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*Research article*

# Fixed-point theorems in complex-valued extended suprametric spaces with applications to fractional integro-differential equations

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**Abstract:** The aim of this work is to introduce the concept of complex-valued extended suprametric spaces and to establish new common fixed-point results for cyclic and interpolative contractive mappings. The obtained theorems significantly extend and unify several well-known fixed-point results in complex-valued suprametric spaces, complex-valued metric spaces, and classical suprametric spaces as particular cases. To illustrate the effectiveness of the developed theory, nontrivial examples are provided. As an application of the main results, we study the existence and uniqueness of solutions to a nonlinear fractional integro-differential initial value problem involving the Atangana-Baleanu-Caputo derivative of order $0 < \alpha < 1$ by employing fixed-point methods within the proposed complex-valued extended suprametric framework.

**Keywords:** common fixed points; cyclic contractions; self mappings; interpolative contractions; complex-valued extended suprametric spaces; nonlinear fractional integro-differential initial value problem
**Mathematics Subject Classification:** 34A08, 47H10, 54H25

## 1. Introduction

Fixed-point (FP) theory plays a central role in nonlinear analysis due to its wide-ranging applications in differential equations, integral equations, optimization theory, dynamical systems, and applied sciences. Among the many results in this area, contraction-type FP theorems in metric and normed spaces form the backbone of both theoretical developments and practical algorithms. Over the years, researchers have progressively weakened classical contraction conditions in order to enlarge the class of admissible mappings while preserving the existence, uniqueness, and convergence of FPs.

The Banach contraction principle (BCP) [1], established in the early twentieth century, represents the foundational result in this direction. It asserts that every strict contraction mapping on a complete metric space (CMS) admits a unique FP and that the Picard iterative sequence converges to this point. Beyond guaranteeing existence and uniqueness, the BCP provides an effective iterative scheme, making it indispensable in numerical analysis and the study of nonlinear functional equations. Despite its strength, the requirement of a uniform Lipschitz constant strictly less than one is often too restrictive for many nonlinear models arising in applications.

To overcome this limitation, several generalizations of the BCP have been proposed. One of the most influential among them is the Kannan FP theorem [2], which replaces the classical Lipschitz-type condition with an inequality involving distances between points and their images. Remarkably, Kannan contractions do not require continuity of the mapping, yet they still ensure the existence and uniqueness of an FP in a CMS. This result demonstrated that contractiveness can be formulated in terms of self-distance control rather than direct comparison of distinct points, thereby significantly expanding the scope of FP theory. Closely related to Kannan's work is the Chatterjea FP theorem [3], which introduces a symmetric contraction condition based on cross distances between points and images. Chatterjea's approach further illustrates that the contraction mechanism may arise from the interaction between points and images rather than from their mutual distance alone. Together, the Kannan and Chatterjea theorems form a fundamental class of nonlinear contractions that are independent of Banach's classical framework while preserving convergence and uniqueness properties.

In a different but complementary direction, Fisher's FP theorem [4] considers mappings that fall outside the classical Banach contraction framework by introducing rational-type contractive conditions. Instead of relying solely on linear Lipschitz inequalities, Fisher employed contractive expressions involving rational combinations of metric distances.

Another important extension of contraction theory arises from problems with inherent structural constraints. The Kirk FP theorem [5] for cyclic contractions considers mappings defined on a union of subsets that are visited cyclically rather than globally invariant. This framework reflects many real-world iterative processes where the domain is partitioned into phases or components. Kirk's results ensure the existence of FPs or periodic points under suitable cyclic contractive conditions, thereby extending classical FP theory to structured and multidomain settings. Recently, significant progress has been made through the introduction of interpolative contraction conditions, notably by Karapınar [6]. These theorems unify and generalize several known contraction principles by combining different distance expressions through adjustable parameters. These interpolative conditions allow a smooth transition between Banach, Kannan, Chatterjea, and related contractions, offering a flexible framework capable of handling highly nonlinear mappings.

In all the aforementioned classical FP theorems, the analysis is carried out within the framework of metric spaces (MSs), which play a fundamental role in guaranteeing the convergence of iterative sequences. The formal notion of an MS was introduced by Fréchet [7] in 1906, providing a precise mathematical structure for capturing the notion of distance between elements of a nonempty set. An MS is defined by a distance function satisfying essential properties such as positivity, symmetry, and the triangle inequality. This framework has since become a cornerstone of modern analysis.

Over time, however, it became evident that the classical metric structure is sometimes too restrictive for certain theoretical investigations and real-world applications. As a result, numerous generalizations have been developed to extend the applicability of metric-based techniques. One important extension is the idea of a $b$-MS, pioneered by Bakhtin [8], in which the triangle inequality is relaxed through embedding a constant coefficient. This modification allows the treatment of problems where the standard triangular property does not hold, while still preserving many essential features required for FP analysis. Dwivedi [9] developed FP theorems for cyclic contractive mappings within the framework of $b$-MS. Further expanding this line of research, Berzig [10] proposed the notion of supra metric spaces (SMSs), which impose even weaker constraints on the triangle inequality. This flexible structure provides a broader setting for studying FP problems and has proven particularly effective in the analysis of nonlinear integral equations and matrix equations. The development of SMSs thus illustrates the ongoing effort to refine and generalize distance structures in order to accommodate increasingly complex mathematical models. Later, Panda et al. [11] further generalized the framework of SMSs by substituting the fixed constant in the triangle inequality with a control function, thereby introducing greater flexibility into the underlying distance structure.

In a closely related line of research, Azam et al. [12] proposed the framework of complex-valued metric spaces (C-VMSs) as a meaningful extension of classical MSs. In this approach, the distance between two elements is taken to be a complex number, which enables the incorporation of additional structural information and provides a broader analytical setting. Rouzkard et al. [13] extended the contractive inequality given by Azam et al. [12] and obtained FP results in C-VMSs. Abbas et al. [14] established FP results for cocyclic mappings in the setting of C-VMSs. Moreover, Fulga [15] established FP theorems for interpolative contractions involving rational expressions, and Alharbi et al. [16] extended these results to C-VMSs. Building upon this idea, Panda et al. [17] combined the ideas of SMSs and C-VMSs to introduce complex-valued suprametric spaces (C-VSMSs). In this unified setting, they established several common fixed point (CFP) results under rational-type contractive conditions and showed the relevance of their theory through complex nonlinear integral equations. Subsequently, Abdou [18] employed the C-VSMS framework to derive further CFP results for more generalized contractive inequalities. Additional extensions and related investigations along these lines may be found in the literature.

In this research article, we define the notion of C-VESMSs and establish CFP results under novel cyclic and interpolative contractive conditions. To demonstrate the significance and novelty of our findings, some illustrative examples are presented. Additionally, we apply these results to discuss the existence and uniqueness of solutions to a nonlinear fractional integro-differential initial value problem involving the Atangana–Baleanu–Caputo derivative of order $0 < \alpha < 1$.

## 2. Preliminaries

In this section, we recall some fundamental definitions and auxiliary results that will be used throughout the paper. Let $\mathbb{N}, \mathbb{R}$, and $\mathbb{R}^+$ denote the set of positive integers, a set of real numbers, and a set of positive real numbers, respectively.

**Definition 1.** Let $\Omega \neq \emptyset$. A function $d : \Omega \times \Omega \to \mathbb{R}^+$ is called a metric if, for all $\varkappa, \varsigma, \nu \in \Omega$, the following conditions hold:

(i) $0 \leq d(\varkappa, \varsigma)$, and $d(\varkappa, \varsigma) = 0 \Longleftrightarrow \varkappa = \varsigma$;

(ii) $d(\varkappa, \varsigma) = d(\varsigma, \varkappa)$;
(iii) $d(\varkappa, \varsigma) \leq d(\varkappa, \nu) + d(\nu, \varsigma)$.

Then, the pair $(\Omega, d)$ is called an MS.

**Definition 2.** Let $(\Omega, d)$ be a CMS, and $\aleph : \Omega \to \Omega$.

- $\aleph$ is called a contraction if there exists a constant $\sigma \in [0, 1)$ such that

$$d(\aleph\varkappa, \aleph\varsigma) \leq \sigma d(\varkappa, \varsigma)$$

for all $\varkappa, \varsigma \in \Omega$.
- $\aleph$ is called a Kannan's contraction if there exists a constant $\sigma \in [0, \frac{1}{2})$ such that

$$d(\aleph\varkappa, \aleph\varsigma) \leq \sigma \left(d(\varkappa, \aleph\varkappa) + d(\varsigma, \aleph\varsigma)\right)$$

for all $\varkappa, \varsigma \in \Omega$.
- $\aleph$ is called a Chatterjea's contraction if there exists a constant $\sigma \in [0, \frac{1}{2})$ such that

$$d(\aleph\varkappa, \aleph\varsigma) \leq \sigma \left(d(\varkappa, \aleph\varsigma) + d(\varsigma, \aleph\varkappa)\right)$$

for all $\varkappa, \varsigma \in \Omega$.
- $\aleph$ is called a Fisher's contraction if there exist constants $\sigma_1, \sigma_2 \in [0, 1)$ such that $\sigma_1 + \sigma_2 < 1$, and

$$d(\aleph\varkappa, \aleph\varsigma) \leq \sigma_1 d(\varkappa, \varsigma) + \sigma_2 \frac{d(\varkappa, \aleph\varkappa) d(\varsigma, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)}$$

for all $\varkappa, \varsigma \in \Omega$.
- $\aleph$ is said to be an interpolative contraction if there exist constants $\sigma \in [0, 1)$ and $p \in (0, 1)$ such that

$$d(\aleph\varkappa, \aleph\varsigma) \leq \sigma \left[d(\varkappa, \aleph\varkappa)\right]^p \left[d(\varsigma, \aleph\varsigma)\right]^{1-p}$$

for all $\varkappa, \varsigma \in \Omega \backslash Fix\{\aleph\}$.
- $\aleph : A \cup B \to A \cup B$ is called a cyclic contraction if there exist two closed subsets $A, B$ of CMS $(\Omega, d)$ and a constant $\sigma \in [0, 1)$ such that

$$\aleph(A) \subset B \text{ and } \aleph(B) \subset A$$

and

$$d(\aleph\varkappa, \aleph\varsigma) \leq \sigma d(\varkappa, \varsigma)$$

for all $\varkappa \in A, \varsigma \in B$.

Berzig [10] proposed the SMS based on the following procedure.

**Definition 3.** [10] Let $\Omega \neq \emptyset$, and $\curlywedge \geq 0$. Let $d : \Omega \times \Omega \to \mathbb{R}^+ \cup \{0\}$ denote a function that meets the below-mentioned criteria:

(i) $0 \leq d(\varkappa, \varsigma)$, and $d(\varkappa, \varsigma) = 0 \Longleftrightarrow \varkappa = \varsigma$;
(ii) $d(\varkappa, \varsigma) = d(\varsigma, \varkappa)$;

(iii) $d(\varkappa,\varsigma) \leq d(\varkappa,\nu) + d(\nu,\varsigma) + \curlywedge d(\varkappa,\nu)d(\nu,\varsigma)$

for all $\varkappa,\varsigma,\nu \in \Omega$. Then, $(\Omega, d)$ is called an SMS.

Let us define a partial ordering $\precsim$ on $\mathbb{C}$ in this way:

$$z_1 \precsim z_2 \ \Leftrightarrow \ \operatorname{Re}(z_1) \leqslant \operatorname{Re}(z_2), \ \operatorname{Im}(z_1) \leqslant \operatorname{Im}(z_2).$$

$\forall z_1, z_2 \in \mathbb{C}$. Consequently,

$$z_1 \precsim z_2,$$

provided that at least one of the following conditions is satisfied:

$$\begin{aligned} \text{(a)} \operatorname{Re}(z_1) &= \operatorname{Re}(z_2), \ \operatorname{Im}(z_1) < \operatorname{Im}(z_2); \\ \text{(b)} \operatorname{Re}(z_1) &< \operatorname{Re}(z_2), \ \operatorname{Im}(z_1) = \operatorname{Im}(z_2); \\ \text{(c)} \operatorname{Re}(z_1) &< \operatorname{Re}(z_2), \ \operatorname{Im}(z_1) < \operatorname{Im}(z_2); \\ \text{(d)} \operatorname{Re}(z_1) &= \operatorname{Re}(z_2), \ \operatorname{Im}(z_1) = \operatorname{Im}(z_2). \end{aligned}$$

Azam et al. [12] defined the idea of C-VMS in this fashion.

**Definition 4.** [12] Let $\Omega \neq \emptyset$, and $d : \Omega \times \Omega \to \mathbb{C}$ satisfy

(i) $0 \precsim d(\varkappa,\varsigma)$ and $d(\varkappa,\varsigma) = 0 \Longleftrightarrow \varkappa = \varsigma$;
(ii) $d(\varkappa,\varsigma) = d(\varsigma,\varkappa)$;
(iii) $d(\varkappa,\varsigma) \precsim d(\varkappa,\nu) + d(\nu,\varsigma)$;

$\forall\ \varkappa,\varsigma,\nu \in \Omega$. Then, $(\Omega, d)$ is called a C-VMS.

Panda et al. [17] proposed the definition of a C-VSMS in this form.

**Definition 5.** [17] Let $\Omega \neq \emptyset$, $\curlywedge \geq 0$, and $d : \Omega \times \Omega \to \mathbb{C}$ satisfy

(i) $0 \precsim d(\varkappa,\varsigma)$ and $d(\varkappa,\varsigma) = 0 \Longleftrightarrow \varkappa = \varsigma$;
(ii) $d(\varkappa,\varsigma) = d(\varsigma,\varkappa)$;
(iii) $d(\varkappa,\varsigma) \precsim d(\varkappa,\nu) + d(\nu,\varsigma) + \curlywedge d(\varkappa,\nu)d(\nu,\varsigma)$

for all $\varkappa,\varsigma,\nu \in \Omega$. It follows that $(\Omega, d)$ constitutes a C-VSMS.

**Example 1.** Let $\Omega = \{0, 1, 2\}$, and let $d : \Omega \times \Omega \to \mathbb{C}$ be defined as

$$d(\varkappa,\varsigma) = \begin{cases} 0, \ \text{if } \varkappa = \varsigma; \\ 1 + i, \ \text{if } \varkappa \neq \varsigma \end{cases}$$

for all $\varkappa,\varsigma \in \Omega$. Then, $(\Omega, d)$ is a C-VSMS.

In this manner, we introduce the concept of a complex-valued extended suprametric space (C-VESMS).

**Definition 6.** Let $\Omega \neq \emptyset$, $\curlywedge : \Omega \times \Omega \to [0, +\infty)$, and $d : \Omega \times \Omega \to \mathbb{C}$ satisfy

(i) $0 \precsim d(\varkappa,\varsigma)$ and $d(\varkappa,\varsigma) = 0 \Longleftrightarrow \varkappa = \varsigma$;
(ii) $d(\varkappa,\varsigma) = d(\varsigma,\varkappa)$;

(iii) $d(\varkappa, \varsigma) \precsim d(\varkappa, \nu) + d(\nu, \varsigma) + \curlywedge(\varkappa, \varsigma) d(\varkappa, \nu) d(\nu, \varsigma)$

for all $\varkappa, \varsigma, \nu \in \Omega$. It follows that $(\Omega, d)$ is said to be a C-VESMS.

**Remark 1.** The C-VESMS introduced above provides a genuine extension of the C-VSMS by replacing the constant coefficient $\curlywedge$ in the triangle-type inequality with a non-negative control function $\curlywedge : \Omega \times \Omega \to [0, +\infty)$. This point-dependent control allows the geometry of the space to vary according to the particular pair of points under consideration, thereby providing greater flexibility than the constant-coefficient setting. In particular, the admissible upper bound for $d(\varkappa, \varsigma)$ is no longer governed by a single global parameter, but may vary from one pair of points to another. Consequently, a C-VESMS can accommodate structures for which no single constant $\curlywedge$ is suitable for describing the triangle-type relation.

