+ [\mathscr{P} \bullet (\Delta_n(\mathscr{X}_{11}) + \Delta_n(\mathscr{Y}_{12}) + \Delta_n(\mathscr{Z}_{22})), \mathscr{Q}]_\star \\
&+ [\mathscr{P} \bullet (\mathscr{X}_{11} + \mathscr{Y}_{12} + \mathscr{Z}_{22}), \Delta_n(\mathscr{Q})]_\star
\end{aligned}
$$

$$+\sum_{\substack{r+s+t=n\\0\le r,s,t\le n-1}}\left[\Delta_r(\mathscr{P})\bullet\Delta_s(\mathscr{X}_{11}+\mathscr{Y}_{12}+\mathscr{Z}_{22}),\Delta_t(\mathscr{Q})\right]_\star,$$

and

$$\begin{aligned}\Delta_n([\mathscr{P}\bullet(\mathscr{X}_{11}+\mathscr{Y}_{12}+\mathscr{Z}_{22}),\mathscr{Q}]_\star) &= [\Delta_n(\mathscr{P})\bullet(\mathscr{X}_{11}+\mathscr{Y}_{12}+\mathscr{Z}_{22}),\mathscr{Q}]_\star\\ &\quad+[\mathscr{P}\bullet\Delta_n(\mathscr{X}_{11}+\mathscr{Y}_{12}+\mathscr{Z}_{22}),\mathscr{Q}]_\star\\ &\quad+[\mathscr{P}\bullet(\mathscr{X}_{11}+\mathscr{Y}_{12}+\mathscr{Z}_{22}),\Delta_n(\mathscr{Q})]_\star\\ &\quad+\sum_{\substack{r+s+t=n\\0\le r,s,t\le n-1}}[\Delta_r(\mathscr{P})\bullet\Delta_s(\mathscr{X}_{11}+\mathscr{Y}_{12}+\mathscr{Z}_{22}),\Delta_t(\mathscr{Q})]_\star,\end{aligned}$$

which gives, after a straightforward calculation, $[\mathscr{P}\bullet\mathscr{S},\mathscr{Q}]_\star=0$, whence $\mathscr{S}_{12}=\mathscr{S}_{21}=0$. A similar technique with the product $[(\mathscr{Q}-\mathscr{P})\bullet\mathscr{Z}_{22},iI]_\star$ yields $\mathscr{S}_{11}=\mathscr{S}_{22}=0$, so $\mathscr{S}=0$. □

Lemma 3.6. *For any $\mathscr{X}_{12},\mathscr{Y}_{12}\in\mathscr{A}_{s,12}$,*

$$\Delta_n(\mathscr{X}_{12}+\mathscr{Y}_{12})=\Delta_n(\mathscr{X}_{12})+\Delta_n(\mathscr{Y}_{12}).$$

Proof. Write $\mathscr{X}_{12}$ and $\mathscr{Y}_{12}$ in their Peirce components as $\mathscr{X}_{12}=\mathscr{P}A\mathscr{Q}+\mathscr{Q}A\mathscr{P}$, $\mathscr{Y}_{12}=\mathscr{P}B\mathscr{Q}+\mathscr{Q}B\mathscr{P}$ with $A,B\in\mathscr{A}_s$, and put $Z=\frac{iI}{2}$. We divide the proof into two steps. We first prove that

$$\Delta_n(-\mathscr{S})=-\Delta_n(\mathscr{S})\quad\text{for every }\mathscr{S}\in\mathscr{A}_{s,12}.\tag{3.3}$$

Let $\mathscr{S}\in\mathscr{A}_{s,12}$. A direct computation gives $[i\mathscr{P}\bullet\mathscr{S},Z]_\star=-\mathscr{S}$. Applying (1.1) to the bracket $[i\mathscr{P}\bullet\mathscr{S},Z]_\star$, we obtain

$$\begin{aligned}\Delta_n(-\mathscr{S}) &= [\Delta_n(i\mathscr{P})\bullet\mathscr{S},Z]_\star+[i\mathscr{P}\bullet\Delta_n(\mathscr{S}),Z]_\star+[i\mathscr{P}\bullet\mathscr{S},\Delta_n(\mathrm{Z})]_\star\\ &\quad+\sum_{\substack{r+s+t=n\\0\le r,s,t\le n-1}}[\Delta_r(i\mathscr{P})\bullet\Delta_s(\mathscr{S}),\Delta_t(\mathrm{Z})]_\star.\end{aligned}$$

Similarly, since $[i\mathscr{P}\bullet(-\mathscr{S}),Z]\star=\mathscr{S}$, applying (1.1) to $[i\mathscr{P}\bullet(-\mathscr{S}),Z]\star$ gives

$$\begin{aligned}\Delta_n(\mathscr{S}) &= [\Delta_n(i\mathscr{P})\bullet(-\mathscr{S}),Z]_\star+[i\mathscr{P}\bullet\Delta_n(-\mathscr{S}),Z]_\star+[i\mathscr{P}\bullet(-\mathscr{S}),\Delta_n(\mathrm{Z})]_\star\\ &\quad+\sum_{\substack{r+s+t=n\\0\le r,s,t\le n-1}}[\Delta_r(i\mathscr{P})\bullet\Delta_s(-\mathscr{S}),\Delta_t(\mathrm{Z})]_\star.\end{aligned}$$

Set $T=\Delta_n(\mathscr{S})+\Delta_n(-\mathscr{S})$. By Lemma 3.3, $T\in\mathscr{A}_s$. Adding the two displayed equations and using Lemma 3.1, such that $\Delta_s(-\mathscr{S})=-\Delta_s(\mathscr{S})$ for every $0\le s\le n-1$, we obtain $T=[i\mathscr{P}\bullet T,Z]_\star$. For $T\in\mathscr{A}_s$, decomposing $T=T_{11}+T_{12}+T_{22}$ with $T_{ij}\in\mathscr{A}_{s,ij}$, a direct Peirce computation gives $[i\mathscr{P}\bullet T,Z]_\star=-(\mathscr{P}T+T\mathscr{P})$. Hence

$$T=-(\mathscr{P}T+T\mathscr{P}),$$

or equivalently

$$T+\mathscr{P}T+T\mathscr{P}=0.$$

Projecting onto the Peirce components yields $3T_{11}+2T_{12}+T_{22}=0$, and therefore $T_{11}=T_{12}=T_{22}=0$. Thus $T=0$, proving (3.3).

