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Research article

Deteriorating inventory under carbon cap-and-trade: profit–emission trade-offs between no-shortage and backlogging policies

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Abstract: This study investigated profit–emission trade-offs in deteriorating inventory systems under a carbon cap-and-trade mechanism with exponentially time-varying demand and a constant deterioration rate. Two profit-maximizing models were developed for a retailer: a no-shortage policy and a complete backlogging policy. The analysis established the existence and uniqueness of the optimal replenishment decisions, including the replenishment cycle, order quantity, and shortage interval where applicable. Numerical results showed that, compared with systems without carbon regulation, both models reduce carbon emissions per unit time under the cap-and-trade mechanism, confirming its operational effectiveness. The results further indicate that the complete backlogging policy yields higher profits, but it also produces higher emissions and greater sensitivity to key cost parameters. In particular, shortage cost plays an important role in balancing profitability and environmental performance. These findings provide practical guidance for designing sustainable inventory strategies un-der carbon cap-and-trade regulation.

Keywords: carbon cap-and-trade; backlogging policy; deteriorating inventory; profit-emission trade-off; low carbon inventory management
Mathematics Subject Classification: 90B06, 90C90

1. Introduction

Inventory management has experienced a fundamental shift in its operational paradigm. Traditionally, greenhouse gas emissions appeared as externalities that did not impose a direct financial cost on firms [1]. However, in response to the urgency of climate change and in accordance with the Paris Agreement, increasingly stringent carbon policies have been implemented globally. This encompasses mandatory regulatory mechanisms such as carbon taxes and carbon caps, which enforce explicit economic repercussions on emission-intensive activities. Consequently, inventory management decisions must now clearly integrate carbon emissions as a fundamental aspect of operational planning. Specifically, carbon credits and emission allowances should be systematically regarded as decision variables in inventory models, necessitating that firms concurrently evaluate profitability and environmental performance instead of perceiving environmental impact as an optional corporate social responsibility endeavor [2–4]. Considering that the transportation and warehousing sectors rep-resent a substantial portion of global energy consumption and greenhouse gas emissions, carbon-aware inventory models have become crucial for legal compliance, sustaining investor confidence, and ensuring long-term competitiveness [4].

Carbon pricing methods have shifted from theoretical frameworks to extensive global implementation, with Emissions Trading Systems (ETS) and carbon tax schemes operating in more than 35 systems worldwide. These programs account for around 18% of global greenhouse gas emissions and generate tens of billions of U.S. dollars each year for climate and innovation funds [5]. The major emission trading programs are the European Union ETS and China's national ETS, both of which considerably impact the cost structures of production and logistics operations. The European Union has enacted the EU Carbon Border Adjustment Mechanism (CBAM), requiring importers of certain goods to report the embedded carbon emissions attached to such products and pay an associated carbon charge. This system imposes additional carbon-related fees on imported goods and integrates carbon expenses as a critical factor in production planning, order quantity determination, and inventory management within global supply chains [6]. Furthermore, leading firms have shown that restructuring their distribution networks, modifying transportation methods, and implementing circular or reverse logistics can substantially reduce transport emissions while maintaining or improving profitability [7,8]. Concurrently, operational alternatives such as Just-in-Time (JIT), which reduce inventory levels while boosting transit frequency, may lead to increasing emissions. Therefore, a comprehensive assessment is essential. This highlights the necessity of integrating inventory and transportation models that explicitly account for carbon emissions to develop more appropriate and sustainable operational guidelines.

The integration of carbon policy into inventory management has seen methodological advancements, shifting from the traditional Economic Order Quantity (EOQ) framework to sustainable EOQ models that explicitly include carbon-related factors. These variables are included as either direct cost components or as constraints on total carbon emissions [9]. In the initial stage of this research stream, carbon emissions resulting from production, storage, and transportation were generally regarded as linear functions of order quantity or transportation distance, with carbon taxes or penalty costs directly integrated into the total cost structure. Subsequent research extended model complexity to reflect real-world practices, including deteriorating products and expanding items in agri-food supply chains, frequently under time-varying or price-sensitive demand [10]. Simultaneously, carbon-pricing mechanisms were integrated into inventory and production models.

These advancements have resulted in optimization frameworks that determine optimal lot sizing, replenishment cycles, and investments in green technology across various regulatory frameworks and emission control policies [11,12].

Despite substantial advancements in research literature on low-carbon inventory management, significant gaps remain in relation to the evolving carbon regulations. Many recent models employ simplified representations of demand (e.g., constant or linear), deterioration, and emission processes, consequently constraining the model's capability to properly capture the combined effects of time-varying demand, deteriorating items, and operational strategies such as complete backlogging under explicit carbon regulation [10,13]. Several inventory models with time-varying demand and deteriorating products either disregard carbon regulation or incorporate emissions through simplified tax-like cost parameters instead of employing cap-and-trade quotas with tradable allowances [10]. Second, previous studies have focused on the effects of carbon taxes compared to cap-and-trade on lot sizing or price, with limited analysis of how shortage and backlogging policies might serve as implicit mechanisms for carbon mitigation [12]. Third, there are few studies that (i) confirm the existence and uniqueness of optimal replenishment policies and (ii) propose a computationally efficient algorithm with sensitivity analysis to illustrate how critical parameters, such as shortage cost and carbon trading price, affect optimal profit and emissions [13,14].

Collectively, this paper addresses these limitations by developing a unified profit- maximization framework for deteriorating inventory with exponentially time-varying demand under a carbon cap-and-trade mechanism. The model explicitly incorporates fixed and variable emission components and analytically compares two structurally distinct shortage policies (no shortage versus complete backlogging) into a car-bon-regulated setting. Therefore, this study extends prior cap-and-trade and carbon-tax inventory models that typically rely on simplified demand assumptions or do not analytically contrast different shortage regimes.

The remainder of this paper is organized as follows: Section 2 presents the relevant literature. Section 3 introduces the notation and assumptions. Section 4 presents the model formulation. Section 5 provides the model analysis. Section 6 discusses the numerical simulation and sensitivity analysis. Finally, Section 7 concludes the paper and discusses the managerial implications.

2. Literature review

2.1. Classical inventory models with time-varying demand

Deteriorating inventory models with time-varying demand build on the traditional EOQ framework by integrating time-varying variability in both product quality and demand. To account for seasonal and product life-cycle effects, preliminary studies often assume a constant deterioration rate and a time-varying demand. In this framework, the optimal order quantities and replenishment cycles are determined to minimize total costs, while taking into consideration shortages and salvage values [15–17]. Furthermore, the exponential demand pattern has shown effectiveness in modeling real-world situations [18,19].

Recent studies introduce more complicated demand structures, including power-form or price and stock-dependent demand, together with non-instantaneous or Weibull-type deterioration, indicating that demand patterns significantly influence the optimal cycle length, shortage time, and backlogging decisions in deteriorating systems [20,21]. A concurrent research domain integrates time-varying demand within more complex frameworks, such as two-warehouse systems, trade-credit conditions, and partial or complete backlogging policies, emphasizing the interaction between demand patterns and deterioration in influencing inventory levels and service performance [22,23].

2.2. Carbon regulation mechanisms in inventory systems

The existing literature investigated the interaction between carbon regulation and inventory decisions, emphasizing essential economic–environmental trade-offs across various carbon regulations, including cap, carbon taxes, and cap-and-offset [24,25]. Most studies focused on the influence of a carbon price on traditional EOQ/EPQ decisions by internalizing the costs of emissions associated with production, storage, and transportation. Early studies often include a carbon tax as a linear cost term proportional to total emissions, demonstrating that higher tax rates reduce optimal order quantities and promote smaller, more frequent shipments or environmentally friendly technologies, though at the cost of higher explicit carbon expenses in the objective function [3,9,26].

The cap-and-trade system, on the other hand, sets an emissions limit, known as an emissions cap, and allows firms to trade allowances. This makes it a piecewise cost structure in which firms either incur costs for exceeding the emissions cap or generate income from selling surplus permits [25]. This policy provides firms with an initial amount of tradable emission permits on the carbon market. Although surplus permits can be sold, scarcity requires extra purchases to assure compliance [27,28]. While car-bon quotas do not affect the optimal replenishment or emission reduction strategy, increased quotas may boost profit through expanded permit trading [29]. In multi-echelon supply chains, increasing the carbon cap lowers total supply chain cost without altering safety stock levels and emissions, suggesting that the carbon cap affects only the constant term of the objective function rather than the structure of the optimal solution [30]. However, when emissions exceed the cap, total cost increases with the carbon price, where safety stock costs are non-decreasing and carbon emission is non-increasing in carbon price [31].

