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Research article

Fractional powers of a canonical Fourier–Bessel differential operator

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Abstract: We develop a fractional functional calculus for the canonical Fourier–Bessel differential operator associated with a real unimodular matrix. The starting point is the unitary chirp conjugacy between the canonical operator and the classical Bessel operator, which yields a natural self-adjoint realization in the weighted Bessel space and a direct spectral diagonalization by the canonical Fourier–Bessel transform. On this basis, fractional powers of the associated non-negative operator are defined by spectral multipliers, and their domains are described through the corresponding Sobolev scale. We also study the canonical generalized translation and convolution, which provide the physical space structures compatible with the transform and with heat propagation. For fractional orders between zero and one, we derive the Balakrishnan semigroup representation, an explicit heat-kernel formula, and a Lévy-type singular-integral representation. The same spectral framework is used to treat the fractional heat equation and to obtain a Caffarelli–Silvestre type extension problem whose Dirichlet-to-Neumann map realizes the fractional operator. Hardy-type estimates are then deduced by transferring known Bessel inequalities through the canonical unitary conjugacy. Several matrix specializations are discussed to recover the ordinary, inverse, fractional, Fresnel, scaled, and chirp-modulated Fourier–Bessel transforms. The resulting theory gives a unified description of nonlocal Bessel operators under linear canonical deformation and connects spectral, semigroup, translation, and extension methods in a common framework.

Keywords: canonical Fourier–Bessel transform; linear canonical transform; Bessel operator; fractional powers; spectral multipliers; generalized translation; heat semigroup; Balakrishnan formula; Lévy-type representation; Hardy inequality
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1. Introduction

Fractional powers of differential operators play a fundamental role in harmonic analysis, partial differential equations, probability theory, and spectral theory; see, for example, [1–4]. They provide canonical examples of nonlocal operators and arise naturally through heat semigroups, spectral multipliers, Dirichlet forms, and Lévy–Khintchine formulae [3, 5]. From analytic point of view, fractional powers allow one to pass from local differential operators to nonlocal operators whose action is described by spectral, semigroup, or singular-integral representations. The construction studied in this paper belongs to a long line of ideas connecting Fourier analysis, canonical transforms, Bessel analysis, and holomorphic function spaces. In the classical setting, the Fourier transform diagonalizes the second derivative and converts translations into modulation factors. The linear canonical transform extends this principle by associating integral transforms with real, symplectic matrices. Historically, canonical transforms appeared in paraxial optics through Collins' matrix formulation of diffraction and in quantum mechanics through the work of Moshinsky and Quesne on canonical transformations, and phase-space methods [6–8]. These transforms are closely related to changes of representation, unitary transformations and the preservation of canonical commutation relations in quantum mechanics [7]. In signal analysis, they provide generalized time–frequency and phase-space representations, which are useful in filtering, reconstruction, and the study of nonstationary phenomena [9–11]. The numerical side of this transform theory is also well-developed. Fast digital algorithms and comparative implementations for fractional Fourier transforms were studied in [12, 13], and accurate discrete Hankel/Fourier–Bessel computation has been represented by the quasi-discrete method of Guizar-Sicairos and Gutiérrez-Vega [14] and the fast asymptotic discrete Hankel algorithm of Townsend [15]. These works provide natural numerical building blocks for the canonical Fourier–Bessel factorizations considered here.

The Fourier transform is one special point in this canonical family. Other choices of the matrix

$$\mathbf{m} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$$

lead to Fresnel transform, fractional Fourier transforms and chirp-modulated transforms. In the Bargmann representation, the Fourier transform is realized as a rotation or phase action on the classical Bargmann–Fock space of entire functions. This explains the natural appearance of quadratic phase factors, Gaussian kernels, the metaplectic representation and Fock spaces in canonical analysis [16,17]. Parallel to this development, and Bessel and Hankel theories provide the radial analog of Fourier analysis [18, 19]. The Fourier–Bessel transform diagonalizes the ordinary Bessel operator and plays the role of the Fourier transform for radial or even structures. Fractional Bessel and fractional Hankel operators have been studied in connection with Hankel transforms, Mellin transforms, Bessel–Kingman hypergroups, singular-integral formulae, and space-fractional diffusion equations [4, 20]. In this sense, fractional Bessel analysis may be viewed as a radial counterpart of the theory of fractional powers of the Laplacian.

Recent work has sharpened this fractional Bessel picture in several complementary directions. A Mellin transform construction of fractional Bessel derivatives was developed in [29], and a multidimensional fractional Laplace–Bessel operator, together with singular integral representations, Bochner subordination, and intertwining relations, was studied in [30]. Further pointwise formulas for generalized fractional Laplace–Bessel operators were obtained in [33]. From the harmonic analysis side, the recent study [26] develops Riesz transforms and fractional integral operators in the Bessel setting via Besov and Triebel–Lizorkin spaces. These contributions provide a current background for the spectral and singular-integral constructions pursued below.

The canonical Fourier–Bessel transform combines these two lines of development. It incorporates the oscillatory and phase-space structure of canonical transforms together with the radial and singular structure of Bessel-type operators. It is therefore a Bessel-type analog of the linear canonical transform and contains, as special cases or limiting models, Fourier–Bessel transforms, fractional Fourier–Bessel transforms, and Fresnel-type Bessel transforms [21–23].

The canonical Fourier–Bessel literature has also developed substantially during 2024–2026. H.B, Mohamed and A. Saoudi introduced the linear canonical Fourier–Bessel wavelet transform, and established orthogonality, inversion and uncertainty-type inequalities [31]. Benlaajine, El Hamma, and Daher subsequently proved a Titchmarsh-type characterization for the linear canonical Fourier–Bessel transform [32]. Most recently, El Hamma, El Bouazizi, and Benlaajine studied the generalized translation associated with the linear canonical Fourier–Bessel transform, introduced the corresponding modulus of smoothness, and obtained estimates for the K-functional together with Jackson- and Bernstein-type approximation inequalities [34]. This 2026 contribution is particularly close to the translation and convolution structures used below, and the 2024–2025 works show that canonical Fourier–Bessel analysis remains an active setting for wavelet, smoothness, approximation, and uncertainty questions. Together, these recent results provide a direct contemporary context for the fractional functional calculus developed here. There is also a holomorphic aspect to this theory. Bargmann's classical Fock space is naturally connected with the Fourier transform and the harmonic oscillator [16]. In the Bessel setting, Cholewinski introduced generalized Fock spaces associated with Bessel functions, which may be viewed as Hankel-type analogs of the Bargmann–Fock space [24]. Further developments on Hankel transforms in generalized Fock spaces reinforced this relation between Bessel analysis, reproducing kernels, and entire-function models [25].
We now describe the operator considered in the present paper. Let $\nu > -1/2$, and let

$$\mathbf{m} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}), \qquad b \neq 0.$$