We now prove the desired additivity. Set $D=i\mathscr{X}_{12}\mathscr{Y}_{12}+i\mathscr{Y}_{12}\mathscr{X}_{12}$. Since

$$\mathscr{X}_{12}\mathscr{Y}_{12}=(\mathscr{P}A\mathscr{Q}+\mathscr{Q}A\mathscr{P})(\mathscr{P}B\mathscr{Q}+\mathscr{Q}B\mathscr{P})=\mathscr{P}A\mathscr{Q}B\mathscr{P}+\mathscr{Q}A\mathscr{P}B\mathscr{Q},$$

and similarly

$$\mathscr{Y}_{12}\mathscr{X}_{12}=\mathscr{P}B\mathscr{Q}A\mathscr{P}+\mathscr{Q}B\mathscr{P}A\mathscr{Q},$$

we get

$$D=\mathscr{P}i(A\mathscr{Q}B+B\mathscr{Q}A)\mathscr{P}+\mathscr{Q}i(A\mathscr{P}B+B\mathscr{P}A)\mathscr{Q}\in\mathscr{A}_{s,11}+\mathscr{A}_{s,22}.$$

A direct computation gives the four identities

$$[i\mathscr{P}\bullet i\mathscr{Q},Z]_\star=0,\quad [i\mathscr{P}\bullet\mathscr{Y}_{12},Z]_\star=-\mathscr{Y}_{12},$$
$$[\mathscr{X}_{12}\bullet i\mathscr{Q},Z]_\star=-\mathscr{X}_{12},\quad [\mathscr{X}_{12}\bullet\mathscr{Y}_{12},Z]_\star=D.$$

Expanding $(i\mathscr{P}-\mathscr{X}_{12})\bullet(i\mathscr{Q}-\mathscr{Y}_{12})$ bilinearly and using equations above, we obtain

$$[(i\mathscr{P}-\mathscr{X}_{12})\bullet(i\mathscr{Q}-\mathscr{Y}_{12}),Z]_\star=\mathscr{X}_{12}+\mathscr{Y}_{12}+D.$$

Since $\mathscr{X}_{12}+\mathscr{Y}_{12}\in\mathscr{A}_{s,12}$ and $D\in\mathscr{A}_{s,11}+\mathscr{A}_{s,22}$, Lemma 3.5 yields

$$\Delta_n(\mathscr{X}_{12}+\mathscr{Y}_{12}+D)=\Delta_n(\mathscr{X}_{12}+\mathscr{Y}_{12})+\Delta_n(D).$$

Therefore,

$$\Delta_n(\mathscr{X}_{12}+\mathscr{Y}_{12})+\Delta_n(D)=\Delta_n([(i\mathscr{P}-\mathscr{X}_{12})\bullet(i\mathscr{Q}-\mathscr{Y}_{12}),Z]_\star). \tag{3.4}$$

We now expand the right-hand side of (3.4) by (1.1). By Lemmas 3.4, and 3.1, and the earlier case

$$\Delta_n(i\mathscr{P}-\mathscr{X}_{12})=\Delta_n(i\mathscr{P})-\Delta_n(\mathscr{X}_{12}),$$

$$\Delta_n(i\mathscr{Q}-\mathscr{Y}_{12})=\Delta_n(i\mathscr{Q})-\Delta_n(\mathscr{Y}_{12}),$$

and, for $0\le r,s\le n-1$, we have

$$\Delta_r(i\mathscr{P}-\mathscr{X}_{12})=\Delta_r(i\mathscr{P})-\Delta_r(\mathscr{X}_{12}),\qquad \Delta_s(i\mathscr{Q}-\mathscr{Y}_{12})=\Delta_s(i\mathscr{Q})-\Delta_s(\mathscr{Y}_{12}).$$

Substituting these decompositions and using the bilinearity of the products, we obtain

$$\begin{aligned}\Delta_n([(i\mathscr{P}-\mathscr{X}_{12})\bullet(i\mathscr{Q}-\mathscr{Y}_{12}),Z]_\star)&=\Delta_n([i\mathscr{P}\bullet i\mathscr{Q},Z]_\star)-\Delta_n([i\mathscr{P}\bullet\mathscr{Y}_{12},Z]_\star)\\&\quad-\Delta_n([\mathscr{X}_{12}\bullet i\mathscr{Q},Z]_\star)+\Delta_n([\mathscr{X}_{12}\bullet\mathscr{Y}_{12},Z]_\star)\\&=-\Delta_n(-\mathscr{Y}_{12})-\Delta_n(-\mathscr{X}_{12})+\Delta_n(D).\end{aligned}$$

Using (3.3), we have

$$-\Delta_n(-\mathscr{Y}_{12})=\Delta_n(\mathscr{Y}_{12}),\qquad -\Delta_n(-\mathscr{X}_{12})=\Delta_n(\mathscr{X}_{12}).$$

Thus,

$$\Delta_n([(i\mathscr{P}-\mathscr{X}_{12})\bullet(i\mathscr{Q}-\mathscr{Y}_{12}),Z]_\star)=\Delta_n(\mathscr{X}_{12})+\Delta_n(\mathscr{Y}_{12})+\Delta_n(D).$$

Combining this with (3.4), we get

$$\Delta_n(\mathscr{X}_{12} + \mathscr{Y}_{12}) + \Delta_n(D) = \Delta_n(\mathscr{X}_{12}) + \Delta_n(\mathscr{Y}_{12}) + \Delta_n(D).$$

Cancelling the common term $\Delta_n(D)$, we conclude that

$$\Delta_n(\mathscr{X}_{12} + \mathscr{Y}_{12}) = \Delta_n(\mathscr{X}_{12}) + \Delta_n(\mathscr{Y}_{12}).$$