Recent sustainable inventory models extended the carbon regulation framework by incorporating product deterioration and fluctuating demand over time. These models show that variations in demand affect both economic performance and emission outcomes [32]. Product deterioration results in higher emissions due to ineffective resource utilization and waste, as replacing deteriorated products requires additional procurement and increases environmental impacts [33]. Furthermore, by integrating deterioration, time-dependent demand, and emission-related holding costs, the results indicate that carbon costs affect key decisions, including cycle time, backlogged demand, and service levels [15]. In particular, higher deterioration rates shorten cycle time and reduce total profit while increasing order quantities. Under certain conditions, carbon emissions may decline, though possibly at the cost of reduced profitability [34]. In the context of imperfect production with deteriorating items and time-varying demand, Mukherjee et al. [35] incorporated four carbon regulation mechanisms—strict emission cap, carbon tax, cap-and-trade, and cap-and-offset—into a fuzzy-random EPQ framework. Each policy induces a different cost–emission trade-off and leads to distinct production and inventory decisions when both deterioration and fluctuating demand are present. Their results emphasize that the choice of carbon regulation mechanism is crucial for balancing economic performance and environmental impact in stochastic, deteriorating inventory systems.

At a broader decision-making level, integrated models jointly optimize replenishment policy, pricing, and green technology investment under alternative carbon policies. Comparative analyses reveal that green investment levels and optimal replenishment cycles are highly sensitive to both regulatory mechanisms and demand characteristics [12]. In production–inventory contexts, incorporating direct and indirect emissions from production, warehousing, and transportation under cap-and-trade demonstrates that carbon caps and allowance prices significantly affect optimal production quantity and total costs, while operational factors such as delivery distance and fuel standards remain critical for emission reduction [11]. Extensions of deteriorating inventory models with time varying demand under carbon emission regulations further confirm that low carbon policies can alter both cycle length and backlogging behavior, and thereby change the economic–environmental trade-off faced by decision-makers [13].

Comparative analyses indicate that the cap-and-trade policy generally leads to a reduction in total costs [32] and enhanced flexibility relative to the carbon tax program. Low-carbon firms gain benefit from trading, whereas high-emission firms are motivated to adjust inventory and implement cleaner technologies [11,36,37]. This approach may drive inventory decision-makers to decrease carbon emissions by reducing inventory levels or allowing more shortages [38]. Furthermore, the regulation could be advantageous for firms, as supply chains may generate revenue by selling surplus quotas [39].

2.3. Shortage and backlogging policies in low carbon systems

Shortage and backlogging policies play an important role when deciding on service levels and environmental performance in low-carbon inventory systems [40,41]. In a classical model with deteriorating items, allowing shortages with complete or partial backlogging is a standard mechanism to balance inventory holding costs against shortage and delay costs, often improving profit at the expense of lower immediate service levels [20]. When carbon emissions are incorporated, backlogging decisions pro-vide an additional environmental dimension. Specifically, lower on-hand inventory can help reduce storage-related emissions; however, prolonged shortages may lead to more frequent or urgent replenishments, resulting in an increase in transport-related emissions and energy consumption in production [7,13].

These studies, especially in the context of deteriorating products and time-varying demand, illustrate that emission-related costs impact optimal policies by altering the trade-offs among replenishment cycle time, backlogged demand, and service levels [13]. Subsequent extensions incorporating environmentally sensitive demand indicate that environmentally oriented policies can be compatible with positive service levels, provided that policymakers carefully manage backlog rates and replenishment cycles [10,42]. Moreover, models that include controlled emissions and partial backlogging reveal that investments in emission reduction technologies can improve both profitability and environmental performance, which shows the synergistic impact of shortage decisions and carbon control levers [43,44].

A second stream of research in low carbon inventory problems focuses on shortage and backlogging policies. In multi-echelon settings, models with deteriorating products show that allowing backlogging can facilitate the coordination of production and shipments. Hence, it leads to a reduction in both emission costs and inventory waste but at the cost of lower short-term service levels [45]. Furthermore, when customers are willing to wait for products, a complete backlogging policy can serve as a mechanism to balance holding and emission costs while preserving long-run profitability [33,46]. Additionally, models under cap-and-trade regulation with deteriorating items show that both the emission cap and allowance price significantly influence firms' decisions. Firms can choose either to provide a high service level with minimal backlogging or to adopt more aggressive backlogging policies to reduce on-hand inventory and associated emissions. As a result, a clear profit–emission trade-off emerges, influenced by shortage and backlogging decisions [12,13]. Extending this line of research to a multi-objective production setting, Mukherjee et al. [47] studied a fuzzy-random EPQ model that explicitly allows shortages and jointly optimizes economic performance and sustainability. The results indicate that appropriately designed shortage and backlogging policies can improve the combined economic–environmental performance compared with traditional single-objective models.

Previous studies suggest that shortage and backlogging decisions should not be evaluated solely on economic criteria, since they also affect service levels, emissions, and exposure to carbon costs. While backlogging can improve profitability, it may reduce emissions by lowering on-hand inventory. However, overly aggressive backlogging policies can increase transport emissions and weaken service performance. Therefore, careful adjustment is essential under both carbon tax and cap-and-trade regimes.

3. Notation and assumptions

For the purpose of developing the mathematical model of this research, the following notations are used throughout this paper (see Table 1).

Table 1. Notation and parameter definitions.

Notation	Definition
Decision variables:	
T	Length of replenishment cycle
t_s	Duration of positive inventory period within a cycle, $0 \leq t_s \leq T$
Model parameters:	
A	Fixed ordering cost per replenishment
p	Unit selling price
c	Unit purchasing cost
h	Unit holding cost per unit time
s	Unit shortage cost per unit time
$D(t)$	Demand rate, an exponential function of time
$I(t)$	Inventory level at time t
Q_1	Order quantity per replenishment cycle for Model 1
Q_2	Order quantity per replenishment cycle for Model 2
θ	A constant deterioration rate of inventory items per unit time, $0 \leq \theta \leq 1$
E	Carbon emission allowance (cap) per replenishment cycle by regulatory authorities
e^A	Fixed carbon emissions generated per replenishment order
e^c	Variable carbon emissions per unit item ordered
e^h	Carbon emissions per unit item held in inventory per unit time
c_e	Unit trading price of carbon emission rights
$TP_1(T)$	Total profit function with no shortage for Model 1
$TP_2(t_s,T)$	Total profit function with complete backlogging for Model 2

The inventory model developed in this study is based on the following assumptions and model descriptions, including the characterization of carbon emissions in the inventory system.

A1. There is a single product in this model.
A2. The demand rate is an exponential function of time, given by $D(t) = D_0 e^{\lambda t}$, where D_0 is the initial demand at time $t = 0$ and λ is the constant demand growth rate.
A3. Replenishment is instantaneous with zero lead time.
A4. Product is perishable with a constant deterioration rate θ.
A5. The retailer operates under a carbon cap-and-trade regulation in which the government allocates an initial carbon emission quota E. If the retailer's actual emissions exceed the allocated quota, additional emission rights must be purchased at the prevailing carbon price. Conversely, if emissions fall below the quota, the retailer can sell the surplus emission rights to obtain additional revenue.
A6. Following Dye and Yang [15], total carbon emissions consist of three components: (1) fixed emissions per order, arising from activities such as transportation, and order processing; (2) emissions linearly proportional to the order quantity, generated by activities such as manufacturing and packaging; and (3) holding-related emissions produced by unsold items during the storage period, associated with activities such as refrigeration and warehouse energy consumption.
A7. The selling price is assumed to exceed the total costs associated with inventory operations and product procurement, thereby ensuring the retailer's profitability and operational stability.
A8. Shortages are allowed and fully backlogged. All backorders accumulated during the shortage period $(T - t_s)$ are satisfied immediately at the beginning of the next replenishment cycle.
A9. In each replenishment cycle, the shortage period is assumed to be no longer than the sales period.
A10. The exponential demand growth rate is assumed to be lower than the product deterioration rate, ensuring that demand remains manageable and does not exceed the rate of quality decay; that is, $\lambda < \theta$.
A11. All cost parameters (holding cost h, shortage cost s, fixed ordering cost A, and carbon emission cost c_e) and emission parameters (E, e^A, e^c, e^h) are known, deterministic, and constant over the planning horizon.
A12. The unit selling price satisfies

$$p < c + c_e e^c + \frac{s + c\theta + h + c_e e^c \theta + c_e e^h}{2\lambda}$$

ensuring the retailer's margin does not grow disproportionately fast relative to demand growth rate λ, deterioration rate θ, and the associated holding, shortage, and carbon emission costs.