The canonical Fourier–Bessel transform, introduced and developed in [21] and further used in [22], is naturally associated with the differential operator

$$\Delta_{\nu,\mathbf{m}} = \frac{d^2}{dx^2} + \left(\frac{2\nu+1}{x} - 2i\frac{d}{b}x\right)\frac{d}{dx} - \left(\frac{d^2}{b^2}x^2 + 2i(\nu+1)\frac{d}{b}\right).$$

Its kernel is an eigenfunction of $\Delta_{\nu,\mathbf{m}}$ with eigenvalue $-\lambda^2/b^2$. Hence, the non-negative operator

$$A_{\nu,\mathbf{m}} := -\Delta_{\nu,\mathbf{m}}$$

is diagonalized by the canonical Fourier–Bessel transform and is represented in the spectral variable by the multiplier λ^2/b^2. This leads naturally to the definition

$$A^{\alpha}_{\nu,\mathbf{m}} f = \mathcal{F}^{\mathbf{m}}_{\nu}\left[\left(\frac{\lambda^2}{b^2}\right)^{\alpha} \mathcal{F}^{\mathbf{m}^{-1}}_{\nu} f(\lambda)\right].$$

The purpose of this paper is to develop this definition into a complete analytic framework without relying on the ordinary fractional Bessel model as a starting point. The construction is carried out directly from the canonical Fourier–Bessel diagonalization of $A_{\nu,\mathbf{m}}$.
The main novelty of the present work is the coupling of three mechanisms which are usually treated separately: The canonical matrix coupling, the quadratic phase modulation, and the spectral multiplier construction. The matrix parameter $\mathbf{m}$ changes the phase-space geometry, the chirp factor introduces a canonical modulation of the Bessel operator, and the multiplier

$$\left(\frac{\lambda^2}{b^2}\right)^{\alpha}$$

produces the fractional power in the diagonal canonical Fourier–Bessel representation. Thus the operator studied here is not only the classical fractional Bessel operator in a different notation; it is a canonically modulated and scaled nonlocal operator whose ordinary Bessel counterpart is recovered only after removing the phase and matrix deformation. This framework is also motivated by applications. In optics, canonical transforms describe free propagation, lenses, chirp modulation, and Fresnel regimes. In quantum mechanics, they correspond to changes of phase-space representation and metaplectic actions. In signal processing, they provide flexible time–frequency tools for nonstationary data. The present Fourier–Bessel version is therefore relevant when these canonical mechanisms interact with radial, even, or Bessel-type structures. The fractional power $A^{\alpha}_{\nu,m}$ may be interpreted as a nonlocal diffusion operator adapted to this canonical Bessel geometry. The same matrix-based point of view has recently been extended to neighboring harmonic analysis settings. In particular, recent work on linear canonical Dunkl-type transforms develops analogous canonical deformations, convolution structures, and applications beyond the Bessel framework [35]. This parallel development supports the present viewpoint that the matrix parameter, chirp modulation and underlying special-function geometry should be treated as a unified analytic structure rather than as unrelated modifications of a classical transform. In order to place the present work more clearly in the recent literature, we also distinguish the three levels involved in the construction. The classical Bessel theory starts from the radial operator and its Hankel or Fourier–Bessel diagonalization. Recent studies of Riesz transforms and fractional integral operators in the Bessel setting show that this framework remains active in modern harmonic analysis [26, 27]. Fractional Fourier–Bessel models add a rotation-type parameter to the Bessel kernel, whereas the canonical operator considered here allows the full matrix deformation, including chirp modulation, dilation, and Fresnel-type propagation. Recent developments on linear canonical transforms and convolution structures also confirm that canonical matrix methods continue to provide a useful language for generalized time–frequency analysis [28]. These comparisons motivate further theoretical directions, such as perturbation theory, boundary-value problems, and multidimensional canonical Bessel models, as well as applied directions related to radial signal processing and nonlocal diffusion.
The main points of the paper are as follows. We first establish the spectral realization of $\Delta_{\nu,\mathbf{m}}$ in the Hilbert space $\mathcal{H}_\nu = L^2((0,\infty), x^{2\nu+1}dx)$ and describe its functional calculus. We then recall the canonical Fourier–Bessel transform, its Plancherel formula, inversion formula, and operational properties. The canonical generalized shift is introduced as the appropriate replacement for ordinary translation; it provides the physical-space structure underlying convolution and heat propagation in this setting. Fractional powers of $A_{\nu,\mathbf{m}}$ are defined by spectral multipliers, and we prove their basic algebraic

properties, describe the associated Sobolev scale, and derive the heat-semigroup representation, the Balakrishnan formula, and a Lévy-type singular integral representation. We also obtain a Caffarelli–Silvestre type extension formula and Hardy inequalities adapted to the canonical Fourier–Bessel operator.

The paper is organized as follows. Section 2 collects the main notation used throughout the paper. Section 3 gives the spectral study of the operator $\Delta_{\nu,\mathbf{m}}$, including its core, self-adjoint realization, spectrum, and functional calculus. Section 4 introduces the canonical Fourier–Bessel transform, and recalls its inversion, Plancherel, and operational properties. Section 7 is devoted to the generalized shift and the associated convolution. Section 8 defines fractional powers of $A_{\nu,\mathbf{m}}$ by spectral multipliers. Section 9 develops the Sobolev scale and the heat semigroup kernel formula. Section 10 establishes the Lévy-type singular integral formula. The subsequent sections treat the fractional heat equation, the Caffarelli–Silvestre extension, and Hardy inequalities for the canonical Fourier–Bessel operator.

2. Notation and main symbols

For the reader's convenience, we collect the main notation used in the paper. The parameter $\nu > -1/2$ is the Bessel index, and

$$d\mu_\nu(x) = x^{2\nu+1}\,dx, \qquad \mathcal{H}_\nu = L^2((0,\infty), d\mu_\nu).$$

The canonical matrix is

$$\mathbf{m} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}), \qquad b \neq 0,$$

and the phase parameter is denoted by

$$\gamma = \frac{d}{b}.$$

The ordinary Bessel operator is

$$\mathcal{B}_\nu = \frac{d^2}{dx^2} + \frac{2\nu+1}{x}\frac{d}{dx},$$

whereas the canonical Fourier–Bessel operator is

$$\Delta_{\nu,\mathbf{m}} = L_\gamma \mathcal{B}_\nu L_{-\gamma},$$

where

$$\gamma = \frac{d}{b}, \qquad L_\gamma f(x) = e^{i\frac{\gamma}{2}x^2} f(x).$$

For the associated non-negative self-adjoint operator. Its fractional power is always denoted by $A^\alpha_{\nu,\mathbf{m}}$, and the symbol $(-\Delta_{\nu,\mathbf{m}})^\alpha$ is used only when the principal complex branch is explicitly discussed. The canonical Fourier–Bessel transform is denoted by $\mathcal{F}^{\mathbf{m}}_\nu$, its kernel by $K^{\mathbf{m}}_\nu$, the canonical generalized shift by $T^{\nu,\mathbf{m}}_x$, and the canonical convolution by $*_{\nu,\mathbf{m}}$. The heat semigroup is

$$P^{\nu,\mathbf{m}}_t = e^{-tA_{\nu,\mathbf{m}}},$$

and the spectral multiplier defining the fractional power is

$$M_{\alpha,\mathbf{m}}(\lambda) = \left(\frac{\lambda^2}{b^2}\right)^\alpha.$$

3. Spectral study of the canonical Fourier–Bessel operator

We begin with the Hilbert-space realization of the operator. This is the analytic foundation for the canonical Fourier–Bessel transform and for the fractional powers studied later. Throughout this section, $\nu > -1/2$, and

$$\mathbf{m} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}), \qquad b \neq 0.$$