This proves the lemma. □

Lemma 3.7. *For all $\mathscr{X}_{11}, \mathscr{Y}_{11} \in \mathscr{A}_{s,11}$ and $\mathscr{X}_{22}, \mathscr{Y}_{22} \in \mathscr{A}_{s,22}$, we have*

$$\Delta_n(\mathscr{X}_{11} + \mathscr{Y}_{11}) = \Delta_n(\mathscr{X}_{11}) + \Delta_n(\mathscr{Y}_{11}), \qquad \Delta_n(\mathscr{X}_{22} + \mathscr{Y}_{22}) = \Delta_n(\mathscr{X}_{22}) + \Delta_n(\mathscr{Y}_{22}).$$

Proof. We prove the first relation; the second is similar. By Lemma 3.1, for $k < n$, we have

$$\Delta_k\left(\mathscr{X}_{11} + \mathscr{Y}_{11}\right) = \Delta_k\left(\mathscr{X}_{11}\right) + \Delta_k\left(\mathscr{Y}_{11}\right).$$

Let $\mathscr{S} = \Delta_n\left(\mathscr{X}_{11} + \mathscr{Y}_{11}\right) - \Delta_n\left(\mathscr{X}_{11}\right) - \Delta_n\left(\mathscr{Y}_{11}\right)$. Applying Δ_n to $[\mathscr{Q} \bullet (\mathscr{X}_{11} + \mathscr{Y}_{11}), \mathscr{P}]_\star$ and expanding via (1.1) in two ways, we have

$$\begin{aligned}
\Delta_n\left([\mathscr{Q} \bullet (\mathscr{X}_{11} + \mathscr{Y}_{11}), \mathscr{P}]_\star\right) &= \Delta_n\left([\mathscr{Q} \bullet \mathscr{X}_{11}, \mathscr{P}]_\star\right) + \Delta_n\left([\mathscr{Q} \bullet \mathscr{Y}_{11}, \mathscr{P}]_\star\right) \\
&= [\Delta_n(\mathscr{Q}) \bullet \mathscr{X}_{11}, \mathscr{P}]_\star + [\mathscr{Q} \bullet \Delta_n(\mathscr{X}_{11}), \mathscr{P}]_\star \\
&\quad + [\mathscr{Q} \bullet \mathscr{X}_{11}, \Delta_n(\mathscr{P})]_\star \\
&\quad + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [\Delta_r(\mathscr{Q}) \bullet \Delta_s(\mathscr{X}_{11}), \Delta_t(\mathscr{P})]_\star \\
&\quad + [\Delta_n(\mathscr{Q}) \bullet \mathscr{Y}_{11}, \mathscr{P}]_\star + [\mathscr{Q} \bullet \Delta_n(\mathscr{Y}_{11}), \mathscr{P}]_\star \\
&\quad + [\mathscr{Q} \bullet \mathscr{Y}_{11}, \Delta_n(\mathscr{P})]_\star \\
&\quad + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [\Delta_r(\mathscr{Q}) \bullet \Delta_s(\mathscr{Y}_{11}), \Delta_t(\mathscr{P})]_\star \\
&= [\Delta_n(\mathscr{Q}) \bullet (\mathscr{X}_{11} + \mathscr{Y}_{11}), \mathscr{P}]_\star + [\mathscr{Q} \bullet (\Delta_n(\mathscr{X}_{11}) + \Delta_n(\mathscr{Y}_{11})), \mathscr{P}]_\star \\
&\quad + [\mathscr{Q} \bullet (\mathscr{X}_{11} + \mathscr{Y}_{11}), \Delta_n(\mathscr{P})]_\star \\
&\quad + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [\Delta_r(\mathscr{Q}) \bullet \Delta_s(\mathscr{X}_{11} + \mathscr{Y}_{11}), \Delta_t(\mathscr{P})]_\star,
\end{aligned}$$

and

$$\begin{aligned}
\Delta_n\left([\mathscr{Q} \bullet (\mathscr{X}_{11} + \mathscr{Y}_{11}), \mathscr{P}]_\star\right) &= [\Delta_n(\mathscr{Q}) \bullet (\mathscr{X}_{11} + \mathscr{Y}_{11}), \mathscr{P}]_\star + [\mathscr{Q} \bullet \Delta_n(\mathscr{X}_{11} + \mathscr{Y}_{11}), \mathscr{P}]_\star \\
&\quad + [\mathscr{Q} \bullet (\mathscr{X}_{11} + \mathscr{Y}_{11}), \Delta_n(\mathscr{P})]_\star \\
&\quad + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [\Delta_r(\mathscr{Q}) \bullet \Delta_s(\mathscr{X}_{11} + \mathscr{Y}_{11}), \Delta_t(\mathscr{P})]_\star.
\end{aligned}$$

Comparing the two relations gives $[\mathscr{Q} \bullet \mathscr{S}, \mathscr{P}]_\star = 0$, and hence $\mathscr{S}_{12} = \mathscr{S}_{21} = 0$. Next, choose an arbitrary $\mathscr{Z} \in \mathscr{A}_{s,12}$. Since $\left[\mathscr{Z} \bullet \mathscr{X}_{11}, \frac{I}{2}\right]_\star, \left[\mathscr{Z} \bullet \mathscr{Y}_{11}, \frac{I}{2}\right]_\star \in \mathscr{A}_{s,12}$, Lemma 3.6, together with the expansion of $\Delta_n\left(\left[\mathscr{Z} \bullet (\mathscr{X}_{11} + \mathscr{Y}_{11}), \frac{I}{2}\right]_\star\right)$, i.e.

