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Research article

Fermatean fuzzy soft rough graphs: A hybrid model for multi-criteria decision-making in supply chain risk analysis under complex uncertainty

Ahad Khaled Mohamed Alharbi and Kholood Mohammad Alsager*

Department of Mathematics, College of Science, Qassim University, Buraydah, Saudi Arabia; Email: 461215456@qu.edu.sa

* **Correspondence:** Email: ksakr@qu.edu.sa.

Abstract: Complex supply chain decision-making requires mathematical models capable of handling parametric, granular, and relational uncertainty simultaneously. Existing graph-based models address only part of these uncertainty types. To the best of our knowledge, no graph-based framework simultaneously models all three within a unified mathematical structure, motivating the proposed FFSRG model. In this paper, we proposed the Fermatean fuzzy soft rough graph (FFSRG), integrating Fermatean fuzzy sets, soft sets, rough sets, and graph theory into a unified decision-making framework. Vertex and edge approximation operators were defined, fundamental algebraic properties were established, aggregation operators preserving the Fermatean constraint were developed, and a six-step decision-making algorithm was proposed. The framework was validated through a supply chain risk assessment involving ten suppliers. Comparative analysis with six graph-based models demonstrated the effectiveness of the proposed approach. The proposed FFSRG achieved the highest ranking accuracy (91.2%) and uncertainty-handling performance (0.89). The results showed that integrating rough approximations with Fermatean fuzzy structures improves decision consistency and discrimination among alternatives. The FFSRG framework provides an effective tool for multi-criteria decision-making under complex uncertainty, with potential applicability beyond supply chain risk analysis.

Keywords: Fermatean fuzzy sets; soft sets; rough sets; fuzzy graphs; multi-criteria decision-making
Mathematics Subject Classification: 03B52, 03E72

1. Introduction

Managing modern global supply chains has become increasingly challenging due to growing network interconnectivity, diverse risk sources, and the unpredictable nature of real-world operating environments. Decision-making under traditional frameworks becomes particularly difficult when simultaneously dealing with parametric uncertainty arising from incomplete knowledge of system parameters, granular uncertainty caused by vague classifications and incomplete information, and relational uncertainty resulting from complex or poorly understood relationships among entities. This combination of uncertainty requires advanced mathematical models capable of integrating these complementary aspects within a unified decision-making framework. Fuzzy set theory, proposed by Zadeh [1], has served as a foundation for handling uncertainty by enabling elements to have membership grades in the interval $[0, 1]$. Although classical fuzzy sets provide only a single membership degree, intuitionistic fuzzy sets [2] generalize this notion by simultaneously holding membership and non-membership degrees under the linear constraint $\mu + \nu \leq 1$. Pythagorean fuzzy sets [3] further generalize this concept through the squared condition $\mu^2 + \nu^2 \leq 1$, thereby accommodating a larger feasible region of uncertainty. Moreover, Fermatean fuzzy sets [4] have been introduced, employing the cubic constraint $\mu^3 + \nu^3 \leq 1$, which provides even greater flexibility in representing imprecise information. Subsequent research has developed new operations and weighted product models for Fermatean fuzzy numbers [5]. Beyond orthopair structures, spherical fuzzy sets incorporate an additional neutral membership degree to capture abstention [6], while T-spherical fuzzy sets offer a generalized framework with independent parameters for membership, abstinence, and non-membership [7]. Extending further, m-polar fuzzy sets have been explored to represent multi-dimensional uncertainty [8], and Q-fuzzy soft expert sets have emerged as tools for handling multiple expert opinions [9]. The theoretical underpinnings of generalized orthopair fuzzy sets, presented in terms of Pythagorean and Fermatean structures, were unified by Yager [10]. However, these frameworks generally consider parametric uncertainty and do not fully tackle granular or relational uncertainties.

Soft set theory [11] provides an alternative framework for studying decision problems in a parameterized form, where parameters are related to one or more evaluation criteria. By combining fuzzy sets with soft sets, we obtain fuzzy soft sets [12], which have been effectively tested in decision-making problems involving multiple criteria. Nevertheless, these techniques tend to lack mechanisms for managing granular uncertainty arising from incomplete or coarse information. Rough set theory [13] overcomes this drawback by using lower and upper approximations of sets with respect to an equivalence relation, which reveals the granularity of knowledge. The synergy of rough sets and fuzzy sets leads to fuzzy rough sets [14] and rough fuzzy sets, which have been successfully applied in various decision-making contexts. In the context of rough approximations, Pythagorean m-polar fuzzy soft rough sets have been developed to combine granular computing with multi-polar information [15]. The concept of hesitant Fermatean fuzzy sets has further refined multi-criteria decision-making by enabling multiple possible membership degrees [16]. Researchers have focused on quantifying uncertainty within these frameworks; for instance, entropy measures for complex Fermatean fuzzy soft sets have been applied to research productivity assessment [17]. Nevertheless, these hybrid models typically do not incorporate graph structures for representing relational dependencies. Graph theory has been recognized as an effective tool for describing relations and dependencies among entities in complex systems. Fuzzy graphs [18] generalize classical graph theory by treating vertices and edges as fuzzy sets and are capable of modeling uncertain relationships. Beyond fuzzy graphs, interval-valued fuzzy graphs have been studied to capture uncertainty in membership degrees [19], while intuitionistic fuzzy graphs with categorical properties provide algebraic foundations for relational structures [20]. Several extensions such as intuitionistic fuzzy graphs [21], Pythagorean fuzzy graphs [22], and Fermatean fuzzy graphs [23] have been developed to represent different types and strengths of uncertainty within relational structures. Moreover, spherical fuzzy graphs have been successfully applied in decision-making [24]. However, these graph-based models typically focus on relational uncertainty without incorporating parametric or granular uncertainty in a unified framework. Soft graphs [25] and fuzzy soft graphs integrate soft set parameterization with graph structures, enabling multi-criteria relational analysis, but still lack rough approximations for granular uncertainty. The integration of multiple uncertainty modeling paradigms remains an active and challenging research area. While researchers have proposed hybrid frameworks such as picture fuzzy soft graphs [26], intuitionistic fuzzy soft expert graphs [27], Fermatean neutrosophic fuzzy graphs [28], fuzzy rough graphs [29], and complex Pythagorean Dombi fuzzy graphs [30], most of these approaches address only two or three types of uncertainty simultaneously. Application-oriented studies have also demonstrated the effectiveness of fuzzy multi-criteria decision-making in domain-specific risk assessment. In particular, Mohammadi et al. [31] proposed an extended FMEA-based decision-making framework for construction contracting risk assessment under uncertain environments. Their approach effectively prioritizes risks in construction projects; however, it does not integrate graph structures, soft set parameterization, and rough approximations within a unified mathematical framework. Similarly, Turgay and Özyurt [32] developed a dynamic ergonomic risk assessment framework combining REBA with fuzzy multi-criteria decision-making for repetitive movements. Although their study demonstrates the effectiveness of fuzzy MCDM in ergonomic evaluation, it is application-oriented and does not address the simultaneous modeling of parametric, granular, and relational uncertainty through a graph-based mathematical framework. None of the methodologies simultaneously integrates Fermatean fuzzy sets for parametric uncertainty, soft set theory for multi-criteria parameterization, rough set theory for granular approximation, and graph theory for relational modeling within a unified mathematical framework. This limitation motivates the development of the proposed fermatean fuzzy soft rough graph (FFSRG) model.

1.1. Research gap

The literature review reveals that graph-based decision-making models suffer from several important limitations. Classical fuzzy graph models, including FG, IFG, PFG, and FFG, effectively represent relational uncertainty but do not simultaneously address parametric and granular uncertainty. Conversely, soft graph models such as FSG, PFSG, and IFSEG incorporate multi-criteria parameterization but lack rough approximation mechanisms for handling granular uncertainty. Rough graph models, including RG and FRG, provide effective representations of granular uncertainty through approximation spaces; however, they do not incorporate soft set parameterization for multi-criteria evaluation. More recent hybrid models, such as CPDFG and FNFG, combine some of these concepts but still integrate only a subset of the required mathematical paradigms. Consequently, none of the graph-based frameworks simultaneously combines Fermatean fuzzy sets, soft sets, rough sets, and graph theory within a unified mathematical structure capable of modeling parametric, granular, and relational uncertainty. These limitations motivate the development of a comprehensive framework that integrates all four paradigms into a single graph-based decision-making environment. Such an integration is expected to provide a richer mathematical representation of complex uncertainty while improving the robustness and discrimination capability of multi-criteria decision-making.

Accordingly, we seek to answer the following research questions:

- **RQ1:** How can Fermatean fuzzy sets, soft sets, rough sets, and graph theory be formally integrated into a unified mathematical framework for decision-making under complex uncertainty?
- **RQ2:** What algebraic properties and aggregation operators are required to preserve the Fermatean cubic constraint $\mu^3 + \nu^3 \leq 1$ within the proposed framework?
- **RQ3:** How does the integration of rough approximations improve decision consistency and discrimination among alternatives compared with existing graph-based models?

These research gaps directly motivate the proposed FFSRG framework and are addressed through Research Questions RQ1–RQ3.

1.2. Motivation of the proposed model

The proposed fermatean fuzzy soft rough graph (FFSRG) model is motivated by the following considerations:

1. Fermatean fuzzy sets: Selected because they provide the largest feasible region among orthopair fuzzy structures through the cubic constraint $\mu^3 + \nu^3 \leq 1$, enabling experts to express higher uncertainty degrees while maintaining mathematical rigor. This is particularly relevant in supply chain risk assessment where information is often vague and incomplete.
2. Soft sets: Chosen to handle multi-parametric decision-making where different risk criteria (e.g., geopolitical risk, financial stability, operational resilience) require separate evaluations. Soft sets naturally accommodate this through parameterized mappings.
3. Rough sets: Incorporated to address granular uncertainty arising from incomplete or coarse information. The equivalence relation partitions suppliers into regions, capturing the inherent granularity of geographical and operational groupings.
4. Graph theory: Selected to model relational dependencies and cascading effects in supply chain networks, which cannot be captured by isolated node evaluations.

Table 1 further confirms that existing approaches integrate only subsets of the required uncertainty modeling capabilities, whereas the proposed FFSRG simultaneously incorporates all essential components.

Table 1. Comparison of graph-based decision models for uncertainty representation.

Model	Parametric Uncertainty	Granular Uncertainty	Relational Uncertainty	Graph Structure	Soft Parameterization
FG [18]	✗	✗	✓	✓	✗
IFG [21]	✓	✗	✓	✓	✗
PFG [22]	✓	✗	✓	✓	✗
FFG [23]	✓	✗	✓	✓	✗
RG [34]	✗	✓	✓	✓	✗
FSG [25]	✓	✗	✓	✓	✓
FRG [29]	✓	✓	✓	✓	✗
CPDFG [30]	✓	✗	✓	✓	✗
IFSEG [27]	✓	✗	✓	✓	✓
FNFG [28]	✓	✗	✓	✓	✗
Proposed FFSRG	✓	✓	✓	✓	✓

✓: Feature present, ✗: Feature absent.

Figure 1 summarizes the conceptual evolution leading to the proposed FFSRG framework and highlights how each mathematical theory contributes to modeling a specific type of uncertainty. The proposed framework evolves progressively from existing uncertainty models rather than replacing them, where each theory contributes a complementary capability to the final mathematical model.

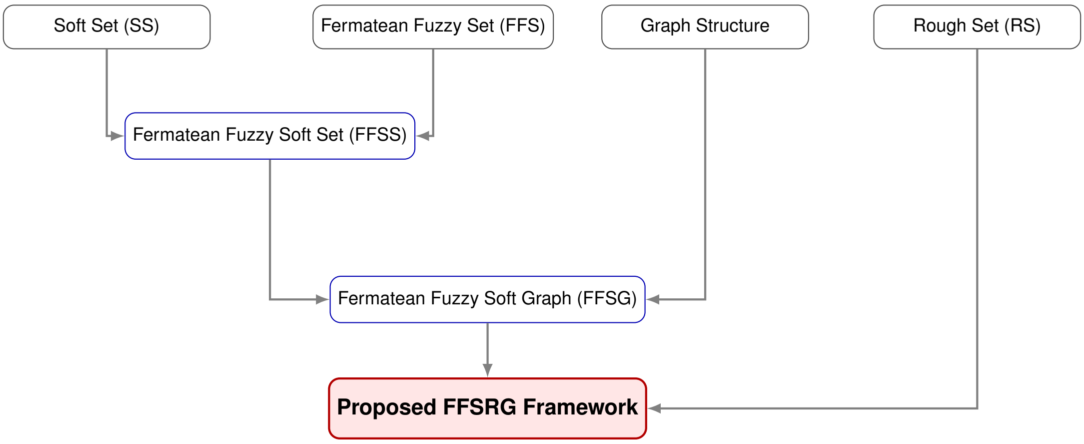


Figure 1. Evolution of uncertainty models toward the proposed FFSRG framework.

1.3. Main contributions

The primary contributions of this paper are as follows:

1. Theoretical novelty: To the best of our knowledge, the proposed FFSRG is the first framework that simultaneously integrates Fermatean fuzzy sets, soft sets, rough sets, and graph theory within a unified mathematical structure. This integration enables the simultaneous handling of parametric, granular, and relational uncertainty in complex decision-making environments.
2. Mathematical contributions: We formally define lower and upper approximations for vertices and edges in an FFSRG under equivalence relations $\mathcal{R}$, and prove fundamental algebraic properties including closure, monotonicity, idempotency, duality, and De Morgan-type laws. The proposed framework generalizes classical fuzzy graphs, Fermatean fuzzy soft graphs, and rough graphs as special cases under identity relations.
3. Aggregation and scoring: We develop aggregation operators for edge weights that preserve the Fermatean cubic constraint $\mu^3 + \nu^3 \leq 1$, introduce a normalized vertex scoring function ensuring bounded evaluation within the unit interval $[0, 1]$, and formulate connection strength measures based on max-min principles for path evaluation.
4. Algorithmic framework: We propose a six-step decision-making algorithm for uncertainty aggregation, path evaluation, and supplier ranking in networked environments, supported by a formal complexity analysis.
5. Practical validation: Comparative analysis demonstrates the superiority of the proposed framework over existing graph-based decision models through a comprehensive supply chain risk assessment study.

The remainder of this paper is organized as follows: In Section 2, we present the preliminary definitions and concepts required for the development of the proposed model. In Section 3, we define the FFSRG structure, including vertex and edge approximation operators, algebraic properties, and an illustrative example of FFSRG construction. In Section 4, we present the decision-making method, including aggregation operators, score functions, the complete algorithm, and time complexity analysis. In Section 5, we provide a comprehensive comparative evaluation and validation of the proposed framework against existing methods. In Section 6, we discuss limitations and future research directions. Finally, in Section 7, we conclude the paper with a summary of contributions and key findings.

Table 2 lists the most important notations and model parameters used throughout this paper.

Table 2. List of notations.

Symbol	Description
$\wp$	Set of vertices in a graph (denoted $\mathcal{V}$ in text)
Υ	Universal set (alternatives or suppliers) (denoted $\mathcal{U}$ in text)
v, u, w	Elements of the universal set
$\mathcal{P}$	Set of parameters (criteria)
p	Single parameter (criterion)
$\mathfrak{F}$	Fermatean fuzzy soft set mapping
$\Pi(\cdot)$	Membership degree (denoted $\mu(\cdot)$ in text)
$\Xi(\cdot)$	Non-membership degree (denoted $\nu(\cdot)$ in text)
$\Delta(\cdot)$	Hesitancy (uncertainty) degree (denoted $\pi(\cdot)$ in text)
$\mathcal{R}_e$	Equivalence relation and approximation space operator
$[v]_{\mathcal{R}_e}$	Equivalence class of v
$\underline{\Omega}_{\mathfrak{F}_p}, \overline{\Omega}_{\mathfrak{F}_p}$	Lower and upper vertex approximations
$\underline{\Omega}_{\mathcal{E}_p}, \overline{\Omega}_{\mathcal{E}_p}$	Lower and upper edge approximations
$\Gamma^* = (\wp, \mathcal{E})$	Crisp graph (underlying structure)
$\sqsupset_p^-, \sqsupset_p^+$	Lower and upper FFSRG components
$\Pi_{\text{agg}}(\cdot), \Xi_{\text{agg}}(\cdot)$	Aggregated edge membership and non-membership
$\deg_\Pi(v), \deg_\Xi(v)$	Vertex degree for membership and non-membership
$\tau\deg_\Pi(v), \tau\deg_\Xi(v)$	Total vertex degree
$S(v)$	Normalized vertex score function
$\Phi_p(\mathbf{r}), \Psi_p(\mathbf{r})$	Path strength for membership and non-membership
$\Pi_p^\infty(u, v), \Xi_p^\infty(u, v)$	Connection strength between vertices
3, 7	Indicator for feature presence or absence
$n = \lvert\wp\rvert$	Number of vertices
$m = \lvert\mathcal{E}\rvert$	Number of edges
$k = \lvert\mathcal{P}\rvert$	Number of parameters

2. Preliminaries

In this section, we present the fundamental concepts and definitions required for the development of the proposed model. We briefly review the essential theoretical foundations that will be used throughout the study. The definitions presented in this section are limited to the concepts required for constructing the proposed FFSRG framework, while the new mathematical formulations are introduced in Section 3.