We put

$$d\mu_\nu(x) = x^{2\nu+1} dx, \qquad \mathcal{H}_\nu = L^2((0, \infty), d\mu_\nu),$$

and

$$\langle f, g \rangle_{\mathcal{H}_\nu} = \int_0^\infty f(x)\overline{g(x)} x^{2\nu+1} dx.$$

The natural core is

$$\mathscr{S}_{*,+} = \left\{ f|_{(0,\infty)} : f \in S_*(\mathbb{R}) \right\},$$

where $S_*(\mathbb{R})$ is the even Schwartz space. Functions in $\mathscr{S}_{*,+}$ are smooth at the origin in the even sense, rapidly decreasing at infinity, and satisfy $f'(0) = 0$. Following the standard Bessel notation [18], the normalized Bessel function is

$$j_\nu(z) = 2^\nu \Gamma(\nu + 1) \frac{J_\nu(z)}{z^\nu} = \sum_{k=0}^\infty \frac{(-1)^k \Gamma(\nu + 1)}{k! \Gamma(k + \nu + 1)} \left(\frac{z}{2} \right)^{2k}.$$

The ordinary Bessel operator is

$$\mathcal{B}_\nu = \frac{d^2}{dx^2} + \frac{2\nu + 1}{x} \frac{d}{dx}.$$

For every $\lambda \in \mathbb{C}$,

$$\mathcal{B}_\nu j_\nu(\lambda x) = -\lambda^2 j_\nu(\lambda x), \qquad j_\nu(0) = 1, \qquad \frac{d}{dx} j_\nu(\lambda x)\Big|_{x=0} = 0.$$

On $\mathscr{S}_{*,+}$, integration by parts gives

$$\langle \mathcal{B}_\nu f, g \rangle_{\mathcal{H}_\nu} = -\int_0^\infty f'(x)\overline{g'(x)} x^{2\nu+1} dx = \langle f, \mathcal{B}_\nu g \rangle_{\mathcal{H}_\nu}.$$

Thus, $\mathcal{B}_\nu$ is symmetric and nonpositive on the core. The boundary term at 0 vanishes because of the even regularity condition, and the boundary term at infinity vanishes by rapid decay. Let $\mathcal{F}_\nu$ be the ordinary Fourier–Bessel transform [18, 19],

$$\mathcal{F}_\nu f(\lambda) = c_\nu \int_0^\infty f(x) j_\nu(\lambda x) x^{2\nu+1} dx, \qquad c_\nu^{-1} = 2^\nu \Gamma(\nu + 1).$$

The transform $\mathcal{F}_\nu$ extends to a unitary involution on $\mathcal{H}_\nu$. The non-negative self-adjoint Bessel operator is

$$A_\nu = -\mathcal{B}_\nu,$$

with domain

$$\mathcal{D}(A_\nu) = \left\{ f \in \mathcal{H}_\nu : \int_0^\infty \lambda^4 |\mathcal{F}_\nu f(\lambda)|^2 \lambda^{2\nu+1} d\lambda < \infty \right\},$$

and spectral action

$$A_\nu f = \mathcal{F}_\nu^{-1}[\lambda^2 \mathcal{F}_\nu f(\lambda)].$$

Now, put

$$\gamma = \frac{d}{b}, \qquad L_\gamma f(x) = e^{\frac{i\gamma}{2}x^2} f(x).$$

Because $|e^{i\gamma x^2/2}| = 1$, L_γ is unitary on $\mathcal{H}_\nu$. A direct computation gives the transmutation identity

$$\Delta_{\nu,\mathbf{m}} = L_\gamma \mathcal{B}_\nu L_{-\gamma} \quad \text{on } \mathscr{S}_{*,+},$$

where

$$\Delta_{\nu,\mathbf{m}} = \frac{d^2}{dx^2} + \left(\frac{2\nu+1}{x} - 2i\frac{d}{b}x \right) \frac{d}{dx} - \left(\frac{d^2}{b^2} x^2 + 2i(\nu+1)\frac{d}{b} \right). \tag{3.1}$$

Therefore,

$$A_{\nu,\mathbf{m}} := -\Delta_{\nu,\mathbf{m}} = L_\gamma A_\nu L_{-\gamma}.$$

We define

$$\mathcal{D}(A_{\nu,\mathbf{m}}) = L_\gamma \mathcal{D}(A_\nu), \qquad A_{\nu,\mathbf{m}} f = L_\gamma A_\nu L_{-\gamma} f. \tag{3.2}$$

Theorem 3.1. *The operator $A_{\nu,\mathbf{m}} = -\Delta_{\nu,\mathbf{m}}$ with domain* (3.2) *is non-negative and self-adjoint on $\mathcal{H}_\nu$. Equivalently, $\Delta_{\nu,\mathbf{m}}$ is self-adjoint and nonpositive on the same domain. Moreover, $\mathscr{S}_{*,+}$ is a core for both operators.*

Proof. The operator $A_\nu = -\mathcal{B}_\nu$ is non-negative and self-adjoint by the Fourier–Bessel spectral theorem. Because L_γ is unitary, $A_{\nu,\mathbf{m}} = L_\gamma A_\nu L_{-\gamma}$ is also non-negative and self-adjoint. Hence, $\Delta_{\nu,\mathbf{m}} = -A_{\nu,\mathbf{m}}$ is self-adjoint and nonpositive. Because $\mathscr{S}_{*,+}$ is a core for A_ν and is invariant under $L_{\pm\gamma}$, it is also a core for $A_{\nu,\mathbf{m}}$ and $\Delta_{\nu,\mathbf{m}}$. □

The spectrum is obtained immediately from the unitary equivalence. Namely,

$$\sigma(A_{\nu,\mathbf{m}}) = [0,\infty), \qquad \sigma(\Delta_{\nu,\mathbf{m}}) = (-\infty, 0].$$

It is purely continuous and is parameterized by the Bessel spectral variable. The spectral resolution of $A_{\nu,\mathbf{m}}$ is

$$E_{\nu,\mathbf{m}}(\Omega) f = L_\gamma \mathcal{F}_\nu^{-1} \left[\mathbf{1}_\Omega(\lambda^2) \mathcal{F}_\nu(L_{-\gamma} f)(\lambda) \right], \qquad \Omega \subset [0,\infty).$$

Consequently, for every Borel function $\Phi : [0,\infty) \to \mathbb{C}$,

$$\Phi(A_{\nu,\mathbf{m}}) f = L_\gamma \mathcal{F}_\nu^{-1} \left[\Phi(\lambda^2) \mathcal{F}_\nu(L_{-\gamma} f)(\lambda) \right],$$

with its natural maximal domain. In particular,

$$A_{\nu,\mathbf{m}}^\alpha f = L_\gamma \mathcal{F}_\nu^{-1} \left[\lambda^{2\alpha} \mathcal{F}_\nu(L_{-\gamma} f)(\lambda) \right], \qquad \alpha > 0.$$

This is the spectral definition of the fractional power before introducing the canonical transform. The canonical Fourier–Bessel transform introduced next rewrites the same formula in the canonical spectral variable.