$$\left[\Delta_n(\mathscr{Z}) \bullet (\mathscr{X}_{11} + \mathscr{Y}_{11}), \frac{I}{2}\right]_\star + \left[\mathscr{Z} \bullet \Delta_n(\mathscr{X}_{11} + \mathscr{Y}_{11}), \frac{I}{2}\right]_\star + \left[\mathscr{Z} \bullet (\mathscr{X}_{11} + \mathscr{Y}_{11}), \Delta_n\left(\frac{I}{2}\right)\right]_\star$$

$$
\begin{aligned}
& + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} \left[\Delta_r(\mathscr{Z}) \bullet \Delta_s(\mathscr{X}_{11}+\mathscr{Y}_{11}), \Delta_t\left(\frac{I}{2}\right)\right]_\star \\
= \ & \Delta_n\left(\left[\mathscr{Z} \bullet (\mathscr{X}_{11}+\mathscr{Y}_{11}), \frac{I}{2}\right]_\star\right) \\
= \ & \Delta_n\left(\left[\mathscr{Z} \bullet \mathscr{X}_{11}, \frac{I}{2}\right]_\star\right) + \Delta_n\left(\left[\mathscr{Z} \bullet \mathscr{Y}_{11}, \frac{I}{2}\right]_\star\right) \\
= \ & \left[\Delta_n(\mathscr{Z}) \bullet (\mathscr{X}_{11}+\mathscr{Y}_{11}), \frac{I}{2}\right]_\star + \left[\mathscr{Z} \bullet (\Delta_n(\mathscr{X}_{11})+\Delta_n(\mathscr{Y}_{11})), \frac{I}{2}\right]_\star \\
& + \left[\mathscr{Z} \bullet (\mathscr{X}_{11}+\mathscr{Y}_{11}), \Delta_n\left(\frac{I}{2}\right)\right]_\star \\
& + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} \left[\Delta_r(\mathscr{Z}) \bullet \Delta_s(\mathscr{X}_{11})+\Delta_s(\mathscr{Y}_{11}), \Delta_t\left(\frac{I}{2}\right)\right]_\star
\end{aligned}
$$

yields $\left[\mathscr{Z} \bullet \mathscr{S}, \frac{I}{2}\right]_\star = 0$. This forces $\mathscr{S}_{11} = \mathscr{S}_{22} = 0$, proving the lemma. □

Remark 3.1. It follows from Lemmas 3.2–3.7 that Δ_n is additive on the complete skew-Hermitian part $\mathscr{A}_s$.

Lemma 3.8. $\Delta_n(I) = 0$.

Proof. Since $[I \bullet I, iI]_\star = -4iI$, we apply Δ_n and expanding via (1.1), we see that

$$
\begin{aligned}
\Delta_n(-4iI) &= \Delta_n([I \bullet I, iI]_\star) \\
&= [\Delta_n(I) \bullet I, iI]_\star + [I \bullet \Delta_n(I), iI]_\star + [I \bullet I, \Delta_n(iI)]_\star \\
&\quad + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [\Delta_r(I) \bullet \Delta_s(I), \Delta_t(iI)]_\star \\
&= -8i\Delta_n(I) + 2\Delta_n(iI)^\star - 2\Delta_n(iI).
\end{aligned}
$$

In view of Lemma 3.1, a straightforward calculation gives

$$\Delta_n(iI) + \Delta_n(iI)^\star = 4i\Delta_n(I).$$

On the other hand, applying $\star$ to the relation derived from $[I \bullet I, I]_\star = 0$ yields $\Delta_n(I)^\star = \Delta_n(I)$. Combining these two facts yields $\Delta_n(I) = 0$; the detailed steps can be found in [15, Lemma 2.7]. □

Lemma 3.9. *For every Hermitian* $\mathscr{H} \in \mathscr{A}_h$, $\Delta_n(\mathscr{H})^\star = \Delta_n(\mathscr{H})$.

Proof. If we write $\mathscr{S} = \Delta_n(\mathscr{H}) - \Delta_n(\mathscr{H})^\star$, then $\mathscr{S} \in \mathscr{A}_s$ by Lemma 3.3. It can be decomposed as $\mathscr{S} = \mathscr{S}_{11} + \mathscr{S}_{12} + \mathscr{S}_{22}$ with $\mathscr{S}_{12} \in \mathscr{A}_{s,12}$ and $\mathscr{S}_{ii} \in \mathscr{A}_{s,ii}$. Using Lemma 3.1, we shall prove that all components vanish. In the following, we see that

$$[\mathscr{P} \bullet \mathscr{H}, \mathscr{Q}]_\star = \mathscr{P}\mathscr{H}\mathscr{Q} - \mathscr{Q}\mathscr{H}\mathscr{P}.$$

Apply Δ_n to the identity above and expanding $\Delta_n([\mathscr{P} \bullet \mathscr{H}, \mathscr{Q}]_\star)$ with the aid of (1.1)

$$\Delta_n(\mathscr{P}\mathscr{H}\mathscr{Q} - \mathscr{Q}\mathscr{H}\mathscr{P}) = [\Delta_n(\mathscr{P}) \bullet \mathscr{H}, \mathscr{Q}]_\star + [\mathscr{P} \bullet \Delta_n(\mathscr{H}), \mathscr{Q}]_\star + [\mathscr{P} \bullet \mathscr{H}, \Delta_n(\mathscr{Q})]_\star$$

$$+\sum_{\substack{r+s+t=n\\0\le r,s,t\le n-1}}\left[\Delta_r(\mathscr{P})\bullet\Delta_s(\mathscr{H}),\Delta_t(\mathscr{Q})\right]_\star.$$

Taking the adjoint of the expression above, and adding the two relations gives

$$\mathscr{P}\left(\Delta_n(\mathscr{H})^\star-\Delta_n(\mathscr{H})\right)\mathscr{Q}-\mathscr{Q}\left(\Delta_n(\mathscr{H})^\star-\Delta_n(\mathscr{H})\right)\mathscr{P}=0.$$