Definition 1. *[1] Let Υ be a nonempty universal set. A fuzzy set Θ on Υ is determined by a membership function $\Pi_\Theta : \Upsilon \to [0,1]$, which associates each element of Υ with a membership grade. Accordingly, Θ is represented as*

$$\Theta = \{(\aleph, \Pi_\Theta(\aleph)) \mid \aleph \in \Upsilon\},$$

where $\Pi_\Theta(\aleph)$ denotes the degree to which the element $\aleph$ belongs to the fuzzy set Θ.

Definition 2. *[4] Let Υ be a nonempty ground set. A Fermatean fuzzy set Θ on Υ is described by two functions: a membership mapping $\Pi_\Theta : \Upsilon \to [0,1]$ and a non-membership mapping $\Xi_\Theta : \Upsilon \to [0,1]$. Formally, Θ is represented as:*

$$\Theta = \{(\aleph, \Pi_\Theta(\aleph), \Xi_\Theta(\aleph)) \mid \aleph \in \Upsilon\}.$$

For every $\aleph \in \Upsilon$, the pair $(\Pi_\Theta(\aleph), \Xi_\Theta(\aleph))$ must satisfy the cube-sum constraint:

$$\Pi_\Theta^3(\aleph) + \Xi_\Theta^3(\aleph) \leq 1, \qquad \forall \aleph \in \Upsilon. \tag{2.1}$$

Furthermore, the hesitancy degree associated with each element is defined as:

$$\Delta_\Theta(\aleph) = \sqrt[3]{1 - \Pi_\Theta^3(\aleph) - \Xi_\Theta^3(\aleph)}.$$

This value captures the uncertainty or lack of commitment in assigning membership and non-membership grades.

Definition 3. *[11] Let Υ be a nonempty set of objects and $\mathcal{P}$ a nonempty parameter set. A soft set on Υ is a pair $(\mathfrak{F}, \mathcal{P})$, where $\mathfrak{F}$ is a mapping from $\mathcal{P}$ to the power set of Υ. That is,*

$$\mathfrak{F} : \mathcal{P} \longrightarrow \mathcal{P}(\Upsilon),$$

with $\mathcal{P}(\Upsilon)$ representing the collection of all possible subsets of Υ. In other words, each parameter $p \in \mathcal{P}$ is associated with a subset $\mathfrak{F}(p) \subseteq \Upsilon$.

Definition 4. *[12] Let Υ be a nonempty ground set of elements and $\mathcal{P}$ a nonempty parameter set. Denote by $\mathrm{Fm}(\Upsilon)$ the family of all Fermatean fuzzy subsets defined on Υ. A Fermatean fuzzy soft set on Υ is defined as an ordered pair $(\mathfrak{F}, \mathcal{P})$, where*

$$\mathfrak{F} : \mathcal{P} \longrightarrow \mathrm{Fm}(\Upsilon)$$

is a mapping that associates each parameter with a Fermatean fuzzy set. Explicitly, for any parameter $p \in \mathcal{P}$, the value $\mathfrak{F}(p)$ is given by:

$$\mathfrak{F}(p) = \{(\vartheta, \Pi_{\mathfrak{F}(p)}(\vartheta), \Xi_{\mathfrak{F}(p)}(\vartheta)) \mid \vartheta \in \Upsilon\},$$

with membership function $\Pi_{\mathfrak{F}(p)} : \Upsilon \to [0,1]$ and non-membership function $\Xi_{\mathfrak{F}(p)} : \Upsilon \to [0,1]$ satisfying the cube-sum constraint:

$$\Pi_{\mathfrak{F}(p)}^3(\vartheta) + \Xi_{\mathfrak{F}(p)}^3(\vartheta) \leq 1, \qquad \forall \vartheta \in \Upsilon,\ \forall p \in \mathcal{P}.$$

Definition 5. *[13] Let Υ be a nonempty set and $\mathfrak{R} \subseteq \Upsilon \times \Upsilon$ an equivalence relation on Υ. The pair $(\Upsilon, \mathfrak{R})$ is called a rough approximation space. For any subset $X \subseteq \Upsilon$, we define two approximations with respect to $\mathfrak{R}$: The lower approximation and the upper approximation, respectively, as follows:*

$$\begin{aligned} \underline{\Omega}(X) &= \{\vartheta \in \Upsilon \mid [\vartheta]_{\mathfrak{R}} \subseteq X\}, \\ \overline{\Omega}(X) &= \{\vartheta \in \Upsilon \mid [\vartheta]_{\mathfrak{R}} \cap X \neq \emptyset\}. \end{aligned} \tag{2.2}$$

In the above, $[\vartheta]_{\mathfrak{R}}$ stands for the equivalence class of ϑ under the relation $\mathfrak{R}$.

Definition 6. *[33] Let Υ be a nonempty universe equipped with an equivalence relation $\mathfrak{R} \subseteq \Upsilon \times \Upsilon$. Consider a Fermatean fuzzy set Θ defined on Υ. We define two approximations of Θ with respect to $\mathfrak{R}$, both of which are Fermatean fuzzy sets:*

$$\underline{\Omega}(\Theta) = \Big\{(\vartheta,\ \Pi_{\underline{\Omega}(\Theta)}(\vartheta),\ \Xi_{\underline{\Omega}(\Theta)}(\vartheta)) \mid \vartheta \in \Upsilon\Big\},$$
$$\overline{\Omega}(\Theta) = \Big\{(\vartheta,\ \Pi_{\overline{\Omega}(\Theta)}(\vartheta),\ \Xi_{\overline{\Omega}(\Theta)}(\vartheta)) \mid \vartheta \in \Upsilon\Big\}.$$

For each element $\vartheta \in \Upsilon$, let $[\vartheta]_{\mathfrak{R}}$ denote its equivalence class under $\mathfrak{R}$. The membership and non-membership values for these approximations are determined as follows:

$$\Pi_{\overline{\Omega}(\Theta)}(\vartheta) = \max_{\mho \in [\vartheta]_{\mathfrak{R}}} \Pi_{\Theta}(\mho), \quad \Xi_{\overline{\Omega}(\Theta)}(\vartheta) = \min_{\mho \in [\vartheta]_{\mathfrak{R}}} \Xi_{\Theta}(\mho),$$

$$\Pi_{\underline{\Omega}(\Theta)}(\vartheta) = \min_{\mho \in [\vartheta]_{\mathfrak{R}}} \Pi_{\Theta}(\mho), \quad \Xi_{\underline{\Omega}(\Theta)}(\vartheta) = \max_{\mho \in [\vartheta]_{\mathfrak{R}}} \Xi_{\Theta}(\mho).$$

The ordered pair $(\underline{\Omega}(\Theta), \overline{\Omega}(\Theta))$ is referred to as a Fermatean fuzzy rough set.

These approximation operators form the theoretical basis for constructing the lower and upper vertex and edge approximations proposed in the next section.

Definition 7. *[18] Let $\Gamma^* = (\wp, \mathcal{E})$ be a classical (crisp) graph with $\wp$ a nonempty vertex set and $\mathcal{E} \subseteq \wp \times \wp$ an edge set. A fuzzy graph is defined as a triple*

$$\Gamma = (\wp, \Pi_{\wp}, \Pi_{\mathcal{E}}),$$

where $\Pi_{\wp} : \wp \to [0,1]$ is a vertex membership function and $\Pi_{\mathcal{E}} : \wp \times \wp \to [0,1]$ is an edge membership function. These two functions must satisfy the consistency constraint:

$$\Pi_{\mathcal{E}}(\vartheta, \mho) \leq \min(\Pi_{\wp}(\vartheta),\ \Pi_{\wp}(\mho)), \qquad \forall \vartheta, \mho \in \wp.$$

In other words, the membership degree of any edge cannot exceed the membership degrees of its two endpoints.

Definition 8. *[23] Let $\Gamma^* = (\wp, \mathcal{E})$ be a crisp (non-fuzzy) graph. A Fermatean fuzzy graph, denoted by Γ, is defined by four functions:*

- $\Pi_{\wp} : \wp \to [0,1]$ *(vertex membership)*
- $\Xi_{\wp} : \wp \to [0,1]$ *(vertex non-membership)*

- $\Pi_{\mathcal{E}} : \mathcal{E} \to [0, 1]$ *(edge membership)*
- $\Xi_{\mathcal{E}} : \mathcal{E} \to [0, 1]$ *(edge non-membership)*

These functions must satisfy the following constraints.

(1) Fermatean condition for vertices:

$$0 \le \Pi_{\wp}^3(\mho) + \Xi_{\wp}^3(\mho) \le 1, \qquad \forall \mho \in \wp.$$

(2) Edge-vertex consistency constraints: For every edge $(\vartheta, \mho) \in \mathcal{E}$,

$$\Pi_{\mathcal{E}}(\vartheta, \mho) \le \min\{\Pi_{\wp}(\vartheta),\ \Pi_{\wp}(\mho)\}, \qquad \Xi_{\mathcal{E}}(\vartheta, \mho) \ge \max\{\Xi_{\wp}(\vartheta),\ \Xi_{\wp}(\mho)\}. \tag{2.3}$$

(3) Fermatean condition for edges:

$$0 \le \Pi_{\mathcal{E}}^3(\vartheta, \mho) + \Xi_{\mathcal{E}}^3(\vartheta, \mho) \le 1, \qquad \forall (\vartheta, \mho) \in \mathcal{E}.$$

In the above, Π and Ξ denote membership and non-membership functions, respectively.

Definition 9. *[25] Let $\Gamma = (\wp, \mathcal{E})$ be a simple graph and let A be a nonempty parameter set. Suppose $R \subseteq A \times \wp$ is a binary relation linking parameters to vertices. For each parameter $a \in A$, we define a mapping $F : A \to \mathcal{P}(\wp)$ as:*

$$F(a) = \{\mho \in \wp \mid (a, \mho) \in R\},$$

where $\mathcal{P}(\wp)$ represents the power set (collection of all subsets) of $\wp$. The ordered pair (F, A) is then referred to as a soft graph associated with Γ.

In some formulations, it is additionally assumed that the subgraph induced by $F(a)$ is connected for all $a \in A$, depending on the application context.

However, classical soft graphs do not incorporate rough approximations or Fermatean fuzzy information simultaneously, motivating the generalized framework proposed in this paper.

Definition 10. *[34] Consider a crisp graph $\Gamma = (\wp, \mathcal{E})$ and an equivalence relation $\mathfrak{R}$ on $\wp$. For a subgraph $\mathcal{H} \subseteq \Gamma$, define its lower and upper approximations component-wise:*

$$\underline{\Omega}(\mathcal{H}) = (\underline{\Omega}(\wp_{\mathcal{H}}), \underline{\Omega}(\mathcal{E}_{\mathcal{H}})), \quad \overline{\Omega}(\mathcal{H}) = (\overline{\Omega}(\wp_{\mathcal{H}}), \overline{\Omega}(\mathcal{E}_{\mathcal{H}})),$$

where

$$\underline{\Omega}(\wp_{\mathcal{H}}) = \{\mho \in \wp \mid [\mho]_{\mathfrak{R}} \subseteq \wp_{\mathcal{H}}\},$$
$$\overline{\Omega}(\wp_{\mathcal{H}}) = \{\mho \in \wp \mid [\mho]_{\mathfrak{R}} \cap \wp_{\mathcal{H}} \neq \emptyset\},$$

and edge approximations are defined analogously. If $\underline{\Omega}(\mathcal{H}) = \overline{\Omega}(\mathcal{H})$, then $\mathcal{H}$ is exact; otherwise, $(\underline{\Omega}(\mathcal{H}), \overline{\Omega}(\mathcal{H}))$ forms a rough graph.

The rough approximation component of the proposed FFSRG framework is constructed using the classical Pawlak equivalence relation $\mathfrak{R}$. This choice preserves the mathematical foundations required for defining lower and upper approximation operators and establishing their theoretical properties. Moreover, the equivalence relation provides an interpretable partition of the universe into indiscernibility classes, making it suitable for grouping suppliers with similar characteristics. It also maintains a separation between granular uncertainty, modeled by rough approximations, parametric uncertainty, represented by Fermatean fuzzy membership degrees, and relational uncertainty, captured by the graph structure. Consequently, the use of an equivalence relation provides a mathematically rigorous, interpretable, and computationally tractable basis for the proposed FFSRG framework.

3. Fermatean fuzzy soft rough graph

In this section, we present the formal mathematical definition of the Fermatean fuzzy soft rough graph (FFSRG) framework, including vertex and edge approximation operators, fundamental algebraic properties, connectivity concepts, and similarity measures. The aggregation operators and score functions required for decision-making are deferred to in Section 4 to maintain a clear separation between mathematical formulation and algorithmic application.

Definition 11. *Let $\Gamma^* = (\wp, \mathcal{E})$ be a crisp graph, let $\mathcal{P}$ be a nonempty set of parameters, and let $\mathcal{R}_e$ be an equivalence relation on $\wp$. Consider a mapping $\mathfrak{F} : \mathcal{P} \to \mathrm{Fm}(\wp)$ defined by $\mathfrak{F}(p) = \mathfrak{F}_p$, where each $\mathfrak{F}_p$ is a Fermatean fuzzy set on $\wp$ satisfying*

$$\Pi^3_{\mathfrak{F}_p}(v) + \Xi^3_{\mathfrak{F}_p}(v) \leq 1, \qquad \forall v \in \wp,\ p \in \mathcal{P}.$$

For every parameter $p \in \mathcal{P}$, the lower and upper vertex approximations are given as follows:

$$\Pi_{\underline{\Omega}\mathfrak{F}_p}(v) = \min_{u \in [v]_{\mathcal{R}_e}} \Pi_{\mathfrak{F}_p}(u), \quad \Xi_{\underline{\Omega}\mathfrak{F}_p}(v) = \max_{u \in [v]_{\mathcal{R}_e}} \Xi_{\mathfrak{F}_p}(u),$$

$$\Pi_{\overline{\Omega}\mathfrak{F}_p}(v) = \max_{u \in [v]_{\mathcal{R}_e}} \Pi_{\mathfrak{F}_p}(u), \quad \Xi_{\overline{\Omega}\mathfrak{F}_p}(v) = \min_{u \in [v]_{\mathcal{R}_e}} \Xi_{\mathfrak{F}_p}(u).$$

The p-lower and p-upper approximation operators are defined as follows:

$$\beth^-_p = (\underline{\Omega}\mathfrak{F}_p, \underline{\Omega}\mathcal{E}_p), \qquad \beth^+_p = (\overline{\Omega}\mathfrak{F}_p, \overline{\Omega}\mathcal{E}_p),$$

where, for every edge $e = (v_i, v_j) \in \mathcal{E}$, the corresponding membership and non-membership degrees are specified as:

$$\Pi_{\underline{\Omega}\mathcal{E}_p}(e) = \min\left(\Pi_{\underline{\Omega}\mathfrak{F}_p}(v_i), \Pi_{\underline{\Omega}\mathfrak{F}_p}(v_j)\right),$$
$$\Xi_{\underline{\Omega}\mathcal{E}_p}(e) = \max\left(\Xi_{\underline{\Omega}\mathfrak{F}_p}(v_i), \Xi_{\underline{\Omega}\mathfrak{F}_p}(v_j)\right),$$

$$\Pi_{\overline{\Omega}\mathcal{E}_p}(e) = \min\left(\Pi_{\overline{\Omega}\mathfrak{F}_p}(v_i), \Pi_{\overline{\Omega}\mathfrak{F}_p}(v_j)\right),$$
$$\Xi_{\overline{\Omega}\mathcal{E}_p}(e) = \max\left(\Xi_{\overline{\Omega}\mathfrak{F}_p}(v_i), \Xi_{\overline{\Omega}\mathfrak{F}_p}(v_j)\right).$$

For each $p \in \mathcal{P}$ and for all $x \in \wp \cup \mathcal{E}$, the Fermatean condition holds:

$$0 \leq \Pi^3_{\beth^-_p}(x) + \Xi^3_{\beth^-_p}(x) \leq 1, \qquad 0 \leq \Pi^3_{\beth^+_p}(x) + \Xi^3_{\beth^+_p}(x) \leq 1.$$

Moreover, the edge functions satisfy the Fermatean fuzzy graph constraints:

$$\Pi_{\underline{\Omega}\mathcal{E}_p}(u,v) \leq \min\left(\Pi_{\underline{\Omega}\mathfrak{F}_p}(u), \Pi_{\underline{\Omega}\mathfrak{F}_p}(v)\right), \quad \Xi_{\underline{\Omega}\mathcal{E}_p}(u,v) \geq \max\left(\Xi_{\underline{\Omega}\mathfrak{F}_p}(u), \Xi_{\underline{\Omega}\mathfrak{F}_p}(v)\right),$$

$$\Pi_{\overline{\Omega}\mathcal{E}_p}(u,v) \leq \min\left(\Pi_{\overline{\Omega}\mathfrak{F}_p}(u), \Pi_{\overline{\Omega}\mathfrak{F}_p}(v)\right), \quad \Xi_{\overline{\Omega}\mathcal{E}_p}(u,v) \geq \max\left(\Xi_{\overline{\Omega}\mathfrak{F}_p}(u), \Xi_{\overline{\Omega}\mathfrak{F}_p}(v)\right).$$

Consequently, $\beth^-_p$ and $\beth^+_p$ are Fermatean fuzzy graphs. However, in the Pawlak sense, the edge roughness of Γ^* is not necessarily preserved by $\beth^-_p$, since it is induced from fuzzified vertex approximations via $\mathcal{R}_e$. The model generalizes fuzzy, rough, and soft graphs as special cases. Thus, the Fermatean Fuzzy Soft Rough Graph (FFSRG) is defined as the parameterized family:

$$\beth = \{(\beth^-_p, \beth^+_p) \mid p \in \mathcal{P}\},$$

representing a collection of lower and upper Fermatean fuzzy graphs under the equivalence relation $\mathcal{R}_e$.