The role of this functional control is also relevant to convergence and FP theory. Along an iterative sequence $\varkappa_j$, the quantities $\curlywedge(\varkappa_j, \varkappa_{j+1})$ may vary with $j$, allowing the triangle-type estimate to adapt to the local behavior of the sequence. Thus, the C-VESMS framework provides a more flexible setting for establishing convergence and FP results under contractive conditions that may not be naturally formulated in a C-VSMS. The classical C-VSMS is recovered as the special case

$$\curlywedge(\varkappa, \varsigma) = \curlywedge,$$

where $\curlywedge \geq 0$ is constant.

**Example 2.** Let $\Omega = \mathbb{R}$. Define $d : \Omega \times \Omega \to \mathbb{C}$ as

$$d(\varkappa, \varsigma) = |\varkappa - \varsigma| + i\,|\varkappa - \varsigma|$$

and $\curlywedge : \Omega \times \Omega \to [0, +\infty)$ by

$$\curlywedge(\varkappa, \varsigma) = 1$$

for all $\varkappa, \varsigma \in \Omega$. Then, $(\Omega, d)$ is a C-VESMS, as

(i)

$$d(\varkappa, \varsigma) = |\varkappa - \varsigma| + i\,|\varkappa - \varsigma| \succeq 0 + 0i$$

and

$$d(\varkappa, \varsigma) = 0 + 0i$$

imply

$$|\varkappa - \varsigma| + i\,|\varkappa - \varsigma| = 0 + 0i$$

if and only if $\varkappa = \varsigma$.

(ii)

$$d(\varkappa, \varsigma) = |\varkappa - \varsigma| + i\,|\varkappa - \varsigma| = |\varsigma - \varkappa| + i\,|\varsigma - \varkappa| = d(\varsigma, \varkappa).$$

(iii) For all $\varkappa, \varsigma, \nu \in \Omega$, we have

$$\begin{aligned} d(\varkappa, \varsigma) &= |\varkappa - \varsigma|\,(1 + i) \\ &\precsim (|\varkappa - \nu| + |\nu - \varsigma|)\,(1 + i) + 2i\,|\varkappa - \nu|\,|\nu - \varsigma| \\ &= d(\varkappa, \nu) + d(\nu, \varsigma) + \curlywedge(\varkappa, \varsigma) d(\varkappa, \nu) d(\nu, \varsigma), \end{aligned}$$

which holds by the triangle inequality and non-negativity. Hence, $(\Omega, d)$ satisfies all conditions of a C-VESMS.

**Example 3.** Let $\Omega = [0, 1]$ and $d : \Omega \times \Omega \to \mathbb{C}$ be defined in this way:

$$d(\varkappa, \varsigma) = |\varkappa - \varsigma| + i\,|\varkappa - \varsigma|$$

for all $\varkappa, \varsigma \in \Omega$. Define $\curlywedge : \Omega \times \Omega \to [0, +\infty)$ by

$$\curlywedge(\varkappa, \varsigma) = |\varkappa - \varsigma|$$

for all $\varkappa, \varsigma \in \Omega$. Conditions (i) and (ii) are verified as in the previous example. For Condition (iii), let $\varkappa, \varsigma, \nu \in \Omega$. Then,

$$\begin{aligned} d(\varkappa, \varsigma) &= |\varkappa - \varsigma|\,(1 + i) \\ &\preceq (|\varkappa - \nu| + |\nu - \varsigma|)\,(1 + i) + 2i\,|\varkappa - \varsigma|\,|\varkappa - \nu|\,|\nu - \varsigma| \\ &= d(\varkappa, \nu) + d(\nu, \varsigma) + \curlywedge(\varkappa, \varsigma) d(\varkappa, \nu) d(\nu, \varsigma). \end{aligned}$$

Hence, Condition (iii) is satisfied, and $(\Omega, d)$ forms a C-VESMS.

**Example 4.** Let $\Omega = C\,([0, 1]\,, \mathbb{R})$, and define $d : \Omega \times \Omega \to \mathbb{C}$ by

$$d(f, g) = \|f - g\|_\infty + i\,\|f - g\|_\infty$$

and $\curlywedge : \Omega \times \Omega \to [0, +\infty)$ by $\curlywedge(\varkappa, \varsigma) = 2$.

(i)

$$d(f, g) = \|f - g\|_\infty\,(1 + i) \succeq 0 + 0i$$

and

$$d(f, g) = 0 + 0i$$

imply

$$\|f - g\|_\infty\,(1 + i) = 0 + 0i$$

if and only if $f = g$.

(ii)

$$d(f, g) = \|f - g\|_\infty\,(1 + i) = \|g - f\|_\infty\,(1 + i) = d(g, f).$$

(iii) For all $f, g, h \in \Omega$, we have

$$\begin{aligned} d(f, g) &= \|f - g\|_\infty\,(1 + i) \\ &\preceq (|f - h| + |h - g|)\,(1 + i) + 4i\,|f - h|\,|h - g| \\ &= d(f, h) + d(h, g) + \curlywedge(\varkappa, \varsigma) d(f, h) d(h, g), \end{aligned}$$

which holds by the triangle inequality (iii). Thus, $(\Omega, d)$ forms a C-VESMS.

**Remark 2.** When $\curlywedge : \Omega\times\Omega \to [0, +\infty)$ by $\curlywedge(\varkappa, \varsigma) = \curlywedge \geq 0$ in Definition 6, the concept of a C-VESMS becomes that of a C-VSMS.

**Lemma 1.** *A sequence $\left\{\varkappa_j\right\}$ in C-VSMS $(\Omega, d)$ is said to converge to $\varkappa$ iff*

$$\left|d(\varkappa_j, \varkappa)\right| \to 0$$

*as $j \to \infty$.*

**Lemma 2.** *A sequence $\left\{\varkappa_j\right\}$ in C-VSMS $(\Omega, d)$ is a Cauchy sequence iff*

$$\left|d(\varkappa_j, \varkappa_{j+m})\right| \to 0$$

*as $j \to \infty$ for any fixed $m \in \mathbb{N}$.*

## 3. Core findings

This section is devoted to presenting the main FP results in the context of complete C-VESMSs. Throughout the section, we assume that $(\Omega, d)$ is a complete C-VESMS with $\curlywedge : \Omega \times \Omega \to [0, +\infty)$.

### 3.1. Rational contractions in complex-valued extended suprametric spaces

In this subsection, we examine rational contractive conditions for pairs of self-mappings defined on a C-VESMS. Motivated by the increasing interest in extending classical contraction principles to generalized metric frameworks, we prove a CFP theorem for self mappings that satisfy a rational-type inequality with nonlinear control terms. This result encompasses several known contraction theorems in SMS, E-SMS, C-VSMS, and C-VMS theory.

**Theorem 1.** *Let $\mathcal{M}, \aleph : \Omega \to \Omega$. Suppose that there exist the constants $\sigma_1, \sigma_2, \sigma_3 \in [0, 1)$ with $\sigma_1 + \sigma_2 + \sigma_3 < 1$ such that for all $\varkappa, \varsigma \in \Omega$,*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \preceq \sigma_1 d(\varkappa, \varsigma) + \sigma_2 \frac{d(\varkappa, \mathcal{M}\varkappa) d(\varsigma, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)} + \sigma_3 \frac{d(\varsigma, \mathcal{M}\varkappa) d(\varkappa, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)}. \tag{3.1}$$

*Furthermore, assume that for any $\varkappa_0 \in \Omega$, the sequence $\{\varkappa_j\}$ defined by*

$$\varkappa_{2j+1} = \mathcal{M}\varkappa_{2j} \ \text{ and } \ \varkappa_{2j+2} = \aleph\varkappa_{2j+1}, \tag{3.2}$$

$\forall\ j \in \mathbb{N} \cup \{0\}$ *satisfies*

$$\sup_{j,m \in \mathbb{N}\cup\{0\}} \curlywedge\left(\varkappa_j, \varkappa_m\right) < \infty.$$

*Then, $\mathcal{M}$, $\aleph$ have a unique CFP in $\Omega$.*

*Proof.* For an arbitrary initial point $\varkappa_0 \in \Omega$, define a sequence $\{\varkappa_j\}$ according to (3.2). By (3.1) with $\varkappa = \varkappa_{2j}$ and $\varsigma = \varkappa_{2j+1}$, we get

$$\begin{aligned} d(\varkappa_{2j+1}, \varkappa_{2j+2}) &= d(\mathcal{M}\varkappa_{2j}, \aleph\varkappa_{2j+1}) \preceq \sigma_1 d(\varkappa_{2j}, \varkappa_{2j+1}) \\ &\quad + \sigma_2 \frac{d(\varkappa_{2j}, \mathcal{M}\varkappa_{2j}) d(\varkappa_{2j+1}, \aleph\varkappa_{2j+1})}{1 + d(\varkappa_{2j}, \varkappa_{2j+1})} \\ &\quad + \sigma_3 \frac{d(\varkappa_{2j+1}, \mathcal{M}\varkappa_{2j}) d(\varkappa_{2j}, \aleph\varkappa_{2j+1})}{1 + d(\varkappa_{2j}, \varkappa_{2j+1})} \\ &= \sigma_1 d(\varkappa_{2j}, \varkappa_{2j+1}) \\ &\quad + \sigma_2 \frac{d(\varkappa_{2j}, \varkappa_{2j+1}) d(\varkappa_{2j+1}, \varkappa_{2j+2})}{1 + d(\varkappa_{2j}, \varkappa_{2j+1})}, \end{aligned}$$

which implies that

$$\begin{aligned} \left|d(\varkappa_{2j+1}, \varkappa_{2j+2})\right| &\leq \sigma_1 \left|d(\varkappa_{2j}, \varkappa_{2j+1})\right| + \sigma_2 \left|\frac{d(\varkappa_{2j}, \varkappa_{2j+1})}{1 + d(\varkappa_{2j}, \varkappa_{2j+1})}\right| \left|d(\varkappa_{2j+1}, \varkappa_{2j+2})\right| \\ &\leq \sigma_1 \left|d(\varkappa_{2j}, \varkappa_{2j+1})\right| + \sigma_2 \left|d(\varkappa_{2j+1}, \varkappa_{2j+2})\right|; \end{aligned}$$

that is,

$$\left|d(\varkappa_{2j+1}, \varkappa_{2j+2})\right| \leq \frac{\sigma_1}{1 - \sigma_2} \left|d(\varkappa_{2j}, \varkappa_{2j+1})\right|, \tag{3.3}$$

$\forall\ j \in \mathbb{N} \cup \{0\}$. Let $\sigma = \frac{\sigma_1}{1-\sigma_2} < 1$. Then, the inequality (3.3) becomes

$$\left|d(\varkappa_{2j+1}, \varkappa_{2j+2})\right| \leq \sigma \left|d(\varkappa_{2j}, \varkappa_{2j+1})\right|, \tag{3.4}$$

$\forall\ j \in \mathbb{N}\cup\{0\}$. Similarly, applying the contractive condition (3.1) with $\varkappa = \varkappa_{2j+2}$ and $\varsigma = \varkappa_{2j+1}$, we obtain

$$\left|d(\varkappa_{2j+2}, \varkappa_{2j+3})\right| \leq \sigma \left|d(\varkappa_{2j+2}, \varkappa_{2j+1})\right|, \tag{3.5}$$

$\forall\ j \in \mathbb{N} \cup \{0\}$, where $\sigma = \frac{\sigma_1}{1-\sigma_2} < 1$. Finally, combining (3.4) and (3.5), we obtain

$$\left|d(\varkappa_j, \varkappa_{j+1})\right| \leq \sigma \left|d(\varkappa_{j-1}, \varkappa_j)\right|$$

for all $\mathbb{N} \cup \{0\}$. Hence,

$$\left|d(\varkappa_j, \varkappa_{j+1})\right| \leq \sigma \left|d(\varkappa_{j-1}, \varkappa_j)\right| \leq \sigma^2 \left|d(\varkappa_{j-2}, \varkappa_{j-1})\right| \leq \ldots \leq \sigma^j \left|d(\varkappa_0, \varkappa_1)\right| \tag{3.6}$$

$\forall\ \mathbb{N} \cup \{0\}$. Let $m > j$. Then, we have

$$\begin{aligned} \left|d\left(\varkappa_j, \varkappa_m\right)\right| &\leq \left|d\left(\varkappa_j, \varkappa_{j+1}\right)\right| + \left|d\left(\varkappa_{j+1}, \varkappa_m\right)\right| + \curlywedge\left(\varkappa_j, \varkappa_m\right)\left|d\left(\varkappa_j, \varkappa_{j+1}\right)\right|\left|d\left(\varkappa_{j+1}, \varkappa_m\right)\right| \\ &= \left|d\left(\varkappa_j, \varkappa_{j+1}\right)\right| + \left(1 + \curlywedge\left(\varkappa_j, \varkappa_m\right)\left|d\left(\varkappa_j, \varkappa_{j+1}\right)\right|\right)\left|d\left(\varkappa_{j+1}, \varkappa_m\right)\right|. \end{aligned} \tag{3.7}$$

Applying the same inequality recursively to $\left|d\left(\varkappa_{j+1}, \varkappa_m\right)\right|$, we obtain

$$\begin{aligned} \left|d\left(\varkappa_{j+1}, \varkappa_m\right)\right| &\leq \left|d\left(\varkappa_{j+1}, \varkappa_{j+2}\right)\right| + \left|d\left(\varkappa_{j+2}, \varkappa_m\right)\right| + \curlywedge\left(\varkappa_{j+1}, \varkappa_m\right)\left|d\left(\varkappa_{j+1}, \varkappa_{j+2}\right)\right|\left|d\left(\varkappa_{j+2}, \varkappa_m\right)\right| \\ &= \left|d\left(\varkappa_{j+1}, \varkappa_{j+2}\right)\right| + \left(1 + \curlywedge\left(\varkappa_{j+1}, \varkappa_m\right)\left|d\left(\varkappa_{j+1}, \varkappa_{j+2}\right)\right|\right)\left|d\left(\varkappa_{j+2}, \varkappa_m\right)\right| \end{aligned}$$

and similarly for $\left|d\left(\varkappa_{j+2}, \varkappa_m\right)\right|$, and so on, until

$$\begin{aligned} |d(\varkappa_{m-2}, \varkappa_m)| &\leq |d(\varkappa_{m-2}, \varkappa_{m-1})| + |d(\varkappa_{m-1}, \varkappa_m)| + \curlywedge(\varkappa_{m-2}, \varkappa_m)|d(\varkappa_{m-2}, \varkappa_{m-1})||d(\varkappa_{m-1}, \varkappa_m)| \\ &= |d(\varkappa_{m-2}, \varkappa_{m-1})| + (1 + \curlywedge(\varkappa_{m-2}, \varkappa_m)|d(\varkappa_{m-2}, \varkappa_{m-1})|)|d(\varkappa_{m-1}, \varkappa_m)|. \end{aligned}$$

By recursively substituting each inequality into the preceding one in (3.7) and simplifying, we obtain

$$\left|d\left(\varkappa_j, \varkappa_m\right)\right| \leq \sum_{k=j}^{m-1} \left|d\left(\varkappa_j, \varkappa_{j+1}\right)\right| \prod_{i=j}^{k-1} \left(1 + \curlywedge(\varkappa_i, \varkappa_m)|d(\varkappa_i, \varkappa_{i+1})|\right). \tag{3.8}$$

Let

$$C = \sup_{j,m \in \mathbb{N}\cup\{0\}} \curlywedge\left(\varkappa_j, \varkappa_m\right) < \infty.$$