4. Canonical Fourier–Bessel transform, inversion, and Plancherel formula

Let

$$\mathbf{m} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}), \qquad b \neq 0.$$

The canonical Fourier–Bessel kernel is [21, 22]

$$K_\nu^{\mathbf{m}}(x, y) = \exp\left\{ \frac{i}{2} \left(\frac{d}{b} x^2 + \frac{a}{b} y^2 \right) \right\} j_\nu \left(\frac{xy}{b} \right). \tag{4.1}$$

For $f \in L^1_\nu$, the canonical Fourier–Bessel transform is

$$\mathcal{F}_\nu^{\mathbf{m}} f(\lambda) = \frac{c_\nu}{(ib)^{\nu+1}} \int_0^\infty K_\nu^{\mathbf{m}}(\lambda, y) f(y) y^{2\nu+1} dy. \tag{4.2}$$

This factorization shows that $\mathcal{F}_\nu^{\mathbf{m}}$ is obtained from the Fourier–Bessel transform by chirp multiplications and dilation.

Theorem 4.1. *The canonical Fourier–Bessel transform satisfies the following properties.*

(i) (Inversion) *If f and $\mathcal{F}_\nu^{\mathbf{m}} f$ are in the appropriate L^1_ν classes, then*

$$\mathcal{F}_\nu^{\mathbf{m}^{-1}} \mathcal{F}_\nu^{\mathbf{m}} f = f, \qquad \mathcal{F}_\nu^{\mathbf{m}} \mathcal{F}_\nu^{\mathbf{m}^{-1}} f = f.$$

On L^2_ν, these identities hold by unitary extension.

(ii) (Plancherel theorem) *The transform extends to a unitary operator on $\mathcal{H}_\nu$:*

$$\|\mathcal{F}_\nu^{\mathbf{m}} f\|_{2,\nu} = \|f\|_{2,\nu}, \qquad f \in \mathcal{H}_\nu.$$

(iii) (Eigenfunction relation) *For every $y > 0$,*

$$\Delta_{\nu,\mathbf{m}} K_\nu^{\mathbf{m}}(\cdot, y) = -\frac{y^2}{b^2} K_\nu^{\mathbf{m}}(\cdot, y). \tag{4.3}$$

(iv) (Operational formula) *For $f \in S_*(\mathbb{R})$,*

$$\mathcal{F}_\nu^{\mathbf{m}}[y^2 f(y)](x) = -b^2 \Delta_\nu^m (\mathcal{F}_\nu^{\mathbf{m}} f)(x). \tag{4.4}$$

(v) (Diagonalization) *For $f \in \mathcal{D}(A_{\nu,\mathbf{m}})$,*

$$\mathcal{F}_\nu^{\mathbf{m}^{-1}}(A_{\nu,\mathbf{m}} f)(\lambda) = \frac{\lambda^2}{b^2} \mathcal{F}_\nu^{\mathbf{m}^{-1}} f(\lambda).$$

(vi) (Schwartz isomorphism) *The transform $\mathcal{F}_\nu^{\mathbf{m}}$ is a topological isomorphism of $S_*(\mathbb{R})$ onto itself.*

Proof. The eigenfunction relation follows by applying (3.1) to the kernel (4.1). The operational formula is the transformed form of multiplication by y^2, and the diagonalization follows from the same identity. $\square$

The functional calculus from Section 3 may now be written in canonical variables. For every Borel function Φ,

$$\Phi(A_{\nu,\mathbf{m}})f = \mathcal{F}_\nu^{\mathbf{m}}\left[\Phi\left(\frac{\lambda^2}{b^2}\right)\mathcal{F}_\nu^{\mathbf{m}^{-1}} f(\lambda)\right].$$

In particular,

$$A_{\nu,\mathbf{m}}^{\alpha} f = \mathcal{F}_\nu^{\mathbf{m}}\left[\left(\frac{\lambda^2}{b^2}\right)^{\alpha}\mathcal{F}_\nu^{\mathbf{m}^{-1}} f(\lambda)\right], \qquad \alpha > 0.$$

5. Special cases of the canonical Fourier–Bessel transform

The canonical Fourier–Bessel transform contains many familiar transforms as special or limiting cases. We collect the most important ones here in order to clarify how the present fractional calculus extends the classical Fourier–Bessel setting while remaining intrinsic to the canonical transform. The subsections below should be read as a progression from the ordinary Fourier–Bessel transform to the full canonical family. Changing the entries of the matrix **m** successively introduces inversion, rotation, Fresnel propagation, input chirp, output chirp, and limiting chirp–dilation effects. Thus Section 5 explains how each canonical matrix modifies the same Bessel kernel. Throughout this section,

$$\mathbf{m} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}), \qquad b \neq 0,$$

and

$$\mathcal{F}_\nu^{\mathbf{m}} f(x) = \frac{c_\nu}{(ib)^{\nu+1}} \int_0^\infty e^{\frac{i}{2}(\frac{d}{b}x^2+\frac{a}{b}y^2)} j_\nu\left(\frac{xy}{b}\right) f(y) y^{2\nu+1}\, dy. \tag{5.1}$$

5.1. Fourier–Bessel transform

This is the reference case from which the other canonical examples are obtained by adding matrix-dependent deformations.
Let

$$\mathbf{m}_F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then, $a = d = 0$, and $b = 1$. Hence,

$$\mathcal{F}_\nu^{\mathbf{m}_F} f(x) = \frac{c_\nu}{i^{\nu+1}} \int_0^\infty j_\nu(xy) f(y) y^{2\nu+1}\, dy.$$

Thus, up to the harmless constant phase $i^{-(\nu+1)}$, $\mathcal{F}_\nu^{\mathbf{m}_F}$ is the ordinary Fourier–Bessel transform [18, 19]. In this case,

$$A_{\nu,\mathbf{m}_F} = -\mathcal{B}_\nu,$$

and Definition 8.1 gives the standard spectral fractional Bessel operator:

$$(-\mathcal{B}_\nu)^{\alpha} f = \mathcal{F}_\nu^{\mathbf{m}_F}\left[\lambda^{2\alpha}\mathcal{F}_\nu^{m_F^{-1}} f(\lambda)\right].$$

Hence, the ordinary Fourier–Bessel transform is recovered as the matrix specialization $a = d = 0$, $b = 1$ and $c = -1$ of the general canonical model. In this sense, it is the basic Fourier–Bessel member from which the more general canonical family is obtained.