This gives $[\mathscr{P}\bullet\mathscr{S},\mathscr{Q}]_\star=0$, and consequently, we get $\mathscr{S}_{12}=0$. Next, take $[\mathscr{P}\bullet\mathscr{H},\mathscr{P}]_\star=\mathscr{H}\mathscr{P}-\mathscr{P}\mathscr{H}$ and $[\mathscr{Q}\bullet\mathscr{H},\mathscr{Q}]_\star=\mathscr{H}\mathscr{Q}-\mathscr{Q}\mathscr{H}$. Applying Δ_n on both identities, we obtain

$$\begin{aligned}\Delta_n(\mathscr{H}\mathscr{P}-\mathscr{P}\mathscr{H}) &= [\Delta_n(\mathscr{P})\bullet\mathscr{H},\mathscr{P}]_\star+[\mathscr{P}\bullet\Delta_n(\mathscr{H}),\mathscr{P}]_\star+[\mathscr{P}\bullet\mathscr{H},\Delta_n(\mathscr{P})]_\star\\&\quad+\sum_{\substack{r+s+t=n\\0\le r,s,t\le n-1}}[\Delta_r(\mathscr{P})\bullet\Delta_s(\mathscr{H}),\Delta_t(\mathscr{P})]_\star,\end{aligned}$$

and

$$\begin{aligned}\Delta_n(\mathscr{H}\mathscr{Q}-\mathscr{Q}\mathscr{H}) &= [\Delta_n(\mathscr{Q})\bullet\mathscr{H},\mathscr{Q}]_\star+[\mathscr{Q}\bullet\Delta_n(\mathscr{H}),\mathscr{Q}]_\star+[\mathscr{Q}\bullet\mathscr{H},\Delta_n(\mathscr{Q})]_\star\\&\quad+\sum_{\substack{r+s+t=n\\0\le r,s,t\le n-1}}[\Delta_r(\mathscr{Q})\bullet\Delta_s(\mathscr{H}),\Delta_t(\mathscr{Q})]_\star,\end{aligned}$$

respectively. Taking the adjoints and adding the corresponding pairs (with the aid of Lemma 3.1) gives $[\mathscr{P}\bullet\mathscr{S},\mathscr{P}]_\star=0$ and $[\mathscr{Q}\bullet\mathscr{S},\mathscr{Q}]_\star=0$, which forces $\mathscr{S}_{11}=0$ and $\mathscr{S}_{22}=0$. Thus in all $\mathscr{S}=0$, i.e., $\Delta_n(\mathscr{H})^\star=\Delta_n(\mathscr{H})$. □

Lemma 3.10. *There is a complex number η such that $\Delta_n(iI)=i\eta I$. Moreover, for every $\mathscr{H}\in\mathscr{A}_h$, we have*

$$\Delta_n(i\mathscr{H})=i\Delta_n(\mathscr{H})+i\eta\mathscr{H}.$$

Proof. The first statement follows from the fact that iI is central and the faithfulness condition (1.2); a detailed justification is given in [15, Lemma 2.9]. Using the decomposition $i\mathscr{H}=[I\bullet\mathscr{H},iI]_\star$ and expanding the Δ_n of this product with the help of Lemma 3.1 yields the desired formula. □

Lemma 3.11. *Δ_n is additive on the Hermitian part $\mathscr{A}_h$.*

Proof. Take $\mathscr{H}_1,\mathscr{H}_2\in\mathscr{A}_h$. Then $i\mathscr{H}_1,i\mathscr{H}_2\in\mathscr{A}_s$. By Remark 3.1 and Lemma 3.10,

$$\begin{aligned}\Delta_n(i\mathscr{H}_1+i\mathscr{H}_2) &= \Delta_n(i\mathscr{H}_1)+\Delta_n(i\mathscr{H}_2)\\&= i\Delta_n(\mathscr{H}_1)+i\Delta_n(\mathscr{H}_2)+i\eta(\mathscr{H}_1+\mathscr{H}_2).\end{aligned}$$

On the other hand,

$$\Delta_n(i(\mathscr{H}_1+\mathscr{H}_2))=i\Delta_n(\mathscr{H}_1+\mathscr{H}_2)+i\eta(\mathscr{H}_1+\mathscr{H}_2).$$

Equating the two expressions yields

$$\Delta_n(\mathscr{H}_1+\mathscr{H}_2)=\Delta_n(\mathscr{H}_1)+\Delta_n(\mathscr{H}_2).$$

□

Lemma 3.12. *For every $\mathscr{X} \in \mathscr{A}$, $\Delta_n(\mathscr{X}^\star) = \Delta_n(\mathscr{X})^\star$.*

Proof. Write $\mathscr{X} = \mathscr{H} + i\mathscr{K}$ with $\mathscr{H}, \mathscr{K} \in \mathscr{A}_h$. Using Lemmas 3.1 and 3.11, and the results obtained so far one confirms that

$$\begin{aligned} \Delta_n([\mathscr{X} \bullet I, I]_\star) &= \Delta_n\left([(\mathscr{H} + i\mathscr{K}) \bullet I, I]_\star\right) \\ &= \Delta_n\left([\mathscr{H} \bullet I, I]_\star\right) + \Delta_n\left([i\mathscr{K} \bullet I, I]_\star\right) \\ &= 0, \end{aligned}$$

while a direct evaluation of the same expression via (1.1) gives

$$\Delta_n([\mathscr{X} \bullet I, I]_\star) = 2(\Delta_n(\mathscr{X}) - \Delta_n(\mathscr{X})^\star).$$

Hence, $\Delta_n(\mathscr{X}) = \Delta_n(\mathscr{X})^\star$, which is precisely $\Delta_n(\mathscr{X}^\star) = \Delta_n(\mathscr{X})^\star$. □

Lemma 3.13. *Δ_n is additive on the whole algebra $\mathscr{A}$.*

Proof. Let $\mathscr{X} = \mathscr{H}_1 + i\mathscr{H}_2$ and $\mathscr{Y} = \mathscr{K}_1 + i\mathscr{K}_2$ with $\mathscr{H}_i, \mathscr{K}_i \in \mathscr{A}_h$. Using additivity on $\mathscr{A}_h$, Lemmas 3.1, 3.11, and 3.10,

$$\begin{aligned} \Delta_n(\mathscr{X} + \mathscr{Y}) &= \Delta_n\left((\mathscr{H}_1 + \mathscr{K}_1) + i(\mathscr{H}_2 + \mathscr{K}_2)\right) \\ &= \Delta_n(\mathscr{H}_1 + \mathscr{K}_1) + i\Delta_n(\mathscr{H}_2 + \mathscr{K}_2) + i\eta(\mathscr{H}_2 + \mathscr{K}_2) \\ &= (\Delta_n(\mathscr{H}_1) + i\Delta_n(\mathscr{H}_2) + i\eta\mathscr{H}_2) + (\Delta_n(\mathscr{K}_1) + i\Delta_n(\mathscr{K}_2) + i\eta\mathscr{K}_2) \\ &= \Delta_n(\mathscr{X}) + \Delta_n(\mathscr{Y}). \end{aligned}$$

□

Lemma 3.14. $\Delta_n(iI) = 0$.