Using the inequality (3.6) in (3.8), it follows that

$$\left|d\left(\varkappa_j, \varkappa_m\right)\right| \leq \sum_{k=j}^{m-1} \sigma^j |d(\varkappa_0, \varkappa_1)| \prod_{i=j}^{k-1} \left(1 + C\sigma^i |d(\varkappa_0, \varkappa_1)|\right). \tag{3.9}$$

Let

$$P_j = \prod_{i=j}^{\infty} \left(1 + C\sigma^i |d(\varkappa_0, \varkappa_1)|\right).$$

Because

$$\sum_{i=0}^{\infty} \sigma^i |d(\varkappa_0, \varkappa_1)| < \infty,$$

the infinite product converges to a finite positive number $P_j \leq P_0 < \infty$, independent of $m$. Hence, for all $m > j$, we have

$$\prod_{i=j}^{k-1} \left(1 + C\sigma^i |d(\varkappa_0, \varkappa_1)|\right) \leq P_j.$$

Thus,

$$\left|d\left(\varkappa_j, \varkappa_m\right)\right| \leq P_j \sum_{k=j}^{m-1} \sigma^k |d(\varkappa_0, \varkappa_1)| \leq P_j \cdot \frac{\sigma^k}{1-\sigma} |d(\varkappa_0, \varkappa_1)|.$$

Because $\sigma^j \to 0$ as $j \to \infty$, we have $\left|d\left(\varkappa_j, \varkappa_m\right)\right| \to 0$ as $j, m \to \infty$. Therefore, $\left\{\varkappa_j\right\}$ is a Cauchy sequence. As $\Omega$ is complete, $\exists \varkappa^* \in \Omega$ such that

$$\lim_{j\to\infty} \varkappa_j = \varkappa^*. \tag{3.10}$$

From (3.11) and (3.13) with $\varkappa = \varkappa^* \in \Psi_1$ and $\varsigma = \varkappa_{2j+1} \in \Psi_2$, we have

$$\begin{aligned} d(\varkappa^*, \mathcal{M}\varkappa^*) &\leq d\left(\varkappa^*, \varkappa_{2j+2}\right) + d\left(\varkappa_{2j+2}, \mathcal{M}\varkappa^*\right) + \wedge(\varkappa^*, \mathcal{M}\varkappa^*)\, d\left(\varkappa^*, \varkappa_{2j+2}\right) d\left(\varkappa_{2j+2}, \mathcal{M}\varkappa^*\right) \\ &= d\left(\varkappa^*, \varkappa_{2j+2}\right) + d\left(\aleph\varkappa_{2j+1}, \mathcal{M}\varkappa^*\right) + \wedge(\varkappa^*, \mathcal{M}\varkappa^*)\, d\left(\varkappa^*, \varkappa_{2j+2}\right) d\left(\aleph\varkappa_{2j+1}, \mathcal{M}\varkappa^*\right) \\ &= d\left(\varkappa^*, \varkappa_{2j+2}\right) + d\left(\mathcal{M}\varkappa^*, \aleph\varkappa_{2j+1}\right) + \wedge(\varkappa^*, \mathcal{M}\varkappa^*)\, d\left(\varkappa^*, \varkappa_{2j+2}\right) d\left(\mathcal{M}\varkappa^*, \aleph\varkappa_{2j+1}\right) \\ &\leq d\left(\varkappa^*, \varkappa_{2j+2}\right) + \sigma_1 d\left(\varkappa^*, \varkappa_{2j+1}\right) + \sigma_2 \frac{d(\varkappa^*, \mathcal{M}\varkappa^*) d(\varkappa_{2j+1}, \aleph\varkappa_{2j+1})}{1 + d\left(\varkappa^*, \varkappa_{2j+1}\right)} \\ &\quad + \sigma_3 \frac{d(\varkappa_{2j+1}, \mathcal{M}\varkappa^*) d(\varkappa^*, \aleph\varkappa_{2j+1})}{1 + d\left(\varkappa^*, \varkappa_{2j+1}\right)} \\ &\quad + \wedge(\varkappa^*, \mathcal{M}\varkappa^*)\, d\left(\varkappa^*, \varkappa_{2j+2}\right) \left( \begin{array}{c} \sigma_1 d\left(\varkappa^*, \varkappa_{2j+1}\right) + \sigma_2 \frac{d(\varkappa^*, \mathcal{M}\varkappa^*) d(\varkappa_{2j+1}, \aleph\varkappa_{2j+1})}{1 + d\left(\varkappa^*, \varkappa_{2j+1}\right)} \\ + \sigma_3 \frac{d(\varkappa_{2j+1}, \mathcal{M}\varkappa^*) d(\varkappa^*, \aleph\varkappa_{2j+1})}{1 + d\left(\varkappa^*, \varkappa_{2j+1}\right)} \end{array} \right); \end{aligned}$$

that is,

$$\begin{aligned} d(\varkappa^*, \mathcal{M}\varkappa^*) &\leq d\left(\varkappa^*, \varkappa_{2j+2}\right) + \sigma_1 d\left(\varkappa^*, \varkappa_{2j+1}\right) + \sigma_2 \frac{d(\varkappa^*, \mathcal{M}\varkappa^*) d(\varkappa_{2j+1}, \varkappa_{2j+2})}{1 + d\left(\varkappa^*, \varkappa_{2j+1}\right)} \\ &\quad + \sigma_3 \frac{d(\varkappa_{2j+1}, \mathcal{M}\varkappa^*) d(\varkappa^*, \varkappa_{2j+2})}{1 + d\left(\varkappa^*, \varkappa_{2j+1}\right)} \\ &\quad + \wedge(\varkappa^*, \mathcal{M}\varkappa^*)\, d\left(\varkappa^*, \varkappa_{2j+2}\right) \left( \begin{array}{c} \sigma_1 d\left(\varkappa^*, \varkappa_{2j+1}\right) + \sigma_2 \frac{d(\varkappa^*, \mathcal{M}\varkappa^*) d(\varkappa_{2j+1}, \varkappa_{2j+2})}{1 + d\left(\varkappa^*, \varkappa_{2j+1}\right)} \\ + \sigma_3 \frac{d(\varkappa_{2j+1}, \mathcal{M}\varkappa^*) d(\varkappa^*, \varkappa_{2j+2})}{1 + d\left(\varkappa^*, \varkappa_{2j+1}\right)} \end{array} \right), \end{aligned}$$

which entails that

$$\begin{aligned}
|d(\varkappa^*, \mathcal{M}\varkappa^*)| &\leq \left|d\left(\varkappa^*, \varkappa_{2j+2}\right)\right| + \sigma_1 \left|d\left(\varkappa^*, \varkappa_{2j+1}\right)\right| \\
&\quad +\sigma_2 \left|d(\varkappa^*,\mathcal{M}\varkappa^*)\right| \left|\frac{d(\varkappa_{2j+1}, \varkappa_{2j+2})}{1 + d\left(\varkappa^*, \varkappa_{2j+1}\right)}\right| \\
&\quad +\sigma_3 \left|d(\varkappa_{2j+1},\mathcal{M}\varkappa^*)\right| \left|\frac{d(\varkappa^*, \varkappa_{2j+2})}{1 + d\left(\varkappa^*, \varkappa_{2j+1}\right)}\right| \\
&\quad + \curlywedge\left(\varkappa^*, \mathcal{M}\varkappa^*\right) \left|d\left(\varkappa^*, \varkappa_{2j+2}\right)\right| \left(\begin{array}{c} \sigma_1 \left|d\left(\varkappa^*, \varkappa_{2j+1}\right)\right| \\ +\sigma_2 \left|d(\varkappa^*,\mathcal{M}\varkappa^*)\right| \left|\frac{d(\varkappa_{2j+1},\varkappa_{2j+2})}{1+d\left(\varkappa^*,\varkappa_{2j+1}\right)}\right| \\ +\sigma_3 \left|d(\varkappa_{2j+1},\mathcal{M}\varkappa^*)\right| \left|\frac{d(\varkappa^*,\varkappa_{2j+2})}{1+d\left(\varkappa^*,\varkappa_{2j+1}\right)}\right| \end{array}\right).
\end{aligned}$$

By taking the limit as $j \to +\infty$ in the preceding inequality, we deduce the following:

$$|d(\varkappa^*, \mathcal{M}\varkappa^*)| = 0,$$

which implies $\varkappa^* = \mathcal{M}\varkappa^*$. Thus, $\varkappa^*$ is a FP of $\mathcal{M}$. Likewise, $\varkappa^*$ is a FP of $\aleph$. Consequently, $\varkappa^*$ is a CFP of $\aleph$ and $\mathcal{M}$ in $\Omega$. To establish the uniqueness of $\varkappa^*$, assume to the contrary that there is one more CFP $\varkappa'$ of $\aleph$ and $\mathcal{M}$ in $\Psi_1 \cap \Psi_2$; that is, $\varkappa' = \aleph\varkappa' = \mathcal{M}\varkappa'$ with $\varkappa^* \neq \varkappa'$. Then, from (3.1) with $\varkappa = \varkappa^*$ and $\varsigma = \varkappa'$, we have

$$\begin{aligned}
d(\varkappa^*, \varkappa') &= d(\mathcal{M}\varkappa^*, \aleph\varkappa') \leq \sigma_1 d\left(\varkappa^*, \varkappa'\right) + \sigma_2 \frac{d(\varkappa^*,\mathcal{M}\varkappa^*) d(\varkappa', \aleph\varkappa')}{1 + d(\varkappa^*, \varkappa')} + \sigma_3 \frac{d(\varkappa',\mathcal{M}\varkappa^*) d(\varkappa^*, \aleph\varkappa')}{1 + d(\varkappa^*, \varkappa')} \\
&= \sigma_1 d\left(\varkappa^*, \varkappa'\right) + \sigma_3 \frac{d(\varkappa',\varkappa^*) d(\varkappa^*, \varkappa')}{1 + d(\varkappa^*, \varkappa')},
\end{aligned}$$

which implies

$$\begin{aligned}
\left|d(\varkappa^*, \varkappa')\right| &\leq \sigma_1 \left|d(\varkappa^*, \varkappa')\right| + \sigma_3 \left|d(\varkappa',\varkappa^*)\right| \left|\frac{d(\varkappa^*, \varkappa')}{1 + d(\varkappa^*, \varkappa')}\right| \\
&\leq \sigma_1 \left|d(\varkappa^*, \varkappa')\right| + \sigma_3 \left|d(\varkappa',\varkappa^*)\right| \\
&= (\sigma_1 + \sigma_3) \left|d(\varkappa^*, \varkappa')\right|,
\end{aligned}$$

which implies

$$(1 - \sigma_1 - \sigma_3) \left|d(\varkappa^*, \varkappa')\right| \leq 0.$$

Because $(1 - \sigma_1 - \sigma_3) \neq 0$, thus $\left|d(\varkappa^*, \varkappa')\right| = 0$, which implies $\varkappa^* = \varkappa'$. Hence, the CFP is unique in $\Omega$. □

**Remark 3.** The condition

$$\sup_{j,m \in \mathbb{N}\cup\{0\}} \curlywedge\left(\varkappa_j, \varkappa_m\right) < \infty$$

is a boundedness assumption on the generalized metric coefficient along the constructed iterative sequence. It is used in the proof to obtain a finite constant $C$, which is required to control the generalized triangle inequality in establishing the Cauchy property of the sequence. We emphasize that this condition is only a sufficient assumption for the present FP result; its necessity and sharpness are not claimed. The investigation of weaker or intrinsic conditions on $\curlywedge$ that do not depend explicitly on the generated sequence is an interesting direction for future research.

**Example 5.** Let $\Omega = [0,1] \subset \mathbb{R}$. Define $d : \Omega \times \Omega \to \mathbb{C}$ by

$$d(\varkappa, \varsigma) = |\varkappa - \varsigma| + i\,|\varkappa - \varsigma|$$

for all $\varkappa, \varsigma \in \Omega$ and $i^2 = -1$. Then, $(\Omega, d)$ is a complete C-VSMS with $\curlywedge(\varkappa, \varsigma) = 1$. Define

$$\mathcal{M}(\varkappa) = \frac{\varkappa}{4}$$

and

$$\aleph(\varkappa) = \frac{\varkappa}{6}.$$

For any $\varkappa, \varsigma \in \Omega$, we have

$$\begin{aligned} d(\mathcal{M}\varkappa, \aleph\varsigma) &\leq \left(\left|\frac{\varkappa}{4} - \frac{\varsigma}{6}\right|\right)(1+i) \\ &\leq \left(\left|\frac{\varkappa}{4} - \frac{\varsigma}{4}\right|\right)(1+i) \\ &= \frac{1}{4} d(\varkappa, \varsigma). \end{aligned}$$

For any $\varkappa_0 \in \Psi_1$, define

$$\varkappa_{2j+1} = \mathcal{M}\varkappa_{2j} = \frac{\varkappa_{2j}}{4} \text{ and } \varkappa_{2j+2} = \aleph\varkappa_{2j+1} = \frac{\varkappa_{2j+1}}{6},$$

$\forall\ j \in \mathbb{N} \cup \{0\}$. Then,

$$0 \leq \varkappa_j \leq 1, \quad \text{for all } j.$$

The sequence is then bounded, and thus,

$$\sup_{j,m \in \mathbb{N}\cup\{0\}} \curlywedge\left(\varkappa_j, \varkappa_m\right) < \infty.$$

All hypotheses of Theorem 1 are satisfied. Therefore, there exists a unique CFP $\varkappa^* = 0 \in \Omega$ of mappings $\mathcal{M}$ and $\aleph$.