Based on assumptions A8 and A9, it follows that $t_s \le T$ and $T - t_s \le t_s$ which imply that $t_s \le T \le 2t_s$.

4. Model formulation

This study considers two inventory strategies: 1) no stockouts and 2) stockouts with complete backlogging. Two inventory models are formulated to maximize the retailer's total profit. The models are subsequently analyzed to determine the optimal replenishment cycle and shortage time. Specifically, Model 1 focuses solely on the replenishment cycle under a no-stockout policy, whereas Model 2 jointly optimizes the shortage period and replenishment cycle.

4.1. An inventory model with no shortage

Under the carbon cap-and-trade mechanism, an inventory model without stock-outs is established based on the assumptions of product deterioration and time-varying demand. As shown in Figure 1, the inventory level decreases gradually over time due to two factors: product deterioration and demand. Thus, the rate of change in inventory levels equals the combined depletion caused by deterioration and demand. Therefore, the inventory level $I(t)$ satisfies the following differential equation:

$$\frac{dI(t)}{dt} = -\theta I(t) - D(t), \quad 0 \le t \le T. \tag{1}$$

As the sales cycle ends, the inventory level is depleted to zero. Therefore, with the boundary condition $I(T) = 0,$ the inventory level function can be obtained as

$$I(t) = \int_t^T e^{\theta(x-t)} D(x)dx, \quad 0 \le t \le T. \tag{2}$$

The order quantity is fully stored in the warehouse at the beginning of each replenishment cycle, serving as the initial inventory level Q_1. Hence, the order quantity is equal to the initial inventory level, that is,

$$Q_1 = I(0) = \int_0^T e^{\theta x} D(x)dx. \tag{3}$$

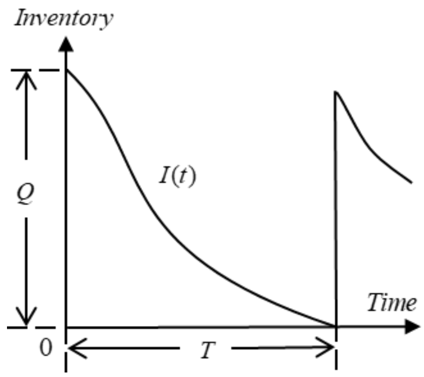

Figure 1. Inventory levels change over time in Model 1.

Based on the inventory level function and order quantity, the retailer's revenue and costs over the sales cycle [*0,T*] can be determined as follows:

First, the retailer's sales revenue (P_1) is obtained by multiplying the unit selling price by the total sales quantity during the sales period. Thus, the sales revenue is given by

$$P_1 = p\int_0^T D(x)dx. \tag{4}$$

Second, the fixed ordering cost (A) represents the one-time cost incurred each time a replenishment order is placed. This cost is independent of the order quantity. Consequently, smaller order quantities and more frequent orders lead to higher total ordering cost, so firms must balance ordering cost against inventory holding cost when determining an appropriate order quantity, as in classical EOQ inventory models.

Third, the product purchasing cost is

$$C_1 = cQ_1 = c\int_0^T e^{\theta x} D(x)dx. \tag{5}$$

Fourth, the holding cost (H_1) is calculated as the product of the unit holding cost and the cumulative inventory held during the sales cycle. Let X denote the cumulative inventory over the cycle, which can be expressed as

$$\begin{aligned} X &= \int_0^T I(t)dt = \int_0^T \int_t^T e^{\theta(x-t)} D(x)dxdt \\ &= \int_0^T \int_0^x e^{\theta(x-t)} D(x)dtdx = \theta^{-1}\int_0^T (e^{\theta x}-1)D(x)dx. \end{aligned} \tag{6}$$

Therefore, the holding cost can be written as

$$H_1 = hX = h\theta^{-1}\int_0^T (e^{\theta x}-1)D(x)dx. \tag{7}$$

Finally, the carbon emission charge is determined by multiplying the carbon trading price by the net amount of carbon emissions. According to the assumptions, there are three components contributing to carbon emissions: fixed emissions per order, variable emissions associated with ordering activities, and emissions generated during inventory holding. The latter two components are directly linked to order quantity and the cumulative holding, respectively. Since each firm is allocated a predetermined fixed carbon emission quota assigned by the government, the actual cost is calculated based on the difference between the total emission and the allocated carbon allowance. Therefore, the carbon emission costs (CEC_1) over the sale period can be expressed as

$$\begin{aligned} CEC_1 &= c_e \cdot CE_1 = c_e\left[e^A + e^c Q_1 + e^h X - E\right] \\ &= c_e\left[e^A + e^c\int_0^T e^{\theta x}D(x)dx + e^h\theta^{-1}\int_0^T (e^{\theta x}-1)D(x)dx - E\right], \end{aligned} \tag{8}$$

where CE_1 denotes the total amount of carbon emissions over the cycle, and CEC_1 denotes the corresponding carbon emission cost.

The retailer's total profit per cycle includes sales revenue, ordering cost, purchasing cost, holding cost, and carbon emission cost. Thus, the total profit function can be written as

$$\begin{aligned} TP_1(T) &= P_1 - A - C_1 - H_1 - CEC_1 \\ &= p\int_0^T D(x)dx - A - (c + c_e e^c)\int_0^T e^{\theta x}D(x)dx \\ &\quad -\theta^{-1}(h + c_e e^h)\int_0^T (e^{\theta x}-1)D(x)dx - c_e(e^A - E). \end{aligned} \tag{9}$$

The objective of this inventory model is to determine the optimal replenishment cycle that maximizes the retailer's total profit. Accordingly, the optimization problem is formulated as

$$
\begin{aligned} M1: \quad \max \quad & TP_1(T) \\ s.t. \quad & 0 < T. \end{aligned} \tag{10}
$$

4.2. An inventory model with completely backlogged

The complete backlogging strategy not only reduces inventory pressure but also ensures that customer demand is eventually satisfied. Similar to Model 1 (M1), this model is formulated to maximize the retailer's total profit; however, the decision variables include both the optimal replenishment cycle and the optimal shortage period.

During the sales period [*0,* t_s], the inventory level must satisfy demand and deterioration requirements. During the shortage period [t_s ,T], demand continues to occur and is completely backlogged until the next replenishment arrives (see Figure 2).

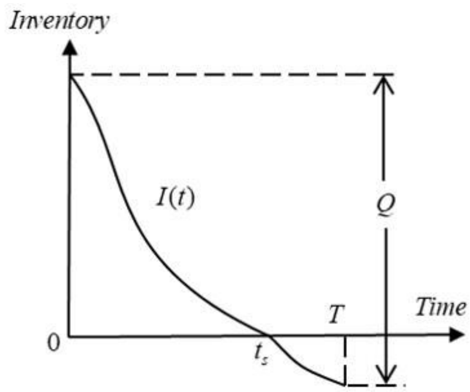

Figure 2. Inventory levels change over time in Model 2.