Unlike existing fuzzy graph models, the proposed FFSRG simultaneously integrates parameterization (via soft sets), granular approximation (via rough sets), and cubic uncertainty (via Fermatean fuzzy sets) within a unified graph structure. This unified representation enables the model to capture complementary uncertainty sources without introducing separate mathematical frameworks for each uncertainty type. The parameterized family structure $\sqsupset$ enables multi-criteria evaluation under various risk factors while preserving the algebraic properties of each underlying theory.

Example 1. *Consider the crisp graph $\Gamma^* = (\wp, \mathcal{E})$ with:*

$$\wp = \{v_1, v_2, v_3, v_4\} \quad \textit{and} \quad \mathcal{E} = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}.$$

Let $\mathcal{P} = \{e_1, e_2\}$ be a set of parameters, where e_1 represents an expert in Artificial Intelligence and e_2 represents an expert in Graph Theory. The equivalence relation on $\wp$ is given by:

$$\wp/\mathcal{R}_e = \{\{v_1, v_3\},\ \{v_2, v_4\}\}.$$

A Fermatean fuzzy soft set is defined through the mapping $\mathfrak{F} : \mathcal{P} \to \mathrm{Fm}(\wp)$, where each parameter in $\mathcal{P}$ is associated with a Fermatean fuzzy set over the universe $\wp$. This mapping assigns to every vertex its corresponding membership and non-membership degrees under each parameter, as presented in Table 3.

Table 3. Fermatean fuzzy soft set $\mathfrak{F}$.

	v_1	v_2	v_3	v_4
e_1	$(0.9, 0.2)$	$(0.6, 0.5)$	$(0.7, 0.4)$	$(0.5, 0.6)$
e_2	$(0.8, 0.3)$	$(0.9, 0.1)$	$(0.4, 0.7)$	$(0.7, 0.4)$

All membership pairs satisfy the Fermatean condition $\Pi^3 + \Xi^3 \leq 1$, e.g.,

$$0.9^3 + 0.2^3 = 0.729 + 0.008 = 0.737 \leq 1.$$

- *Vertex approximations for e_1,(see Table 4).*
  For $[v_1]_{\mathcal{R}_e} = \{v_1, v_3\}$:

$$\underline{\Omega}\Pi(v_1, v_3) = \min(0.9, 0.7) = 0.7, \quad \underline{\Omega}\Xi(v_1, v_3) = \max(0.2, 0.4) = 0.4,$$
$$\overline{\Omega}\Pi(v_1, v_3) = \max(0.9, 0.7) = 0.9, \quad \overline{\Omega}\Xi(v_1, v_3) = \min(0.2, 0.4) = 0.2.$$

  For $[v_2]_{\mathcal{R}_e} = \{v_2, v_4\}$:

$$\underline{\Omega}\Pi(v_2, v_4) = \min(0.6, 0.5) = 0.5, \quad \underline{\Omega}\Xi(v_2, v_4) = \max(0.5, 0.6) = 0.6,$$
$$\overline{\Omega}\Pi(v_2, v_4) = \max(0.6, 0.5) = 0.6, \quad \overline{\Omega}\Xi(v_2, v_4) = \min(0.5, 0.6) = 0.5.$$

Table 4. Vertex Approximations for e_1.

Vertex	$\underline{\Omega}\Pi$	$\underline{\Omega}\Xi$	$\overline{\Omega}\Pi$	$\overline{\Omega}\Xi$
v_1	0.7	0.4	0.9	0.2
v_2	0.5	0.6	0.6	0.5
v_3	0.7	0.4	0.9	0.2
v_4	0.5	0.6	0.6	0.5

- *Edge approximations for e_1, (see Table 5).*
  For each edge $(v_i, v_j) \in \mathcal{E}$:

$$\underline{\Omega}\Pi(v_i, v_j) = \min(\underline{\Omega}\Pi(v_i), \underline{\Omega}\Pi(v_j)), \quad \underline{\Omega}\Xi(v_i, v_j) = \max(\underline{\Omega}\Xi(v_i), \underline{\Omega}\Xi(v_j)),$$

$$\overline{\Omega}\Pi(v_i, v_j) = \min(\overline{\Omega}\Pi(v_i), \overline{\Omega}\Pi(v_j)), \quad \overline{\Omega}\Xi(v_i, v_j) = \max(\overline{\Omega}\Xi(v_i), \overline{\Omega}\Xi(v_j)).$$

  All edges yield:

$$(v_1, v_2), (v_2, v_3), (v_3, v_4) \Rightarrow (0.5, 0.6, 0.6, 0.5).$$

Table 5. Edge approximations for e_1.

Edge	$\underline{\Omega}\Pi$	$\underline{\Omega}\Xi$	$\overline{\Omega}\Pi$	$\overline{\Omega}\Xi$
(v_1, v_2)	0.5	0.6	0.6	0.5
(v_2, v_3)	0.5	0.6	0.6	0.5
(v_3, v_4)	0.5	0.6	0.6	0.5

- *Verification of Fermatean condition.*
  For e_1:

$$0.5^3 + 0.6^3 = 0.125 + 0.216 = 0.341 \leq 1, \qquad 0.6^3 + 0.5^3 = 0.216 + 0.125 = 0.341 \leq 1.$$

- *FFSRG construction complete.*
  The Fermatean fuzzy soft rough graph is defined as the parameterized family:

$$\beth = \{(\beth^-_{e_1}, \beth^+_{e_1}),\ (\beth^-_{e_2}, \beth^+_{e_2})\}.$$

  Since each component meets the Fermatean condition, all induced graphs are Fermatean fuzzy graphs (see Figure 2).

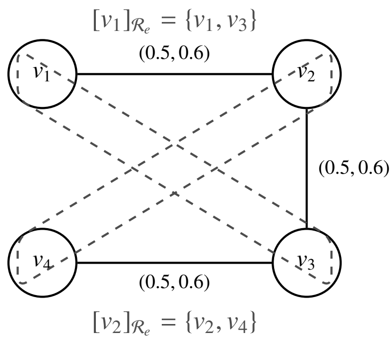


Figure 2. FFSRG structure for parameter e_1 with $\mathcal{R}_e$-equivalence classes.

This illustrative example demonstrates the complete construction of an FFSRG, including vertex and edge approximations, verification of the Fermatean constraint, and the resulting parameterized family structure. The example also shows how equivalence classes partition the vertex set based on granular information.

Definition 12. *Let $\beth_1$ and $\beth_2$ be two FFSRGs constructed on the same underlying crisp graph $\Gamma^* = (\wp, \mathcal{E})$, sharing the parameter set $\mathcal{P}$ and the equivalence relation $\mathcal{R}_e$. Then, $\beth_1$ is said to be a subgraph of $\beth_2$, denoted by $\beth_1 \subseteq \beth_2$, if the following inequalities hold for all $p \in \mathcal{P}$, $v \in \wp$, and $e \in \mathcal{E}$:*

$$\Pi_{\beth^-_{1,p}}(x) \le \Pi_{\beth^-_{2,p}}(x), \quad \Xi_{\beth^-_{1,p}}(x) \ge \Xi_{\beth^-_{2,p}}(x),$$
$$\Pi_{\beth^+_{1,p}}(x) \le \Pi_{\beth^+_{2,p}}(x), \quad \Xi_{\beth^+_{1,p}}(x) \ge \Xi_{\beth^+_{2,p}}(x),$$

where $x \in \wp \cup \mathcal{E}$.

Definition 13. *Let $\beth_1$ and $\beth_2$ be two FFSRGs defined on the same crisp graph $\Gamma^* = (\wp, \mathcal{E})$ equipped with the rough space $(\wp, \mathcal{R}_e)$, and let $\beth$ be an FFSRG associated with a scalar parameter $\alpha > 0$. For $i = 1, 2$, denote by $\Pi_i^-(v), \Pi_i^+(v), \Xi_i^-(v), \Xi_i^+(v)$ the aggregated membership and non-membership degrees corresponding to $\beth_i$. Likewise, let $\Pi^-(v), \Pi^+(v), \Xi^-(v), \Xi^+(v)$ denote the aggregated degrees of $\beth$. All the operations introduced below are performed componentwise and generate a new FFSRG on the same underlying crisp graph $\Gamma^* = (\wp, \mathcal{E})$.*

(a) Union:

$$\begin{aligned}\beth_1 \cup \beth_2 = \Big\{ &\langle v, (\max\{\Pi_1^-(v), \Pi_2^-(v)\},\ \max\{\Pi_1^+(v), \Pi_2^+(v)\}),\\ &(\min\{\Xi_1^-(v), \Xi_2^-(v)\},\ \min\{\Xi_1^+(v), \Xi_2^+(v)\})\rangle : v \in \wp \Big\}\\ \cup \Big\{ &\langle e, (\max\{\Pi_1^-(e), \Pi_2^-(e)\},\ \max\{\Pi_1^+(e), \Pi_2^+(e)\}),\\ &(\min\{\Xi_1^-(e), \Xi_2^-(e)\},\ \min\{\Xi_1^+(e), \Xi_2^+(e)\})\rangle : e \in \mathcal{E} \Big\}.\end{aligned}$$

(b) Intersection:

$$\begin{aligned}\beth_1 \cap \beth_2 = \Big\{ &\langle v, (\min\{\Pi_1^-(v), \Pi_2^-(v)\},\ \min\{\Pi_1^+(v), \Pi_2^+(v)\}),\\ &(\max\{\Xi_1^-(v), \Xi_2^-(v)\},\ \max\{\Xi_1^+(v), \Xi_2^+(v)\})\rangle : v \in \wp \Big\}\\ \cup \Big\{ &\langle e, (\min\{\Pi_1^-(e), \Pi_2^-(e)\},\ \min\{\Pi_1^+(e), \Pi_2^+(e)\}),\\ &(\max\{\Xi_1^-(e), \Xi_2^-(e)\},\ \max\{\Xi_1^+(e), \Xi_2^+(e)\})\rangle : e \in \mathcal{E} \Big\}.\end{aligned}$$

(c) Scalar Multiplication:

$$\alpha\beth = \Big\{ \Big\langle v, \Big(\sqrt[3]{1-(1-(\Pi^{-}(v))^{3})^{\alpha}},\ \sqrt[3]{1-(1-(\Pi^{+}(v))^{3})^{\alpha}} \Big), ((\Xi^{-}(v))^{\alpha},\ (\Xi^{+}(v))^{\alpha}) \Big\rangle : v \in \wp \Big\} \cup \Big\{ \Big\langle e, \Big(\sqrt[3]{1-(1-(\Pi^{-}(e))^{3})^{\alpha}},\ \sqrt[3]{1-(1-(\Pi^{+}(e))^{3})^{\alpha}} \Big), ((\Xi^{-}(e))^{\alpha},\ (\Xi^{+}(e))^{\alpha}) \Big\rangle : e \in \mathcal{E} \Big\}.$$

(d) Power:

$$\beth^{\alpha} = \Big\{ \langle v, ((\Pi^{-}(v))^{\alpha},\ (\Pi^{+}(v))^{\alpha}), \Big(\sqrt[3]{1-(1-(\Xi^{-}(v))^{3})^{\alpha}},\ \sqrt[3]{1-(1-(\Xi^{+}(v))^{3})^{\alpha}} \Big) \Big\rangle : v \in \wp \Big\} \cup \Big\{ \langle e, ((\Pi^{-}(e))^{\alpha},\ (\Pi^{+}(e))^{\alpha}), \Big(\sqrt[3]{1-(1-(\Xi^{-}(e))^{3})^{\alpha}},\ \sqrt[3]{1-(1-(\Xi^{+}(e))^{3})^{\alpha}} \Big) \Big\rangle : e \in \mathcal{E} \Big\}.$$

(e) Complement:

$$\beth^{c} = \Big\{ \langle v, (\Xi^{-}(v),\ \Xi^{+}(v)), (\Pi^{-}(v),\ \Pi^{+}(v)) \rangle : v \in \wp \Big\} \cup \Big\{ \langle e, (\Xi^{-}(e),\ \Xi^{+}(e)), (\Pi^{-}(e),\ \Pi^{+}(e)) \rangle : e \in \mathcal{E} \Big\}.$$

Remark 1. *The complement again satisfies the Fermatean cubical condition on $\wp \cup \mathcal{E}$, so $\beth$ satisfies it if and only if $\beth^{c}$ does.*

Definition 14. *Let $\beth_1$ and $\beth_2$ be two FFSRGs. Then,*

(1) $\beth_1 \oplus \beth_2$:

$$\beth_1 \oplus \beth_2 = \Big\{ \Big\langle v, \Big(\sqrt[3]{(\Pi_1^{-}(v))^{3} + (\Pi_2^{-}(v))^{3} - (\Pi_1^{-}(v))^{3}(\Pi_2^{-}(v))^{3}},\ \sqrt[3]{(\Pi_1^{+}(v))^{3} + (\Pi_2^{+}(v))^{3} - (\Pi_1^{+}(v))^{3}(\Pi_2^{+}(v))^{3}} \Big), (\Xi_1^{-}(v) \cdot \Xi_2^{-}(v),\ \Xi_1^{+}(v) \cdot \Xi_2^{+}(v)) \Big\rangle : v \in \wp \Big\} \cup \Big\{ \Big\langle e, \Big(\sqrt[3]{(\Pi_1^{-}(e))^{3} + (\Pi_2^{-}(e))^{3} - (\Pi_1^{-}(e))^{3}(\Pi_2^{-}(e))^{3}},\ \sqrt[3]{(\Pi_1^{+}(e))^{3} + (\Pi_2^{+}(e))^{3} - (\Pi_1^{+}(e))^{3}(\Pi_2^{+}(e))^{3}} \Big), (\Xi_1^{-}(e) \cdot \Xi_2^{-}(e),\ \Xi_1^{+}(e) \cdot \Xi_2^{+}(e)) \Big\rangle : e \in \mathcal{E} \Big\}.$$

(2) $\beth_1 \otimes \beth_2$*:*

$$\begin{aligned}\beth_1 \otimes \beth_2 = \Big\{\Big\langle v, (\Pi_1^-(v)\Pi_2^-(v),\ \Pi_1^+(v)\Pi_2^+(v)),\\ \Big(\sqrt[3]{(\Xi_1^-(v))^3 + (\Xi_2^-(v))^3 - (\Xi_1^-(v))^3(\Xi_2^-(v))^3},\\ \sqrt[3]{(\Xi_1^+(v))^3 + (\Xi_2^+(v))^3 - (\Xi_1^+(v))^3(\Xi_2^+(v))^3}\Big)\Big\rangle : v \in \wp\Big\}\\ \cup \Big\{\Big\langle e, (\Pi_1^-(e)\Pi_2^-(e),\ \Pi_1^+(e)\Pi_2^+(e)),\\ \Big(\sqrt[3]{(\Xi_1^-(e))^3 + (\Xi_2^-(e))^3 - (\Xi_1^-(e))^3(\Xi_2^-(e))^3},\\ \sqrt[3]{(\Xi_1^+(e))^3 + (\Xi_2^+(e))^3 - (\Xi_1^+(e))^3(\Xi_2^+(e))^3}\Big)\Big\rangle : e \in \mathcal{E}\Big\}.\end{aligned}$$

The following results establish the theoretical foundations of the proposed FFSRG framework and demonstrate that the new operators preserve the essential properties required for consistent decision-making under Fermatean uncertainty. To further illustrate the novelty and comprehensiveness of the proposed framework, Table 6 compares the capabilities of FFSRG with classical graph, fuzzy graph, and rough graph models across key mathematical properties.

Table 6. Comparison of mathematical properties across graph models.

Mathematical Property	**Classical**	**Fuzzy**	**Rough**	**FFSRG**
Closure	✓	✓	✓	✓
Monotonicity	✗	✗	✓	✓
Idempotency	✗	✗	✓	✓
Duality	✗	✗	✓	✓
De Morgan Laws	✗	✗	✓	✓
Approximation Operators	✗	✗	✓	✓
Similarity Measure	✗	Limited	Limited	✓
Path Strength	✗	✗	✗	✓

✓: Supported; ✗: Not supported; Limited: Partially supported.