**Corollary 1.** *Let $\aleph : \Omega \to \Omega$. Suppose that there exist the constants $\sigma_1, \sigma_2, \sigma_3 \in [0,1)$ with $\sigma_1 + \sigma_2 + \sigma_3 < 1$ such that for all $\varkappa, \varsigma \in \Omega$,*

$$d(\aleph\varkappa, \aleph\varsigma) \leq \sigma_1 d(\varkappa, \varsigma) + \sigma_2 \frac{d(\varkappa, \aleph\varkappa) d(\varsigma, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)} + \sigma_3 \frac{d(\varsigma, \aleph\varkappa) d(\varkappa, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)}.$$

*Furthermore, assume that for any $\varkappa_0 \in \Omega$, the sequence $\{\varkappa_j\}$ defined by*

$$\varkappa_{j+1} = \aleph\varkappa_j;$$

$\forall\ j \in \mathbb{N} \cup \{0\}$ *satisfies*

$$\sup_{j,m \in \mathbb{N}\cup\{0\}} \curlywedge\left(\varkappa_j, \varkappa_m\right) < \infty.$$

*Then, $\aleph$ has a unique FP in $\Omega$.*

*Proof.* Take $\mathcal{M} = \aleph$ in Theorem 1. □

### 3.2. Common fixed-point results for Kannan type cyclic contraction

**Theorem 2.** *Let $\Psi_1$ and $\Psi_2$ be nonempty closed subsets of $\Omega$. Consider the mappings $\mathcal{M},\aleph : \Psi_1\cup\Psi_2 \to \Psi_1 \cup \Psi_2$ satisfying*

$$\mathcal{M}(\Psi_1) \subseteq \Psi_2,\ \mathcal{M}(\Psi_2) \subseteq \Psi_1 \text{ and } \aleph(\Psi_1) \subseteq \Psi_2,\ \aleph(\Psi_2) \subseteq \Psi_1.$$

*Suppose that there is a constant $\sigma \in [0, \frac{1}{2})$ such that*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \leq \sigma (d(\varkappa,\mathcal{M}\varkappa) + d(\varsigma, \aleph\varsigma)) \tag{3.11}$$

*holds for all $\varkappa \in \Psi_1, \varsigma \in \Psi_2$. Furthermore, assume that for any $\varkappa_0 \in \Psi_1$, the sequence $\{\varkappa_j\}$ defined by*

$$\varkappa_{2j+1} = \mathcal{M}\varkappa_{2j} \in \Psi_2 \ \text{ and } \ \varkappa_{2j+2} = \aleph\varkappa_{2j+1} \in \Psi_1, \tag{3.12}$$

*$\forall\ j \in \mathbb{N} \cup \{0\}$ satisfies*

$$\sup_{j,m\in\mathbb{N}\cup\{0\}} \wedge \left(\varkappa_j, \varkappa_m\right) < \infty.$$

*Then, $\mathcal{M}$, $\aleph$ have a unique CFP in $\Omega$.*

*Proof.* Let $\varkappa_0 \in \Psi_1$ be given. Define the sequence $\{\varkappa_j\}$ as in (3.12). By (3.11) with $\varkappa = \varkappa_{2j} \in \Psi_1$ and $\varsigma = \varkappa_{2j+1} \in \Psi_2$, we get

$$\begin{aligned} d(\varkappa_{2j+1}, \varkappa_{2j+2}) &= d(\mathcal{M}\varkappa_{2j}, \aleph\varkappa_{2j+1}) \leq \sigma \left(d(\varkappa_{2j},\mathcal{M}\varkappa_{2j}) + d(\varkappa_{2j+1}, \aleph\varkappa_{2j+1})\right) \\ &= \sigma \left(d(\varkappa_{2j},\varkappa_{2j+1}) + d(\varkappa_{2j+1}, \varkappa_{2j+2})\right); \end{aligned}$$

that is,

$$d(\varkappa_{2j+1}, \varkappa_{2j+2}) \leq \frac{\sigma}{1-\sigma} d(\varkappa_{2j},\varkappa_{2j+1}),$$

which implies that

$$\left|d(\varkappa_{2j+1}, \varkappa_{2j+2})\right| \leq \frac{\sigma}{1-\sigma} \left|d(\varkappa_{2j},\varkappa_{2j+1})\right|, \tag{3.13}$$

$\forall\ j \in \mathbb{N} \cup \{0\}$. Similarly, using (3.11) with $\varkappa = \varkappa_{2j+2} \in \Psi_1$ and $\varsigma = \varkappa_{2j+1} \in \Psi_2$, we get

$$\left|d(\varkappa_{2j+2}, \varkappa_{2j+3})\right| \leq \frac{\sigma}{1-\sigma} \left|d(\varkappa_{2j+1}, \varkappa_{2j+2})\right| \tag{3.14}$$

for all $j \in \mathbb{N} \cup \{0\}$. Let $\lambda = \frac{\sigma}{1-\sigma} < 1$. Then, the inequalities (3.13) and (3.14) become

$$\left|d(\varkappa_{2j+1}, \varkappa_{2j+2})\right| \leq \lambda \left|d(\varkappa_{2j},\varkappa_{2j+1})\right| \tag{3.15}$$

and

$$\left|d(\varkappa_{2j+2}, \varkappa_{2j+3})\right| \leq \lambda \left|d(\varkappa_{2j+1}, \varkappa_{2j+2})\right| \tag{3.16}$$

for all $j \in \mathbb{N} \cup \{0\}$. Finally, combining (3.15) and (3.16), we obtain

$$\left|d(\varkappa_j, \varkappa_{j+1})\right| \leq \sigma \left|d(\varkappa_{j-1}, \varkappa_j)\right|$$

for all $\mathbb{N} \cup \{0\}$. Hence,

$$\left|d(\varkappa_j, \varkappa_{j+1})\right| \leq \sigma \left|d(\varkappa_{j-1}, \varkappa_j)\right| \leq \sigma^2 \left|d(\varkappa_{j-2}, \varkappa_{j-1})\right| \leq ... \leq \sigma^j \left|d(\varkappa_0, \varkappa_1)\right| \tag{3.17}$$

$\forall\ \mathbb{N} \cup \{0\}$. Following the same procedure as in Theorem 1, we can show that the sequence $\left\{\varkappa_j\right\}$ is a Cauchy sequence in $\Omega$. As $\Omega$ is complete, $\exists \varkappa^* \in \Omega$ such that

$$\lim_{j\to\infty} \varkappa_j = \varkappa^*. \tag{3.18}$$

Because $\{\varkappa_{2j}\}$ is a subsequence in $\Psi_1$ and $\Psi_1$ is closed, its limit $\varkappa^*$ belongs to $\overline{\Psi_1} = \Psi_1$. Similarly, $\{\varkappa_{2j+1}\}$ is in $\Psi_2$, and $\Psi_2$ is closed, so $\varkappa^* \in \overline{\Psi_2} = \Psi_2$. Consequently, $\varkappa^* \in \Psi_1 \cap \Psi_2$. From (3.11) with $\varkappa = \varkappa^* \in \Psi_1$ and $\varsigma = \varkappa_{2j+1} \in \Psi_2$, we have

$$\begin{aligned}
d(\varkappa^*, \mathcal{M}\varkappa^*) &\leq d\left(\varkappa^*, \varkappa_{2j+2}\right) + d\left(\varkappa_{2j+2}, \mathcal{M}\varkappa^*\right) + \curlywedge(\varkappa^*, \mathcal{M}\varkappa^*)\, d\left(\varkappa^*, \varkappa_{2j+2}\right) d\left(\varkappa_{2j+2}, \mathcal{M}\varkappa^*\right)\\
&= d\left(\varkappa^*, \varkappa_{2j+2}\right) + d\left(\aleph\varkappa_{2j+1}, \mathcal{M}\varkappa^*\right) + \curlywedge(\varkappa^*, \mathcal{M}\varkappa^*)\, d\left(\varkappa^*, \varkappa_{2j+2}\right) d\left(\aleph\varkappa_{2j+1}, \mathcal{M}\varkappa^*\right)\\
&= d\left(\varkappa^*, \varkappa_{2j+2}\right) + d\left(\mathcal{M}\varkappa^*, \aleph\varkappa_{2j+1}\right) + \curlywedge(\varkappa^*, \mathcal{M}\varkappa^*)\, d\left(\varkappa^*, \varkappa_{2j+2}\right) d\left(\mathcal{M}\varkappa^*, \aleph\varkappa_{2j+1}\right)\\
&\leq d\left(\varkappa^*, \varkappa_{2j+2}\right) + \sigma\left(d(\varkappa^*, \mathcal{M}\varkappa^*) + d\left(\varkappa_{2j+1}, \aleph\varkappa_{2j+1}\right)\right)\\
&\quad +\sigma \curlywedge(\varkappa^*, \mathcal{M}\varkappa^*)\, d\left(\varkappa^*, \varkappa_{2j+2}\right)\left(d(\varkappa^*, \mathcal{M}\varkappa^*) + d\left(\varkappa_{2j+1}, \aleph\varkappa_{2j+1}\right)\right)\\
&= d\left(\varkappa^*, \varkappa_{2j+2}\right) + \sigma\left(d(\varkappa^*, \mathcal{M}\varkappa^*) + d\left(\varkappa_{2j+1}, \varkappa_{2j+2}\right)\right)\\
&\quad +\sigma \curlywedge(\varkappa^*, \mathcal{M}\varkappa^*)\, d\left(\varkappa^*, \varkappa_{2j+2}\right)\left(d(\varkappa^*, \mathcal{M}\varkappa^*) + d\left(\varkappa_{2j+1}, \varkappa_{2j+2}\right)\right),
\end{aligned}$$

which entails that

$$\begin{aligned}
|d(\varkappa^*, \mathcal{M}\varkappa^*)| &\leq \left|d\left(\varkappa^*, \varkappa_{2j+2}\right)\right| + \sigma\left|d(\varkappa^*, \mathcal{M}\varkappa^*)\right| + \sigma\left|d\left(\varkappa_{2j+1}, \varkappa_{2j+2}\right)\right|\\
&\quad +\sigma \curlywedge(\varkappa^*, \mathcal{M}\varkappa^*)\left|d\left(\varkappa^*, \varkappa_{2j+2}\right)\right|\left(\left|d(\varkappa^*, \mathcal{M}\varkappa^*)\right| + \left|d\left(\varkappa_{2j+1}, \varkappa_{2j+2}\right)\right|\right).
\end{aligned}$$

Taking the limit as $j \to +\infty$ in the preceding inequality, we deduce the following

$$|d(\varkappa^*, \mathcal{M}\varkappa^*)| \leq \sigma\left|d(\varkappa^*, \mathcal{M}\varkappa^*)\right|,$$

which implies

$$(1-\sigma)\left|d(\varkappa^*, \mathcal{M}\varkappa^*)\right| \leq 0.$$

Because $(1-\sigma) \neq 0$, $|d(\varkappa^*, \mathcal{M}\varkappa^*)| = 0$, which implies $\varkappa^* = \mathcal{M}\varkappa^*$. Hence, $\varkappa^*$ is an FP of $\mathcal{M}$. Likewise, $\varkappa^*$ is an FP of $\aleph$. Consequently, $\varkappa^*$ is a CFP of $\aleph$ and $\mathcal{M}$ in $\Psi_1 \cap \Psi_2$. To establish the uniqueness of $\varkappa^*$, assume to the contrary that there is one more CFP $\varkappa'$ of $\aleph$ and $\mathcal{M}$ in $\Psi_1 \cap \Psi_2$, that is, $\varkappa' = \aleph\varkappa' = \mathcal{M}\varkappa'$ with $\varkappa^* \neq \varkappa'$. Then, from (3.11) with $\varkappa = \varkappa^* \in \Psi_1$ and $\varkappa = \varkappa' \in \Psi_2$, we have

$$\begin{aligned}
d(\varkappa^*, \varkappa') &= d(\mathcal{M}\varkappa^*, \aleph\varkappa') \leq \sigma d(\varkappa^*, \mathcal{M}\varkappa^*) d(\varkappa', \aleph\varkappa')\\
&= 0,
\end{aligned}$$

which implies

$$\left|d(\varkappa^*, \varkappa')\right| = 0.$$

Hence, $\varkappa^* = \varkappa'$. Thus, CFP is unique in $\Psi_1 \cap \Psi_2$. □

**Example 6.** Let $\Omega = [0,2] \subset \mathbb{R}$. Define $d : \Omega \times \Omega \to \mathbb{C}$ by

$$d(\varkappa, \varsigma) = |\varkappa - \varsigma| + i\,|\varkappa - \varsigma|\,,$$

for all $\varkappa, \varsigma \in \Omega$ and $i^2 = -1$. Then, $(\Omega, d)$ is a complete C-VSMS with $\curlywedge(\varkappa, \varsigma) = 1$. Let

$$\Psi_1 = [0,1]\,, \text{ and } \Psi_2 = [1,2]\,.$$

Clearly, $\Psi_1$ and $\Psi_2$ are nonempty and closed, and $\Psi_1 \cap \Psi_2 = \{1\}$. Define $\mathcal{M}, \aleph : \Psi_1 \cup \Psi_2 \to \Psi_1 \cup \Psi_2$ by

$$\mathcal{M}(\varkappa) = \begin{cases} 1 + \frac{\varkappa - 1}{4}, & \text{if } \varkappa \in \Psi_1, \\ 1 - \frac{\varkappa - 1}{4}, & \text{if } \varkappa \in \Psi_2 \end{cases}$$

and

$$\aleph(\varkappa) = \begin{cases} 1 + \frac{\varkappa - 1}{8}, & \text{if } \varkappa \in \Psi_1, \\ 1 - \frac{\varkappa - 1}{8}, & \text{if } \varkappa \in \Psi_2. \end{cases}$$

Then,

$$\mathcal{M}(\Psi_1) \subseteq \Psi_2, \ \mathcal{M}(\Psi_2) \subseteq \Psi_1$$

and

$$\aleph(\Psi_1) \subseteq \Psi_2, \ \aleph(\Psi_2) \subseteq \Psi_1.$$

The cyclic condition is thus satisfied. Now, for any $\varkappa \in \Psi_1, \varsigma \in \Psi_2$,

$$\begin{aligned} d(\mathcal{M}\varkappa, \aleph\varsigma) &\le \left(\frac{1}{4}|\varkappa - 1| + \frac{1}{8}|\varsigma - 1|\right)(1+i) \\ &\le \left(\frac{1}{4}|\varkappa - 1| + \frac{1}{4}|\varsigma - 1|\right)(1+i) \\ &= \sigma\left(d(\varkappa, \mathcal{M}\varkappa) + d(\varsigma, \aleph\varsigma)\right) \end{aligned}$$

holds for any $\sigma = \frac{1}{4} \in [0, \frac{1}{2})$. For any $\varkappa_0 \in \Psi_1$, define

$$\varkappa_{2j+1} = \mathcal{M}\varkappa_{2j} \in \Psi_2 \ \text{ and } \ \varkappa_{2j+2} = \aleph\varkappa_{2j+1} \in \Psi_1,$$

$\forall\, j \in \mathbb{N} \cup \{0\}$. Because $\{\varkappa_j\} \subset [0,2]$, the sequence is bounded, and thus,

$$\sup_{j,m \in \mathbb{N} \cup \{0\}} \curlywedge\left(\varkappa_j, \varkappa_m\right) < \infty.$$

All hypotheses of the theorem are satisfied. Therefore, there exists a unique CFP $\varkappa^* = 1 \in \Psi_1 \cap \Psi_2$ of mappings $\mathcal{M}$ and $\aleph$.

**Corollary 2.** *Let $\Psi_1$ and $\Psi_2$ be nonempty, closed subsets of $\Omega$. Consider the mappings $\aleph : \Psi_1 \cup \Psi_2 \to \Psi_1 \cup \Psi_2$ satisfying*

$$\aleph(\Psi_1) \subseteq \Psi_2, \ \aleph(\Psi_2) \subseteq \Psi_1.$$

*Suppose that there is a constant $\sigma \in [0, \frac{1}{2})$ such that*

$$d(\aleph\varkappa, \aleph\varsigma) \le \sigma\left(d(\varkappa, \aleph\varkappa) + d(\varsigma, \aleph\varsigma)\right)$$

*holds for all* $\varkappa \in \Psi_1, \varsigma \in \Psi_2$. *Furthermore, assume that for any* $\varkappa_0 \in \Psi_1$, *the sequence* $\{\varkappa_j\}$ *defined by*

$$\varkappa_{2j+1} = \aleph\varkappa_{2j} \in \Psi_2 \text{ and } \varkappa_{2j+2} = \aleph\varkappa_{2j+1} \in \Psi_1,$$

$\forall\ j \in \mathbb{N} \cup \{0\}$ *satisfies*

$$\sup_{j,m\in\mathbb{N}\cup\{0\}} \wedge\left(\varkappa_j, \varkappa_m\right) < \infty.$$

*Then,* $\aleph$ *has a unique FP in* $\Omega$.

*Proof.* Take $\mathcal{M} = \aleph$ in Theorem 2. □

**Corollary 3.** *Let* $\mathcal{M}, \aleph : \Omega \to \Omega$. *Suppose that there is a constant* $\sigma \in [0, \frac{1}{2})$ *such that*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \leq \sigma\left(d(\varkappa, \mathcal{M}\varkappa) + d(\varsigma, \aleph\varsigma)\right)$$

*holds for all* $\varkappa, \varsigma \in \Omega$. *Furthermore, assume that for any* $\varkappa_0 \in \Omega$, *the sequence* $\{\varkappa_j\}$ *defined by*

$$\varkappa_{2j+1} = \mathcal{M}\varkappa_{2j} \text{ and } \varkappa_{2j+2} = \aleph\varkappa_{2j+1},$$

$\forall\ j \in \mathbb{N} \cup \{0\}$ *satisfies*

$$\sup_{j,m\in\mathbb{N}\cup\{0\}} \wedge\left(\varkappa_j, \varkappa_m\right) < \infty.$$

*Then,* $\mathcal{M}, \aleph$ *admit a unique CFP in* $\Omega$.