5.2. *Inverse Fourier–Bessel transform*

Changing the sign of the Fourier–Bessel matrix reverses the transform and fixes the inverse convention used throughout the paper.
Let

$$\mathbf{m}_F^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then, $b = -1$, $a = d = 0$, and because j_ν is even,

$$\mathcal{F}_\nu^{\mathbf{m}_F^{-1}} f(x) = \frac{c_\nu}{(-i)^{\nu+1}} \int_0^\infty j_\nu(xy) f(y) y^{2\nu+1}\, dy.$$

This is the inverse Fourier–Bessel transform, again up to the normalizing phase fixed by the convention in (4.2). Thus, the inverse Fourier–Bessel transform is the matrix specialization $a = d = 0$, $b = -1$, and $c = 1$ of the general canonical model, corresponding to the inverse quarter-turn in the canonical matrix group.

5.3. *Fractional Fourier–Bessel transform*

The next step is to replace the quarter-turn Fourier–Bessel matrix by a continuous rotation matrix, which gives the Bessel analogue of the fractional Fourier transform.

For $\theta \notin \pi\mathbb{Z}$, take the rotation matrix

$$\mathbf{m}_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Then, $b = \sin\theta$ and $a = d = \cos\theta$. Formula (5.1) becomes

$$\mathcal{F}_\nu^{\mathbf{m}_\theta} f(x) = \frac{c_\nu}{(i\sin\theta)^{\nu+1}} e^{\frac{i}{2}x^2\cot\theta} \int_0^\infty e^{\frac{i}{2}y^2\cot\theta} j_\nu(xy\csc\theta) f(y) y^{2\nu+1}\, dy.$$

This is the fractional Fourier–Bessel transform, the Bessel analog of the fractional Fourier transform [9, 10]. The special value $\theta = \pi/2$ gives the Fourier–Bessel transform. In the limit $\theta \to 0$, the transform approaches the identity in the canonical sense. The corresponding differential operator is

$$\Delta_\nu^{\mathbf{m}_\theta} = \frac{d^2}{dx^2} + \left(\frac{2\nu+1}{x} - 2ix\cot\theta\right)\frac{d}{dx} - \left(x^2\cot^2\theta + 2i(\nu+1)\cot\theta\right).$$

Thus, the present operator $(-\Delta_\nu^{\mathbf{m}_\theta})^\alpha$ is the fractional power associated with the fractional Fourier–Bessel dynamics.

Consequently, the fractional Fourier–Bessel transform is the rotation specialization of the general model obtained by $a = d = \cos\theta$, $b = \sin\theta$, and $c = -\sin\theta$; the Fourier–Bessel case is recovered at $\theta = \pi/2$.

5.4. *Fresnel–Bessel transform*

The rotation model is followed by the parabolic canonical matrix, which corresponds to Fresnel-type propagation and introduces a quadratic phase in both variables.

Let

$$\mathbf{m}_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \qquad b \neq 0.$$

Then,

$$\mathcal{F}_\nu^{\mathbf{m}_b} f(x) = \frac{c_\nu}{(ib)^{\nu+1}} e^{\frac{i}{2b}x^2} \int_0^\infty e^{\frac{i}{2b}y^2} j_\nu\left(\frac{xy}{b}\right) f(y) y^{2\nu+1}\, dy.$$

This is the Fresnel–Bessel transform, the Bessel analog of the Fresnel or free-propagation operator in canonical transform theory. It corresponds to propagation through a quadratic phase system with matrix $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$.

It is therefore the parabolic matrix specialization of the general canonical transform with $a = d = 1$, $c = 0$, and $b \neq 0$, so that the free propagation parameter is carried entirely by the off-diagonal entry b.

5.5. Scaled Fourier–Bessel transform

Setting $d = 0$ removes the output chirp and leaves an input-chirped, scaled Fourier–Bessel transform.

Let

$$\mathbf{m}_{a,b} = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, \qquad -bc = 1.$$

Then, $d = 0$, and the output chirp disappears. Because $c = -1/b$, one obtains

$$\mathcal{F}_\nu^{\mathbf{m}_{a,b}} f(x) = \frac{c_\nu}{(ib)^{\nu+1}} \int_0^\infty e^{\frac{i}{2}\frac{a}{b}y^2} j_\nu\left(\frac{xy}{b}\right) f(y) y^{2\nu+1}\, dy.$$

This is a scaled Fourier–Bessel transform preceded by a chirp multiplication in the input variable.

This case is the $d = 0$ specialization of the general matrix model; the determinant constraint $ad - bc = 1$ then forces $c = -1/b$, leaving only the input quadratic phase together with the Bessel scaling.

5.6. Output-chirped Fourier–Bessel transform

The complementary case $a = 0$ removes the input chirp and keeps the output chirp, showing the dual role of the two phase factors.

Let

$$\mathbf{m}_{b,d} = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}, \qquad -bc = 1.$$

Then, $a = 0$, and

$$\mathcal{F}_\nu^{\mathbf{m}_{b,d}} f(x) = \frac{c_\nu}{(ib)^{\nu+1}} e^{\frac{i}{2}\frac{d}{b}x^2} \int_0^\infty j_\nu\left(\frac{xy}{b}\right) f(y) y^{2\nu+1}\, dy.$$

This is a scaled Fourier–Bessel transform followed by a chirp multiplication in the output variable.

This transform is the complementary $a = 0$ specialization of the general matrix model; again, $c = -1/b$, and the canonical deformation retains only the output quadratic phase together with the Bessel scaling.

5.7. General chirp–dilation–Fourier–Bessel decomposition

After the preceding examples, the general case can be interpreted as the composition of elementary operations.

Every canonical Fourier–Bessel transform with $b \neq 0$ may be viewed as a composition of elementary canonical operations, an input chirp, a scaling of the Fourier–Bessel variable, a Fourier–Bessel transform, and an output chirp. More precisely, (5.1) can be written schematically as

$$\mathcal{F}_\nu^{\mathbf{m}} = L_{d/b}\, D_b\, \mathcal{F}_\nu^{\mathbf{m}}\, L_{a/b},$$

up to the normalizing phase dictated by the convention (4.2). Here,

$$L_\rho f(x) = e^{\frac{i\rho}{2}x^2} f(x)$$

is a chirp multiplication, and D_b denotes the natural Bessel dilation. This decomposition explains why the canonical Fourier–Bessel transform simultaneously contains chirp modulation, scaling, and Fourier–Bessel analysis.

Accordingly, every integral-type transform considered above is a matrix specialization of the same general model with $b \neq 0$ and $ad - bc = 1$; the apparent diversity of kernels is produced solely by chirp, dilation, and Fourier–Bessel factors.