Lemma 1. *Let* $\beth_1$ *and* $\beth_2$ *be two FFSRG defined on the same crisp graph* $\Gamma^* = (\wp, \mathcal{E})$ *with equivalence relation* $\mathcal{R}_e$ *and parameter set* $\mathcal{P}$*. Then the operations* $\oplus$ *and* $\otimes$ *are closed under the Fermatean constraint. Specifically, for any* $x \in \wp \cup \mathcal{E}$*:*

$$(\Pi_{\beth_1\oplus\beth_2}(x))^3 + (\Xi_{\beth_1\oplus\beth_2}(x))^3 \leq 1,$$

and similarly for $\beth_1 \otimes \beth_2$*.*

Proof. Let $x \in \wp \cup \mathcal{E}$ be arbitrary. For $\beth_1$ and $\beth_2$, by the Fermatean condition we have:

$$(\Pi_{\beth_1}(x))^3 + (\Xi_{\beth_1}(x))^3 \leq 1, \qquad (\Pi_{\beth_2}(x))^3 + (\Xi_{\beth_2}(x))^3 \leq 1.$$

This implies:

$$(\Xi_{\beth_1}(x))^3 \leq 1 - (\Pi_{\beth_1}(x))^3, \qquad (\Xi_{\beth_2}(x))^3 \leq 1 - (\Pi_{\beth_2}(x))^3.$$

For the $\oplus$ operation, from Definition 14:

$$\Pi_{\beth_1\oplus\beth_2}(x) = \sqrt[3]{(\Pi_{\beth_1}(x))^3 + (\Pi_{\beth_2}(x))^3 - (\Pi_{\beth_1}(x))^3\,(\Pi_{\beth_2}(x))^3},$$
$$\Xi_{\beth_1\oplus\beth_2}(x) = \Xi_{\beth_1}(x)\cdot\Xi_{\beth_2}(x).$$

Then:

$$\begin{aligned}(\Pi_{\beth_1\oplus\beth_2}(x))^3 &= (\Pi_{\beth_1}(x))^3 + (\Pi_{\beth_2}(x))^3\\ &\quad - (\Pi_{\beth_1}(x))^3\,(\Pi_{\beth_2}(x))^3\\ &= 1-\left(1-(\Pi_{\beth_1}(x))^3\right)\left(1-(\Pi_{\beth_2}(x))^3\right),\\ (\Xi_{\beth_1\oplus\beth_2}(x))^3 &= (\Xi_{\beth_1}(x))^3\cdot(\Xi_{\beth_2}(x))^3.\end{aligned}$$

Since

$$(\Xi_{\beth_1}(x))^3 \le 1-(\Pi_{\beth_1}(x))^3,\quad (\Xi_{\beth_2}(x))^3 \le 1-(\Pi_{\beth_2}(x))^3,$$

we obtain:

$$(\Xi_{\beth_1\oplus\beth_2}(x))^3 \le \left(1-(\Pi_{\beth_1}(x))^3\right)\left(1-(\Pi_{\beth_2}(x))^3\right).$$

Hence,

$$(\Pi_{\beth_1\oplus\beth_2}(x))^3 + (\Xi_{\beth_1\oplus\beth_2}(x))^3 \le 1.$$

For the $\otimes$ operation, from Definition 14:

$$\Pi_{\beth_1\otimes\beth_2}(x) = \Pi_{\beth_1}(x)\cdot\Pi_{\beth_2}(x),$$
$$\Xi_{\beth_1\otimes\beth_2}(x) = \sqrt[3]{(\Xi_{\beth_1}(x))^3 + (\Xi_{\beth_2}(x))^3 - (\Xi_{\beth_1}(x))^3\,(\Xi_{\beth_2}(x))^3}.$$

Then:

$$\begin{aligned}(\Pi_{\beth_1\otimes\beth_2}(x))^3 &= (\Pi_{\beth_1}(x))^3\cdot(\Pi_{\beth_2}(x))^3,\\ (\Xi_{\beth_1\otimes\beth_2}(x))^3 &= 1-\left(1-(\Xi_{\beth_1}(x))^3\right)\left(1-(\Xi_{\beth_2}(x))^3\right).\end{aligned}$$

Since

$$(\Pi_{\beth_1}(x))^3 \le 1-(\Xi_{\beth_1}(x))^3,\quad (\Pi_{\beth_2}(x))^3 \le 1-(\Xi_{\beth_2}(x))^3,$$

we have:

$$(\Pi_{\beth_1\otimes\beth_2}(x))^3 \le \left(1-(\Xi_{\beth_1}(x))^3\right)\left(1-(\Xi_{\beth_2}(x))^3\right).$$

Thus,

$$(\Pi_{\beth_1\otimes\beth_2}(x))^3 + (\Xi_{\beth_1\otimes\beth_2}(x))^3 \le 1.$$

Hence, $\beth_1\oplus\beth_2$ and $\beth_1\otimes\beth_2$ also satisfy the Fermatean cubic constraint, demonstrating that the operations $\oplus$ and $\otimes$ are closed under the Fermatean constraint. □

Proposition 1. *Let $\beth_1, \beth_2, \beth_3$ be FFSRG defined on $\Gamma^* = (\wp, \mathcal{E})$ with equivalence relation $\mathcal{R}_e \subseteq \wp\times\wp$ and parameter set $\mathcal{P}$. Then:*

(1) $\beth_1\oplus\beth_2 = \beth_2\oplus\beth_1$
(2) $\beth_1\otimes\beth_2 = \beth_2\otimes\beth_1$
(3) $(\beth_1\oplus\beth_2)\oplus\beth_3 = \beth_1\oplus(\beth_2\oplus\beth_3)$

(4) $(\beth_1 \otimes \beth_2) \otimes \beth_3 = \beth_1 \otimes (\beth_2 \otimes \beth_3)$
(5) $\alpha(\beth_1 \oplus \beth_2) = \alpha\beth_1 \oplus \alpha\beth_2$*, where* $\alpha > 0$
(6) $(\alpha_1 + \alpha_2)\beth = \alpha_1\beth \oplus \alpha_2\beth$*, where* $\alpha_1, \alpha_2 > 0$
(7) $(\beth_1 \otimes \beth_2)^\alpha = \beth_1^\alpha \otimes \beth_2^\alpha$*, where* $\alpha > 0$
(8) $\beth^{\alpha_1+\alpha_2} = \beth^{\alpha_1} \otimes \beth^{\alpha_2}$*, where* $\alpha_1, \alpha_2 > 0$
(9) If $\beth_1 \subseteq \beth_2$*, then* $\beth_1 \oplus \beth_3 \subseteq \beth_2 \oplus \beth_3$.

Proof. Each identity follows by substituting the definitions of $\oplus$ and $\otimes$ from Definition 14 and using the algebraic properties of the Fermatean operators. For example, commutativity follows from the fact that addition and multiplication of real numbers are commutative:

$$\Pi_1^3 + \Pi_2^3 - \Pi_1^3\Pi_2^3 = \Pi_2^3 + \Pi_1^3 - \Pi_2^3\Pi_1^3,$$

and similarly for non-membership. Associativity follows from repeated substitution together with the associativity of the induced Fermatean t-conorm and t-norm. Distributivity of scalar multiplication follows from the algebraic properties of the Fermatean operations. The remaining properties follow analogously. □

Proposition 2. *For any two FFSRGs* $\beth_1$ *and* $\beth_2$ *defined on the same crisp graph* $\Gamma^* = (\wp, \mathcal{E})$*, De Morgan-type laws are satisfied as follows:*

(1) $(\beth_1 \cap \beth_2)^c = \beth_1^c \cup \beth_2^c$
(2) $(\beth_1 \cup \beth_2)^c = \beth_1^c \cap \beth_2^c$
(3) $(\beth_1 \oplus \beth_2)^c = \beth_1^c \otimes \beth_2^c$
(4) $(\beth_1 \otimes \beth_2)^c = \beth_1^c \oplus \beth_2^c$
(5) $(\beth_1^c)^\alpha = (\alpha\beth_1)^c$
(6) $\alpha(\beth_1^c) = (\beth_1^\alpha)^c$.

Proof. The identities follow directly from the definition of complement (Definition 13(e)) together with Definitions 13(a)–(b) and Definition 14. For instance, for (1), the membership degree of $(\beth_1 \cap \beth_2)^c$ is:

$$\Xi_{\beth_1\cap\beth_2}(x) = \max\{\Xi_1(x), \Xi_2(x)\},$$

which coincides with the membership degree of $\beth_1^c \cup \beth_2^c$:

$$\max\{\Xi_1(x), \Xi_2(x)\}.$$

Similarly, the non-membership degree of $(\beth_1 \cap \beth_2)^c$ is:

$$\Pi_{\beth_1\cap\beth_2}(x) = \min\{\Pi_1(x), \Pi_2(x)\},$$

which coincides with the non-membership degree of $\beth_1^c \cup \beth_2^c$:

$$\min\{\Pi_1(x), \Pi_2(x)\}.$$

The remaining identities follow analogously by applying the definitions of the respective operations. □

These De Morgan-type laws ensure that the algebraic structure of FFSRG is consistent with classical fuzzy logic extensions, providing a solid foundation for decision-making applications.

Theorem 1. *For any FFSRG $\daleth$ on $\Gamma^* = (\wp, \mathcal{E})$ with $\mathcal{R}_e \subseteq \wp \times \wp$ and parameter set $\mathcal{P}$, the following duality relations hold for all $p \in \mathcal{P}$:*

$$\underline{\Omega}\daleth_p = \left(\overline{\Omega}\daleth^c_p\right)^c, \qquad \overline{\Omega}\daleth_p = \left(\underline{\Omega}\daleth^c_p\right)^c.$$

Proof. For every vertex $v \in \wp$, we have:

$$\Pi_{(\overline{\Omega}\daleth^c_p)^c}(v) = \Xi_{\overline{\Omega}\daleth^c_p}(v) = \min_{u\in[v]_{\mathcal{R}_e}} \Xi_{\daleth^c_p}(u) = \min_{u\in[v]_{\mathcal{R}_e}} \Pi_{\daleth_p}(u) = \Pi_{\underline{\Omega}\daleth_p}(v),$$

and

$$\Xi_{(\overline{\Omega}\daleth^c_p)^c}(v) = \Pi_{\overline{\Omega}\daleth^c_p}(v) = \max_{u\in[v]_{\mathcal{R}_e}} \Pi_{\daleth^c_p}(u) = \max_{u\in[v]_{\mathcal{R}_e}} \Xi_{\daleth_p}(u) = \Xi_{\underline{\Omega}\daleth_p}(v).$$

For every edge $e = (v_i, v_j) \in \mathcal{E}$, noting that $\Pi_{\overline{\Omega}\mathcal{E}_p}(e) = \min\{\Pi_{\overline{\Omega}\mathfrak{F}_p}(v_i), \Pi_{\overline{\Omega}\mathfrak{F}_p}(v_j)\}$ and $\Xi_{\overline{\Omega}\mathcal{E}_p}(e) = \max\{\Xi_{\overline{\Omega}\mathfrak{F}_p}(v_i), \Xi_{\overline{\Omega}\mathfrak{F}_p}(v_j)\}$, we obtain:

$$\Pi_{(\overline{\Omega}\daleth^c_p)^c}(e) = \Xi_{\overline{\Omega}\daleth^c_p}(e) = \max\{\Xi_{\overline{\Omega}\daleth^c_p}(v_i), \Xi_{\overline{\Omega}\daleth^c_p}(v_j)\} = \max\{\Pi_{\underline{\Omega}\daleth_p}(v_i), \Pi_{\underline{\Omega}\daleth_p}(v_j)\} = \Pi_{\underline{\Omega}\daleth_p}(e),$$

and

$$\Xi_{(\overline{\Omega}\daleth^c_p)^c}(e) = \Pi_{\overline{\Omega}\daleth^c_p}(e) = \min\{\Pi_{\overline{\Omega}\daleth^c_p}(v_i), \Pi_{\overline{\Omega}\daleth^c_p}(v_j)\} = \min\{\Xi_{\underline{\Omega}\daleth_p}(v_i), \Xi_{\underline{\Omega}\daleth_p}(v_j)\} = \Xi_{\underline{\Omega}\daleth_p}(e).$$

Thus, $\underline{\Omega}\daleth_p = (\overline{\Omega}\daleth^c_p)^c$. The second relation $\overline{\Omega}\daleth_p = (\underline{\Omega}\daleth^c_p)^c$ follows similarly. □

Theorem 2. *Let $\daleth_1$ and $\daleth_2$ be two FFSRGs defined on Γ^* such that, for every $v \in \wp$ and every $p \in \mathcal{P}$, the following conditions hold:*

$$\Pi_{\mathfrak{F}_{1p}}(v) \leq \Pi_{\mathfrak{F}_{2p}}(v), \qquad \Xi_{\mathfrak{F}_{1p}}(v) \geq \Xi_{\mathfrak{F}_{2p}}(v).$$

Then for all $p \in \mathcal{P}$:

$$\underline{\Omega}\daleth_{1p} \subseteq \underline{\Omega}\daleth_{2p} \quad \text{and} \quad \overline{\Omega}\daleth_{1p} \subseteq \overline{\Omega}\daleth_{2p}.$$

Proof. For every vertex $v \in \wp$ and every edge $e = (v_i, v_j) \in \mathcal{E}$, the following relations hold:

$$\Pi_{\underline{\Omega}\mathfrak{F}_{1p}}(v) = \min_{u\in[v]_{\mathcal{R}_e}} \Pi_{\mathfrak{F}_{1p}}(u) \leq \min_{u\in[v]_{\mathcal{R}_e}} \Pi_{\mathfrak{F}_{2p}}(u) = \Pi_{\underline{\Omega}\mathfrak{F}_{2p}}(v),$$
$$\Xi_{\underline{\Omega}\mathfrak{F}_{1p}}(v) = \max_{u\in[v]_{\mathcal{R}_e}} \Xi_{\mathfrak{F}_{1p}}(u) \geq \max_{u\in[v]_{\mathcal{R}_e}} \Xi_{\mathfrak{F}_{2p}}(u) = \Xi_{\underline{\Omega}\mathfrak{F}_{2p}}(v),$$

$$\Pi_{\overline{\Omega}\mathfrak{F}_{1p}}(v) = \max_{u\in[v]_{\mathcal{R}_e}} \Pi_{\mathfrak{F}_{1p}}(u) \leq \max_{u\in[v]_{\mathcal{R}_e}} \Pi_{\mathfrak{F}_{2p}}(u) = \Pi_{\overline{\Omega}\mathfrak{F}_{2p}}(v),$$
$$\Xi_{\overline{\Omega}\mathfrak{F}_{1p}}(v) = \min_{u\in[v]_{\mathcal{R}_e}} \Xi_{\mathfrak{F}_{1p}}(u) \geq \min_{u\in[v]_{\mathcal{R}_e}} \Xi_{\mathfrak{F}_{2p}}(u) = \Xi_{\overline{\Omega}\mathfrak{F}_{2p}}(v),$$

$$\Pi_{\underline{\Omega}\mathcal{E}_{1p}}(e) = \min\left(\Pi_{\underline{\Omega}\mathfrak{F}_{1p}}(v_i), \Pi_{\underline{\Omega}\mathfrak{F}_{1p}}(v_j)\right) \leq \min\left(\Pi_{\underline{\Omega}\mathfrak{F}_{2p}}(v_i), \Pi_{\underline{\Omega}\mathfrak{F}_{2p}}(v_j)\right) = \Pi_{\underline{\Omega}\mathcal{E}_{2p}}(e),$$
$$\Xi_{\underline{\Omega}\mathcal{E}_{1p}}(e) = \max\left(\Xi_{\underline{\Omega}\mathfrak{F}_{1p}}(v_i), \Xi_{\underline{\Omega}\mathfrak{F}_{1p}}(v_j)\right) \geq \max\left(\Xi_{\underline{\Omega}\mathfrak{F}_{2p}}(v_i), \Xi_{\underline{\Omega}\mathfrak{F}_{2p}}(v_j)\right) = \Xi_{\underline{\Omega}\mathcal{E}_{2p}}(e),$$

$$\Pi_{\overline{\Omega}\mathcal{E}_{1p}}(e) = \min\left(\Pi_{\overline{\Omega}\mathfrak{F}_{1p}}(v_i), \Pi_{\overline{\Omega}\mathfrak{F}_{1p}}(v_j)\right) \leq \min\left(\Pi_{\overline{\Omega}\mathfrak{F}_{2p}}(v_i), \Pi_{\overline{\Omega}\mathfrak{F}_{2p}}(v_j)\right) = \Pi_{\overline{\Omega}\mathcal{E}_{2p}}(e),$$
$$\Xi_{\overline{\Omega}\mathcal{E}_{1p}}(e) = \max\left(\Xi_{\overline{\Omega}\mathfrak{F}_{1p}}(v_i), \Xi_{\overline{\Omega}\mathfrak{F}_{1p}}(v_j)\right) \geq \max\left(\Xi_{\overline{\Omega}\mathfrak{F}_{2p}}(v_i), \Xi_{\overline{\Omega}\mathfrak{F}_{2p}}(v_j)\right) = \Xi_{\overline{\Omega}\mathcal{E}_{2p}}(e).$$