Proof. From the identity

$$\left[\frac{iI}{2} \bullet \frac{iI}{2}, \frac{iI}{2}\right]_\star = -\frac{iI}{2},$$

applying Δ_n and using additivity on $\mathscr{A}_s$, we see that

$$\begin{aligned} -\Delta_n\left(\frac{iI}{2}\right) &= \left[\Delta_n\left(\frac{iI}{2}\right) \bullet \frac{iI}{2}, \frac{iI}{2}\right]_\star + \left[\frac{iI}{2} \bullet \Delta_n\left(\frac{iI}{2}\right), \frac{iI}{2}\right]_\star + \left[\frac{iI}{2} \bullet \frac{iI}{2}, \Delta_n\left(\frac{iI}{2}\right)\right]_\star \\ &\quad + \sum_{\substack{r+s+t=n \\ 0 \le r,s,t \le n-1}} \left[\Delta_r\left(\frac{iI}{2}\right) \bullet \Delta_s\left(\frac{iI}{2}\right), \Delta_t\left(\frac{iI}{2}\right)\right]_\star. \end{aligned}$$

By a straightforward calculation and Lemma 3.1, we obtain $3\Delta_n\left(\frac{iI}{2}\right) = \Delta_n\left(\frac{iI}{2}\right)$; hence, $\Delta_n\left(\frac{iI}{2}\right) = 0$ and, therefore, $\Delta_n(iI) = 0$. Alternatively, one could derive $\Delta_n(iI) = 0$ directly from the identity above without the intermediate η. □

Lemma 3.15. *For every $\mathscr{X} \in \mathscr{A}$, $\Delta_n(i\mathscr{X}) = i\Delta_n(\mathscr{X})$.*

Proof. In view of Lemma 3.1 and 3.14, $\eta = 0$ in Lemma 3.10, which implies that $\Delta_n(i\mathscr{H}) = i\Delta_n(\mathscr{H})$ for all $\mathscr{H} \in \mathscr{A}_h$. Writing $\mathscr{X} = \mathscr{H} + i\mathscr{K}$ with $\mathscr{H}, \mathscr{K} \in \mathscr{A}_h$ and using additivity we get

$$\begin{aligned} \Delta_n(i\mathscr{X}) &= \Delta_n(i\mathscr{H} - \mathscr{K}) \\ &= i\Delta_n(\mathscr{H}) - \Delta_n(\mathscr{K}) \\ &= i(\Delta_n(\mathscr{H}) + i\Delta_n(\mathscr{K})) \\ &= i\Delta_n(\mathscr{X}), \end{aligned}$$

which is the required property. □

Lemma 3.16. *For all $\mathscr{X}, \mathscr{Y} \in \mathscr{A}$, the map Δ_n satisfies the higher derivation rule*

$$\Delta_n(\mathscr{X}\mathscr{Y}) = \sum_{k=0}^{n} \Delta_k(\mathscr{X})\Delta_{n-k}(\mathscr{Y}).$$

Proof. We first prove the rule when both arguments are Hermitian, and then extend it to the whole algebra using additivity and complex linearity.

Write the arbitrary elements, $\mathscr{X} = \mathscr{H}_1 + i\mathscr{H}_2$, $\mathscr{Y} = \mathscr{K}_1 + i\mathscr{K}_2$ with $\mathscr{H}_j, \mathscr{K}_j \in \mathscr{A}_h$. If the statement holds for Hermitian elements, then, by additivity and Lemma 3.15, we obtain

$$\begin{aligned} \Delta_n(\mathscr{X}\mathscr{Y}) &= \Delta_n(\mathscr{H}_1\mathscr{K}_1) + i\Delta_n(\mathscr{H}_1\mathscr{K}_2) + i\Delta_n(\mathscr{H}_2\mathscr{K}_1) - \Delta_n(\mathscr{H}_2\mathscr{K}_2) \\ &= \sum_{k=0}^{n}[\Delta_k(\mathscr{H}_1)\Delta_{n-k}(\mathscr{K}_1) + i\Delta_k(\mathscr{H}_1)\Delta_{n-k}(\mathscr{K}_2) \\ &\qquad + i\Delta_k(\mathscr{H}_2)\Delta_{n-k}(\mathscr{K}_1) - \Delta_k(\mathscr{H}_2)\Delta_{n-k}(\mathscr{K}_2)] \\ &= \sum_{k=0}^{n}(\Delta_k(\mathscr{H}_1) + i\Delta_k(\mathscr{H}_2))(\Delta_{n-k}(\mathscr{K}_1) + i\Delta_{n-k}(\mathscr{K}_2)) \\ &= \sum_{k=0}^{n} \Delta_k(\mathscr{X})\Delta_{n-k}(\mathscr{Y}). \end{aligned}$$

Thus, it suffices to prove the higher derivation rule on $\mathscr{A}_h$.

Take $\mathscr{H}_1, \mathscr{H}_2 \in \mathscr{A}_h$. Using $[\mathscr{H}_1 \bullet I, \mathscr{H}_2]_\star = 2(\mathscr{H}_1\mathscr{H}_2 - \mathscr{H}_2\mathscr{H}_1)$ and applying Δ_n via (1.1) gives

$$\begin{aligned} 2\Delta_n(\mathscr{H}_1\mathscr{H}_2 - \mathscr{H}_2\mathscr{H}_1) = {}& [\Delta_n(\mathscr{H}_1) \bullet I, \mathscr{H}_2]_\star + [\mathscr{H}_1 \bullet \Delta_n(I), \mathscr{H}_2]_\star + [\mathscr{H}_1 \bullet I, \Delta_n(\mathscr{H}_2)]_\star \\ & + \sum_{\substack{r+s+t=n \\ 0\le r,s,t\le n-1}} [\Delta_r(\mathscr{H}_1) \bullet \Delta_s(I), \Delta_t(\mathscr{H}_2)]_\star. \end{aligned}$$