Accordingly, during the sales period, the inventory level I(t) satisfies the following differential equation:

$$
\frac{dI(t)}{dt} = -\theta I(t) - D(t), \quad 0 \le t \le t_s. \tag{11}
$$

With the boundary condition $I(t_s) = 0,$ the inventory level during the sales period can be obtained as

$$
I(t) = \int_t^{t_s} e^{\theta(x-t)} D(x)dx, \quad 0 \le t \le t_s. \tag{12}
$$

During the shortage period [t_s ,T], the inventory level *I(t)* satisfies

$$
\frac{dI(t)}{dt} = -D(t), \quad t_s \le t \le T. \tag{13}
$$

Using the boundary condition $I(t_s) = 0,$ the inventory level during the shortage period can be derived as

$$I(t) = -\int_{t_s}^{t} D(x)dx, \quad t_s \le t \le T. \tag{14}$$

As the replenishment cycle consists of the sales period [*0, t_s*] and the shortage period [*t_s ,T*], the order quantity per cycle equals the sum of the initial inventory (*I(0)*) supplied during the sales period and the fully backlogged demand accumulated during the shortage period (*−I(T)*). Therefore, the order quantity per cycle can be calculated as follows:

$$Q_2 = I(0) + (-I(T)) = \int_0^{t_s} e^{\theta x} D(x)dx + \int_{t_s}^{T} D(x)dx. \tag{15}$$

Based on the inventory level function and order quantity, the retailer's revenue and costs can be determined as follows:

First, sales revenue is determined by multiplying the unit selling price by the total quantity sold, which equals the cumulative demand during the cycle [*0, T*]. Hence, sales revenue (P_2) can be expressed as

$$P_2 = p\int_0^{T} D(x)dx. \tag{16}$$

Second, the purchasing cost (C_2) can be given by

$$C_2 = cQ_2 = c(\int_0^{t_s} e^{\theta x} D(x)dx + \int_{t_s}^{T} D(x)dx). \tag{17}$$

Third, the holding cost (H_2) is calculated as the product of the unit holding cost and the cumulative inventory during the sales period [*0, t_s*]. Let X_2 denote the cumulative inventory over this period. Thus, $X_2 = \int_0^{t_s} I(t)dt = \theta^{-1}\int_0^{t_s} (e^{\theta x} - 1)D(x)dx.$ The holding cost (H_2) can be expressed as

$$H_2 = hX_2 = h\theta^{-1}\int_0^{t_s} (e^{\theta x} - 1)D(x)dx. \tag{18}$$

Fourth, during the shortage period [*t_s ,T*], a shortage cost is incurred for each backordered item per unit time. The shortage cost (SC_2) is therefore given by

$$\begin{aligned} SC_2 &= s\int_{t_s}^{T} [-I(t)]dt = s\int_{t_s}^{T}\int_{t_s}^{t} D(x)dxdt \\ &= s\int_{t_s}^{T}\left[\int_x^{T} D(x)dt\right]dx = s\int_{t_s}^{T} D(x)\left[\int_x^{T} dt\right]dx \\ &= s\int_{t_s}^{T} D(x)(T-x)dx. \end{aligned} \tag{19}$$

Finally, based on the model assumptions, total carbon emissions consist of three components: fixed emissions per order, emissions associated with ordering activities, and emissions generated during inventory storage. The corresponding carbon emission cost (CEC_2) is expressed as

$$\begin{aligned} CEC_2 &= c_e g CE_2 = c_e\left[e^A + e^c Q_2 + e^h X_2 - E\right] \\ &= c_e[e^A + e^c(\int_0^{t_s} e^{\theta x} D(x)dx + \int_{t_s}^{T} D(x)dx) + e^h\theta^{-1}\int_0^{t_s}(e^{\theta x}-1)D(x)dx - E], \end{aligned} \tag{20}$$

where CE_2 denotes the total amount of carbon emissions over the cycle, and CEC_2 denotes the corresponding carbon emission cost.

Accordingly, the retailer's total profit per cycle can be written as

$$\begin{aligned} TP_2(t_s,T) &= P_2 - A - C_2 - H_2 - SC_2 - CEC_2 \\ &= p\int_0^T D(x)dx - A - c(\int_0^{t_s} e^{\theta x} D(x)dx + \int_{t_s}^T D(x)dx) \\ &\quad -h\theta^{-1}\int_0^{t_s}(e^{\theta x}-1)D(x)dx - s\int_{t_s}^T (T-x)D(x)dx \\ &\quad -c_e[e^A + e^c(\int_0^{t_s} e^{\theta x}D(x)dx + \int_{t_s}^T D(x)dx) + e^h\theta^{-1}\int_0^{t_s}(e^{\theta x}-1)D(x)dx - E]. \end{aligned} \tag{21}$$

The objective of this inventory model is to determine the optimal shortage period and replenishment cycle that maximize the retailer's total profit. The model can therefore be described as

$$\begin{aligned} M2: \quad \max \quad & TP_2(t_s,T) \\ s.t. \quad & 0 < t_s \le T. \end{aligned} \tag{22}$$

5. Model analysis

5.1. Analysis of model 1

To determine the optimal replenishment period for Model 1 (M1), the first-order derivative with respect to the replenishment cycle is derived and analyzed. The existence and uniqueness of the optimal replenishment period are then established based on the properties of the objective function. From Eq (9), we obtain

$$\frac{dTP_1(T)}{dT} = D(T)\varphi(T), \tag{23}$$

$$where \quad \varphi(T) = p - (c + c_e e^c)e^{\theta T} - \theta^{-1}(h + c_e e^h)(e^{\theta T} - 1). \tag{24}$$

The following theorem establishes the optimal solution for Model 1.

Theorem 1. *There exists a unique optimal solution T* for Model 1 that maximizes the profit function $TP_1(T)$, where* $T^* = \theta^{-1}\ln\left(\dfrac{p + \theta^{-1}(h + c_e e^h)}{c + c_e e^c + \theta^{-1}(h + c_e e^h)}\right)$.

Proof. Since the demand function satisfies $D(t) > 0$, the optimal solution of $TP_1(T)$ depends on $\varphi(T)$. By taking the limits and derivative of $\varphi(T)$, we obtain

(1) $\lim_{T\to 0}\varphi(T) = p - (c + c_e e^c) > 0$ (25)

(2) $\lim_{T\to +\infty}\varphi(T) = p - (c + c_e e^c)e^{\theta T} - \theta^{-1}(h + c_e e^h)(e^{\theta T} - 1) < 0$ (26)

(3) $\varphi'(T) = -\left[\theta(c + c_e e^c) + (h + c_e e^h)\right]e^{\theta T} < 0.$ (27)

According to Eq (27), the function $\varphi(T)$ exhibits a strictly decreasing behavior. Combined with Eqs (25) and (26), this implies the existence of a unique $T^* > 0$ that satisfies equation $\varphi(T^*) = 0$. Therefore, T^* is the unique solution of $TP_1(T)$, where

$$T^* = \theta^{-1} \ln\left(\frac{p + \theta^{-1}(h + c_e e^h)}{c + c_e e^c + \theta^{-1}(h + c_e e^h)} \right). \qquad \square$$

Based on Theorem 1, the solution procedure for Model 1 is as follows: Given all model parameters, the optimal replenishment cycle T^* is first computed using Theorem 1. Subsequently, the order quantity Q_1, carbon emission CE_1, and total profit $TP_1(T)$ are determined using Eqs (3), (8), and (9), respectively. Finally, the carbon emissions per unit time CE_1/T and total profit per unit time TP_1/T are evaluated.

5.2. Analysis of model 2

For Model 2, where complete backlogging is allowed, the optimal replenishment cycle and shortage period are determined to maximize the retailer's total profit. Taking the first-order partial derivative of the objective function TP_2 with respect to the replenishment cycle T, we obtain

$$\frac{\partial TP_2(t_s, T)}{\partial T} = D(T)\phi_1(T), \tag{28}$$

where

$$\phi_1(T) = p - c - c_e e^c - s\lambda^{-1} + s\lambda^{-1} e^{-\lambda(T - t_s)}. \tag{29}$$

The optimal solution can be derived based on the analysis in Lemmas 1–3.

Lemma 1. *For any given t_s, the optimal replenishment cycle T^0 for the objective function $TP_2(t_s, T)$ satisfies*

$$T^0 = \begin{cases} 2t_s & \text{if},\ p - c - c_e e^c - s\lambda^{-1} \ge 0 \\ x_0 + t_s & \text{otherwise}, \end{cases}$$

$$\text{where } x_0 = -\lambda^{-1} \ln[(c + c_e e^c + s\lambda^{-1} - p)/s\lambda^{-1}].$$

Proof. When $p - c - c_e e^c - s\lambda^{-1} \ge 0$, it follows from Eq (29) that $\phi_1(T) > 0$. Hence, $\frac{\partial TP_2(t_s, T)}{\partial T} > 0$, which indicates that $TP_2(t_s, T)$ is an increasing function with respect to the replenishment period T. Consequently, the optimal replenishment period is infinite. According to Assumptions A8 and A9, $t_s \le T \le 2t_s$. Thus, $T^0 = 2t_s$.

When $p - c - c_e e^c - s\lambda^{-1} < 0$, we have $\lim_{T \to t_s} \phi_1(T) = p - c - c_e e^c - s\lambda^{-1} + s\lambda^{-1} = p - c - c_e e^c > 0$. Moreover, $\phi_1'(T) = -se^{-\lambda(T - t_s)}$, which implies that $\phi_1(T)$ is a monotone decreasing function. Therefore, the function $\phi_1(T) = 0$ has a unique solution. Let $x = T - t_s$. Then, the solution of $\phi_1(x) = 0$ is $x_0 = -\lambda^{-1} \ln[(c + c_e e^c + s\lambda^{-1} - p)/s\lambda^{-1}]$. $\square$

The solution obtained from Lemma 1 is then substituted into the objective function to determine the optimal replenishment cycle and shortage period through the following lemmas.