*Proof.* Take $\Psi_1 = \Psi_2 = \Omega$ in Theorem 2. □

**Corollary 4.** *Let* $\aleph : \Omega \to \Omega$. *Suppose that there is a constant* $\sigma \in [0, \frac{1}{2})$ *such that*

$$d(\aleph\varkappa, \aleph\varsigma) \leq \sigma\left(d(\varkappa, \aleph\varkappa) + d(\varsigma, \aleph\varsigma)\right)$$

*holds for all* $\varkappa, \varsigma \in \Omega$. *Furthermore, assume that for any* $\varkappa_0 \in \Omega$, *the sequence* $\{\varkappa_j\}$ *defined by* $\varkappa_{j+1} = \aleph\varkappa_j, \forall\ j \in \mathbb{N} \cup \{0\}$ *satisfies*

$$\sup_{j,m\in\mathbb{N}\cup\{0\}} \wedge\left(\varkappa_j, \varkappa_m\right) < \infty.$$

*Then,* $\aleph$ *admits a unique FP in* $\Omega$.

*Proof.* Take $\mathcal{M} = \aleph$ in the above corollary. □

### 3.3. Fixed-point results for interpolative–rational hybrid contractions

**Theorem 3.** *Let* $\mathcal{M}, \aleph : \Omega \to \Omega$. *Suppose that there exist constants* $\sigma \in [0, 1)$ *and* $p, q, r \in (0, 1)$ *with* $p + q + r = 1$ *such that*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \leq \sigma\,[d(\varkappa, \mathcal{M}\varkappa)]^p \cdot [d(\varsigma, \aleph\varsigma)]^q \cdot \left[\frac{d(\varkappa, \mathcal{M}\varkappa)d(\varsigma, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)}\right]^r \tag{3.19}$$

holds for all $\varkappa, \varsigma \in \Omega \backslash Fix\{\mathcal{M}, \aleph\}$. Furthermore, assume that for any $\varkappa_0 \in \Omega$, the sequence $\{\varkappa_j\}$ defined by

$$\varkappa_{2j+1} = \mathcal{M}\varkappa_{2j} \text{ and } \varkappa_{2j+2} = \aleph\varkappa_{2j+1},$$

$\forall\ j \in \mathbb{N} \cup \{0\}$ satisfies

$$\sup_{j,m\in\mathbb{N}\cup\{0\}} \wedge\left(\varkappa_j, \varkappa_m\right) < \infty.$$

Then, $\mathcal{M}, \aleph$ admit a CFP in $\Omega$.

*Proof.* Let $\varkappa_0 \in \Omega$ be given. Define the sequence $\{\varkappa_j\}$ by

$$\varkappa_{2j+1} = \mathcal{M}\varkappa_{2j} \text{ and } \varkappa_{2j+2} = \aleph\varkappa_{2j+1},$$

$\forall\ j \in \mathbb{N} \cup \{0\}$. If for some $j$, we have $\varkappa_{2j} \in Fix\{\mathcal{M},\aleph\}$, then

$$\varkappa_{2j} = \mathcal{M}\varkappa_{2j} = \aleph\varkappa_{2j}$$

and hence, $\varkappa_{2j}$ is a CFP. Thus, without loss of generality, we assume

$$\varkappa_{2j} \notin Fix\{\mathcal{M},\aleph\},$$

$\forall j \in \mathbb{N} \cup \{0\}$. By (3.19) with $\varkappa = \varkappa_{2j}$ and $\varsigma = \varkappa_{2j+1}$, we have

$$d(\mathcal{M}\varkappa_{2j}, \aleph\varkappa_{2j+1}) \leq \sigma\left[d(\varkappa_{2j},\mathcal{M}\varkappa_{2j})\right]^p \cdot \left[d(\varkappa_{2j+1}, \aleph\varkappa_{2j+1})\right]^q \cdot \left[\frac{d(\varkappa_{2j},\mathcal{M}\varkappa_{2j})d(\varkappa_{2j+1}, \aleph\varkappa_{2j+1})}{1 + d(\varkappa_{2j}, \varkappa_{2j+1})}\right]^r;$$

that is,

$$d(\varkappa_{2j+1}, \varkappa_{2j+2}) \leq \sigma\left[d(\varkappa_{2j},\varkappa_{2j+1})\right]^p \cdot \left[d(\varkappa_{2j+1}, \varkappa_{2j+2})\right]^q \cdot \left[\frac{d(\varkappa_{2j}, \varkappa_{2j+1})d(\varkappa_{2j+1}, \varkappa_{2j+2})}{1 + d(\varkappa_{2j}, \varkappa_{2j+1})}\right]^r,$$

from which it follows that

$$\begin{aligned}\left|d(\varkappa_{2j+1}, \varkappa_{2j+2})\right| &\leq \sigma\left|d(\varkappa_{2j},\varkappa_{2j+1})\right|^p \cdot \left|d(\varkappa_{2j+1}, \varkappa_{2j+2})\right|^q \cdot \left[\left|\frac{d(\varkappa_{2j}, \varkappa_{2j+1})}{1 + d(\varkappa_{2j}, \varkappa_{2j+1})}\right| \left|d(\varkappa_{2j+1}, \varkappa_{2j+2})\right|\right]^r \\ &\leq \sigma\left|d(\varkappa_{2j},\varkappa_{2j+1})\right|^p \cdot \left|d(\varkappa_{2j+1}, \varkappa_{2j+2})\right|^q \cdot \left|d(\varkappa_{2j+1}, \varkappa_{2j+2})\right|^r \\ &= \sigma\left|d(\varkappa_{2j},\varkappa_{2j+1})\right|^p \cdot \left|d(\varkappa_{2j+1}, \varkappa_{2j+2})\right|^{q+r}.\end{aligned}$$

Because $q + r = 1 - p$, we obtain

$$\left|d(\varkappa_{2j+1}, \varkappa_{2j+2})\right| \leq \sigma\left|d(\varkappa_{2j},\varkappa_{2j+1})\right|^p \cdot \left|d(\varkappa_{2j+1}, \varkappa_{2j+2})\right|^{1-p},$$

which implies

$$\left|d(\varkappa_{2j+1}, \varkappa_{2j+2})\right|^p \leq \sigma\left|d(\varkappa_{2j},\varkappa_{2j+1})\right|^p.$$

Taking both sides to power $\frac{1}{p}$ (because $p > 0$), we get

$$\left|d(\varkappa_{2j+1}, \varkappa_{2j+2})\right| \leq \sigma^{\frac{1}{p}}\left|d(\varkappa_{2j},\varkappa_{2j+1})\right|, \tag{3.20}$$

$\forall\ j \in \mathbb{N} \cup \{0\}$. Similarly, from (3.19) with $\varkappa = \varkappa_{2j+2}$ and $\varsigma = \varkappa_{2j+1}$, we have

$$\left|d(\varkappa_{2j+2}, \varkappa_{2j+3})\right| \leq \sigma^{\frac{1}{q}}\left|d(\varkappa_{2j+1}, \varkappa_{2j+2})\right|. \tag{3.21}$$

Let

$$\mu_1 = \sigma^{\frac{1}{p}} \text{ and } \mu_2 = \sigma^{\frac{1}{q}}.$$

Because $\sigma < 1$, and $p, q \in (0, 1)$, we have $\mu_1, \mu_2 < 1$. Then, the inequalities (3.20) and (3.21) become

$$\left|d(\varkappa_{2j+1},\varkappa_{2j+2})\right| \leq \mu_1 \left|d(\varkappa_{2j},\varkappa_{2j+1})\right| \tag{3.22}$$

and

$$\left|d(\varkappa_{2j+2},\varkappa_{2j+3})\right| \leq \mu_2 \left|d(\varkappa_{2j+1},\varkappa_{2j+2})\right|. \tag{3.23}$$

Let $\mu = \max\{\mu_1,\mu_2\} < 1$. Then, from (3.22) and (3.23), we conclude

$$\left|d(\varkappa_j,\varkappa_{j+1})\right| \leq \mu \left|d(\varkappa_{j-1},\varkappa_j)\right|$$

for all $j \in \mathbb{N}$. Hence,

$$\left|d(\varkappa_j,\varkappa_{j+1})\right| \leq \mu \left|d(\varkappa_{j-1},\varkappa_j)\right| \leq \mu^2 \left|d(\varkappa_{j-2},\varkappa_{j-1})\right| \leq ... \leq \mu^j \left|d(\varkappa_0,\varkappa_1)\right|, \tag{3.24}$$

$\forall$ $\mathbb{N}$. By using the same procedure as we have done in Theorem 1, we can prove that $\{\varkappa_j\}$ is a Cauchy sequence in $\Omega$. As $\Omega$ is complete, there exists a point $\varkappa^* \in \Omega$ such that

$$\lim_{j\to\infty} \varkappa_j = \varkappa^*. \tag{3.25}$$

From (3.19) with $\varkappa = \varkappa^*$ and $\varsigma = \varkappa_{2j+1}$, we have

$$\begin{aligned}
d(\varkappa^*,\mathcal{M}\varkappa^*) &\leq d\left(\varkappa^*,\varkappa_{2j+2}\right) + d\left(\varkappa_{2j+2},\mathcal{M}\varkappa^*\right) + \curlywedge(\varkappa^*,\mathcal{M}\varkappa^*)d\left(\varkappa^*,\varkappa_{2j+2}\right)d\left(\varkappa_{2j+2},\mathcal{M}\varkappa^*\right)\\
&= d\left(\varkappa^*,\varkappa_{2j+2}\right) + d\left(\aleph\varkappa_{2j+1},\mathcal{M}\varkappa^*\right) + \curlywedge(\varkappa^*,\mathcal{M}\varkappa^*)d\left(\varkappa^*,\varkappa_{2j+2}\right)d\left(\aleph\varkappa_{2j+1},\mathcal{M}\varkappa^*\right)\\
&= d\left(\varkappa^*,\varkappa_{2j+2}\right) + d\left(\mathcal{M}\varkappa^*,\aleph\varkappa_{2j+1}\right) + \curlywedge(\varkappa^*,\mathcal{M}\varkappa^*)d\left(\varkappa^*,\varkappa_{2j+2}\right)d\left(\mathcal{M}\varkappa^*,\aleph\varkappa_{2j+1}\right)\\
&\leq d\left(\varkappa^*,\varkappa_{2j+2}\right) + \sigma\left[d(\varkappa^*,\mathcal{M}\varkappa^*)\right]^p\cdot\left[d(\varkappa_{2j+1},\aleph\varkappa_{2j+1})\right]^q\\
&\quad \cdot\left[\frac{d(\varkappa^*,\mathcal{M}\varkappa^*)d(\varkappa_{2j+1},\aleph\varkappa_{2j+1})}{1+d(\varkappa^*,\varkappa_{2j+1})}\right]^r\\
&\quad + \curlywedge(\varkappa^*,\mathcal{M}\varkappa^*)\sigma d\left(\varkappa^*,\varkappa_{2j+2}\right)\left[d(\varkappa^*,\mathcal{M}\varkappa^*)\right]^p\cdot\left[d(\varkappa_{2j+1},\aleph\varkappa_{2j+1})\right]^q\\
&\quad \cdot\left[\frac{d(\varkappa^*,\mathcal{M}\varkappa^*)d(\varkappa_{2j+1},\aleph\varkappa_{2j+1})}{1+d(\varkappa^*,\varkappa_{2j+1})}\right]^r\\
&= d\left(\varkappa^*,\varkappa_{2j+2}\right) + \sigma\left[d(\varkappa^*,\mathcal{M}\varkappa^*)\right]^p\cdot\left[d(\varkappa_{2j+1},\varkappa_{2j+2})\right]^q\\
&\quad \cdot\left[\frac{d(\varkappa^*,\mathcal{M}\varkappa^*)d(\varkappa_{2j+1},\varkappa_{2j+2})}{1+d(\varkappa^*,\varkappa_{2j+1})}\right]^r\\
&\quad + \curlywedge(\varkappa^*,\mathcal{M}\varkappa^*)\sigma d\left(\varkappa^*,\varkappa_{2j+2}\right)\left[d(\varkappa^*,\mathcal{M}\varkappa^*)\right]^p\cdot\left[d(\varkappa_{2j+1},\varkappa_{2j+2})\right]^q\\
&\quad \cdot\left[\frac{d(\varkappa^*,\mathcal{M}\varkappa^*)d(\varkappa_{2j+1},\varkappa_{2j+2})}{1+d(\varkappa^*,\varkappa_{2j+1})}\right]^r,
\end{aligned} \tag{3.26}$$

which implies

$$\begin{aligned}
\left|d(\varkappa^*,\mathcal{M}\varkappa^*)\right| &\leq \left|d\left(\varkappa^*,\varkappa_{2j+2}\right)\right| + \sigma\left|d(\varkappa^*,\mathcal{M}\varkappa^*)\right|^p\cdot\left|d(\varkappa_{2j+1},\varkappa_{2j+2})\right|^q\\
&\quad \cdot\left|d(\varkappa^*,\mathcal{M}\varkappa^*)\right|^r\left|\frac{d(\varkappa_{2j+1},\varkappa_{2j+2})}{1+d(\varkappa^*,\varkappa_{2j+1})}\right|^r
\end{aligned}$$

$$+ \curlywedge (\varkappa^*, \mathcal{M}\varkappa^*)\sigma \left| d\left(\varkappa^*, \varkappa_{2j+2}\right) \right| |d(\varkappa^*, \mathcal{M}\varkappa^*)|^p \cdot \left| d(\varkappa_{2j+1}, \varkappa_{2j+2}) \right|^q$$
$$\cdot |d(\varkappa^*, \mathcal{M}\varkappa^*)|^r \left| \frac{d(\varkappa_{2j+1}, \varkappa_{2j+2})}{1 + d(\varkappa^*, \varkappa_{2j+1})} \right|^r . \tag{3.27}$$

Taking the limit as $j \to +\infty$ in inequality (3.27), we get

$$|d(\varkappa^*, \mathcal{M}\varkappa^*)| = 0;$$

that is, $\varkappa^* = \mathcal{M}\varkappa^*$. Similarly, $\varkappa^*$ is an FP of $\aleph$. Hence, $\varkappa^*$ is a CFP of the mappings $\mathcal{M}$ and $\aleph$. □

**Example 7.** Let $\Omega = \{0, \frac{1}{4}, \frac{1}{2}, 1\}$. Define $d : \Omega \times \Omega \to \mathbb{C}$ by

$$d(\varkappa, \varsigma) = |\varkappa - \varsigma| \, e^{ik}$$

for all $\varkappa, \varsigma \in \Omega$, and $i^2 = -1$, where $k = 2\pi$. Define $\curlywedge : \Omega \times \Omega \to [0, +\infty)$ by

$$\curlywedge(\varkappa, \varsigma) = \frac{1}{2}$$

for all $\varkappa, \varsigma \in \Omega$. Then, $(\Omega, d)$ is a complete C-VESMS. Define $\mathcal{M}, \aleph : \Omega \to \Omega$ by

$$\mathcal{M}(\varkappa) = \begin{cases} \frac{1}{4}, & \text{if } \varkappa \in \left\{0, \frac{1}{4}, 1\right\}, \\ \frac{1}{2}, & \text{if } \varkappa = \frac{1}{2} \end{cases}$$

and $\aleph = \mathcal{M}$ for all $\varkappa \in \Omega$. Because

$$Fix\{\mathcal{M}, \aleph\} = \{\frac{1}{4}, \frac{1}{2}\},$$

we have

$$\Omega \backslash Fix\{\aleph\} = \{0, 1\}.$$

Let

$$p = 0.2, \; q = 0.3, \; r = 0.5,$$

then,

$$p + q + r = 1.$$

Now, let $\varkappa, \varsigma \in \Omega \backslash Fix\{\mathcal{M}, \aleph\}$, and then, $(\varkappa, \varsigma) \in \{(0, 1), (1, 0)\}$. Then, the inequality (3.19) holds for $\sigma = \frac{4}{5} \in [0, 1)$. Moreover, for any $\varkappa_0 \in \Omega$, defining the sequence $\{\varkappa_j\}$ by

$$\varkappa_{2j+1} = \mathcal{M}\varkappa_{2j} \text{ and } \varkappa_{2j+2} = \aleph\varkappa_{2j+1},$$

$\forall \; j \in \mathbb{N} \cup \{0\}$ satisfies

$$\sup_{j,m \in \mathbb{N} \cup \{0\}} \curlywedge \left(\varkappa_j, \varkappa_m\right) = \frac{1}{2} < \infty.$$

Thus, all the hypotheses of Theorem 3 are satisfied. Consequently, the mappings $\mathcal{M}$ and $\aleph$ have CFPs that are $\frac{1}{4}$ and $\frac{1}{2}$.