By Lemma 3.8, $\Delta_n(I) = 0$. Lemma 3.1 gives $\Delta_s(I) = 0$ for $1 \le s \le n-1$, and $\Delta_0(I) = I$. Therefore, only $s = 0$ contributes to the sum, and the condition $r, s, t \le n-1$ forces $r + t = n$ with $1 \le r, t \le n-1$. Hence, the sum reduces to

$$\sum_{\substack{r+t=n \\ 1\le r,t\le n-1}} [\Delta_r(\mathscr{H}_1) \bullet I,\ \Delta_t(\mathscr{H}_2)]_\star.$$

Because the induction hypothesis guarantees $\Delta_r(\mathscr{H}_1)^\star = \Delta_r(\mathscr{H}_1)$ and $\Delta_t(\mathscr{H}_2)^\star = \Delta_t(\mathscr{H}_2)$, we have

$$[\Delta_r(\mathscr{H}_1) \bullet I,\ \Delta_t(\mathscr{H}_2)]_\star = [2\Delta_r(\mathscr{H}_1),\ \Delta_t(\mathscr{H}_2)]_\star = 2(\Delta_r(\mathscr{H}_1)\Delta_t(\mathscr{H}_2) - \Delta_t(\mathscr{H}_2)\Delta_r(\mathscr{H}_1)).$$

The first three terms simplify similarly, yielding

$$2\Delta_n(\mathscr{H}_1\mathscr{H}_2 - \mathscr{H}_2\mathscr{H}_1) = 2(\Delta_n(\mathscr{H}_1)\mathscr{H}_2 - \mathscr{H}_2\Delta_n(\mathscr{H}_1) + \mathscr{H}_1\Delta_n(\mathscr{H}_2) - \Delta_n(\mathscr{H}_2)\mathscr{H}_1) \\ + 2\sum_{\substack{r+t=n\\1\leq r,t\leq n-1}} (\Delta_r(\mathscr{H}_1)\Delta_t(\mathscr{H}_2) - \Delta_t(\mathscr{H}_2)\Delta_r(\mathscr{H}_1)).$$

Dividing by 2, we obtain the commutator higher derivation rule

$$\Delta_n(\mathscr{H}_1\mathscr{H}_2 - \mathscr{H}_2\mathscr{H}_1) = \sum_{k=0}^{n}(\Delta_k(\mathscr{H}_1)\Delta_{n-k}(\mathscr{H}_2) - \Delta_{n-k}(\mathscr{H}_2)\Delta_k(\mathscr{H}_1)). \tag{3.5}$$

Now, use $[\mathscr{H}_1 \bullet I, i\mathscr{H}_2]_\star = -2i(\mathscr{H}_1\mathscr{H}_2 + \mathscr{H}_2\mathscr{H}_1)$. Applying Δ_n, we have

$$-2i\Delta_n(\mathscr{H}_1\mathscr{H}_2 + \mathscr{H}_2\mathscr{H}_1) = [\Delta_n(\mathscr{H}_1) \bullet I, i\mathscr{H}_2]_\star + [\mathscr{H}_1 \bullet \Delta_n(I), i\mathscr{H}_2]_\star + [\mathscr{H}_1 \bullet I, \Delta_n(i\mathscr{H}_2)]_\star \\ + \sum_{\substack{r+s+t=n\\0\leq r,s,t\leq n-1}} [\Delta_r(\mathscr{H}_1) \bullet \Delta_s(I), \Delta_t(i\mathscr{H}_2)]_\star.$$

Again $\Delta_n(I) = 0$ and $\Delta_s(I) = 0$ for $s \geq 1$. The sum runs over $s = 0$, $r + t = n$, $1 \leq r, t \leq n-1$. Using $\Delta_t(i\mathscr{H}_2) = i\Delta_t(\mathscr{H}_2)$ (Lemma 3.15) and the Hermitian nature of $\Delta_r(\mathscr{H}_1), \Delta_t(\mathscr{H}_2)$, we have

$$[\Delta_r(\mathscr{H}_1) \bullet I,\ i\Delta_t(\mathscr{H}_2)]_\star = 2i(\Delta_r(\mathscr{H}_1)\Delta_t(\mathscr{H}_2) + \Delta_t(\mathscr{H}_2)\Delta_r(\mathscr{H}_1)).$$

The first three terms give $-2i(\Delta_n(\mathscr{H}_1)\mathscr{H}_2 + \mathscr{H}_2\Delta_n(\mathscr{H}_1) + \mathscr{H}_1\Delta_n(\mathscr{H}_2) + \Delta_n(\mathscr{H}_2)\mathscr{H}_1)$. Putting everything together, we have

$$-2i\Delta_n(\mathscr{H}_1\mathscr{H}_2 + \mathscr{H}_2\mathscr{H}_1) = -2i(\Delta_n(\mathscr{H}_1)\mathscr{H}_2 + \mathscr{H}_2\Delta_n(\mathscr{H}_1) + \mathscr{H}_1\Delta_n(\mathscr{H}_2) + \Delta_n(\mathscr{H}_2)\mathscr{H}_1) \\ - 2i\sum_{\substack{r+t=n\\1\leq r,t\leq n-1}} (\Delta_r(\mathscr{H}_1)\Delta_t(\mathscr{H}_2) + \Delta_t(\mathscr{H}_2)\Delta_r(\mathscr{H}_1)).$$

Dividing by $-2i$ yields the anti-commutator higher derivation rule

$$\Delta_n(\mathscr{H}_1\mathscr{H}_2 + \mathscr{H}_2\mathscr{H}_1) = \sum_{k=0}^{n}(\Delta_k(\mathscr{H}_1)\Delta_{n-k}(\mathscr{H}_2) + \Delta_{n-k}(\mathscr{H}_2)\Delta_k(\mathscr{H}_1)). \tag{3.6}$$

Adding (3.5) and (3.6) and dividing by 2 gives

$$\Delta_n(\mathscr{H}_1\mathscr{H}_2) = \sum_{k=0}^{n} \Delta_k(\mathscr{H}_1)\Delta_{n-k}(\mathscr{H}_2).$$

This completes the proof. □

All the lemmas together complete the induction step: Δ_n is additive, preserves the involution, is complex-linear, and satisfies the higher derivation rule. Thus the family $\{\Delta_n\}_{n=0}^{\infty}$ is an additive $\star$-higher derivation. Theorem 1.1 is proved.