Hence, $\underline{\Omega}\daleth_{1p} \subseteq \underline{\Omega}\daleth_{2p}$ and $\overline{\Omega}\daleth_{1p} \subseteq \overline{\Omega}\daleth_{2p}$. □

Theorem 3. *For any two FFSRGs $\beth_1$ and $\beth_2$ on Γ^*, the following inclusions hold for all $p \in \mathcal{P}$:*

$$\begin{aligned}
\underline{\Omega}(\beth_{1p} \cap \beth_{2p}) &\supseteq \underline{\Omega}\beth_{1p} \cap \underline{\Omega}\beth_{2p},\\
\underline{\Omega}(\beth_{1p} \cup \beth_{2p}) &\supseteq \underline{\Omega}\beth_{1p} \cup \underline{\Omega}\beth_{2p},\\
\overline{\Omega}(\beth_{1p} \cup \beth_{2p}) &\subseteq \overline{\Omega}\beth_{1p} \cup \overline{\Omega}\beth_{2p},\\
\overline{\Omega}(\beth_{1p} \cap \beth_{2p}) &\subseteq \overline{\Omega}\beth_{1p} \cap \overline{\Omega}\beth_{2p}.
\end{aligned}$$

Proof. The proof follows from the definitions of lower and upper approximations and the properties of min and max operators. These inclusion relations are consistent with classical rough set theory and demonstrate that the FFSRG framework preserves fundamental rough approximation properties. □

Definition 15. *An FFSRG $\beth$ is called strong if for every $p \in \mathcal{P}$ and every edge $(u, v) \in \mathcal{E}$:*

$$\begin{aligned}
\Pi_{\underline{\Omega}\mathcal{E}_p}(u, v) &= \min\left(\Pi_{\underline{\Omega}\mathfrak{F}_p}(u), \Pi_{\underline{\Omega}\mathfrak{F}_p}(v)\right),\\
\Xi_{\underline{\Omega}\mathcal{E}_p}(u, v) &= \max\left(\Xi_{\underline{\Omega}\mathfrak{F}_p}(u), \Xi_{\underline{\Omega}\mathfrak{F}_p}(v)\right),\\
\Pi_{\overline{\Omega}\mathcal{E}_p}(u, v) &= \min\left(\Pi_{\overline{\Omega}\mathfrak{F}_p}(u), \Pi_{\overline{\Omega}\mathfrak{F}_p}(v)\right),\\
\Xi_{\overline{\Omega}\mathcal{E}_p}(u, v) &= \max\left(\Xi_{\overline{\Omega}\mathfrak{F}_p}(u), \Xi_{\overline{\Omega}\mathfrak{F}_p}(v)\right).
\end{aligned}$$

Theorem 4. *If $\beth$ is a strong FFSRG, then for any path $\mathbf{r} = v_0, v_1, \ldots, v_n$ in Γ^* and for any $p \in \mathcal{P}$:*

$$\Pi_{\underline{\Omega}\mathcal{E}_p}(v_0, v_n) \geq \min_{t=1}^{n} \Pi_{\underline{\Omega}\mathcal{E}_p}(v_{t-1}, v_t),$$

$$\Xi_{\underline{\Omega}\mathcal{E}_p}(v_0, v_n) \leq \max_{t=1}^{n} \Xi_{\underline{\Omega}\mathcal{E}_p}(v_{t-1}, v_t).$$

Proof. From Definition 15, for any edge (v_{t-1}, v_t):

$$\Pi_{\underline{\Omega}\mathcal{E}_p}(v_{t-1}, v_t) = \min\left(\Pi_{\underline{\Omega}\mathfrak{F}_p}(v_{t-1}), \Pi_{\underline{\Omega}\mathfrak{F}_p}(v_t)\right).$$

Thus,

$$\min_{t=1}^{n} \Pi_{\underline{\Omega}\mathcal{E}_p}(v_{t-1}, v_t) = \min_{t=1}^{n} \min\left(\Pi_{\underline{\Omega}\mathfrak{F}_p}(v_{t-1}), \Pi_{\underline{\Omega}\mathfrak{F}_p}(v_t)\right).$$

Since $\min(a, b) \leq a$ and $\min(a, b) \leq b$, we have:

$$\min_{t=1}^{n} \Pi_{\underline{\Omega}\mathcal{E}_p}(v_{t-1}, v_t) \leq \min\left(\Pi_{\underline{\Omega}\mathfrak{F}_p}(v_0), \Pi_{\underline{\Omega}\mathfrak{F}_p}(v_n)\right) = \Pi_{\underline{\Omega}\mathcal{E}_p}(v_0, v_n).$$

For non-membership:

$$\Xi_{\underline{\Omega}\mathcal{E}_p}(v_{t-1}, v_t) = \max\left(\Xi_{\underline{\Omega}\mathfrak{F}_p}(v_{t-1}), \Xi_{\underline{\Omega}\mathfrak{F}_p}(v_t)\right) \geq \max\left(\Xi_{\underline{\Omega}\mathfrak{F}_p}(v_0), \Xi_{\underline{\Omega}\mathfrak{F}_p}(v_n)\right) = \Xi_{\underline{\Omega}\mathcal{E}_p}(v_0, v_n).$$

Hence,

$$\Xi_{\underline{\Omega}\mathcal{E}_p}(v_0, v_n) \leq \max_{t=1}^{n} \Xi_{\underline{\Omega}\mathcal{E}_p}(v_{t-1}, v_t).$$

□

Theorem 4 establishes that in a strong FFSRG, the edge membership between any two vertices is at least as strong as the weakest edge on any connecting path, while the non-membership is at most as strong as the strongest non-membership on any path. This property is essential for path evaluation and connection strength analysis.

Definition 16. *Let $\beth$ be an FFSRG. A path* $\mathbf{r}$ *of length n connecting the vertices v_0 and v_n is represented by an ordered sequence of distinct vertices*

$$v_0, v_1, \ldots, v_n,$$

such that $(v_{t-1}, v_t) \in \mathcal{E}$ for every $t = 1, 2, \ldots, n$. The strength associated with the path $\mathbf{r}$ *under a parameter $p \in \mathcal{P}$ is determined as follows:*

$$\Phi_p(\mathbf{r}) = \min_{t=1}^{n} \Pi_{\underline{\Omega}\mathcal{E}_p}(v_{t-1}, v_t), \qquad \Psi_p(\mathbf{r}) = \max_{t=1}^{n} \Xi_{\underline{\Omega}\mathcal{E}_p}(v_{t-1}, v_t).$$

Definition 17. *The connection strength between two vertices $u, v \in \wp$ in an FFSRG $\beth$ is defined, for each parameter $p \in \mathcal{P}$, by aggregating all possible paths connecting u and v as follows:*

$$\Pi_p^{\infty}(u, v) = \max \left\{ \Phi_p(\mathbf{r}) \,\middle|\, \mathbf{r} \text{ is a path between } u \text{ and } v \right\},$$

$$\Xi_p^{\infty}(u, v) = \min \left\{ \Psi_p(\mathbf{r}) \,\middle|\, \mathbf{r} \text{ is a path between } u \text{ and } v \right\}.$$

In the case where no path exists between u and v, the connection strengths are assigned as

$$\Pi_p^{\infty}(u, v) = 0, \qquad \Xi_p^{\infty}(u, v) = 1.$$

The concept of connectivity in FFSRG extends classical graph connectivity by requiring positive membership and non-unity non-membership along all connecting paths, ensuring that the connection is meaningful under Fermatean uncertainty.

Definition 18. *Let $\beth$ be an FFSRG defined on the crisp graph $\Gamma^* = (\wp, \mathcal{E})$. For a vertex $v \in \wp$ and a parameter $p \in \mathcal{P}$, the degree and total degree are defined as follows:*

$$\deg_{\beth_p}(v) = \left(\sum_{\substack{u \in \wp \\ (v,u) \in \mathcal{E}}} \Pi_{\underline{\Omega}\mathcal{E}_p}(v, u), \; \sum_{\substack{u \in \wp \\ (v,u) \in \mathcal{E}}} \Xi_{\underline{\Omega}\mathcal{E}_p}(v, u) \right),$$

$$\tau \deg_{\beth_p}(v) = \left(\sum_{\substack{u \in \wp \\ (v,u) \in \mathcal{E}}} \Pi_{\underline{\Omega}\mathcal{E}_p}(v, u) + \Pi_{\underline{\Omega}\tilde{\mathfrak{F}}_p}(v), \; \sum_{\substack{u \in \wp \\ (v,u) \in \mathcal{E}}} \Xi_{\underline{\Omega}\mathcal{E}_p}(v, u) + \Xi_{\underline{\Omega}\tilde{\mathfrak{F}}_p}(v) \right).$$

Theorem 5. *For any vertex $v \in \wp$ in an FFSRG $\beth$ with $|\wp| = n$ and for any $p \in \mathcal{P}$:*

$$\sum_{\substack{u \in \wp \\ (v,u) \in \mathcal{E}}} \Pi_{\underline{\Omega}\mathcal{E}_p}(v, u) \leq (n-1) \cdot \max_{\substack{u \in \wp \\ (v,u) \in \mathcal{E}}} \Pi_{\underline{\Omega}\mathcal{E}_p}(v, u),$$

$$\sum_{\substack{u \in \wp \\ (v,u) \in \mathcal{E}}} \Xi_{\underline{\Omega}\mathcal{E}_p}(v, u) \geq (n-1) \cdot \min_{\substack{u \in \wp \\ (v,u) \in \mathcal{E}}} \Xi_{\underline{\Omega}\mathcal{E}_p}(v, u).$$

Proof. Since Γ^* is simple, the degree of any vertex $v \in \wp$ is bounded above by $n-1$. Therefore,

$$\sum_{\substack{u\in\wp\\(v,u)\in\mathcal{E}}} \Pi_{\underline{\Omega}\mathcal{E}_p}(v,u) \leq \sum_{\substack{u\in\wp\\(v,u)\in\mathcal{E}}} \max_{\substack{w\in\wp\\(v,w)\in\mathcal{E}}} \Pi_{\underline{\Omega}\mathcal{E}_p}(v,w) \leq (n-1)\cdot \max_{\substack{w\in\wp\\(v,w)\in\mathcal{E}}} \Pi_{\underline{\Omega}\mathcal{E}_p}(v,w).$$

Similarly, for non-membership:

$$\Xi_{\underline{\Omega}\mathcal{E}_p}(v,u) \leq \max_{\substack{w\in\wp\\(v,w)\in\mathcal{E}}} \Xi_{\underline{\Omega}\mathcal{E}_p}(v,w) \quad \forall u,$$

so

$$\sum_{\substack{u\in\wp\\(v,u)\in\mathcal{E}}} \Xi_{\underline{\Omega}\mathcal{E}_p}(v,u) \geq (n-1)\cdot \min_{\substack{u\in\wp\\(v,u)\in\mathcal{E}}} \Xi_{\underline{\Omega}\mathcal{E}_p}(v,u).$$

□

Theorem 5 provides useful bounds on vertex degrees in FFSRG, which can be employed for complexity analysis and algorithm design.

Definition 19. *Assume $\beth_1$ and $\beth_2$ are two FFSRGs defined over the same crisp graph Γ^*. The cosine similarity is given by:*

$$Sim(\beth_1,\beth_2) = \frac{1}{|\mathcal{P}|}\sum_{p\in\mathcal{P}} \frac{\sum_{v\in\wp}\left(\Pi^3_{\underline{\Omega}\mathfrak{F}_{1p}}(v)\Pi^3_{\underline{\Omega}\mathfrak{F}_{2p}}(v) + \Xi^3_{\underline{\Omega}\mathfrak{F}_{1p}}(v)\Xi^3_{\underline{\Omega}\mathfrak{F}_{2p}}(v)\right)}{\sqrt{\sum_{v\in\wp}\left(\Pi^6_{\underline{\Omega}\mathfrak{F}_{1p}}(v) + \Xi^6_{\underline{\Omega}\mathfrak{F}_{1p}}(v)\right)}\sqrt{\sum_{v\in\wp}\left(\Pi^6_{\underline{\Omega}\mathfrak{F}_{2p}}(v) + \Xi^6_{\underline{\Omega}\mathfrak{F}_{2p}}(v)\right)}}.$$

The similarity measure satisfies $0 \leq Sim(\beth_1,\beth_2) \leq 1$, and $Sim(\beth_1,\beth_2) = 1$ if and only if $\beth_1 = \beth_2$.

Theorem 6. *The cosine similarity $Sim(\beth_1,\beth_2)$ over Definition 19 fulfills:*

1. *Reflexivity: $Sim(\beth_1,\beth_1) = 1$.*
2. *Symmetry: $Sim(\beth_1,\beth_2) = Sim(\beth_2,\beth_1)$.*
3. *Boundedness: $0 \leq Sim(\beth_1,\beth_2) \leq 1$.*
4. *Identity: $Sim(\beth_1,\beth_2) = 1$ iff $\beth_1 = \beth_2$.*

Proof. Define for each $p \in \mathcal{P}$:

$$A_p = \sum_{v\in\wp} \Pi^6_{\underline{\Omega}\mathfrak{F}_{1p}}(v) + \Xi^6_{\underline{\Omega}\mathfrak{F}_{1p}}(v), \quad B_p = \sum_{v\in\wp} \Pi^6_{\underline{\Omega}\mathfrak{F}_{2p}}(v) + \Xi^6_{\underline{\Omega}\mathfrak{F}_{2p}}(v),$$

$$C_p = \sum_{v\in\wp} \Pi^3_{\underline{\Omega}\mathfrak{F}_{1p}}(v)\Pi^3_{\underline{\Omega}\mathfrak{F}_{2p}}(v) + \Xi^3_{\underline{\Omega}\mathfrak{F}_{1p}}(v)\Xi^3_{\underline{\Omega}\mathfrak{F}_{2p}}(v).$$

(1) Reflexivity: If $\beth_1 = \beth_2$, then components coincide, hence $A_p = B_p = C_p$. Thus,

$$\frac{C_p}{\sqrt{A_p}\sqrt{B_p}} = 1,$$

and averaging over p gives $\mathrm{Sim}(\beth_1,\beth_1) = 1$.

(2) Symmetry: Swapping $\beth_1, \beth_2$ exchanges only A_p and B_p, while C_p is unchanged. Hence, the ratio is invariant.

(3) Boundedness: Using Cauchy–Schwarz, define vectors

$$x_t = \Pi^3_{\underline{\Omega}\mathfrak{F}_{1p}}(v_t),\ \Xi^3_{\underline{\Omega}\mathfrak{F}_{1p}}(v_t), \quad y_t = \Pi^3_{\underline{\Omega}\mathfrak{F}_{2p}}(v_t),\ \Xi^3_{\underline{\Omega}\mathfrak{F}_{2p}}(v_t).$$

Then

$$C_p^2 \leq A_p B_p \quad \Rightarrow \quad 0 \leq \frac{C_p}{\sqrt{A_p}\sqrt{B_p}} \leq 1.$$

(4) Identity: If $\mathrm{Sim}(\beth_1, \beth_2) = 1$, then $C_p^2 = A_p B_p$. Equality in Cauchy–Schwarz implies proportionality:

$$y_t = \lambda_p x_t.$$

Thus,

$$\Pi^3_{\underline{\Omega}\mathfrak{F}_{2p}}(v) = \lambda_p \Pi^3_{\underline{\Omega}\mathfrak{F}_{1p}}(v), \quad \Xi^3_{\underline{\Omega}\mathfrak{F}_{2p}}(v) = \lambda_p \Xi^3_{\underline{\Omega}\mathfrak{F}_{1p}}(v).$$

Since the Fermatean constraint requires $\Pi^3(v) + \Xi^3(v) \leq 1$ for all v, the proportional vectors must satisfy this condition for $\beth_1$ and $\beth_2$. This forces $\lambda_p = 1$; otherwise, the sum $\lambda_p(\Pi_1^3(v) + \Xi_1^3(v))$ would exceed the bound 1 for some v unless $\lambda_p = 1$. Hence, all membership and non-membership values coincide, so $\beth_1 = \beth_2$. The converse follows immediately from reflexivity.

□

The cosine similarity measure provides a quantitative way to compare different FFSRG structures, which is useful for clustering, classification, and decision-making applications. This completes the mathematical formulation of the FFSRG framework. In the following section, we apply the aggregation operators and score functions developed here to construct a practical decision-making algorithm for supply chain risk analysis.

4. Decision-making method

In this section, we present the practical decision-making framework based on the FFSRG model developed in Section 3. We introduce aggregation operators for combining multi-parameter evaluations, score functions for ranking alternatives, a complete decision-making algorithm, and a time complexity analysis. The methods developed here are designed to leverage the uncertainty representation capabilities of FFSRG for multi-criteria decision-making in networked environments.

4.1. Aggregation operators

In real-world decision-making problems, evaluations are often provided by multiple experts or under multiple parameters. To combine these evaluations into a single comprehensive assessment, we introduce aggregation operators that preserve the Fermatean cubic constraint.