First, consider the case when $p-c-c_e e^c - s\lambda^{-1} \geq 0.$ From Lemma 1, we obtain $T = 2t_s$, which can be substituted into $TP_2(t_s, T)$. The first derivative of the objective function $TP_2(t_s, 2t_s)$ with respect to the shortage time t_s is then given by

$$\frac{\partial TP_2(t_s, 2t_s)}{\partial t_s} = D(t_s)\phi_2(t_s), \tag{30}$$

where

$$\phi_2(t_s) = 2(p-c-c_e e^c - s\lambda^{-1})e^{\lambda t_s} + s(2\lambda^{-1} + t_s) - [c + h\theta^{-1} + c_e(e^c + e^h\theta^{-1})](e^{\theta t_s} - 1). \tag{31}$$

The following lemma establishes the existence of the optimal solution to $TP_2(t_s, 2t_s)$.

Lemma 2. *Given that* $p-c-c_e e^c - s\lambda^{-1} \geq 0$, *let* (t_s^*, T^*) *yield the maximum value of* $TP_2(t_s, T)$. *Then* $t_s^* = t_1$, *and* $T^* = 2t_1$ *exist, where* t_1 *is the solution* of $\phi_2(t_s) = 0$.

Proof. By taking the limit of the function $\phi_2(t_s)$ in Eq (31), we obtain $\lim_{t_s \to 0} \phi_2(t_s) = 2(p-c-c_e e^c) > 0$.

Taking the first derivative of $\phi_2(t_s)$, we have

$$\phi_2'(t_s) = 2(p-c-c_e e^c - s\lambda^{-1})\lambda e^{\lambda t_s} + s - [c + h\theta^{-1} + c_e(e^c + e^h\theta^{-1})]\theta e^{\theta t_s}.$$

Let $K_1 = 2(p-c-c_e e^c - s\lambda^{-1})\lambda,$ *and* $K_2 = [c + h\theta^{-1} + c_e(e^c + e^h\theta^{-1})]\theta.$

Then at $t_s=0$, $\phi_2'(0) = K_1 + s - K_2$.

From Assumption A12, we have $2\lambda(p-c-c_e e^c) - s < [c + h\theta^{-1} + c_e e^c + c_e e^h\theta^{-1})]\theta.$

That is, $K_1 + s < K_2$. By substituting into $\phi_2'(0)$, we obtain $\phi_2'(0) < 0$.

Since $\lambda < \theta$ *and* $t_s \geq 0$, it follows that $\lambda t_s \leq \theta t_s$ and so $e^{\lambda t_s} \leq e^{\theta t_s}$.

Because $K_1 \geq 0$, we obtain $K_1 e^{\lambda t_s} \leq K_1 e^{\theta t_s}$, and therefore,

$$\begin{aligned}\phi_2'(t_s) &= K_1 e^{\lambda t_s} + s - K_2 e^{\theta t_s} \\ &\leq K_1 e^{\theta t_s} + s - K_2 e^{\theta t_s} = s - (K_2 - K_1)e^{\theta t_s}.\end{aligned}$$

By Assumption A12, $K_2 - K_1 > s > 0$. Since $e^{\theta t_s} \geq 1$ for all $t_s \geq 0$, we have $(K_2 - K_1)e^{\theta t_s} \geq K_2 - K_1 > s$, which implies $\phi_2'(t_s) \leq s - (K_2 - K_1)e^{\theta t_s} < 0$, for all $t_s \geq 0$.

Thus, $\phi_2(t_s)$ is a decreasing function.

By the existence of the root theorem and the facts that $\lim_{t_s \to 0} \phi_2(t_s) > 0$ and $\phi_2'(t_s) < 0$, there exists a unique t_1 satisfying $\phi_2(t_1) = 0$. Hence, $t_s^* = t_1$, and consequently $T^* = 2t_1$. □

Second, consider the case when $p-c-c_e e^c - s\lambda^{-1} < 0.$ From Lemma 1, we obtain $T = x_0 + t_s$. We then obtain the objective function as $TP_2(t_s, x_0 + t_s)$. The first derivative of this function with respect to the shortage time t_s is then obtained as

$$\frac{\partial TP_2(t_s, x_0+t_s)}{\partial t_s} = D(t_s)\phi_3(t_s), \tag{39}$$

where

$$\phi_3(t_s) = (p - c - c_e e^c - s\lambda^{-1})e^{\lambda x_0} + s(\lambda^{-1} + x_0) - [c + h\theta^{-1} + c_e(e^c + e^h\theta^{-1})](e^{\theta t_s} - 1). \tag{40}$$

The following Lemma establishes the solution of the profit function $TP_2(t_s, x_0+t_s)$ under this case.

Lemma 3. *Given that* $p - c - c_e e^c - s\lambda^{-1} < 0$, *let* (t_s^*, T^*) *yield the maximum value of* $TP_2(t_s, T)$. *Then,*
(1) $t_s^* = t_2$, *where* t_2 *is the solution of the equation* $\phi_3(t_s) = 0$; *and*
(2) $T^* = \begin{cases} 2t_2 & \text{if } t_2 \le x_0 \\ x_0 + t_2 & \text{otherwise.} \end{cases}$

Proof. Taking the limit of the function $\phi_3(t_s)$ in Eq (33), we obtain

$$\lim_{t_s \to 0} \phi_3(t_s) = (p - c - c_e e^c - s\lambda^{-1})e^{\lambda x_0} + s(\lambda^{-1} + x_0). \tag{41}$$

According to Lemma 1, where $x_0 = -\lambda^{-1}\ln[(c + c_e e^c + s\lambda^{-1} - p)/s\lambda^{-1}]$, it follows that $-s\lambda^{-1} = (p - c - c_e e^c - s\lambda^{-1})e^{\lambda x_0}$. Substituting this into Eq (41) yields $\lim_{t_s \to 0}\phi_3(t_s) = sx_0 > 0$. Taking the first derivative of $\phi_3(t_s)$, we obtain $\phi_3'(t_s) = -[c + h\theta^{-1} + c_e(e^c + e^h\theta^{-1})]\theta e^{\theta t_s} < 0$, which shows that $\phi_3(t_s)$ is a monotone de-creasing function. Therefore, there exists a unique t_2 satisfying $\phi_3(t_s) = 0$.

Since $x_0 = T - t_s$, we have $T^* = x_0 + t_s^*$. However, when $t_2 \le x_0$ and considering that the shortage period does not exceed the sales period, the monotonic interval $[t_2, x_0 + t_2]$ is restricted to the feasible interval $[t_2, 2t_2]$ of $TP(t_s, T)$. In other words, the objective function is monotone increasing for $T \in [t_2, 2t_2] \subseteq [t_2, x_0 + t_2]$. Therefore, $T^* = 2t_s^*$. □

Theorem 2. *There exists a unique optimal solution* (t_s^*, T^*) *for Model 2 that maximizes the profit function* $TP_2(t_s, T)$, *where*

$$\left(t_s^*, T^*\right) = \begin{cases} (t_1, 2t_1), & \text{if } p - c - c_e e^c - s\lambda^{-1} \ge 0 \\ (t_2, 2t_2), & \text{if } p - c - c_e e^c - s\lambda^{-1} < 0 \text{ and } t_2 \le x_0 \\ (t_2, x_0 + t_2) & \text{if } p - c - c_e e^c - s\lambda^{-1} < 0 \text{ and } t_2 > x_0. \end{cases}$$

Proof. The result follows directly from Lemmas 1–3. □

Based on Theorem 2, the solution procedure for Model 2 can be summarized as follows: Given all model parameters, first compute x_0 according to Lemma 1. Then, apply Theorem 2 to obtain the optimal solution (t_s^*, T^*). Subsequently, the order quantity Q_2, carbon emissions CE_2, and total profit $TP_2(t_s, T)$ can be calculated using Eqs (15), (20), and (21), respectively. Finally, the carbon emissions per unit time CE_2/T and the total profit per unit time TP_2/T can be evaluated.