**Corollary 5.** *Let $\aleph : \Omega \to \Omega$. Suppose that there exist constants $\sigma \in [0, 1)$ and $p, q, r \in (0, 1)$ with $p + q + r = 1$ such that*

$$d(\aleph\varkappa, \aleph\varsigma) \leq \sigma\,[d(\varkappa,\aleph\varkappa)]^{p} \cdot [d(\varsigma, \aleph\varsigma)]^{q} \cdot \left[\frac{d(\varkappa,\aleph\varkappa)d(\varsigma, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)}\right]^{r}, \tag{3.28}$$

holds for all $\varkappa, \varsigma \in \Omega\backslash Fix\{\aleph\}$. Furthermore, assume that for any $\varkappa_0 \in \Omega$, the sequence $\{\varkappa_j\}$ defined by

$$\varkappa_{j+1} = \aleph\varkappa_j,$$

$\forall\ j \in \mathbb{N} \cup \{0\}$ satisfies

$$\sup_{j,m\in\mathbb{N}\cup\{0\}} \curlywedge\left(\varkappa_j, \varkappa_m\right) < \infty.$$

Then, $\aleph$ admits an FP in $\Omega$.

*Proof.* Take $\mathcal{M} = \aleph$ in Theorem 3. □

**Example 8.** Let $\Omega = \{0, a\}$ with $a > 0$. Define $d : \Omega \times \Omega \to \mathbb{C}$ by

$$d(\varkappa, \varsigma) = |\varkappa - \varsigma| + i\,|\varkappa - \varsigma|\,,$$

for all $\varkappa, \varsigma \in \Omega$ and $i^2 = -1$. Define $\curlywedge : \Omega \times \Omega \to [0, +\infty)$ by

$$\curlywedge(\varkappa, \varsigma) = \frac{1}{2},$$

for all $\varkappa, \varsigma \in \Omega$. Then, $(\Omega, d)$ is a complete C-VESMS. Define $\aleph : \Omega \to \Omega$ by

$$\aleph\,(0) = 0,\ \aleph\,(a) = 0 \neq a.$$

Because

$$Fix\{\aleph\} = \{0\},$$

we have

$$\Omega\backslash Fix\{\aleph\} = \{a\}\,.$$

Let

$$p = 0.2,\ q = 0.3,\ r = 0.5,$$

then,

$$p + q + r = 1$$

and choose

$$\sigma = \frac{1}{2} \in [0, 1).$$

Then, the inequality (3.28) holds for all $\varkappa, \varsigma \in \Omega\backslash Fix\{\aleph\} = \{a\}$. Moreover, for any $\varkappa_0 \in \Omega$, defining the sequence $\{\varkappa_j\}$ by

$$\varkappa_{j+1} = \aleph\varkappa_j,$$

$\forall\ j \in \mathbb{N} \cup \{0\}$ satisfies

$$\sup_{j,m\in\mathbb{N}\cup\{0\}} \curlywedge\left(\varkappa_j, \varkappa_m\right) < \infty.$$

Hence, all the hypotheses of Corollary 5 are satisfied. Then, $\aleph$ has an FP $\varkappa^* = 0$ in $\Omega$.

**Corollary 6.** *Let $\mathcal{M}, \aleph : \Omega \to \Omega$. Suppose that there exist constants $\sigma \in [0, 1)$ and $r \in (0, 1)$ such that*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \;\; \leq \;\; \sigma \left[d(\varkappa,\mathcal{M}\varkappa) \cdot d(\varsigma, \aleph\varsigma)\right]^{\frac{1-r}{2}} \cdot \left[\frac{d(\varkappa,\aleph\varkappa)d(\varsigma, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)}\right]^r$$

holds for all $\varkappa, \varsigma \in \Omega \backslash Fix\{\mathcal{M},\aleph\}$. Furthermore, assume that for any $\varkappa_0 \in \Omega$, the sequence $\{\varkappa_j\}$ defined by

$$\varkappa_{2j+1} = \mathcal{M}\varkappa_{2j} \text{ and } \varkappa_{2j+2} = \aleph\varkappa_{2j+1},$$

$\forall\, j \in \mathbb{N} \cup \{0\}$ satisfies

$$\sup_{j,m\in\mathbb{N}\cup\{0\}} \curlywedge\left(\varkappa_j, \varkappa_m\right) < \infty.$$

Then, $\mathcal{M}$, $\aleph$ admit a CFP in $\Omega$.

*Proof.* Take $p = q = \frac{1-r}{2}$ in Theorem 3. □

## 4. Corollaries of the principal results

In this section, we present the consequences of our main results by first deriving FP theorems in C-VSMSs, followed by their extensions to SMSs.

### *4.1. Complex-valued suprametric spaces and fixed-point results*

If we define the function $\curlywedge : \Omega \times \Omega \to [0, +\infty)$ by $\curlywedge(\varkappa, \varsigma) = \curlywedge \in \mathbb{R}^+ \cup \{0\}$ as in Definition 6, the concept of a C-VESMS reduces to that of a C-VSMS. This simplification allows us to derive the following results in C-VSMSs. Throughout this subsection, we assume that $(\Omega, d)$ is a complete C-VSMS.

**Corollary 7.** *Let $\mathcal{M}, \aleph : \Omega \to \Omega$. Suppose that there exist the constants $\sigma_1, \sigma_2, \sigma_3 \in [0, 1)$ with $\sigma_1 + \sigma_2 + \sigma_3 < 1$ such that for all $\varkappa, \varsigma \in \Omega$,*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \leq \sigma_1 d(\varkappa, \varsigma) + \sigma_2 \frac{d(\varkappa,\mathcal{M}\varkappa)d(\varsigma, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)} + \sigma_3 \frac{d(\varsigma,\mathcal{M}\varkappa)d(\varkappa, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)}.$$

*Then, there exists a unique point $\varkappa^* \in \Omega$ such that $\mathcal{M}\varkappa^* = \aleph\varkappa^* = \varkappa^*$.*

*Proof.* Define $\curlywedge : \Omega \times \Omega \to [0, +\infty)$ by $\curlywedge(\varkappa, \varsigma) = \curlywedge \in \mathbb{R}^+ \cup \{0\}$ in Theorem 1. □

**Corollary 8.** *Let $\Psi_1$ and $\Psi_2$ be nonempty, closed subsets of $\Omega$. Consider the mappings $\mathcal{M}, \aleph : \Psi_1 \cup \Psi_2 \to \Psi_1 \cup \Psi_2$ satisfying*

$$\mathcal{M}(\Psi_1) \subseteq \Psi_2,\; \mathcal{M}(\Psi_2) \subseteq \Psi_1 \textit{ and } \aleph(\Psi_1) \subseteq \Psi_2,\; \aleph(\Psi_2) \subseteq \Psi_1.$$

*Suppose that there is a constant $\sigma \in [0, \frac{1}{2})$ such that*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \leq \sigma\left(d(\varkappa,\mathcal{M}\varkappa) + d(\varsigma, \aleph\varsigma)\right),$$

*holds for all $\varkappa \in \Psi_1, \varsigma \in \Psi_2$. Then, there exists a unique point $\varkappa^* \in \Psi_1 \cap \Psi_2$ such that $\mathcal{M}\varkappa^* = \aleph\varkappa^* = \varkappa^*$.*

*Proof.* Define $\curlywedge : \Omega \times \Omega \to [0, +\infty)$ by $\curlywedge(\varkappa, \varsigma) = \curlywedge \in \mathbb{R}^+ \cup \{0\}$ in Theorem 2. □

**Corollary 9.** *Let $\mathcal{M}, \aleph : \Omega \to \Omega$. Suppose that there exist constants $\sigma \in [0, 1)$ and $p, q, r \in (0, 1)$ with $p + q + r = 1$ such that*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \leq \sigma\,[d(\varkappa, \mathcal{M}\varkappa)]^p \cdot [d(\varsigma, \aleph\varsigma)]^q \cdot \left[\frac{d(\varkappa, \mathcal{M}\varkappa)d(\varsigma, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)}\right]^r$$

holds for all $\varkappa, \varsigma \in \Omega \backslash Fix\{\mathcal{M}, \aleph\}$. Then, $\mathcal{M}$, $\aleph$ admit a CFP in $\Omega$.

*Proof.* Define $\curlywedge : \Omega \times \Omega \to [0, +\infty)$ by $\curlywedge(\varkappa, \varsigma) = \curlywedge \in \mathbb{R}^+ \cup \{0\}$ in Theorem 3. □

**Corollary 10.** *Let $\mathcal{M}, \aleph : \Omega \to \Omega$. Suppose that there exist constants $\sigma \in [0, 1)$ and $r \in (0, 1)$ such that*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \quad \leq \quad \sigma\,[d(\varkappa, \mathcal{M}\varkappa) \cdot d(\varsigma, \aleph\varsigma)]^{\frac{1-r}{2}} \cdot \left[\frac{d(\varkappa, \aleph\varkappa)d(\varsigma, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)}\right]^r$$

holds for all $\varkappa, \varsigma \in \Omega \backslash Fix\{\mathcal{M}, \aleph\}$. Then, $\mathcal{M}$, $\aleph$ admit a CFP in $\Omega$.

*Proof.* Take $p = q = \frac{1-r}{2}$ in Corollary 18. □

### *4.2. Fixed-point analysis in extended suprametric spaces*

If we restrict the set of complex numbers $\mathbb{C}$ to the set of real numbers in Definition 6, by setting the imaginary part to zero, the concept of a C-VESMS is reduced to that of an E-SMS. This simplification allows us to deduce the following outcomes in E-SMSs. Throughout this subsection, $(\Omega, d)$ is assumed to be a complete E-SMS, with $\curlywedge : \Omega \times \Omega \to [0, +\infty)$.

**Corollary 11.** *Let $\mathcal{M}, \aleph : \Omega \to \Omega$. Suppose that there exist the constants $\sigma_1, \sigma_2, \sigma_3 \in [0, 1)$ with $\sigma_1 + \sigma_2 + \sigma_3 < 1$ such that for all $\varkappa, \varsigma \in \Omega$,*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \leq \sigma_1 d(\varkappa, \varsigma) + \sigma_2 \frac{d(\varkappa, \mathcal{M}\varkappa)d(\varsigma, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)} + \sigma_3 \frac{d(\varsigma, \mathcal{M}\varkappa)d(\varkappa, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)}.$$

*Furthermore, assume that for any $\varkappa_0 \in \Omega$, the sequence $\{\varkappa_j\}$ defined by*

$$\varkappa_{2j+1} = \mathcal{M}\varkappa_{2j} \;\; \textit{and} \;\; \varkappa_{2j+2} = \aleph\varkappa_{2j+1},$$

$\forall\ j \in \mathbb{N} \cup \{0\}$ *satisfies*

$$\sup_{j,m \in \mathbb{N} \cup \{0\}} \curlywedge\left(\varkappa_j, \varkappa_m\right) < \infty.$$

*Then, there exists a unique point $\varkappa^* \in \Omega$ such that $\mathcal{M}\varkappa^* = \aleph\varkappa^* = \varkappa^*$.*

**Corollary 12.** *Let $\mathcal{M}, \aleph : \Omega \to \Omega$. Suppose that there exist the constants $\sigma_1, \sigma_2 \in [0, 1)$ with $\sigma_1 + \sigma_2 < 1$ such that for all $\varkappa, \varsigma \in \Omega$,*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \leq \sigma_1 d(\varkappa, \varsigma) + \sigma_2 \frac{d(\varkappa, \mathcal{M}\varkappa)d(\varsigma, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)}.$$

*Furthermore, assume that for any $\varkappa_0 \in \Omega$, the sequence $\{\varkappa_j\}$ defined by*

$$\varkappa_{2j+1} = \mathcal{M}\varkappa_{2j} \;\; \textit{and} \;\; \varkappa_{2j+2} = \aleph\varkappa_{2j+1},$$

$\forall\ j \in \mathbb{N} \cup \{0\}$ *satisfies*

$$\sup_{j,m \in \mathbb{N} \cup \{0\}} \curlywedge \left(\varkappa_j, \varkappa_m\right) < \infty.$$

*Then, there exists a unique point* $\varkappa^* \in \Omega$ *such that* $\mathcal{M}\varkappa^* = \aleph\varkappa^* = \varkappa^*$.

*Proof.* Take $\sigma_3 = 0$ in Corollary 11. □

**Corollary 13.** *Let* $\Psi_1$ *and* $\Psi_2$ *be nonempty, closed subsets of* $\Omega$. *Consider the mappings* $\mathcal{M}, \aleph : \Psi_1 \cup \Psi_2 \to \Psi_1 \cup \Psi_2$ *satisfying*

$$\mathcal{M}(\Psi_1) \subseteq \Psi_2,\ \mathcal{M}(\Psi_2) \subseteq \Psi_1 \text{ and } \aleph(\Psi_1) \subseteq \Psi_2,\ \aleph(\Psi_2) \subseteq \Psi_1.$$

*Suppose that there is a constant* $\sigma \in [0, \frac{1}{2})$ *such that*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \leq \sigma\left(d(\varkappa, \mathcal{M}\varkappa) + d(\varsigma, \aleph\varsigma)\right)$$

*holds for all* $\varkappa \in \Psi_1, \varsigma \in \Psi_2$. *Furthermore, assume that for any* $\varkappa_0 \in \Psi_1$, *the sequence* $\{\varkappa_j\}$ *defined by*

$$\varkappa_{2j+1} = \mathcal{M}\varkappa_{2j} \in \Psi_2 \ \text{ and } \ \varkappa_{2j+2} = \aleph\varkappa_{2j+1} \in \Psi_1,$$

$\forall\ j \in \mathbb{N} \cup \{0\}$ *satisfies*

$$\sup_{j,m \in \mathbb{N} \cup \{0\}} \curlywedge \left(\varkappa_j, \varkappa_m\right) < \infty.$$

*Then, there exists a unique point* $\varkappa^* \in \Psi_1 \cap \Psi_2$ *such that* $\mathcal{M}\varkappa^* = \aleph\varkappa^* = \varkappa^*$.