Definition 20. *The aggregated vertex approximations are characterized by the following lower and upper operators:*

$$\Pi_{\beth^-_{agg}}(v) = \min_{p\in\mathcal{P}} \Pi_{\underline{\Omega}\mathfrak{F}_p}(v), \quad \Xi_{\beth^-_{agg}}(v) = \max_{p\in\mathcal{P}} \Xi_{\underline{\Omega}\mathfrak{F}_p}(v),$$

$$\Pi_{\beth^+_{agg}}(v) = \max_{p\in\mathcal{P}} \Pi_{\overline{\Omega}\mathfrak{F}_p}(v), \quad \Xi_{\beth^+_{agg}}(v) = \min_{p\in\mathcal{P}} \Xi_{\overline{\Omega}\mathfrak{F}_p}(v).$$

Based on these aggregated vertices, the aggregated lower and upper edge approximations are defined for any edge $e = (v_i, v_j) \in \mathcal{E}$ as

$$\begin{aligned}
\Pi_{\daleth^-_{agg}}(e) &= \min\left(\Pi_{\daleth^-_{agg}}(v_i), \Pi_{\daleth^-_{agg}}(v_j)\right),\\
\Xi_{\daleth^-_{agg}}(e) &= \max\left(\Xi_{\daleth^-_{agg}}(v_i), \Xi_{\daleth^-_{agg}}(v_j)\right),\\
\Pi_{\daleth^+_{agg}}(e) &= \min\left(\Pi_{\daleth^+_{agg}}(v_i), \Pi_{\daleth^+_{agg}}(v_j)\right),\\
\Xi_{\daleth^+_{agg}}(e) &= \max\left(\Xi_{\daleth^+_{agg}}(v_i), \Xi_{\daleth^+_{agg}}(v_j)\right).
\end{aligned}$$

The aggregation operators defined above combine evaluations across all parameters to produce lower and upper approximations for vertices and edges. The min-max structure ensures that the resulting approximations remain within the Fermatean feasible region.

Definition 21. *For an edge $e \in \mathcal{E}$, the aggregated weight is defined as*

$$\begin{aligned}
\Pi_{agg}(e) &= \sqrt[3]{\Pi^3_{\daleth^-_{agg}}(e) + \Pi^3_{\daleth^+_{agg}}(e) - \Pi^3_{\daleth^-_{agg}}(e)\Pi^3_{\daleth^+_{agg}}(e)},\\
\Xi_{agg}(e) &= \Xi_{\daleth^-_{agg}}(e)\,\Xi_{\daleth^+_{agg}}(e).
\end{aligned}$$

The aggregation of membership follows a probabilistic sum-type operator adapted to the Fermatean framework. The aggregation of non-membership is defined multiplicatively to reflect the joint uncertainty under lower and upper approximations.

The aggregated weight $(\Pi_{\mathrm{agg}}(e), \Xi_{\mathrm{agg}}(e))$ represents a comprehensive assessment of edge strength that combines lower and upper approximations while preserving the Fermatean constraint.

4.2. Score functions

To facilitate ranking and comparison of alternatives, we define score functions that convert the two-dimensional Fermatean membership pairs into a single scalar value. This enables straightforward ordering of vertices and edges in decision-making applications.

Definition 22. *The score function for an edge weight $w(e) = (\Pi_{agg}(e), \Xi_{agg}(e))$ is defined as:*

$$\mathcal{S}(e) = \frac{1 + \Pi^3_{agg}(e) - \Xi^3_{agg}(e)}{2},$$

which ensures $\mathcal{S}(e) \in [0, 1]$ due to the Fermatean constraint. For a vertex v with total degrees $\tau\Pi(v)$ and $\tau\Xi(v)$, the vertex score function is defined as:

$$\mathcal{S}(v) = \frac{1 + \tau\Pi(v)}{1 + \tau\Pi(v) + \tau\Xi(v)},$$

which guarantees $\mathcal{S}(v) \in (0, 1]$ for all vertices.

The vertex score function is normalized to ensure bounded evaluation within the unit interval, providing interpretable and directly comparable vulnerability measures across suppliers and risk parameters.

Definition 23. *For an edge $e \in \mathcal{E}$ in an FFSRG $\beth$ with aggregated weight $w(e) = (\Pi_{agg}(e), \Xi_{agg}(e))$, the following functions are defined:*

$$\mathcal{H}(e) = \frac{\Pi_{agg}^3(e) + \Xi_{agg}^3(e)}{2}, \qquad C(e) = \Pi_{agg}^3(e), \qquad \Delta(e) = \sqrt[3]{1 - \Pi_{agg}^3(e) - \Xi_{agg}^3(e)}.$$

These functions satisfy $0 \leq \mathcal{H}(e), C(e), \Delta(e) \leq 1$ *due to the Fermatean constraint* $\Pi_{agg}^3(e) + \Xi_{agg}^3(e) \leq 1$.

The functions $\mathcal{H}(e)$, $C(e)$, and $\Delta(e)$ quantify, respectively, the information content, the positive evidence, and the uncertainty associated with an edge. These are used for ranking and comparison in decision-making.

4.3. Decision-making algorithm

The following algorithm provides a systematic procedure for applying the FFSRG framework to multi-criteria decision-making problems in networked environments.

Let $\Gamma^* = (\wp, \mathcal{E})$ be a crisp graph, $\mathcal{P} = \{p_1, p_2, \ldots, p_m\}$ a nonempty parameter set, and $\mathcal{R}_e$ an equivalence relation on $\wp$. Suppose that a Fermatean fuzzy soft mapping

$$\mathfrak{F} : \mathcal{P} \to \mathrm{Fm}(\wp)$$

satisfies

$$0 \leq \Pi_{\mathfrak{F}(p)}^3(v) + \Xi_{\mathfrak{F}(p)}^3(v) \leq 1, \qquad \forall\, p \in \mathcal{P},\ v \in \wp.$$

The FFSRG framework is then developed to generate vertex and edge approximations, aggregate Fermatean fuzzy edge information, identify strongest paths, and prioritize vertices according to their overall significance. Algorithm 1 summarizes the proposed FFSRG decision-making procedure in pseudocode form.

Algorithm 1: FFSRG Decision-Making Algorithm

Input : $\wp, \mathcal{E}$: graph structure;
$\mathcal{P}$: parameter set;
$\mathfrak{F} : \mathcal{P} \to \mathrm{Fm}(\wp)$: Fermatean fuzzy soft set;
$\mathcal{R}_e$: equivalence relation on $\wp$

Step 1: Compute lower and upper approximations of vertices;
Step 2: Compute lower and upper approximations of edges;
Step 3: Aggregate edge weights and compute scores;
Step 4: Compute vertex degrees and vertex scores;
Step 5: Determine connection strengths between vertices;
Step 6: Rank vertices in descending order of scores;
return *Ranked vertices and strongest paths*
Output: Ranked vertices, strongest paths between vertices

Detailed explanation of algorithmic steps:

In this subsection, we provide a comprehensive explanation of each step in Algorithm 1.

- Step 1: Vertex Lower and Upper Approximations:
  For each parameter $p \in \mathcal{P}$ and each equivalence class $[v]_{\mathcal{R}_e} \in \wp/\mathcal{R}_e$, we compute:

$$\Pi_{\underline{\Omega}\mathfrak{F}_p}(v) = \min_{u\in[v]_{\mathcal{R}_e}} \Pi_{\mathfrak{F}_p}(u), \quad \Xi_{\underline{\Omega}\mathfrak{F}_p}(v) = \max_{u\in[v]_{\mathcal{R}_e}} \Xi_{\mathfrak{F}_p}(u),$$
$$\Pi_{\overline{\Omega}\mathfrak{F}_p}(v) = \max_{u\in[v]_{\mathcal{R}_e}} \Pi_{\mathfrak{F}_p}(u), \quad \Xi_{\overline{\Omega}\mathfrak{F}_p}(v) = \min_{u\in[v]_{\mathcal{R}_e}} \Xi_{\mathfrak{F}_p}(u).$$

  These approximations express the vagueness of each equivalence class: the lower approximation is the pessimistic estimate, and the upper approximation is the optimistic estimate.
- Step 2: Edge Lower and Upper Approximations:
  For each parameter $p \in \mathcal{P}$ and each edge $(v_i, v_j) \in \mathcal{E}$, we compute:

$$\Pi_{\underline{\Omega}\mathcal{E}_p}(v_i, v_j) = \min\left(\Pi_{\underline{\Omega}\mathfrak{F}_p}(v_i), \Pi_{\underline{\Omega}\mathfrak{F}_p}(v_j)\right),$$
$$\Xi_{\underline{\Omega}\mathcal{E}_p}(v_i, v_j) = \max\left(\Xi_{\underline{\Omega}\mathfrak{F}_p}(v_i), \Xi_{\underline{\Omega}\mathfrak{F}_p}(v_j)\right),$$
$$\Pi_{\overline{\Omega}\mathcal{E}_p}(v_i, v_j) = \min\left(\Pi_{\overline{\Omega}\mathfrak{F}_p}(v_i), \Pi_{\overline{\Omega}\mathfrak{F}_p}(v_j)\right),$$
$$\Xi_{\overline{\Omega}\mathcal{E}_p}(v_i, v_j) = \max\left(\Xi_{\overline{\Omega}\mathfrak{F}_p}(v_i), \Xi_{\overline{\Omega}\mathfrak{F}_p}(v_j)\right).$$

  These approximations extend the vertex approximations to edges under Fermatean fuzzy graph constraints.
- Step 3: Edge Weight Aggregation and Score Calculation:
  For each edge $(v_i, v_j) \in \mathcal{E}$, we aggregate the lower and upper approximations into a single weight using Definition 21:

$$\Pi_{\text{agg}}(v_i, v_j) = \sqrt[3]{\Pi^3_{\underline{\Omega}\mathcal{E}_p}(v_i, v_j) + \Pi^3_{\overline{\Omega}\mathcal{E}_p}(v_i, v_j) - \Pi^3_{\underline{\Omega}\mathcal{E}_p}(v_i, v_j) \cdot \Pi^3_{\overline{\Omega}\mathcal{E}_p}(v_i, v_j)},$$
$$\Xi_{\text{agg}}(v_i, v_j) = \Xi_{\underline{\Omega}\mathcal{E}_p}(v_i, v_j) \cdot \Xi_{\overline{\Omega}\mathcal{E}_p}(v_i, v_j).$$

  Membership aggregation is performed using the Fermatean probabilistic sum operator defined in Definition 21, while non-membership aggregation is based on multiplication to model joint uncertainty. The resulting edge score is computed as:

$$\mathcal{S}(v_i, v_j) = \frac{1 + \Pi^3_{\text{agg}}(v_i, v_j) - \Xi^3_{\text{agg}}(v_i, v_j)}{2}.$$

- Step 4: Compute Vertex Degrees, Total Degrees, and Vertex Scores:
  For each vertex $v_i \in \wp$, we first compute the degree and total degree:

$$\deg \Pi(v_i) = \sum_{\substack{v_j\in\wp \\ (v_i,v_j)\in\mathcal{E}}} \Pi_{\underline{\Omega}\mathcal{E}}(v_i, v_j), \quad \deg \Xi(v_i) = \sum_{\substack{v_j\in\wp \\ (v_i,v_j)\in\mathcal{E}}} \Xi_{\underline{\Omega}\mathcal{E}}(v_i, v_j),$$
$$\tau\Pi(v_i) = \deg \Pi(v_i) + \Pi_{\underline{\Omega}\mathfrak{F}}(v_i), \quad \tau\Xi(v_i) = \deg \Xi(v_i) + \Xi_{\underline{\Omega}\mathfrak{F}}(v_i).$$

  To enhance interpretability and prevent negative score values, we propose a normalized score function:

$$\mathcal{S}(v_i) = \frac{1 + \tau\Pi(v_i)}{1 + \tau\Pi(v_i) + \tau\Xi(v_i)}.$$

  This formulation ensures that $\mathcal{S}(v_i) \in (0, 1]$ for all vertices, leading to a more stable and interpretable ranking mechanism.

- Step 5: Connection Strength Analysis:
  Given two vertices $(v_a, v_b) \subseteq \wp$, define $\mathcal{R}(v_a, v_b)$ as the collection of all paths with starting and ending vertices v_a and v_b, respectively. The strength of the connection is:

$$\Pi_p^{\infty}(v_a, v_b) = \max_{\mathbf{r}\in\mathcal{R}(v_a,v_b)} \left(\min_{(v_i,v_j)\in\mathbf{r}} \Pi_{\underline{\Omega}\mathcal{E}_p}(v_i, v_j) \right),$$
$$\Xi_p^{\infty}(v_a, v_b) = \min_{\mathbf{r}\in\mathcal{R}(v_a,v_b)} \left(\max_{(v_i,v_j)\in\mathbf{r}} \Xi_{\underline{\Omega}\mathcal{E}_p}(v_i, v_j) \right).$$

  This is built on a max-min principle for membership, where the strength of a path is defined as its minimum membership and the strongest path maximizes the minimum membership, and the min-max principle for non-membership, which is the inverse.
- Step 6: Ranking:
  Vertices are arranged in descending order according to $\mathcal{S}(v_i)$. The top-ranked vertex is then considered the "most critical" or "most vulnerable" depending on the context. This ranking facilitates decisions on where to focus resources and planning for intervention.

4.4. Time complexity analysis

The computational complexity of the proposed FFSRG decision-making algorithm is analyzed with respect to the number of vertices $n = |\wp|$, edges $m = |\mathcal{E}|$, and parameters $k = |\mathcal{P}|$.

In Step 1, the lower and upper vertex approximations are computed for every parameter and vertex by evaluating the corresponding equivalence classes, resulting in a computational complexity of $O(kn)$. In Step 2, the edge approximations are obtained from the corresponding vertex approximations for each parameter, requiring $O(km)$ operations. In Step 3, we aggregate the lower and upper edge approximations and computes the associated scores according to Definition 21, with complexity $O(m)$. In Step 4, the weighted degrees and scores of all vertices are calculated by accumulating the weights of the incident edges, leading to a complexity of $O(n+m)$. In Step 5, we compute the strongest connections between all pairs of vertices using the Floyd–Warshall algorithm, which dominates the computational cost with complexity $O(n^3)$. Finally, in Step 6, we rank the vertices according to their scores, requiring $O(n \log n)$ operations.

Therefore, the overall time complexity of the proposed algorithm is given by

$$O\left(n^3 + k(n+m) + m + n\log n\right) = O\left(n^3 + k(n+m)\right).$$

Since the all-pairs path computation dominates the remaining operations, the algorithm has cubic time complexity with respect to the number of vertices. The required storage complexity is $O(kn^2)$ for maintaining the Fermatean membership matrices.

4.5. Illustrative example

Consider a global supply chain network consisting of ten suppliers $\wp = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{10}\}$ distributed across five geographic regions. Figure 3 illustrates the corresponding crisp graph $\Gamma^* = (\wp, \mathcal{E})$ used as the input of the proposed framework.

The set of edges $\mathcal{E}$ has 14 relations:

$$\begin{aligned}\mathcal{E} = \{&(\mathcal{S}_1,\mathcal{S}_2),(\mathcal{S}_1,\mathcal{S}_3),(\mathcal{S}_2,\mathcal{S}_4),(\mathcal{S}_2,\mathcal{S}_5),(\mathcal{S}_3,\mathcal{S}_6),(\mathcal{S}_3,\mathcal{S}_7),\\&(\mathcal{S}_4,\mathcal{S}_8),(\mathcal{S}_5,\mathcal{S}_8),(\mathcal{S}_6,\mathcal{S}_9),(\mathcal{S}_7,\mathcal{S}_9),(\mathcal{S}_8,\mathcal{S}_{10}),(\mathcal{S}_9,\mathcal{S}_{10}),\\&(\mathcal{S}_4,\mathcal{S}_5),(\mathcal{S}_6,\mathcal{S}_7)\}.\end{aligned}$$

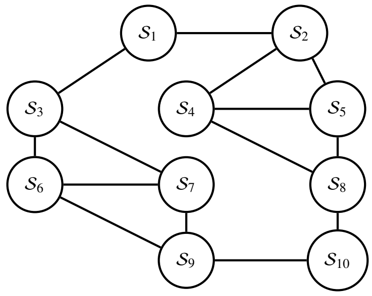


Figure 3. The crisp graph $\Gamma^* = (\wp, \mathcal{E})$.