6. Numerical simulation and sensitivity analysis

This section verifies the feasibility and validity of the proposed models through numerical examples based on theoretical results, along with sensitivity analysis of key parameters. To illustrate the models, parameter values are set according to previous studies. Specifically, Dye and Yang [15] provide empirical inventory data from the textile industry, including real sales data of clothing products. Based on this, the carbon emission coefficients are determined. Furthermore, the parameter settings are refined following the parametric design guidelines proposed by Xu, et al. [13] to ensure the robustness and applicability of the models.

Accordingly, the parameters are set as follows: $D_0 = 1000$, $A = 100$, $p = 15$, $c = 8$, $h = 1$, $s = 1$, $\lambda = 0.2$, $\theta = 0.23$, $c_e = 0.1$, $e^c = 5$, $e^h = 2.5$, $e^A = 250$, $E = 6500$. Based on these values, the optimal replenishment cycle, order quantity, maximum profit, total carbon emissions, and other performance indicators (including those without carbon emission constraints) are computed and shown in Table 2.

Table 2. Optimal replenishment strategy and performance metrics for Model 1.

Model 1	***T***	***Q***	***TP***	***TP/T***	***CE***	***CE/T***
w CC	1.6646	2431.97	6997.38	4203.64	17375.53	10438.47
w/o CC	1.9526	3059.28	7243.95	3709.89	22834.71	11694.46
% Change	−14.75%	−20.51%	−3.40%	13.31%	−23.91%	−10.74%

As shown in Table 2, for the no-shortage model under the carbon cap-and-trade mechanism (with CC), the replenishment cycle, order quantity, total profit, and carbon emissions are reduced to different degrees. Compared with the case without carbon constraints (without CC), the replenishment cycle, the order quantity, the total profit, and carbon emissions decrease. In contrast, the total profit per unit time increases, while carbon emissions per unit time decreases. These results suggest that the carbon cap-and-trade mechanism can effectively reduce carbon emissions while allowing operations to run efficiently.

To further investigate the effects of key parameters on the ordering strategy and carbon emission, a computational analysis is conducted to examine their effects. The corresponding results are presented in Table 3.

As shown in Table 3, a higher selling price leads to a longer replenishment cycle and larger order quantities. As a result, total profit increases, although carbon emissions and carbon emissions per unit time also rise. This indicates a strong positive relationship between selling price and operational decisions.

Table 3. Effects of parameters on ordering strategy indicators.

Parameters	% Change	% Change in					
		T	*Q*	*TP*	*TP/T*	*CE*	*CE/T*
p	−15	−30.12	−38.33	−52.72	−32.34	−43.05	−18.10
	−10	−19.88	−25.99	−37.55	−22.06	−29.73	−12.26
	−5	−9.64	−13.21	−19.98	−11.44	−15.38	−6.22
	0	0.00	0.00	0.00	0.00	0.00	0.00
	5	9.64	13.64	22.36	11.60	16.40	6.38
	10	18.67	27.70	47.10	23.95	33.81	12.91
	15	27.71	42.19	74.20	36.40	52.22	19.59
c	−15	24.10	35.87	48.86	19.96	44.13	16.69
	−10	15.66	22.86	30.89	13.17	27.77	10.67
	−5	7.83	10.95	14.65	6.33	13.12	5.13
	0	0.00	0.00	0.00	0.00	0.00	0.00
	5	−7.23	−10.08	−13.19	−6.43	−11.79	−4.74
	10	−14.46	−19.39	−25.04	−12.37	−22.39	−9.14
	15	−21.08	−27.99	−35.64	−18.44	−31.94	−13.21
h	−15	4.22	5.75	4.46	0.24	6.85	2.70
	−10	3.01	3.76	2.93	−0.08	4.47	1.77
	−5	1.81	1.85	1.44	−0.36	2.19	0.87
	0	0.00	0.00	0.00	0.00	0.00	0.00
	5	−1.20	−1.78	−1.40	−0.20	−2.11	−0.84
	10	−2.41	−3.50	−2.75	−0.35	−4.13	−1.65
	15	−3.61	−5.17	−4.07	−0.47	−6.08	−2.43
c_e	−15	2.41	3.40	2.41	0.00	4.04	1.60
	−10	1.81	2.25	1.59	−0.22	2.67	1.06
	−5	1.20	1.12	0.79	−0.41	1.32	0.52
	0	0.00	0.00	0.00	0.00	0.00	0.00
	5	−0.60	−1.10	−0.77	−0.17	−1.30	−0.52
	10	−1.20	−2.19	−1.52	−0.32	−2.58	−1.03
	15	−1.81	−3.26	−2.26	−0.46	−3.84	−1.53
e^c	−15	1.81	1.98	2.63	0.81	−8.35	−9.63
	−10	1.20	1.32	1.75	0.54	−5.53	−6.41
	−5	0.60	0.66	0.87	0.27	−2.74	−3.20
	0	0.00	0.00	0.00	0.00	0.00	0.00
	5	0.00	−0.65	−0.87	−0.87	2.70	3.19
	10	−0.60	−1.31	−1.73	−1.13	5.36	6.36
	15	−1.20	−1.95	−2.58	−1.39	7.99	9.52
e^h	−15	1.20	1.38	1.08	−0.13	−2.75	−3.70
	−10	1.20	0.92	0.72	−0.48	−1.82	−2.45
	−5	0.60	0.46	0.36	−0.24	−0.90	−1.22
	0	0.00	0.00	0.00	0.00	0.00	0.00
	5	0.00	−0.45	−0.35	−0.35	0.88	1.21
	10	−0.60	−0.90	−0.70	−0.10	1.75	2.41
	15	−0.60	−1.34	−1.05	−0.45	2.60	3.60
E	−15	0.00	0.00	−1.39	−1.39	0.00	0.00
	−10	0.00	0.00	−0.93	−0.93	0.00	0.00
	−5	0.00	0.00	−0.46	−0.46	0.00	0.00
	0	0.00	0.00	0.00	0.00	0.00	0.00
	5	0.00	0.00	0.46	0.46	0.00	0.00
	10	0.00	0.00	0.93	0.93	0.00	0.00
	15	0.00	0.00	1.39	1.39	0.00	0.00

The sensitivity analysis reveals consistent relationships between model parameters and decision variables. The decision indices increase as the unit selling price rises but decline in response to higher purchasing cost, carbon trading price, and unit holding cost. This indicates that higher procurement, carbon trading, and holding costs limit ordering decisions and reduce profitability. These results demonstrate the importance of managing key parameters in achieving an appropriate balance between economic performance and environmental impact. In particular, controlling selling price, procurement cost, and carbon trading price can improve profitability while helping to control carbon emissions.

Additionally, as the carbon emission coefficients for ordering and holding activities rise, the replenishment cycle and total profit decrease. Simultaneously, both carbon emissions and emissions per unit time increase. This reflects a clear trade-off, indicating that emission-related parameters must be carefully chosen to maintain operational efficiency while achieving emission-reduction targets.

The analysis is then extended to Model 2, for which the optimal replenishment strategy and the corresponding performance indicators are computed and summarized in Table 4.

Table 4. Optimal replenishment strategy and performance metrics for Model 2.

Model 2	t_s	***T***	***Q***	***TP***	***TP/T***	***CE***	***CE/T***
w CC	3.9372	7.87	23478.32	42166.54	5357.88	164674.48	20912.37
w/o CC	4.8635	9.73	38258.22	63125.75	6487.74	281245.63	28913.80
% Change	−19.05%	−19.05%	−38.63%	−33.20%	−17.42%	−41.45%	−27.67%

As shown in Table 4, the introduction of the carbon cap-and-trade mechanism (with CC) leads to substantial reductions in key operational and environmental indicators as compared to the unconstrained case (without CC). Specifically, the replenishment and sales cycles are shortened. Although total profit per unit time decreases, carbon emissions per unit time fall more significantly. The findings suggest that the cap-and-trade mechanism effectively reduces emissions; however, it also decreases profits. This shows the trade-off between economic performance and environmental sustainability.

A comparative analysis of the no-shortage and complete backlogging models under the carbon cap-and-trade constraint is presented in Table 5 and graphically illustrated in Figure 3.

Table 5. Comparison of Model 1 and Model 2.

	t_s	***T***	***Q***	***TP***	***TP/T***	***CE***	***CE/T***
Model 1	-	1.66	2431.97	6997.38	4215.29	17375.53	10438.47
Model 2	3.94	7.87	23478.32	42166.54	5357.88	164674.48	20912.37
% diff	-	374.10	865.40	502.60	27.11	847.74	100.34

A comparison of the two models in Table 5 reveals clear differences in how they operate and perform. The complete backlogging model uses a much longer replenishment cycle and much larger order quantities, and this supports higher overall profit. However, this comes with an increase in total carbon emissions. This suggests that, although complete backlogging is effective in enhancing profit, it does not support emission reduction. Instead, it shifts the system toward a higher level of resource use. Therefore, by appropriately increasing the shortage cost, excessive delays in replenishment can be shortened. This helps maintain profitability while keeping carbon emissions under better control.