**Corollary 14.** *Let* $\mathcal{M}, \aleph : \Omega \to \Omega$. *Suppose that there exist constants* $\sigma \in [0, 1)$ *and* $p, q, r \in (0, 1)$ *with* $p + q + r = 1$ *such that*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \leq \sigma\,[d(\varkappa, \mathcal{M}\varkappa)]^p \cdot [d(\varsigma, \aleph\varsigma)]^q \cdot \left[\frac{d(\varkappa, \mathcal{M}\varkappa) d(\varsigma, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)}\right]^r$$

holds for all $\varkappa, \varsigma \in \Omega \backslash Fix\{\mathcal{M}, \aleph\}$. Furthermore, assume that for any $\varkappa_0 \in \Omega$, the sequence $\{\varkappa_j\}$ defined by

$$\varkappa_{2j+1} = \mathcal{M}\varkappa_{2j} \ \text{ and } \ \varkappa_{2j+2} = \aleph\varkappa_{2j+1},$$

$\forall\ j \in \mathbb{N} \cup \{0\}$ satisfies

$$\sup_{j,m \in \mathbb{N} \cup \{0\}} \curlywedge \left(\varkappa_j, \varkappa_m\right) < \infty.$$

Then, $\mathcal{M}, \aleph$ admit a CFP in $\Omega$.

**Corollary 15.** *Let* $\mathcal{M}, \aleph : \Omega \to \Omega$. *Suppose that there exist constants* $\sigma \in [0, 1)$ *and* $r \in (0, 1)$ *such that*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \ \leq \ \sigma\,[d(\varkappa, \mathcal{M}\varkappa) \cdot d(\varsigma, \aleph\varsigma)]^{\frac{1-r}{2}} \cdot \left[\frac{d(\varkappa, \aleph\varkappa) d(\varsigma, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)}\right]^r,$$

holds for all $\varkappa, \varsigma \in \Omega \backslash Fix\{\mathcal{M}, \aleph\}$. Furthermore, assume that for any $\varkappa_0 \in \Omega$, the sequence $\{\varkappa_j\}$ defined by

$$\varkappa_{2j+1} = \mathcal{M}\varkappa_{2j} \ \text{ and } \ \varkappa_{2j+2} = \aleph\varkappa_{2j+1},$$

$\forall\ j \in \mathbb{N} \cup \{0\}$ satisfies

$$\sup_{j,m \in \mathbb{N} \cup \{0\}} \curlywedge \left(\varkappa_j, \varkappa_m\right) < \infty.$$

Then, $\mathcal{M}, \aleph$ admit a CFP in $\Omega$.

*Proof.* Take $p = q = \frac{1-r}{2}$ in Corollary 14. □

### 4.3. On fixed-points in suprametric spaces

If we define the function $\curlywedge : \Omega \times \Omega \to [0, +\infty)$ by $\curlywedge(\varkappa, \varsigma) = \curlywedge \in \mathbb{R}^+ \cup \{0\}$ and restrict the set of complex numbers $\mathbb{C}$ to the set of real numbers in Definition 6, the concept of a C-VESMS reduces to that of an SMS. This simplification allows us to derive the following results in SMSs. Throughout this subsection, we assume that $(\Omega, d)$ is a complete SMS.

**Corollary 16.** *Let $\mathcal{M}, \aleph : \Omega \to \Omega$. Suppose that there exist the constants $\sigma_1, \sigma_2, \sigma_3 \in [0, 1)$ with $\sigma_1 + \sigma_2 + \sigma_3 < 1$ such that for all $\varkappa, \varsigma \in \Omega$,*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \leq \sigma_1 d(\varkappa, \varsigma) + \sigma_2 \frac{d(\varkappa, \mathcal{M}\varkappa) d(\varsigma, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)} + \sigma_3 \frac{d(\varsigma, \mathcal{M}\varkappa) d(\varkappa, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)}.$$

*Then, there exists a unique point $\varkappa^* \in \Omega$ such that $\mathcal{M}\varkappa^* = \aleph\varkappa^* = \varkappa^*$.*

*Proof.* Define $\curlywedge : \Omega \times \Omega \to [0, +\infty)$ by $\curlywedge(\varkappa, \varsigma) = \curlywedge \in \mathbb{R}^+ \cup \{0\}$ in Theorem 1. $\square$

**Corollary 17.** *Let $\Psi_1$ and $\Psi_2$ be nonempty, closed subsets of $\Omega$. Consider the mappings $\mathcal{M}, \aleph : \Psi_1 \cup \Psi_2 \to \Psi_1 \cup \Psi_2$ satisfying*

$$\mathcal{M}(\Psi_1) \subseteq \Psi_2, \ \mathcal{M}(\Psi_2) \subseteq \Psi_1 \text{ and } \aleph(\Psi_1) \subseteq \Psi_2, \ \aleph(\Psi_2) \subseteq \Psi_1.$$

*Suppose that there is a constant $\sigma \in [0, \frac{1}{2})$ such that*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \leq \sigma \left(d(\varkappa, \mathcal{M}\varkappa) + d(\varsigma, \aleph\varsigma)\right)$$

*holds for all $\varkappa \in \Psi_1, \varsigma \in \Psi_2$. Then, there exists a unique point $\varkappa^* \in \Psi_1 \cap \Psi_2$ such that $\mathcal{M}\varkappa^* = \aleph\varkappa^* = \varkappa^*$.*

*Proof.* Define $\curlywedge : \Omega \times \Omega \to [0, +\infty)$ by $\curlywedge(\varkappa, \varsigma) = \curlywedge \in \mathbb{R}^+ \cup \{0\}$ in Theorem 2. $\square$

**Corollary 18.** *Let $\mathcal{M}, \aleph : \Omega \to \Omega$. Suppose that there exist constants $\sigma \in [0, 1)$ and $p, q, r \in (0, 1)$ with $p + q + r = 1$ such that*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \leq \sigma \left[d(\varkappa, \mathcal{M}\varkappa)\right]^p \cdot \left[d(\varsigma, \aleph\varsigma)\right]^q \cdot \left[\frac{d(\varkappa, \mathcal{M}\varkappa) d(\varsigma, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)}\right]^r$$

holds for all $\varkappa, \varsigma \in \Omega \backslash Fix\{\mathcal{M}, \aleph\}$. Then, $\mathcal{M}$, $\aleph$ admit a CFP in $\Omega$.

*Proof.* Define $\curlywedge : \Omega \times \Omega \to [0, +\infty)$ by $\curlywedge(\varkappa, \varsigma) = \curlywedge \in \mathbb{R}^+ \cup \{0\}$ in Theorem 3. $\square$

**Corollary 19.** *Let $\mathcal{M}, \aleph : \Omega \to \Omega$. Suppose that there exist constants $\sigma \in [0, 1)$ and $r \in (0, 1)$ such that*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \quad \leq \quad \sigma \left[d(\varkappa, \mathcal{M}\varkappa) \cdot d(\varsigma, \aleph\varsigma)\right]^{\frac{1-r}{2}} \cdot \left[\frac{d(\varkappa, \aleph\varkappa) d(\varsigma, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)}\right]^r$$

holds for all $\varkappa, \varsigma \in \Omega \backslash Fix\{\mathcal{M}, \aleph\}$. Then, $\mathcal{M}$, $\aleph$ admit a CFP in $\Omega$.

*Proof.* Take $p = q = \frac{1-r}{2}$ in Corollary 18. $\square$

*4.4. Fixed-points in the setting of complex-valued spaces*

If we define the function $\curlywedge : \Omega \times \Omega \to [0, +\infty)$ by $\curlywedge(\varkappa, \varsigma) = 0$ as in Definition 6, the concept of a C-VESMS reduces to that of a C-VMS. This simplification allows us to derive the following results in C-VMSs. Throughout this subsection, we assume that $(\Omega, d)$ is a complete C-VMS.

**Corollary 20.** *Let $\mathcal{M}, \aleph : \Omega \to \Omega$. Suppose that there exist the constants $\sigma_1, \sigma_2, \sigma_3 \in [0, 1)$ with $\sigma_1 + \sigma_2 + \sigma_3 < 1$ such that for all $\varkappa, \varsigma \in \Omega$,*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \leq \sigma_1 d(\varkappa, \varsigma) + \sigma_2 \frac{d(\varkappa, \mathcal{M}\varkappa) d(\varsigma, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)} + \sigma_3 \frac{d(\varsigma, \mathcal{M}\varkappa) d(\varkappa, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)}.$$

*Then, there exists a unique point $\varkappa^* \in \Omega$ such that $\mathcal{M}\varkappa^* = \aleph\varkappa^* = \varkappa^*$.*

*Proof.* Define $\curlywedge : \Omega \times \Omega \to [0, +\infty)$ by $\curlywedge(\varkappa, \varsigma) = 0$ in Theorem 1. □

**Corollary 21.** *Let $\Psi_1$ and $\Psi_2$ be nonempty, closed subsets of $\Omega$. Consider the mappings $\mathcal{M}, \aleph : \Psi_1 \cup \Psi_2 \to \Psi_1 \cup \Psi_2$ satisfying*

$$\mathcal{M}(\Psi_1) \subseteq \Psi_2,\ \mathcal{M}(\Psi_2) \subseteq \Psi_1 \text{ and } \aleph(\Psi_1) \subseteq \Psi_2,\ \aleph(\Psi_2) \subseteq \Psi_1.$$

*Suppose that there is a constant $\sigma \in [0, \frac{1}{2})$ such that*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \leq \sigma \left(d(\varkappa, \mathcal{M}\varkappa) + d(\varsigma, \aleph\varsigma)\right)$$

*holds for all $\varkappa \in \Psi_1, \varsigma \in \Psi_2$. Then, there exists a unique point $\varkappa^* \in \Psi_1 \cap \Psi_2$ such that $\mathcal{M}\varkappa^* = \aleph\varkappa^* = \varkappa^*$.*

*Proof.* Define $\curlywedge : \Omega \times \Omega \to [0, +\infty)$ by $\curlywedge(\varkappa, \varsigma) = 0$ in Theorem 2. □

**Corollary 22.** *Let $\mathcal{M}, \aleph : \Omega \to \Omega$. Suppose that there exist constants $\sigma \in [0, 1)$ and $p, q, r \in (0, 1)$ with $p + q + r = 1$ such that*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \leq \sigma \left[d(\varkappa, \mathcal{M}\varkappa)\right]^p \cdot \left[d(\varsigma, \aleph\varsigma)\right]^q \cdot \left[\frac{d(\varkappa, \mathcal{M}\varkappa) d(\varsigma, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)}\right]^r$$

holds for all $\varkappa, \varsigma \in \Omega \backslash Fix\{\mathcal{M}, \aleph\}$. Then, $\mathcal{M}$, $\aleph$ admit a CFP in $\Omega$.

*Proof.* Define $\curlywedge : \Omega \times \Omega \to [0, +\infty)$ by $\curlywedge(\varkappa, \varsigma) = 0$ in Theorem 3. □

**Corollary 23.** *Let $\mathcal{M}, \aleph : \Omega \to \Omega$. Suppose that there exist constants $\sigma \in [0, 1)$ and $r \in (0, 1)$ such that*

$$d(\mathcal{M}\varkappa, \aleph\varsigma) \quad \leq \quad \sigma \left[d(\varkappa, \mathcal{M}\varkappa) \cdot d(\varsigma, \aleph\varsigma)\right]^{\frac{1-r}{2}} \cdot \left[\frac{d(\varkappa, \aleph\varkappa) d(\varsigma, \aleph\varsigma)}{1 + d(\varkappa, \varsigma)}\right]^r$$

holds for all $\varkappa, \varsigma \in \Omega \backslash Fix\{\mathcal{M}, \aleph\}$. Then, $\mathcal{M}$, $\aleph$ admit a CFP in $\Omega$.

*Proof.* Take $p = q = \frac{1-r}{2}$ in Corollary 22. □

## 5. Applications

Fractional differential operators with nonsingular kernels have attracted considerable attention in recent years due to their ability to describe memory and hereditary effects more realistically than classical fractional derivatives. Their applications to nonlinear problems arising in physics and other applied fields have also motivated both analytical and numerical investigations [19]. Among these operators, the Atangana-Baleanu derivative in the Caputo sense plays a prominent role, as it incorporates the Mittag-Leffler kernel, ensuring nonlocality without singular behavior and providing improved modeling flexibility in applied problems. Motivated by these features, this section is devoted to the study of a nonlinear fractional integro-differential initial value problem governed by the Atangana-Baleanu-Caputo derivative of order $0 < \alpha < 1$.

By employing the associated Atangana-Baleanu fractional integral, the considered problem is transformed into an equivalent integral equation, which allows the use of FP techniques. To this end, an appropriate operator is constructed on the Banach space $C([0,T],\mathbb{R})$, endowed with a complex-valued extended metric structure. Under suitable Lipschitz-type conditions on the nonlinear terms, we establish the existence and uniqueness of the solution by verifying that the associated operator is a contraction. This approach highlights the effectiveness of complex-valued metric methods in the analysis of fractional integro-differential equations involving nonsingular kernels.

For $0 < \alpha < 1$, the Atangana-Baleanu derivative in the Caputo sense (ABC derivative) of a function $\varkappa \in C^1([0,T],\mathbb{R})$ is defined by

$$^{ABC}D_t^{\alpha}(\varkappa(t)) = \frac{B(\alpha)}{1-\alpha}\int_0^t \varkappa'(s)E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-s)^{\alpha}\right)ds,$$

where

- $E_\alpha(\cdot)$ is the Mittag-Leffler function;
- $B(\alpha)$ is a normalization constant with $B(0) = B(1) = 1$.

Consider the initial value problem

$$\begin{cases} ^{ABC}D_t^{\alpha}(\varkappa(t)) = f\left(t, \varkappa(t), \int_0^t K(t,s,\varkappa(s))\,ds\right), \quad t \in [0,T], \\ \varkappa(0) = \varkappa_0. \end{cases} \tag{5.1}$$

Using the associated Atangana–Baleanu fractional integral, the above problem is equivalent to

$$\begin{aligned} \varkappa(t) &= \varkappa_0 + \frac{1-\alpha}{B(\alpha)} f\left(t, \varkappa(t), \int_0^t K(t,s,\varkappa(s))\,ds\right) \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, \varkappa(s), \int_0^s K(s,\tau,\varkappa(\tau))\,d\tau\right)ds. \end{aligned}$$

For more complex nonlinear problems involving the Atangana-Baleanu-Caputo operator, an analytical solution may not be available, and numerical approximation may therefore be required. The above equivalent integral formulation provides a convenient basis for numerical treatment. In particular, a time discretization combined with suitable quadrature or product-integration techniques can be employed to approximate the history-dependent integral terms. Because the kernel $(t-s)^{\alpha-1}$ is weakly singular for $0<\alpha<1$, a quadrature procedure adapted to this fractional kernel is preferable. The nonlinear terms can then be treated by a predictor-corrector or FP iterative procedure at each time step. Such a strategy allows the numerical method to account for both the nonlinear dependence and the accumulated memory effects of the ABC operator. Related numerical treatments of fractional models involving Caputo-type operators have been considered in the literature, including the studies reported in [20–22]. A detailed numerical implementation, together with stability, convergence, and error analysis, is beyond the scope of the present theoretical study and will be considered in future research.