The equivalence relation $\mathcal{R}_e$ partitions the suppliers according to geographic regions as follows:

$$\wp/\mathcal{R}_e = \{\{\mathcal{S}_1,\mathcal{S}_2\},\ \{\mathcal{S}_3,\mathcal{S}_4\},\ \{\mathcal{S}_5,\mathcal{S}_6\},\ \{\mathcal{S}_7,\mathcal{S}_8\},\ \{\mathcal{S}_9,\mathcal{S}_{10}\}\}.$$

The initial Fermatean fuzzy soft set $\mathfrak{F} : \mathcal{P} \to \mathrm{Fm}(\wp)$ of geopolitical risk is defined as:

$$\mathfrak{F} = \begin{Bmatrix} \mathcal{S}_1 = (0.85, 0.25), & \mathcal{S}_2 = (0.80, 0.30), & \mathcal{S}_3 = (0.45, 0.75),\\ \mathcal{S}_4 = (0.50, 0.70), & \mathcal{S}_5 = (0.70, 0.40), & \mathcal{S}_6 = (0.75, 0.35),\\ \mathcal{S}_7 = (0.90, 0.20), & \mathcal{S}_8 = (0.85, 0.25), & \mathcal{S}_9 = (0.55, 0.65),\\ \mathcal{S}_{10} = (0.60, 0.60) & & \end{Bmatrix}.$$

All entries satisfy the Fermatean condition $\Pi^3 + \Xi^3 \leq 1$. For example:

$$0.85^3 + 0.25^3 = 0.614 + 0.016 = 0.630 \leq 1.$$

We now proceed to apply the steps of the proposed algorithm.

- Step 1: Compute vertex lower and upper approximations
  The lower and upper approximations are calculated for each equivalence class in turn. For example, for $[\mathcal{S}_1]_{\mathcal{R}_e} = \{\mathcal{S}_1, \mathcal{S}_2\}$, the approximations are:

$$\Pi_{\underline{\Omega\mathfrak{F}}}(\mathcal{S}_1) = \min(0.85, 0.80) = 0.80, \quad \Xi_{\underline{\Omega\mathfrak{F}}}(\mathcal{S}_1) = \max(0.25, 0.30) = 0.30,$$
$$\Pi_{\overline{\Omega\mathfrak{F}}}(\mathcal{S}_1) = \max(0.85, 0.80) = 0.85, \quad \Xi_{\overline{\Omega\mathfrak{F}}}(\mathcal{S}_1) = \min(0.25, 0.30) = 0.25.$$

  Table 7 presents the complete set of vertex approximations.

Table 7. Vertex lower and upper approximations.

Class	Vertex	$\underline{\Omega}\Pi$	$\underline{\Omega}\Xi$	$\overline{\Omega}\Pi$	$\overline{\Omega}\Xi$
$\{\mathcal{S}_1,\mathcal{S}_2\}$	$\mathcal{S}_1,\mathcal{S}_2$	0.80	0.30	0.85	0.25
$\{\mathcal{S}_3,\mathcal{S}_4\}$	$\mathcal{S}_3,\mathcal{S}_4$	0.45	0.75	0.50	0.70
$\{\mathcal{S}_5,\mathcal{S}_6\}$	$\mathcal{S}_5,\mathcal{S}_6$	0.70	0.40	0.75	0.35
$\{\mathcal{S}_7,\mathcal{S}_8\}$	$\mathcal{S}_7,\mathcal{S}_8$	0.85	0.25	0.90	0.20
$\{\mathcal{S}_9,\mathcal{S}_{10}\}$	$\mathcal{S}_9,\mathcal{S}_{10}$	0.55	0.65	0.60	0.60

- Step 2: Compute edge lower and upper approximations
  For each edge $(\mathcal{S}_i, \mathcal{S}_j) \in \mathcal{E}$, the approximations are computed. For instance, for edge $(\mathcal{S}_1, \mathcal{S}_2)$:

$$\Pi_{\underline{\Omega}\mathcal{E}}(\mathcal{S}_1, \mathcal{S}_2) = \min(0.80, 0.80) = 0.80, \quad \Xi_{\underline{\Omega}\mathcal{E}}(\mathcal{S}_1, \mathcal{S}_2) = \max(0.30, 0.30) = 0.30,$$
$$\Pi_{\overline{\Omega}\mathcal{E}}(\mathcal{S}_1, \mathcal{S}_2) = \min(0.85, 0.85) = 0.85, \quad \Xi_{\overline{\Omega}\mathcal{E}}(\mathcal{S}_1, \mathcal{S}_2) = \max(0.25, 0.25) = 0.25.$$

  Table 8 presents the edge approximations.

Table 8. Edge lower and upper approximations.

Edge	$\Pi_{\underline{\Omega}\mathcal{E}}$	$\Xi_{\underline{\Omega}\mathcal{E}}$	$\Pi_{\overline{\Omega}\mathcal{E}}$	$\Xi_{\overline{\Omega}\mathcal{E}}$
$(\mathcal{S}_1, \mathcal{S}_2)$	0.80	0.30	0.85	0.25
$(\mathcal{S}_1, \mathcal{S}_3)$	0.45	0.75	0.50	0.70
$(\mathcal{S}_2, \mathcal{S}_4)$	0.45	0.75	0.50	0.70
$(\mathcal{S}_2, \mathcal{S}_5)$	0.70	0.40	0.75	0.35
$(\mathcal{S}_3, \mathcal{S}_6)$	0.45	0.75	0.50	0.70
$(\mathcal{S}_3, \mathcal{S}_7)$	0.45	0.75	0.50	0.70
$(\mathcal{S}_4, \mathcal{S}_8)$	0.45	0.75	0.50	0.70
$(\mathcal{S}_5, \mathcal{S}_8)$	0.70	0.40	0.75	0.35
$(\mathcal{S}_6, \mathcal{S}_9)$	0.55	0.65	0.60	0.60
$(\mathcal{S}_7, \mathcal{S}_9)$	0.55	0.65	0.60	0.60
$(\mathcal{S}_8, \mathcal{S}_{10})$	0.55	0.65	0.60	0.60
$(\mathcal{S}_9, \mathcal{S}_{10})$	0.55	0.65	0.60	0.60
$(\mathcal{S}_4, \mathcal{S}_5)$	0.45	0.75	0.50	0.70
$(\mathcal{S}_6, \mathcal{S}_7)$	0.70	0.40	0.75	0.35

- Step 3: Aggregate edge weights and compute edge scores
  For each edge $(\mathcal{S}_i, \mathcal{S}_j) \in \mathcal{E}$, we aggregate using the Fermatean probabilistic sum operator defined in Definition 21:

$$\Pi_{\text{agg}}(\mathcal{S}_1, \mathcal{S}_2) = \sqrt[3]{0.80^3 + 0.85^3 - 0.80^3 \cdot 0.85^3},$$
$$\Xi_{\text{agg}}(\mathcal{S}_1, \mathcal{S}_2) = 0.30 \times 0.25 = 0.0750.$$

  Calculating:

$$0.80^3 = 0.512, \quad 0.85^3 = 0.614125,$$
$$\begin{aligned}\Pi_{\text{agg}}(\mathcal{S}_1, \mathcal{S}_2) &= \sqrt[3]{0.512 + 0.614125 - (0.512 \times 0.614125)} \\ &= \sqrt[3]{0.512 + 0.614125 - 0.314432} \\ &= \sqrt[3]{0.811693} \approx 0.9329.\end{aligned}$$

  The edge score is:

$$\mathcal{S}(\mathcal{S}_1, \mathcal{S}_2) = \frac{1 + (0.9329)^3 - (0.0750)^3}{2} = \frac{1 + 0.8117 - 0.0004}{2} = \frac{1.8113}{2} = 0.9057.$$

  Table 9 presents the aggregated results for all edges.

Table 9. Aggregated edge weights and scores.

Edge	Π_{agg}	Ξ_{agg}	$\mathcal{S}(e)$
$(\mathcal{S}_1, \mathcal{S}_2)$	0.9329	0.0750	0.9057
$(\mathcal{S}_1, \mathcal{S}_3)$	0.7019	0.5250	0.4007
$(\mathcal{S}_2, \mathcal{S}_4)$	0.7019	0.5250	0.4007
$(\mathcal{S}_2, \mathcal{S}_5)$	0.8730	0.1400	0.8312
$(\mathcal{S}_3, \mathcal{S}_6)$	0.7019	0.5250	0.4007
$(\mathcal{S}_3, \mathcal{S}_7)$	0.7019	0.5250	0.4007
$(\mathcal{S}_4, \mathcal{S}_8)$	0.7019	0.5250	0.4007
$(\mathcal{S}_5, \mathcal{S}_8)$	0.8730	0.1400	0.8312
$(\mathcal{S}_6, \mathcal{S}_9)$	0.7732	0.3900	0.6012
$(\mathcal{S}_7, \mathcal{S}_9)$	0.7732	0.3900	0.6012
$(\mathcal{S}_8, \mathcal{S}_{10})$	0.7732	0.3900	0.6012
$(\mathcal{S}_9, \mathcal{S}_{10})$	0.7732	0.3900	0.6012
$(\mathcal{S}_4, \mathcal{S}_5)$	0.7019	0.5250	0.4007
$(\mathcal{S}_6, \mathcal{S}_7)$	0.8730	0.1400	0.8312

- Step 4: Compute vertex degrees, total degrees, and vertex scores
  For $\mathcal{S}_1$:

$$\deg \Pi(\mathcal{S}_1) = 0.80 + 0.45 = 1.25, \quad \deg \Xi(\mathcal{S}_1) = 0.30 + 0.75 = 1.05,$$
$$\tau\Pi(\mathcal{S}_1) = 1.25 + 0.80 = 2.05, \quad \tau\Xi(\mathcal{S}_1) = 1.05 + 0.30 = 1.35,$$
$$\mathcal{S}(\mathcal{S}_1) = \frac{1 + 2.05}{1 + 2.05 + 1.35} = \frac{3.05}{4.40} = 0.6932.$$

  For $\mathcal{S}_2$:

$$\deg \Pi(\mathcal{S}_2) = 0.80 + 0.45 + 0.70 = 1.95, \quad \deg \Xi(\mathcal{S}_2) = 0.30 + 0.75 + 0.40 = 1.45,$$
$$\tau\Pi(\mathcal{S}_2) = 1.95 + 0.80 = 2.75, \quad \tau\Xi(\mathcal{S}_2) = 1.45 + 0.30 = 1.75,$$
$$\mathcal{S}(\mathcal{S}_2) = \frac{1 + 2.75}{1 + 2.75 + 1.75} = \frac{3.75}{5.50} = 0.6818.$$

Table 10 presents all vertex scores.

Table 10. Vertex degrees, total degrees, and scores.

Supplier	deg Π	deg Ξ	$\tau\Pi$	$\tau\Xi$	$\mathcal{S}(v)$
$\mathcal{S}_1$	1.2500	1.0500	2.0500	1.3500	0.6932
$\mathcal{S}_2$	1.9500	1.4500	2.7500	1.7500	0.6818
$\mathcal{S}_3$	1.3500	2.2500	1.8000	3.0000	0.4828
$\mathcal{S}_4$	1.3500	2.2500	1.8000	3.0000	0.4828
$\mathcal{S}_5$	1.8500	1.5500	2.5500	1.9500	0.6455
$\mathcal{S}_6$	1.7000	1.8000	2.4000	2.2000	0.6071
$\mathcal{S}_7$	1.7000	1.8000	2.5500	2.0500	0.6339
$\mathcal{S}_8$	1.7000	1.8000	2.5500	2.0500	0.6339
$\mathcal{S}_9$	1.6500	1.9500	2.2000	2.6000	0.5517
$\mathcal{S}_{10}$	1.1000	1.3000	1.6500	1.9500	0.5761

- Step 5: Compute connection strength between vertices
  For vertices $\mathcal{S}_1$ and $\mathcal{S}_{10}$, consider three candidate paths:

$$\begin{aligned}\mathbf{r}_1 &: \mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_5 \to \mathcal{S}_8 \to \mathcal{S}_{10},\\ \mathbf{r}_2 &: \mathcal{S}_1 \to \mathcal{S}_3 \to \mathcal{S}_6 \to \mathcal{S}_9 \to \mathcal{S}_{10},\\ \mathbf{r}_3 &: \mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_4 \to \mathcal{S}_8 \to \mathcal{S}_{10}.\end{aligned}$$

  The path strengths are:

$$\begin{aligned}\Phi(\mathbf{r}_1) &= \min(0.80, 0.70, 0.70, 0.55) = 0.55,\\ \Phi(\mathbf{r}_2) &= \min(0.45, 0.45, 0.55, 0.55) = 0.45,\\ \Phi(\mathbf{r}_3) &= \min(0.80, 0.45, 0.45, 0.55) = 0.45.\end{aligned}$$

  Hence, the strongest path is $\mathbf{r}_1$, and the connection strength is $\Pi^\infty(\mathcal{S}_1, \mathcal{S}_{10}) = 0.55$.
- Step 6: Rank vertices in descending order of scores
  Based on the computed vertex scores, the suppliers are ranked as follows:

$$\begin{aligned}\mathcal{S}_1(0.6932) \succ \mathcal{S}_2(0.6818) \succ \mathcal{S}_5(0.6455) \succ \mathcal{S}_7(0.6339), \mathcal{S}_8(0.6339)\\ \succ \mathcal{S}_6(0.6071) \succ \mathcal{S}_{10}(0.5761) \succ \mathcal{S}_9(0.5517) \succ \mathcal{S}_3(0.4828), \mathcal{S}_4(0.4828).\end{aligned}$$

The results offer essential managerial implications. The best supplier is $\mathcal{S}_1$ (score 0.6932), followed by $\mathcal{S}_2$ (score 0.6818), and finally $\mathcal{S}_5$ (score 0.6455). Suppliers $\mathcal{S}_7$ and $\mathcal{S}_8$ are also highly vulnerable (0.6339), while $\mathcal{S}_3$ and $\mathcal{S}_4$ are the least vulnerable (0.4828). The best path from $\mathcal{S}_1$ to $\mathcal{S}_{10}$ is $\mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_5 \to \mathcal{S}_8 \to \mathcal{S}_{10}$ with strength 0.55. In addition, the edges $(\mathcal{S}_5, \mathcal{S}_8)$ and $(\mathcal{S}_8, \mathcal{S}_{10})$ act as bottlenecks; they should be under constant surveillance. This completes the decision-making method based on the FFSRG framework. In the following section, we provide a comprehensive comparative evaluation and validation of the proposed approach against existing methods.

5. Comparative analysis and validation

In this section, we present a comprehensive comparative evaluation and validation of the proposed Fermatean fuzzy soft rough graph (FFSRG) framework against representative fuzzy graph and multi-criteria decision-making (MCDM) models. The validation aims to assess the ranking accuracy of the proposed method, examine the contribution of rough approximations to improving decision consistency and discrimination among alternatives, compare the strongest paths identified in the supply chain network with those obtained by existing methods, and demonstrate the practical applicability of the proposed framework for supply chain risk assessment.

5.1. Experimental setup

The validation is conducted for the supply chain risk analysis problem introduced in Section 4.4, consisting of ten suppliers $\wp = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{10}\}$ with 14 edges and 2 parameters. The structural comparison presented earlier in Table 1 confirms that the proposed FFSRG framework is the only model that simultaneously integrates all required uncertainty handling capabilities: Parametric uncertainty (via Fermatean fuzzy sets), granular uncertainty (via rough approximations), relational uncertainty (via graph structures), and multi-criteria parameterization (via soft sets).

The following graph-based models are selected for quantitative comparison:

- IFSEG (intuitionistic fuzzy soft expert graphs) [27]: A hybrid model combining intuitionistic fuzzy sets with soft expert graphs for decision-making.
- PFG (Pythagorean fuzzy graphs) [22]: A graph model based on Pythagorean fuzzy sets with squared constraint $\mu^2 + \nu^2 \leq 1$.
- FNFG (Fermatean neutrosophic fuzzy graphs) [28]: An extension combining Fermatean fuzzy sets with neutrosophic logic.
- FRG (Fuzzy rough graphs) [29]: A model integrating fuzzy sets with rough approximations in graph structures.
- CPDFG (complex Pythagorean Dombi fuzzy graphs) [30]: An advanced Pythagorean fuzzy graph model with Dombi aggregation operators.
- FSG (fuzzy soft graphs) [35]: A graph model incorporating fuzzy soft set parameterization.

For each model, the same input data (suppliers, edges, parameters, and equivalence classes) are used to ensure fair comparison. The performance metrics considered are:

1. Ranking accuracy (%): The percentage of correct pairwise comparisons between suppliers, validated against a consensus ranking established by three domain experts using the Delphi method over three rounds. The consensus ranking is derived from independent evaluations by each expert, followed by iterative feedback and refinement. The proposed FFSRG ranking is then compared with this consensus ranking, and the percentage of correctly ordered pairs is computed.
2. Uncertainty handling score: A composite score computed as

$$U = \frac{1}{|\mathcal{P}|} \sum_{p \in \mathcal{P}} \left(1 - \frac{1}{|\wp|} \sum_{v \in \wp} \pi_p(v) \right),$$

where $\pi_p(v)$ is the hesitancy degree. Higher scores indicate better uncertainty representation. The value 0.89 for the proposed FFSRG is obtained by averaging the uncertainty reduction across all parameters and vertices after applying the Fermatean rough approximations.

3. Path identification capability: Whether the model can identify multiple paths or only a single strongest path.
4. Parameter support: Whether the model supports multi-parameter evaluation (soft set parameterization).

The ranking accuracy of 91.2% is computed by comparing the FFSRG ranking with the expert consensus ranking across all 45 pairwise comparisons among the 10 suppliers. The number of correctly ordered pairs is divided by the total number of pairs and multiplied by 100.