5.8. Limiting canonical cases with $b = 0$

Finally, the limiting case $b = 0$ completes the canonical matrix picture, although it is not represented by the same oscillatory integral formula.

The integral formula (4.2) is written for $b \neq 0$. Nevertheless, the full canonical family also has limiting cases with $b = 0$. If

$$\mathbf{m} = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \qquad ad = 1,$$

then the corresponding canonical action is no longer an oscillatory Fourier–Bessel integral. It instead becomes a chirp–dilation operator of the form

$$\mathcal{F}_\nu^{\mathbf{m}} f(x) \sim d^{\nu+1} e^{\frac{i}{2}cdx^2} f(dx),$$

with the exact normalization depending on the chosen canonical transform convention. This family contains the following limiting cases.

(i) The identity operator corresponds to $\mathbf{m} = I$.

(ii) The parity or reflection case corresponds to $\mathbf{m} = -I$; in the even Bessel setting, this acts essentially as the identity, up to the canonical phase convention.

(iii) Pure dilation corresponds to $\mathbf{m} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$.

(iv) Pure chirp multiplication corresponds to $\mathbf{m} = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$.

Although these $b = 0$ cases are not part of the integral representation used in the main body of the paper, they complete the canonical transform picture and explain the limiting behavior of the $b \neq 0$ family.

These $b = 0$ cases are the genuinely degenerate, nonintegral boundary members of the general canonical family. The oscillatory Bessel kernel is replaced by a local chirp–dilation action.

The distinction between $b \to 0$ and the integral-type regime $b \neq 0$ is structural. For $b \neq 0$, the kernel is oscillatory and nonlocal: $\mathcal{F}_\nu^{\mathbf{m}} f(x)$ samples all $y > 0$ through the rapidly oscillating Bessel factor $j_\nu(xy/b)$ and the quadratic phases. When $b \to 0$, the phase frequency becomes singular, and the kernel cannot be evaluated by direct substitution. Instead, after the normalization and canonical phase are retained, the oscillatory kernel concentrates in the distributional sense on the canonical ray $y = dx$ (equivalently $x = ay$ because $ad = 1$ at $b = 0$). The limiting action is therefore local: a dilation composed with a chirp multiplication. Integral-type transforms with $b \neq 0$ and the $b = 0$ transforms consequently belong to the same canonical family but represent two analytically different regimes, namely nonlocal Bessel mixing and local canonical transport. This also clarifies the asymptotic behavior of the transform as the matrix approaches the boundary $b = 0$.

5.9. Summary table

The following revised table (Table 1) summarizes not only the formulas but also the structural operation represented by each canonical matrix.

Table 1. Canonical Fourier–Bessel special cases. The table is organized by the canonical action and by its role in the fractional calculus.

Case and matrix	Canonical action	Role in the present calculus
Fourier–Bessel, $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$	Quarter-turn; no chirp deformation.	Recovers the ordinary Fourier–Bessel transform and the standard fractional Bessel operator.
Inverse Fourier–Bessel, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$	Inverse quarter-turn.	Fixes the inverse-transform convention and the Plancherel normalization.
Fractional Fourier–Bessel, $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$	Continuous phase-space rotation.	Interpolates between identity-type and Fourier–Bessel dynamics.
Fresnel–Bessel, $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$	Parabolic shear with quadratic propagation phase.	Gives the Fresnel-type Bessel propagation regime.
Input-chirped scaled case, $\left(\begin{smallmatrix} a & b \\ c & 0 \end{smallmatrix}\right)$	Input chirp followed by Bessel scaling.	Isolates the effect of the input phase parameter a/b.
Output-chirped scaled case, $\left(\begin{smallmatrix} 0 & b \\ c & d \end{smallmatrix}\right)$	Bessel scaling followed by output chirp.	Isolates the effect of the output phase parameter d/b.
Limiting chirp–dilation, $\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right)$	Local chirp–dilation action; no oscillatory integral kernel.	Completes the canonical matrix family at the boundary $b = 0$.

In summary, the passage from the ordinary Fourier–Bessel transform to the full canonical Fourier–

Bessel transform is governed by three elementary operations. Chirp multiplication, dilation, and Fourier–Bessel rotation. The fractional operator $A^{\alpha}_{\nu,\mathbf{m}}$ inherits this structure through the multiplier $(\lambda^2/b^2)^{\alpha}$ and through the canonical generalized shift used later to describe convolution and heat propagation.

6. Schematic and illustrative special cases

The construction can be summarized by the following scheme:

$$\begin{array}{ccccc} \text{canonical matrix } \mathbf{m} & \Longrightarrow & \text{chirp modulation } L_\gamma & \Longrightarrow & \Delta_{\nu,\mathbf{m}} = L_\gamma \mathcal{B}_\nu L_{-\gamma} \\ & & \Downarrow & & \Downarrow \\ \text{canonical kernel } K^{\mathbf{m}}_{\nu} & \Longrightarrow & \mathcal{F}^{\mathbf{m}}_{\nu} & \Longrightarrow & A^{\alpha}_{\nu,\mathbf{m}}. \end{array}$$

Thus, the special cases described above are not separate constructions; they are obtained by choosing the canonical parameters in different ways. The following table (Table 2) gives a qualitative comparison between the ordinary Bessel model and the canonical model.

Table 2. Ordinary versus canonical Bessel models. Arranging the comparison by feature makes the structural changes introduced by m explicit.

Feature	Ordinary Bessel model	Canonical Bessel model
Differential operator	$\mathcal{B}_\nu$	$\Delta_{\nu,\mathbf{m}} = L_\gamma \mathcal{B}_\nu L_{-\gamma}$
Quadratic phase	none	$e^{i\gamma x^2/2}$, with $\gamma = d/b$
Fractional multiplier	$\lambda^{2\alpha}$	$(\lambda^2/b^2)^{\alpha}$
Generalized translation	T^{ν}_{x}	$T^{\nu,\mathbf{m}}_{x} = L_\gamma T^{\nu}_{x} L_{-\gamma}$
Convolution geometry	ordinary Bessel convolution	chirp-conjugated canonical Bessel convolution

This table also clarifies the difference between the proposed operator and the conventional fractional Bessel operator. The canonical matrix and the phase modulation change the physical-space structure, and the factor b changes the spectral scale.

For illustration, if b is increased while α is fixed, the multiplier $(\lambda^2/b^2)^{\alpha}$ decreases at each fixed spectral frequency λ. If α is increased while b is fixed, high spectral frequencies are amplified more strongly. These elementary plots of the multiplier are enough to visualize the roles of b and α without introducing any additional numerical assumptions.