Figure 3 shows that both carbon emissions and total profit per unit time decline as the shortage cost increases. Around a narrow interval of the shortage cost, the model switches between two optimal regimes corresponding to different analytical cases, which causes a jump in the optimal shortage and replenishment decisions and produces a spike in the curves. Beyond this transition region, a notable pattern appears: at higher shortage costs, carbon emissions per unit time in Model 2 become lower than those in Model 1, while total profit in Model 2 remains substantially higher. This suggests that, when the shortage cost is set at an appropriate level, firms can gain higher profits while maintaining carbon emissions under control.

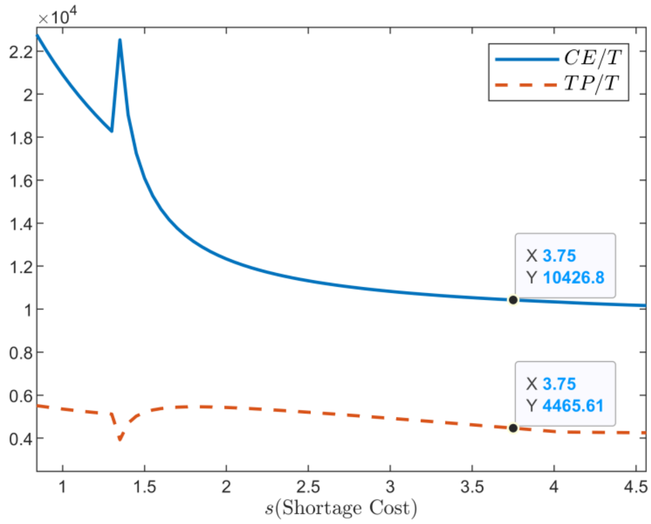

Figure 3. Effects of shortage cost on profit and carbon emissions per unit time.

Building on this result, the two models are next evaluated under different parameter settings to better understand their economic and environmental performance. As shown in Figure 4, Model 2 consistently yields a higher total profit per unit time than Model 1 in all examined cases. This advantage mainly comes from the flexibility of complete backlogging, which allows larger order quantities and longer replenishment cycles, thereby increasing overall profit. In addition, the two models respond differently when key parameters change. Model 2 is more sensitive to economic parameters, particularly the unit selling price and purchasing cost. Its profit increases more when prices are high, but it also drops more quickly when costs increase. By contrast, Model 1 behaves more steadily, with profit that is less sensitive to such fluctuations. For other parameters such as holding cost, carbon trading price, and emission-related factors, both models show only moderate changes, although Model 2 maintains a consistent profit advantage.

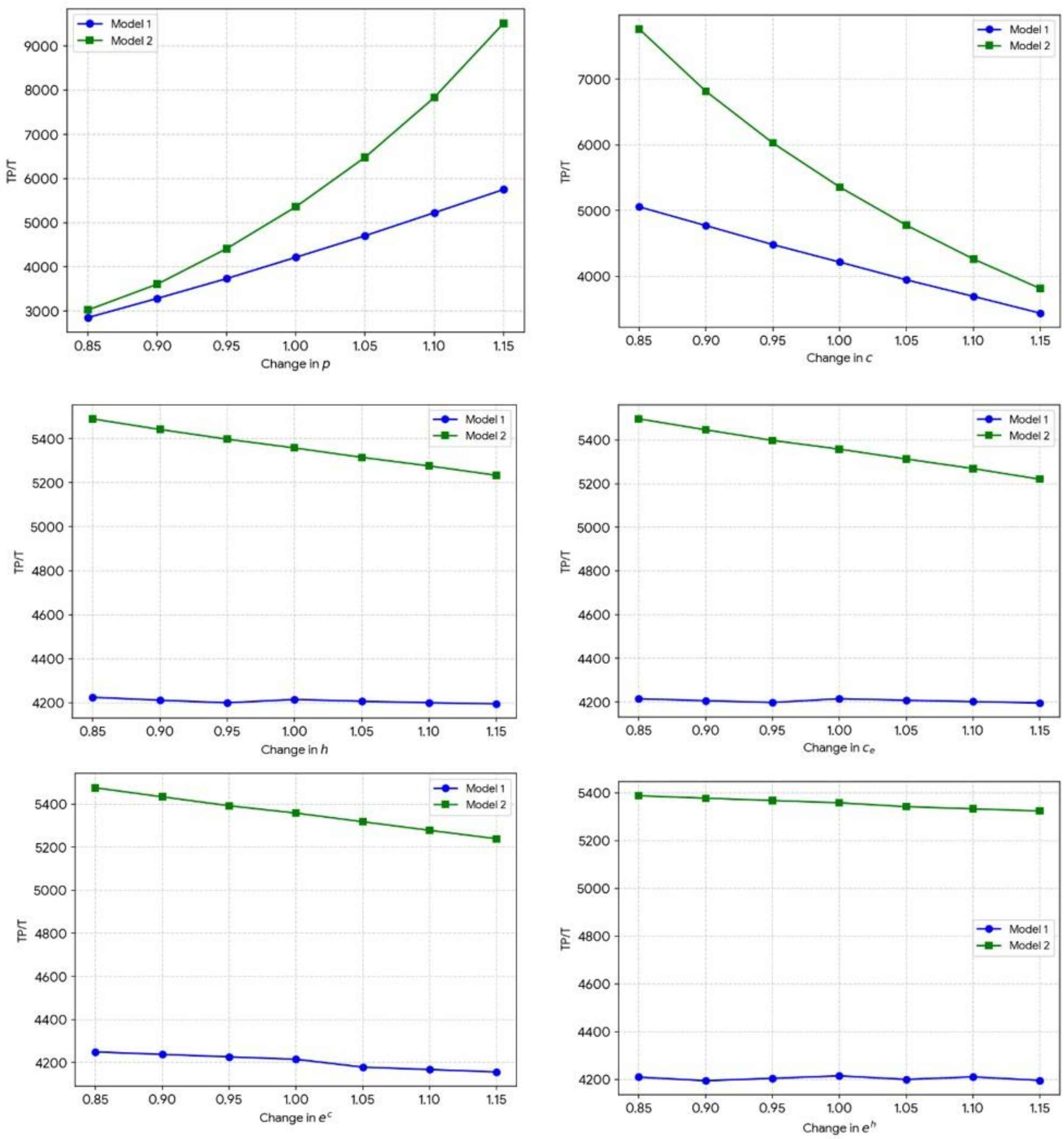

Figure 4. Effects of parameters on total profit per unit time.

From an environmental perspective, Figure 5 shows that Model 2 generates higher carbon emissions per unit time than Model 1 under every parameter setting. This is due to its larger operational scale, since higher order quantities and longer replenishment cycles increase overall system activity. Although both models react in the same direction when parameters change, the size of the change is more pronounced in Model 2. In particular, its emissions respond more strongly to variations in the selling price and the purchasing cost. As a result, Model 1 maintains a lower emission profile, whereas Model 2 attains higher economic performance at the expense of greater environmental impact, highlighting the trade-off between operational scale and sustainability.