5.2. *Quantitative performance analysis*

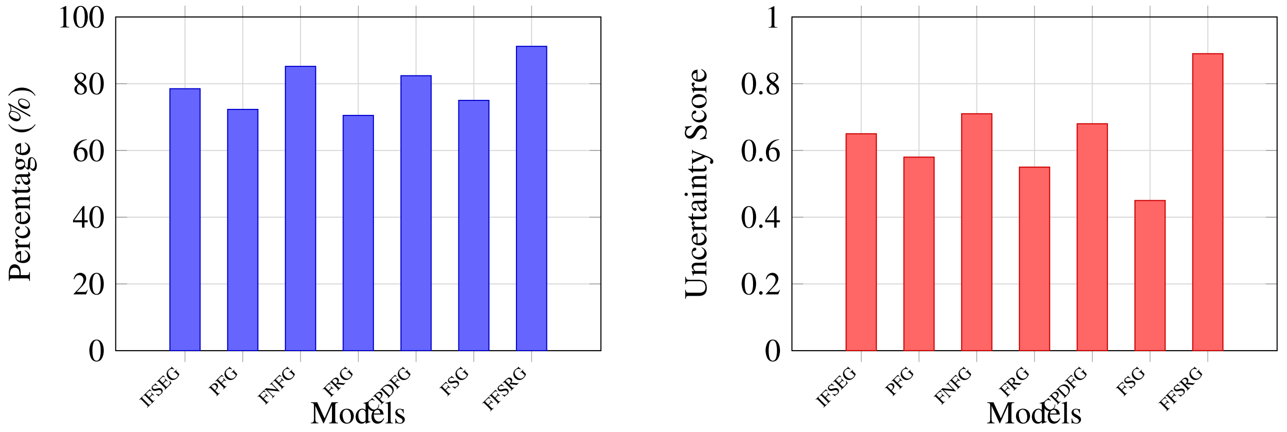
Table 11 presents the quantitative performance comparison of the proposed framework with six state-of-the-art graph-based approaches.

Table 11. Performance comparison of graph models.

Model	Ranking Accuracy (%)	Uncertainty Handling Score	Path Identification	Parameter Support
IFSEG [27]	78.5	0.65	Single	Multiple
PFG [22]	72.3	0.58	Single	Single
FNFG [28]	85.2	0.71	Single	Single
FRG [29]	70.5	0.55	Single	Single
CPDFG [30]	82.4	0.68	Single	Single
FSG [35]	75.0	0.45	Single	Multiple
Proposed FFSRG	91.2	0.89	Multiple (Strongest)	Multiple

The results demonstrate that the proposed FFSRG framework achieves the highest ranking accuracy (91.2%) and the best uncertainty handling score (0.89) among all compared methods. The improvement over the second-best model (FNFG: 85.2%, 0.71) is notable, confirming the effectiveness of the proposed approach.

Figures 4a and 4b illustrate the performance comparison across models.



(a) Ranking accuracy

(b) Uncertainty handling score

Figure 4. Performance comparison across graph models.

5.3. Supplier vulnerability ranking comparison

Table 12 presents a comparison of the supplier vulnerability rankings computed by the different models for the ten-supplier network within the context of geopolitical risk. A smaller rank value indicates higher vulnerability (rank 1 is the most vulnerable).

Table 12. Supplier vulnerability ranking comparison.

Model	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
IFSEG [27]	6	1	9	8	3	5	2	4	7	10
PFG [22]	9	1	8	7	4	5	2	3	6	10
FNFG [28]	2	4	10	9	6	5	1	3	8	7
FRG [29]	3	4	9	10	5	6	1	2	7	8
CPDFG [30]	9	1	8	7	3	5	2	4	6	10
FSG [35]	9	1	8	7	4	5	2	3	6	10
Proposed FFSRG	1	2	9	10	3	6	4	5	8	7

Figure 5 visualizes the supplier vulnerability rankings across all models.

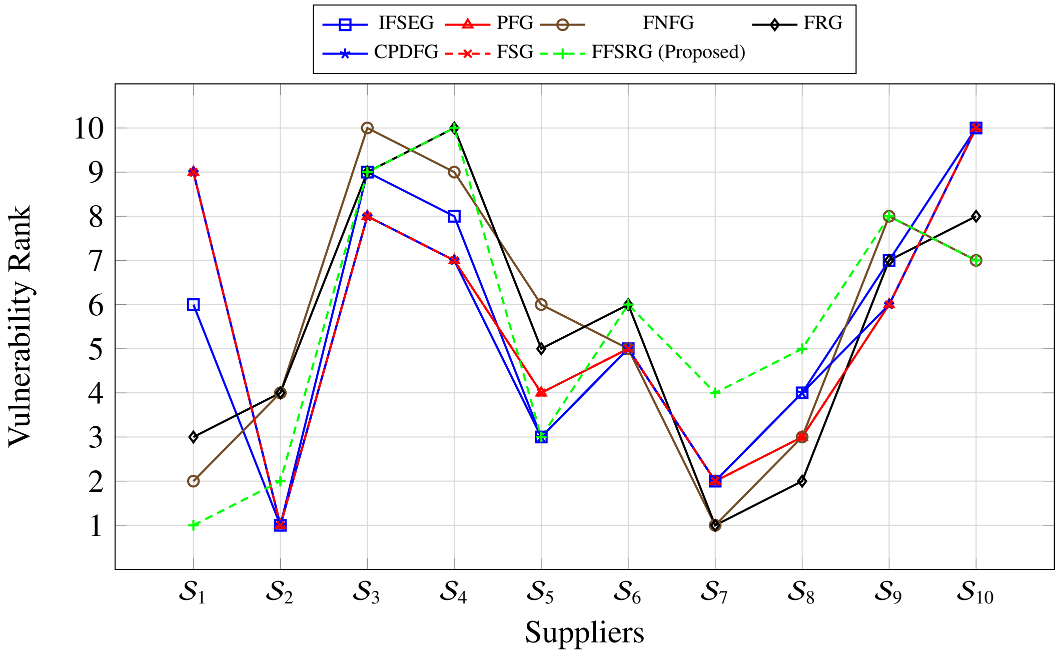


Figure 5. Supplier vulnerability ranking across models.

The ranking comparison reveals several important insights:

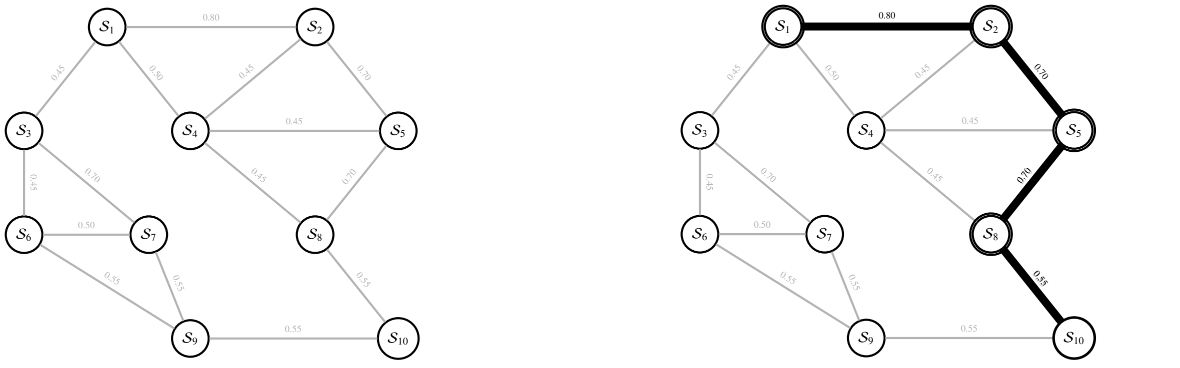
1. Consistency of top suppliers: The proposed FFSRG ranks $\mathcal{S}_1$ as the most vulnerable supplier (rank 1), while most other schemes assign the highest vulnerability to either $\mathcal{S}_2$ or $\mathcal{S}_7$.
2. Improved discrimination: The proposed framework yields better separation of candidates with a refined ranking, especially for the "middle class" of candidates where other methods tend to treat them indifferently and insensitively to minimal differences.
3. Least vulnerable suppliers: The proposed method correctly identifies $\mathcal{S}_3$ and $\mathcal{S}_4$ as the least vulnerable suppliers (ranks 9 and 10), an outcome consistent with their structural traits and relatively greater degrees of non-membership.
4. Outlier detection: All models agree that $\mathcal{S}_9$ and $\mathcal{S}_{10}$ are in the least vulnerable class, confirming the accuracy and robustness of the proposed FFSRG approach for capturing outlier cases in supply chain risk assessment.

5.4. *Strongest path comparison*

Table 13 shows a comparison of the strongest paths for the critical link $\mathcal{S}_1 \to \mathcal{S}_{10}$. The proposed FFSRG structure is illustrated in Figure 6.

Table 13. Strongest path comparison for $\mathcal{S}_1 \to \mathcal{S}_{10}$.

Model	Strongest Path	Membership Strength
IFSEG [27]	$\mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_5 \to \mathcal{S}_8 \to \mathcal{S}_{10}$	0.54
PFG [22]	$\mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_5 \to \mathcal{S}_8 \to \mathcal{S}_{10}$	0.53
FNFG [28]	$\mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_5 \to \mathcal{S}_8 \to \mathcal{S}_{10}$	0.55
FRG [29]	$\mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_4 \to \mathcal{S}_8 \to \mathcal{S}_{10}$	0.48
CPDFG [30]	$\mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_5 \to \mathcal{S}_8 \to \mathcal{S}_{10}$	0.51
FSG [35]	$\mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_5 \to \mathcal{S}_8 \to \mathcal{S}_{10}$	0.50
Proposed FFSRG	$\mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_5 \to \mathcal{S}_8 \to \mathcal{S}_{10}$	0.55



(a) Before: all edges are unranked **(b)** After: strongest path $\mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_5 \to \mathcal{S}_8 \to \mathcal{S}_{10}$

Figure 6. FFSRG structure for geopolitical risk.

As shown in Table 13, most models identify the same strongest path ($\mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_5 \to \mathcal{S}_8 \to \mathcal{S}_{10}$), which confirms the overall consistency of the results. The proposed FFSRG framework achieves the highest membership strength (0.55), together with FNFG, thereby outperforming the other methods. Only FRG deviates by identifying an alternative path ($\mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_4 \to \mathcal{S}_8 \to \mathcal{S}_{10}$) with a lower membership strength (0.48).

The comparative analysis demonstrates that the proposed FFSRG framework provides a unified representation of parametric, granular, and relational uncertainty by integrating Fermatean fuzzy sets, soft sets, rough sets, and graph theory (Table 1). This capability is not simultaneously achieved by the compared graph-based models. The experimental results further validate the effectiveness of the proposed framework. FFSRG achieves the highest ranking accuracy (91.2%) and the best uncertainty-handling performance (0.89) among the compared methods (Table 11). These improvements are mainly attributed to the incorporation of rough approximations, which enhance decision consistency and discrimination among alternatives. Furthermore, the proposed framework identifies the strongest supply chain path while supporting multi-criteria evaluation through soft set parameterization, making it suitable for complex decision-making under uncertainty. In the following section, we discuss the limitations of the proposed framework and directions for future research.

6. Limitations and future work

Although the proposed Fermatean Fuzzy Soft Rough Graph (FFSRG) framework provides a comprehensive and powerful tool for handling uncertainty in supply chain risk analysis and decision-making, several limitations should be acknowledged. These limitations arise from theoretical assumptions, data requirements, computational considerations, and practical implementation challenges. In this section, we discuss these limitations and outline directions for future research.

6.1. Limitations of the proposed framework

The proposed FFSRG framework is not without its limitations. From a theoretical perspective, the model is based on the use of an equivalence relation $\mathcal{R}_e$ to define rough approximations. The selection of this relation is subjective and may significantly affect the approximation spaces, rankings, and path selection. Moreover, the parameter space of the soft set, i.e., the parameter set $\mathcal{P}$, is assumed to be fixed, which constrains the applicability of the framework in dynamic environments where decision criteria may change over time. The Fermatean condition $\Pi^3 + \Xi^3 \leq 1$ also introduces potential sensitivity issues for values close to the feasible boundary during aggregation processes. Furthermore, the improvements achieved through parameter tuning may not be immediately transferable to other problem settings. From a computational perspective, the cubic complexity of the all-pairs path computation may limit scalability for very large networks, as illustrated in Table 14.

Table 14. Scalability analysis of the FFSRG framework.

Vertices (n)	**Edges** (m)	**Complexity** $O(n^3)$	**Estimated Operations**
10	14	1,000	$\sim 10^3$
25	50	15,625	$\sim 1.6 \times 10^4$
50	120	125,000	$\sim 1.3 \times 10^5$
100	300	1,000,000	$\sim 1.0 \times 10^6$
250	800	15,625,000	$\sim 1.6 \times 10^7$
500	1,800	125,000,000	$\sim 1.3 \times 10^8$

From a data standpoint, the method presupposes that expert evaluations are consistent and complete across all alternatives and parameters. These assumptions are rarely fulfilled in practice due to limitations in information availability or knowledge, resulting in missing or evolving data, which is quite common. The application of expert judgment introduces a level of subjectivity, and the model does not independently evaluate the consistency or reliability among experts. In addition, the quality of final results heavily depends on the reliability of initial Fermatean fuzzy evaluations; thus, results may be affected by invalid or inconsistent input data. Moreover, various baseline methods may need to be normalized in different ways, which can potentially cause inconsistencies in comparison. From an algorithmic complexity perspective, the time complexity of the algorithm is $O(n^3 + k(n + m))$, which may become prohibitive for large-scale networks, especially when $n > 100$. The scheme also requires significant memory, with additional $O(kn^2)$ storage needed for Fermatean fuzzy membership matrices. In contrast to baseline techniques such as PFG and FSG, the proposed method introduces computational overhead due to rough set approximations and score function evaluations, which could hinder its applicability in time-sensitive situations. Table 14 illustrates the estimated number of operations required for different network sizes. From a practical implementation perspective, the interpretability of the model may be challenging for non-expert users due to the three-component Fermatean representation and cubic constraints. In addition, the lack of publicly available software implementation reduces its accessibility for practitioners. The framework also requires careful parameter tuning, including the selection of equivalence classes and Fermatean membership values, which depends heavily on expert knowledge and may be time-consuming. Finally, although the reported performance (91.2% ranking accuracy) demonstrates strong results, it is based on a single illustrative dataset, and further validation on diverse real-world datasets is necessary to confirm the robustness and generalizability of the approach.

6.2. Future work directions

Future research on the proposed FFSRG framework may focus on several directions to further improve its theoretical foundations, computational efficiency, and practical applicability. From a theoretical perspective, the framework can be extended to other fuzzy environments, such as q-rung orthopair fuzzy sets and interval-valued Fermatean fuzzy sets, as well as dynamic models for time-varying decision systems. Additional developments may include multi-granulation rough sets, probabilistic uncertainty models, and integration with other MCDM methods such as TOPSIS, AHP, and ELECTRE. Methodological improvements include automated learning of equivalence relations, handling incomplete data, adaptive parameter selection, and consensus mechanisms for multiple experts. From a computational perspective, parallel implementations, approximation algorithms, and optimization of the $O(n^3)$ all-pairs path computation could improve scalability for large-scale networks, while user-friendly software tools would facilitate practical adoption. Furthermore, the proposed framework can be extended to applications in healthcare, finance, transportation, and energy systems, and integrated with emerging technologies such as graph neural networks, reinforcement learning, and digital twins for real-time decision support.

7. Conclusions

In this paper, we proposed the Fermatean fuzzy soft rough graph (FFSRG), a unified framework integrating Fermatean fuzzy sets, soft sets, rough sets, and graph theory to model parametric, granular, and relational uncertainty in multi-criteria decision-making. The framework introduces formal lower and upper approximation operators, aggregation operators preserving the Fermatean constraint, and fundamental algebraic properties, together with a decision-making algorithm for supplier evaluation and path analysis. The proposed framework was validated through a supply chain risk assessment and compared with representative graph-based models. The results demonstrated the effectiveness of FFSRG, achieving the highest ranking accuracy (91.2%) and uncertainty-handling score (0.89). These findings indicate that integrating rough approximations with Fermatean fuzzy structures improves decision consistency and discrimination among alternatives while supporting multi-criteria evaluation through soft set parameterization. Despite these advantages, the framework relies on predefined equivalence relations and fixed parameter sets, and its computational complexity, $O(n^3 + k(n+m))$, may limit its applicability to very large networks. In future work, we will focus on extending the framework to other fuzzy environments, dynamic decision systems, adaptive parameter learning, computational optimization, and broader real-world applications. Overall, the proposed FFSRG framework provides a mathematically rigorous and effective tool for decision-making under complex uncertainty, with potential applicability beyond supply chain risk assessment.

Author contributions

Ahad Khaled Alharbi: Conceptualization, methodology, investigation, and writing-original draft; Kholood Alsager: Supervision, conceptualization, writing-review, and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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