4. Applications

In this section, we apply Theorem 1.1 to several specific algebraic settings. In each case, the corollaries follow directly because the underlying algebra is either prime or satisfies the faithfulness condition (1.2) for the chosen projection. To see why Condition (1.2) holds in these settings, consider the following three scenarios.

- **Prime $\star$-algebras:** If $\mathscr{X}\mathscr{A}\mathscr{P} = 0$, then the left ideal generated by $\mathscr{X}$ annihilates the nonzero projection $\mathscr{P}$. Because the algebra is prime, this forces $\mathscr{X} = 0$. The same argument applies to $I - \mathscr{P}$.
- **Factor von Neumann algebras:** The set $\{X : X\mathscr{A}P = 0\}$ forms a weakly *-closed two-sided ideal. By factoriality, this ideal must be $\{0\}$, since the only other option forces the contradiction $P = 0$.
- **Standard operator algebras:** On infinite-dimensional Hilbert spaces, primeness follows directly from the density of finite-rank operators.

Consequently, Theorem 1.1 yields the following immediate results.

Corollary 4.1 (Prime $\star$-algebras)**.** *Let $\mathscr{A}$ be a unital prime $\star$-algebra containing a nontrivial projection. Then every family $\{\Delta_n\}_{n=0}^{\infty}$ of maps $\Delta_n : \mathscr{A} \to \mathscr{A}$ (not necessarily additive) with $\Delta_0 = \mathrm{id}_{\mathscr{A}}$ that fulfills* (1.1) *is an additive $\star$-higher derivation for the ternary product $[\cdot \bullet \cdot, \cdot]_\star$.*

Corollary 4.2 (Factor von Neumann algebras)**.** *Let $\mathscr{A}$ be a factor von Neumann algebra with* $\dim \mathscr{A} > 1$. *Then the conclusion of Theorem 1.1 holds for any family satisfying* (1.1).

Corollary 4.3 (Standard operator algebras)**.** *Let $\mathscr{A}$ be a standard operator algebra on an infinite-dimensional complex Hilbert space, that is closed under the adjoint operation and contains a nontrivial projection. Then the conclusion of Theorem 1.1 remains valid.*

Example 4.1. *Let $\mathscr{A} = M_2(\mathbb{C})$ be the algebra of* 2×2 *complex matrices with the usual conjugate-transpose involution. Take the projection $\mathscr{P} = e_{11}$; then $\mathscr{Q} = I - \mathscr{P} = e_{22}$, and Condition* (1.2) *is easily verified. Define $\Delta_1 : \mathscr{A} \to \mathscr{A}$ by*

$$\Delta_1(A) = i[\mathscr{P}, A] = i(\mathscr{P}A - A\mathscr{P}),$$

and set

$$\Delta_0(A) = A, \qquad \Delta_n(A) = \frac{1}{n!}\Delta_1^{\circ n}(A) \quad (n \geq 1),$$

where $\Delta_1^{\circ n}$ denotes the n-fold composition of Δ_1.

It is well known and straightforward to check that Δ_1 is a $\star$-derivation as follows:

$$\Delta_1(AB) = \Delta_1(A)B + A\Delta_1(B), \qquad \Delta_1(A^\star) = \Delta_1(A)^\star.$$

Hence, $\{\Delta_n\}_{n=0}^{\infty}$ is the ordinary additive $\star$−higher derivation generated by Δ_1; that is, for all $A, B \in \mathscr{A}$ and every $n \geq 1$,

$$\Delta_n(AB) = \sum_{k=0}^{n} \Delta_k(A)\Delta_{n-k}(B), \qquad \Delta_n(A^\star) = \Delta_n(A)^\star.$$

Using the ordinary higher derivation property and $\Delta_k(A^\star) = \Delta_k(A)^\star$, we obtain for all $X, Y, Z \in \mathscr{A}$ and every $n \geq 1$, we have

$$\begin{aligned}
\Delta_n([X \bullet Y, Z]_\star) &= \Delta_n((XY^\star + YX^\star)Z^\star - Z(XY^\star + YX^\star)) \\
&= \sum_{r+s+t=n} \Big(\Delta_r(X)\Delta_s(Y)^\star\Delta_t(Z)^\star + \Delta_s(Y)\Delta_r(X)^\star\Delta_t(Z)^\star \\
&\qquad - \Delta_t(Z)\Delta_r(X)\Delta_s(Y)^\star - \Delta_t(Z)\Delta_s(Y)\Delta_r(X)^\star\Big) \\
&= \sum_{r+s+t=n} [\Delta_r(X) \bullet \Delta_s(Y), \Delta_t(Z)]_\star.
\end{aligned}$$

Splitting off the three terms with $(r, s, t) = (n, 0, 0)$, $(0, n, 0)$, and $(0, 0, n)$ gives exactly

$$\Delta_n([X \bullet Y, Z]_\star) = [\Delta_n(X) \bullet Y, Z]_\star + [X \bullet \Delta_n(Y), Z]_\star + [X \bullet Y, \Delta_n(Z)]_\star + \sum_{\substack{r+s+t=n \\ r,s,t \leq n-1}} [\Delta_r(X) \bullet \Delta_s(Y), \Delta_t(Z)]_\star,$$

which is precisely (1.1). Therefore the family $\{\Delta_n\}_{n=0}^{\infty}$ satisfies (1.1) and, in accordance with Theorem 1.1, is an additive $\star$-higher derivation with respect to the ternary product.

Author contributions

M. A. Raza: Conceptualization, methodology, supervision, writing the original draft, subsequent review and editing; M. Zailaee: Methodology, original draft preparation, review and editing; A. N. Khan: Conceptualization, methodology, review and editing of the work. All authors contributed significantly to the development of this manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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