**Theorem 4.** *Let* $\Omega = C([0,T],\mathbb{R})$, *and define* $d:\Omega\times\Omega\to\mathbb{C}$ *by*

$$d(\varkappa,\varsigma)=\|\varkappa-\varsigma\|_\infty(1+i)=\sup_{t\in[0,T]}|\varkappa(t)-\varsigma(t)|(1+i).$$

*Then,* $(\Omega,d)$ *is a complete C-VESMS with* $\curlywedge:\Omega\times\Omega\to[0,+\infty)$ *defined by* $\curlywedge(\varkappa,\varsigma)=1$ *for all* $\varkappa,\varsigma\in\Omega$. *Define the operator* $\aleph:\Omega\to\Omega$ *as follows:*

$$\begin{aligned}(\aleph\varkappa)(t) &= \varkappa_0+\frac{1-\alpha}{B(\alpha)}f\left(t,\varkappa(t),\int_0^t K(t,s,\varkappa(s))\,ds\right)\\ &\quad+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f\left(s,\varkappa(s),\int_0^s K(s,\tau,\varkappa(\tau))\,d\tau\right)ds.\end{aligned}$$

*Assume that the following conditions hold.*

(i) $f$ is Lipschitz continuous; that is,

$$\left|f(t,\varkappa,\varsigma)-f\left(t,\varkappa',\varsigma'\right)\right|\le L_f\left(\left|\varkappa-\varkappa'\right|+\left|\varsigma-\varsigma'\right|\right),$$

for some constant $L_f\ge 0$,

(ii) $K$ is Lipschitz in the third variable

$$|K(t,s,\varkappa)-K(t,s,\varsigma)|\le L_K|\varkappa-\varsigma|.$$

(iii) Let

$$\sigma=\left[\frac{1-\alpha}{B(\alpha)}+\frac{\alpha}{B(\alpha)\Gamma(\alpha+1)}\right]L_f(1+TL_K),$$

and assume $\sigma<1$. Then, the initial value problem (5.1) has a unique solution $\varkappa^*\in C([0,T],\mathbb{R})$.

*Proof.* For any $\varkappa,\varsigma\in\Omega$ and fixed $t\in[0,T]$, we have

$$(\aleph\varkappa)(t)-(\aleph\varsigma)(t) = \frac{1-\alpha}{B(\alpha)}\left[f\left(t,\varkappa(t),\int_0^t K(t,s,\varkappa(s))\,ds\right)-f\left(t,\varsigma(t),\int_0^t K(t,s,\varsigma(s))\,ds\right)\right]$$

$$+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[\begin{array}{c} f\left(s,\varkappa(s),\int_0^s K(s,\tau,\varkappa(\tau))\,d\tau\right) \\ -f\left(s,\varsigma(s),\int_0^s K(s,\tau,\varsigma(\tau))\,d\tau\right)\end{array}\right]ds. \tag{5.2}$$

Taking the absolute on both side and using the triangle inequality, we get

$$\begin{aligned}
|(\aleph\varkappa)(t)-(\aleph\varsigma)(t)| &\le \frac{1-\alpha}{B(\alpha)}\left|f\left(t,\varkappa(t),\int_0^t K(t,s,\varkappa(s))\,ds\right)-f\left(t,\varsigma(t),\int_0^t K(t,s,\varsigma(s))\,ds\right)\right| \\
&\quad +\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|\begin{array}{c} f\left(s,\varkappa(s),\int_0^s K(s,\tau,\varkappa(\tau))\,d\tau\right) \\ -f\left(s,\varsigma(s),\int_0^s K(s,\tau,\varsigma(\tau))\,d\tau\right)\end{array}\right|ds.
\end{aligned}$$

By the assumption (i), we have

$$\begin{aligned}
|(\aleph\varkappa)(t)-(\aleph\varsigma)(t)| &\le \frac{1-\alpha}{B(\alpha)}L_f\left(|\varkappa(t)-\varsigma(t)|+\left|\int_0^t K(t,s,\varkappa(s))\,ds-\int_0^t K(t,s,\varsigma(s))\,ds\right|\right) \\
&\quad +\frac{\alpha}{B(\alpha)\Gamma(\alpha)}L_f\int_0^t (t-s)^{\alpha-1}\left(\begin{array}{c} |\varkappa(s)-\varsigma(s)| \\ +\left|\begin{array}{c}\int_0^s K(s,\tau,\varkappa(\tau))\,d\tau \\ -\int_0^s K(s,\tau,\varsigma(\tau))\,d\tau\end{array}\right|\end{array}\right)ds \\
&\le \frac{1-\alpha}{B(\alpha)}L_f\left(|\varkappa(t)-\varsigma(t)|+\int_0^t |K(t,s,\varkappa(s))-K(t,s,\varsigma(s))|\,ds\right) \\
&\quad +\frac{\alpha}{B(\alpha)\Gamma(\alpha)}L_f\int_0^t (t-s)^{\alpha-1}\left(\begin{array}{c} |\varkappa(s)-\varsigma(s)| \\ +\int_0^s\left|\begin{array}{c}K(s,\tau,\varkappa(\tau)) \\ -K(s,\tau,\varsigma(\tau))\end{array}\right|d\tau\end{array}\right)ds. 
\end{aligned} \tag{5.3}$$

By the assumption (ii), we have

$$\begin{aligned}
\int_0^t |K(t,s,\varkappa(s))-K(t,s,\varsigma(s))|\,ds &\le \int_0^t L_K\,|\varkappa(s)-\varsigma(s)|\,ds \\
&\le TL_K\,\|\varkappa-\varsigma\|_\infty .
\end{aligned}$$

Similarly,

$$\int_0^s |K(s,\tau,\varkappa(\tau))-K(s,\tau,\varsigma(\tau))|\,d\tau \le TL_K\,\|\varkappa-\varsigma\|_\infty .$$

Because

$$|\varkappa(t)-\varsigma(t)| \le \sup_{t\in[0,T]}|\varkappa(t)-\varsigma(t)| = \|\varkappa-\varsigma\|_\infty$$

and

$$|\varkappa(s) - \varsigma(s)| \leq \sup_{s\in[0,T]} |\varkappa(s) - \varsigma(s)| = \|\varkappa - \varsigma\|_\infty .$$

Thus, inequality (5.3) becomes

$$\begin{aligned} |(\aleph\varkappa)(t) - (\aleph\varsigma)(t)| &\leq \frac{1-\alpha}{B(\alpha)} L_f \left(\|\varkappa - \varsigma\|_\infty + T L_K \|\varkappa - \varsigma\|_\infty\right) \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} L_f \int_0^t (t-s)^{\alpha-1} \begin{pmatrix} \|\varkappa - \varsigma\|_\infty \\ + T L_K \|\varkappa - \varsigma\|_\infty \end{pmatrix} ds, \end{aligned}$$

which implies that

$$|(\aleph\varkappa)(t) - (\aleph\varsigma)(t)| \leq L_f (1 + T L_K) \|\varkappa - \varsigma\|_\infty \begin{bmatrix} \frac{1-\alpha}{B(\alpha)} \\ + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \end{bmatrix}. \tag{5.4}$$

Now, because

$$\int_0^t (t-s)^{\alpha-1} ds = \frac{t^\alpha}{\alpha} \leq \frac{T^\alpha}{\alpha},$$

we have

$$\frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \leq \frac{T^\alpha}{B(\alpha)\Gamma(\alpha+1)}. \tag{5.5}$$

Taking the supremum over $t \in [0, T]$ and using (5.5) in inequality (5.4), we have

$$\|\aleph\varkappa - \aleph\varsigma\|_\infty = \sup_{t\in[0,T]} |(\aleph\varkappa)(t) - (\aleph\varsigma)(t)| \leq \sigma \|\varkappa - \varsigma\|_\infty ,$$

where

$$\sigma = \left[\frac{1-\alpha}{B(\alpha)} + \frac{\alpha}{B(\alpha)\Gamma(\alpha+1)}\right] L_f (1 + T L_K) < 1,$$

by Hypothesis (iii). Thus,

$$\|\aleph\varkappa - \aleph\varsigma\|_\infty \leq \sigma \|\varkappa - \varsigma\|_\infty .$$

Now, by the definition of $d$,

$$\begin{aligned} d(\aleph\varkappa, \aleph\varsigma) &= \|\aleph\varkappa - \aleph\varsigma\|_\infty (1+i) \leq \sigma \|\varkappa - \varsigma\|_\infty (1+i) \\ &= \sigma d(\varkappa, \varsigma). \end{aligned}$$

Thus, all the conditions of Corollary 1 are satisfied with $\sigma_1 = \sigma$ and $\sigma_2 = \sigma_3 = 0$, and the operator defined by (5.2) has a unique $\varkappa^* \in C([0, T], \mathbb{R})$, which is a solution of the initial value problem (5.1).

□

**Example 9.** Let

$$T = 1, 0 < \alpha = \frac{1}{2} < 1,\ B(\alpha) = 1,$$

and consider the initial value problem

$$\begin{cases} {}^{ABC}D_t^{\alpha}(\varkappa(t)) = f\left(t, \varkappa(t), \int\limits_0^t K(t, s, \varkappa(s))\, ds\right), \quad t \in [0, 1], \\ \varkappa(0) = \varkappa_0, \end{cases} \tag{5.6}$$

where the functions $f$ and $K$ are defined by

$$f(t, \varkappa, \varsigma) = \frac{1}{6}(\varkappa + \varsigma), \quad K(t, s, \varkappa) = \frac{1}{4}\varkappa.$$

Now, for any $\varkappa, \varkappa^{/}, \varsigma, \varsigma^{/} \in \mathbb{R}$, we have

$$\left| f(t, \varkappa, \varsigma) - f\left(t, \varkappa^{/}, \varsigma^{/}\right)\right| = \frac{1}{6}\left(\left|\varkappa - \varkappa^{/}\right| + \left|\varsigma - \varsigma^{/}\right|\right).$$

Hence, $f$ is Lipschitz continuous with $L_f = \frac{1}{6}$. Thus, Assumption (i) is satisfied. Also, for any $\varkappa, \varsigma \in \mathbb{R}$, we have

$$|K(t, s, \varkappa) - K(t, s, \varsigma)| = \frac{1}{4}|\varkappa - \varsigma|.$$

Thus, $L_K = \frac{1}{4}$. Hence, Condition (ii) of above theorem is satisfied. Now, because $T = 1$, and $\alpha = \frac{1}{2}$,

$$\Gamma(\alpha + 1) = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Thus,

$$\begin{aligned} \sigma &= \left[\frac{1-\alpha}{B(\alpha)} + \frac{\alpha}{B(\alpha)\Gamma(\alpha+1)}\right] L_f (1 + T L_K) \\ &= \left[\frac{1}{2} + \frac{\frac{1}{2}}{\Gamma\left(\frac{3}{2}\right)}\right] \frac{1}{6}\left(1 + \frac{1}{6}\right) \\ &= \left[\frac{1}{2} + \frac{1}{\sqrt{\pi}}\right] \frac{1}{6}\left(\frac{5}{4}\right). \end{aligned}$$

Numerically,

$$\sigma \approx (0.5 + 0.564) \times 0.2083 \approx 0.221 < 1.$$

Therefore, the assumption (iii) holds. Thus, all the conditions of Theorem 4 are satisfied. Hence, the operator

$$(\aleph\varkappa)(t) = \varkappa_0 + \frac{1}{2} \cdot \frac{1}{6}\left(\varkappa(t) + \int\limits_0^t \frac{1}{4}\varkappa(s) ds\right) + \frac{\frac{1}{2}}{\Gamma\left(\frac{1}{2}\right)} \int\limits_0^t (t-s)^{-\frac{1}{2}} \left(\varkappa(s) + \int\limits_0^s \frac{1}{4}\varkappa(\tau) d\tau\right) ds$$

has a unique FP in $C([0, 1], \mathbb{R})$.

## 6. Conclusions

In this work, we introduced the concept of C-VESMSs and established some CFP theorems under newly proposed generalized contractive conditions. The developed framework provides a unified setting that encompasses several existing structures as particular cases, including SMSs, E-SMSs, C-VSMSs, and C-VMSs. Consequently, the obtained results extend and generalize a number of existing CFP results available in the literature.

The theoretical results were established under suitable contractive assumptions and were supported by nontrivial examples to illustrate the applicability and effectiveness of the proposed conditions. These examples demonstrate that the obtained CFP theorems can be applied in settings that are not necessarily covered directly by some of the existing FP results. The developed C-VESMS framework therefore provides a flexible mathematical structure for studying CFP problems involving generalized distance-type functions with complex-valued and extended-valued characteristics.

As an application of the main theoretical results, we considered a nonlinear fractional integro-differential initial value problem involving the Atangana-Baleanu-Caputo derivative of order $0 < \alpha < 1$. By employing the corresponding Atangana-Baleanu fractional integral, the considered problem was transformed into an equivalent integral equation. An appropriate operator was then constructed on the underlying function space, and the proposed CFP results were applied within the C-VESMS framework. Under suitable assumptions on the nonlinear terms, the existence and uniqueness of the solution were established. This application demonstrates the usefulness of the proposed FP framework in the analysis of nonlinear fractional problems involving nonsingular memory kernels.

Overall, the results obtained in this study contribute to the development of generalized FP theory in C-VESMSs and demonstrate the theory's applicability to nonlinear fractional integro-differential equations. The proposed framework provides a basis for further investigations involving more general contractive conditions, mapping structures, and applications to nonlinear mathematical models.

## 7. Future work

Several promising directions can be considered for extending the present results and broadening the applicability of the C-VESMS framework. First, an important direction for future research is to investigate the short-memory concept in the fractional integro-differential problem considered in this work. The present formulation is developed under a full-memory setting, where the Atangana-Baleanu-Caputo derivative and the Volterra-type integral term account for the history over the entire interval $[0, T]$. In practical applications, however, the contribution of the distant past may become sufficiently small so that only a finite portion of the recent history needs to be retained. A short-memory formulation could therefore provide a potentially more efficient representation of the memory effects while preserving the dominant contribution of the recent history. Fractional formulations involving memory, hereditary effects, and time-dependent fractional behavior have been investigated in related studies, including Zhou et al. [20], Sumelka et al. [21], and Voyiadjis et al. [22]. These works provide motivation for further investigation of memory-dependent fractional models. In this context, a natural extension of the present work would be to formulate and analyze a short-memory version of the considered problem within the C-VESMS framework, including the corresponding existence and uniqueness results and the relationship between the full-memory and short-memory formulations.

Beyond the memory-related extension, the framework of C-VESMSs can be further developed by incorporating fuzzy structures, leading to the study of complex-valued fuzzy SMSs and corresponding fuzzy CFP results. Such extensions could provide a useful mathematical setting for problems involving vagueness, imprecision, and uncertainty. In addition, other generalized classes of contractions, such as polynomial, rational, implicit, nonlinear, and hybrid-type contractions, can be introduced and investigated within the proposed framework. This may lead to further generalizations and unifications of existing CFP results.

The proposed C-VESMS framework may also be applied to broader classes of nonlinear problems, including nonlinear integral equations, integro-differential equations, fractional integral equations, fractional differential equations, and systems of fractional differential or integro-differential equations. In these applications, the developed FP techniques could be used not only to establish the existence and uniqueness of solutions but also to investigate stability and other qualitative properties.

Another interesting direction is the extension of the present CFP results from single-valued mappings to multivalued or set-valued mappings. Such developments could be useful for mathematical models involving multiple admissible states or solutions. CFP results for multivalued mappings in C-VESMSs could therefore be investigated under suitable generalized contractive conditions.

From an applied perspective, the proposed framework may be explored in physics, engineering, biomechanics, and control theory, particularly for nonlinear models involving memory, hereditary effects, fractional dynamics, or complex-valued quantities. The fractional integro-differential application presented in this work provides a starting point for such investigations. Further applications to models with time-dependent memory and fractional constitutive behavior may also be considered.

Finally, randomized, probabilistic, and fuzzy extensions of C-VESMSs could be developed to address uncertainty and randomness in mathematical modeling. Combining these structures with generalized CFP techniques may provide a unified framework for studying deterministic, uncertain, and stochastic nonlinear problems. Collectively, these extensions, together with the investigation of short-memory formulations, may substantially broaden the theoretical scope and practical applicability of C-VESMSs.

## Author contributions

Amnah Essa Shammaky: Conceptualization, methodology, visualization, writing-original draft, writing-review and editing, supervision, validation; Jamshaid Ahmad: Conceptualization, methodology, writing-original draft, writing-review and editing, supervision. All authors have read and agreed to the published version of the manuscript.

## Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflicts of interest.

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