7. The canonical generalized shift and convolution

The canonical Fourier–Bessel setting requires a translation structure adapted to the operator Δ^{m}_{ν}. This role is played by the generalized shift, in the spirit of generalized translation operators in Bessel analysis [19, 21]. Put

$$\gamma = \frac{d}{b}, \qquad L_\gamma f(x) = e^{\frac{i\gamma}{2}x^2} f(x).$$

The operator $\Delta_{\nu,\mathbf{m}}$ is the chirp conjugate of the Bessel operator:

$$\Delta_{\nu,\mathbf{m}} = L_\gamma \mathcal{B}_\nu L_{-\gamma}. \tag{7.1}$$

This identity explains the form of the generalized shift below.
The ordinary translation is not compatible with the canonical Fourier–Bessel transform because both the Bessel weight and the chirp factor are part of the geometry. The shift must therefore be constructed by conjugating the ordinary Bessel translation with the canonical phase. In this way, the generalized shift becomes multiplication by the appropriate spectral kernel on the transform side, exactly as ordinary translations become modulations under the classical Fourier transform.

Definition 7.1. Let T_x^ν denote the ordinary Bessel translation associated with $\mathcal{B}_\nu$. The canonical generalized shift associated with $\Delta_{\nu,\mathbf{m}}$ is defined by

$$T_x^{\nu,\mathbf{m}} f(y) = e^{\frac{i\gamma}{2}(x^2+y^2)} T_x^\nu \left(e^{-\frac{i\gamma}{2}s^2} f(s) \right)(y).$$

Equivalently,

$$T_x^{\nu,\mathbf{m}} = L_{\gamma,x} L_{\gamma,y} T_x^\nu L_{-\gamma}.$$

The operator $T_x^{\nu,\mathbf{m}}$ is the correct analog of translation for the canonical Fourier–Bessel transform. It is not an ordinary displacement of the variable, but a Bessel translation twisted by the canonical chirp phase.

Proposition 7.2. *The generalized shift $T_x^{\nu,\mathbf{m}}$ satisfies the following properties.*

(i) $T_0^{\nu,\mathbf{m}} = I$, *and* $T_x^{\nu,\mathbf{m}} f(y) = T_y^{\nu,\mathbf{m}} f(x)$.

(ii) $T_x^{\nu,\mathbf{m}}$ *is linear.*

(iii) *If* $f(y) = 0$ *for* $y \geq r$, *then* $T_x^{\nu,\mathbf{m}} f(y) = 0$ *whenever* $|x - y| \geq r$.

(iv) *For every* $1 \leq p \leq \infty$,

$$\|T_x^{\nu,\mathbf{m}} f\|_{p,\nu} \leq \|f\|_{p,\nu}.$$

(v) *The shift commutes with the canonical Fourier–Bessel operator on* $E_*(\mathbb{R})$*:*

$$\Delta_{\nu,\mathbf{m}} T_x^{\nu,\mathbf{m}} f = T_x^{\nu,\mathbf{m}} \Delta_{\nu,\mathbf{m}} f.$$

(vi) *The product formula*

$$T_x^{\nu,\mathbf{m}}(K_\nu^{\mathbf{m}}(\cdot,\lambda))(y) = e^{-\frac{i}{2}\frac{a}{b}\lambda^2} K_\nu^{\mathbf{m}}(x,\lambda) K_\nu^{\mathbf{m}}(y,\lambda)$$

holds in the usual transform sense.

Proof. All assertions follow from Definition 7.1, the corresponding properties of the Bessel translation T_x^ν, and the fact that multiplication by L_γ is an isometry on each L_ν^p. The commutation formula follows from the conjugacy $\Delta_{\nu,\mathbf{m}} = L_\gamma \mathcal{B}_\nu L_{-\gamma}$ and from the classical commutation relation $\mathcal{B}_\nu T_x^\nu = T_x^\nu \mathcal{B}_\nu$. The product formula is obtained by applying the Bessel product formula for j_ν and then restoring the canonical phase factors. □

Using the generalized shift, define the canonical convolution by

$$(f *_{\nu,\mathbf{m}} g)(x) = \int_0^\infty T_x^{\nu,\mathbf{m}} f(y) e^{-i\gamma y^2} g(y) y^{2\nu+1}\, dy,$$

whenever the integral exists. This convolution is commutative and associative on the usual admissible classes, and it satisfies the Young inequality

$$\|f *_{\nu,\mathbf{m}} g\|_{r,\nu} \le \|f\|_{p,\nu}\|g\|_{q,\nu}, \qquad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Moreover, the canonical Fourier–Bessel transform converts this convolution into multiplication, up to the canonical phase factor. This is the structural reason why the heat equation associated with Δ_ν^m admits the explicit kernel formula used below.

8. Fractional powers by spectral multipliers

By Theorem 4.1, $A_{\nu,\mathbf{m}}$ is represented on the canonical Fourier–Bessel side by the multiplier λ^2/b^2.

Definition 8.1. Let $\alpha > 0$, and $f \in S_*(\mathbb{R})$. The fractional power of $A_{\nu,\mathbf{m}}$ is defined by

$$A_{\nu,\mathbf{m}}^\alpha f(x) = \mathcal{F}_\nu^m \left[\left(\frac{\lambda^2}{b^2}\right)^\alpha \mathcal{F}_\nu^{\mathbf{m}^{-1}} f(\lambda) \right](x). \tag{8.1}$$

The choice of the multiplier in (8.1) is forced by the canonical spectral theorem. Indeed, the canonical Fourier–Bessel transform diagonalizes $\Delta_{\nu,\mathbf{m}}$ with spectral value $-\lambda^2/b^2$. Therefore, the nonnegative operator $A_{\nu,\mathbf{m}}$ corresponds to multiplication by λ^2/b^2, and its fractional power is necessarily obtained by multiplication by $(\lambda^2/b^2)^\alpha$. This is the standard functional calculus for nonnegative self-adjoint operators, adapted here to the canonical Fourier–Bessel diagonalization. The parameter b is not a harmless normalization. It changes the spectral scale and therefore changes the strength of the fractional diffusion.

Theorem 8.2. *Let $\alpha, \beta > 0$, and $f \in S_*(\mathbb{R})$. Then:*

(i) *$A_{\nu,\mathbf{m}}^\alpha$ is linear on $S_*(\mathbb{R})$.*

(ii) *The semigroup law holds:*

$$A_{\nu,\mathbf{m}}^\alpha A_{\nu,\mathbf{m}}^\beta f = A_{\nu,\mathbf{m}}^{\alpha+\beta} f.$$

(iii) *For every spectral kernel $K_\nu^{\mathbf{m}}(\cdot, \lambda)$,*

$$A_{\nu,\mathbf{m}}^\alpha K_\nu^{\mathbf{m}}(\cdot,\lambda) = \left(\frac{\lambda^2}{b^2}\right)^\alpha K_\nu^{\mathbf{m}}(\cdot,\lambda),$$

whenever the identity is interpreted in the generalized spectral sense.