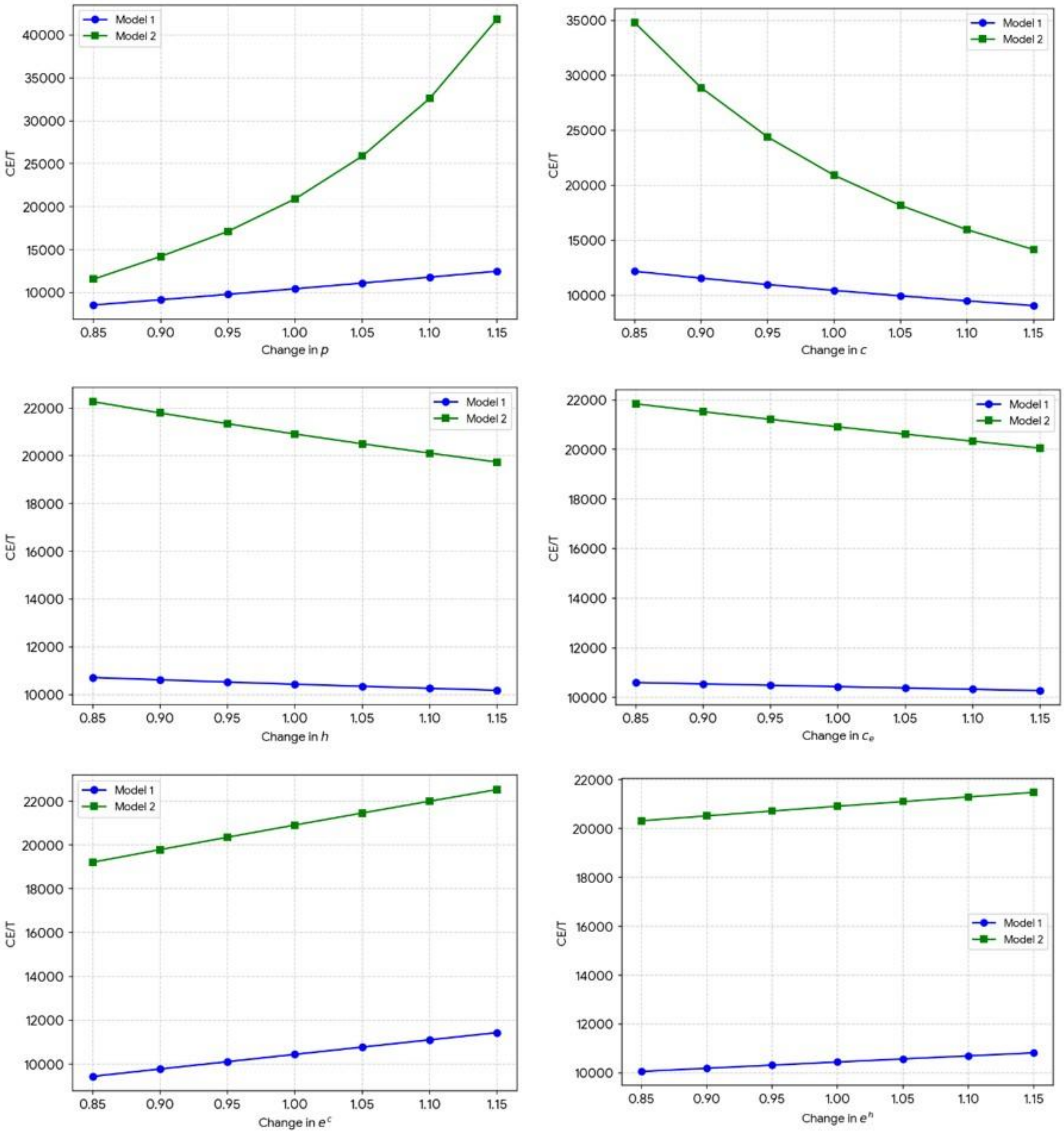

Figure 5. Effects of parameters on carbon emissions per unit time.

The sensitivity analysis for Model 2 shows that it depends strongly on a few key economic parameters. The unit selling price and the purchasing cost have the greatest impact on both profit and emissions. Other parameters have a more moderate effect, allowing the model to adjust to changes without significant behavioral modifications. Overall, Model 2 responds strongly to market conditions, offering higher profit potential but also greater sensitivity to parameter changes, whereas Model 1 delivers a more stable and conservative performance. In addition, Model 2 includes an extra parameter, which is the shortage cost, which plays an important role in shaping system behavior. The sensitivity analysis with respect to this parameter is reported in Table 6.

Table 6. Effects of changes in shortage cost on ordering strategy indicators.

% Change in shortage cost *s*	**% Change in**						
	t_s	T	Q	TP	TP/T	CE	CE/T
−15	5.84	5.84	13.25	8.76	2.78	14.68	8.38
−10	3.81	3.81	8.49	5.67	1.80	9.39	5.39
−5	1.78	1.91	4.09	2.75	0.88	4.51	2.60
0	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	−1.78	−1.78	−3.80	−2.60	−0.83	−4.18	−2.44
10	−3.55	−3.43	−7.34	−5.06	−1.62	−8.06	−4.73

Table 6 shows that the shortage cost has a negative impact on all key performance measures. In particular, a higher shortage cost results in a shorter replenishment cycle and, in turn, a smaller order quantity. The results also indicate that the shortage cost has an important influence on the trade-off between economic performance and environmental impact. As the shortage cost rises, the retailer tends to choose more conservative replenishment policies. This reduces profitability but also lowers carbon emissions. These findings suggest that shortage cost can be used as a practical decision variable to balance operational efficiency and sustainability goals. When this parameter is set appropriately, inventory decisions can be directed toward reducing emissions while still maintaining an acceptable level of profit.

7. Conclusions and managerial implications

This study analyzes two deteriorating-inventory models under a carbon cap-and-trade scheme: a no-shortage policy (Model 1) and a complete backlogging policy (Model 2). Model 2 shows distinct operational behavior and consistently generates higher profit per unit time. This is mainly because backlogging allows demand to be postponed, so the system can use larger order quantities and longer replenishment cycles. However, this larger operating scale results in higher carbon emissions and makes the model more sensitive to key parameters, especially the unit selling price and purchasing cost. In contrast, as Model 1 yields shorter cycles and smaller order quantities, it earns less profit but also produces lower emissions and shows more stable performance. Both models react in the same direction when parameters change, but the effect is stronger in Model 2. Overall, the findings show a clear trade-off between profitability and environmental impact, with shortage cost playing an important role in shaping this balance.

In addition, compared with the cases without carbon constraints, both models under the cap-and-trade scheme show a clear reduction in carbon emissions per unit time. However, this reduction comes with lower profit, which indicates that the carbon policy successfully limits emissions but also creates economic costs. The results suggest that the cap-and-trade mechanism meaningfully reshapes inventory decisions by encouraging firms to operate at a more controlled scale.

The results provide practical implications for both retailers and policymakers in managing inventory systems under carbon constraints, as follows:

1) Although a complete backlogging policy can increase profit by allowing larger order quantities and longer replenishment cycles, it also raises emissions and makes the system more sensitive to changes in cost. Retailers should avoid relying too heavily on backlogging and instead adopt a balanced policy that moderates order size, replenishment frequency, and shortage level. This strategy helps sustain profit while keeping emissions and operational risk under control.

2) A shortage-allowing policy is not always more profitable than a no-shortage policy. Allowing backorders is attractive when shortage costs are moderate and the savings from reduced holding, deterioration, and carbon emissions outweigh the penalty of delayed supply. When shortage costs become very high, the optimal decisions move close to a no-shortage regime, and the benefit of permitting backlogs largely disappears.
3) The findings also show that carbon cap-and-trade policies shape operational decisions by discouraging excessive expansion of the system. Policymakers should set carbon prices at a level that restrains overly large order quantities and long cycles. Well-designed carbon pricing can reduce emissions effectively without severely harming firms' economic performance.
4) Key parameters such as selling price, purchasing cost, and shortage cost are important in determining both profit and emissions. In particular, shortage cost can function as a control lever for managing replenishment behavior. By carefully adjusting this cost, decision-makers can influence the scale of operations and achieve a more appropriate balance between profitability and emissions.

Future research may extend these models by incorporating more realistic demand patterns, such as stochastic or non-exponential demand, and by incorporating partial backlogging in shortage situations. In addition, future studies may examine carbon regulation schemes that are more likely to be implemented in practice, such as carbon tax and cap-and-trade policies, to further enrich the scientific development of inventory modeling under environmental regulations. Since this paper develops an analytical inventory optimization model under a cap-and-trade scheme, related future work may also employ mathematical modeling and operations research techniques, including nonlinear optimization, numerical methods, and sensitivity analysis. When uncertainty or richer empirical data are introduced, simulation and statistical modeling may further complement the analysis.

Author contributions

Zhonghui Li: Conceptualization, Formal analysis, Investigation, Methodology, Validation, Writing–original draft; Panida Chamchang: Conceptualization, Investigation, Methodology, Validation, Supervision, Writing–original draft, Writing–review & editing; Wipada Sompong: Writing–review & editing; Lili Niu: Visualization, Writing–original draft, Writing–review & editing.

Use of Generative-AI tools declaration

The authors would like to disclose that SciSpace and ChatGPT were used to support the review of related literature, as well as to improve the grammatical accuracy and clarity of writing in this manuscript.

Conflict of interest

The authors declare no conflicts of interest in this paper.

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