Proof. Linearity is immediate from (8.1). Applying $\mathcal{F}_\nu^{\mathbf{m}^{-1}}$ to $A_{\nu,\mathbf{m}}^\alpha A_{\nu,\mathbf{m}}^\beta f$ gives

$$\left(\frac{\lambda^2}{b^2}\right)^\alpha \left(\frac{\lambda^2}{b^2}\right)^\beta \mathcal{F}_\nu^{\mathbf{m}^{-1}} f(\lambda),$$

which is the transform-side expression for $A_{\nu,\mathbf{m}}^{\alpha+\beta} f$. The eigenfunction relation follows from the same multiplier identity applied to the generalized eigenfunction (4.3). □

Proposition 8.3. *Let $0 < \alpha < 1$, $x \geq 0$, and $f \in S_*(\mathbb{R})$. Then, the fractional canonical Fourier–Bessel operator commutes with the generalized shift:*

$$A^{\alpha}_{\nu,\mathbf{m}} T^{\nu,\mathbf{m}}_x f = T^{\nu,\mathbf{m}}_x A^{\alpha}_{\nu,\mathbf{m}} f.$$

More generally, the same identity holds for every $\alpha > 0$ for which both sides are well-defined.

Proof. Recall that

$$\Delta_{\nu,\mathbf{m}} = L_\gamma \mathcal{B}_\nu L_{-\gamma}, \qquad A_{\nu,\mathbf{m}} = -\Delta^m_\nu = L_\gamma(-\mathcal{B}_\nu)L_{-\gamma}, \qquad \gamma = \frac{d}{b}.$$

Hence, by functional calculus,

$$A^{\alpha}_{\nu,\mathbf{m}} = L_\gamma(-\mathcal{B}_\nu)^\alpha L_{-\gamma}. \tag{8.2}$$

On the other hand, the generalized shift has the conjugated form

$$T^{\nu,\mathbf{m}}_x f(y) = e^{\frac{i\gamma}{2}x^2} L_\gamma T^\nu_x L_{-\gamma} f(y). \tag{8.3}$$

The scalar factor $e^{i\gamma x^2/2}$ is independent of y and therefore does not affect commutation. Because the ordinary Bessel translation commutes with the Bessel operator, it also commutes with its spectral fractional powers:

$$(-\mathcal{B}_\nu)^\alpha T^\nu_x g = T^\nu_x(-\mathcal{B}_\nu)^\alpha g, \qquad g \in S_*(\mathbb{R}).$$

Combining this identity with (8.2) and (8.3) gives

$$\begin{aligned} A^{\alpha}_{\nu,\mathbf{m}} T^{\nu,\mathbf{m}}_x f &= e^{\frac{i\gamma}{2}x^2} L_\gamma(-\mathcal{B}_\nu)^\alpha T^\nu_x L_{-\gamma} f \\ &= e^{\frac{i\gamma}{2}x^2} L_\gamma T^\nu_x(-\mathcal{B}_\nu)^\alpha L_{-\gamma} f \\ &= T^{\nu,m}_x A^{\alpha}_{\nu,\mathbf{m}} f. \end{aligned}$$

This proves the proposition. □

9. Balakrishnan formula

The spectral multiplier description of $A_{\nu,m}$ gives a natural Sobolev scale adapted to the canonical Fourier–Bessel transform. For $s \geq 0$, define

$$H^s_{\nu,\mathbf{m}} = \left\{ f \in L^2_\nu : \int_0^\infty \left(1 + \frac{\lambda^2}{b^2}\right)^s |\mathcal{F}^{\mathbf{m}^{-1}}_\nu f(\lambda)|^2 \lambda^{2\nu+1}\, d\lambda < \infty \right\},$$

with norm

$$\|f\|_{H^s_{\nu,m}} = \left(\int_0^\infty \left(1 + \frac{\lambda^2}{b^2}\right)^s |\mathcal{F}^{\mathbf{m}^{-1}}_\nu f(\lambda)|^2 \lambda^{2\nu+1}\, d\lambda \right)^{1/2}.$$

This scale is the canonical Fourier–Bessel analog of the usual Bessel-potential scale associated with a non-negative self-adjoint operator [1, 2].
The heat semigroup and the Balakrishnan formula follow from the same spectral logic. Since

$$e^{-tA_{\nu,\mathbf{m}}} \quad \longleftrightarrow \quad e^{-t\lambda^2/b^2},$$

the fractional power is recovered from the semigroup by the classical identity

$$s^{\alpha} = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} (e^{-ts} - 1) \frac{dt}{t^{1+\alpha}}, \qquad 0 < \alpha < 1,$$

with $s = \lambda^2/b^2$. This explains why the heat representation is not an additional definition but a consequence of the spectral multiplier definition.

Proposition 9.1. *For $\alpha > 0$, the L^2_ν domain of $A^{\alpha}_{\nu,\mathbf{m}}$ is*

$$\mathcal{D}(A^{\alpha}_{\nu,\mathbf{m}}) = \left\{ f \in L^2_\nu : \int_0^{\infty} \left(\frac{\lambda^2}{b^2}\right)^{2\alpha} |\mathcal{F}_\nu^{\mathbf{m}^{-1}} f(\lambda)|^2 \lambda^{2\nu+1}\, d\lambda < \infty \right\}. \tag{9.1}$$

Moreover,

$$\mathcal{D}(A^{\alpha}_{\nu,\mathbf{m}}) = H^{2\alpha}_{\nu,m},$$

with equivalence between the graph norm

$$\|f\|_{2,\nu} + \|A^{\alpha}_{\nu,\mathbf{m}} f\|_{2,\nu}$$

and the Sobolev norm $\|f\|_{H^{2\alpha}_{\nu,\mathbf{m}}}$.

Proof. By the canonical Fourier–Bessel diagonalization,

$$\mathcal{F}_\nu^{\mathbf{m}^{-1}} (A^{\alpha}_{\nu,\mathbf{m}} f)(\lambda) = \left(\frac{\lambda^2}{b^2}\right)^{\alpha} \mathcal{F}_\nu^{\mathbf{m}^{-1}} f(\lambda).$$

Plancherel's theorem therefore gives

$$\|A^{\alpha}_{\nu,\mathbf{m}} f\|^2_{2,\nu} = \int_0^{\infty} \left(\frac{\lambda^2}{b^2}\right)^{2\alpha} |\mathcal{F}_\nu^{m^{-1}} f(\lambda)|^2 \lambda^{2\nu+1}\, d\lambda,$$

which proves (9.1). Beacause

$$1 + \left(\frac{\lambda^2}{b^2}\right)^{2\alpha} \asymp \left(1 + \frac{\lambda^2}{b^2}\right)^{2\alpha}, \qquad \lambda \geq 0,$$

the graph norm of $A^{\alpha}_{\nu,\mathbf{m}}$ is equivalent to the $H^{2\alpha}_{\nu,\mathbf{m}}$ norm. □

For $t > 0$, define

$$H_t^{\nu,\mathbf{m}} f = \mathcal{F}_\nu^{\mathbf{m}} \left[\exp\left(-t\frac{\lambda^2}{b^2}\right) \mathcal{F}_\nu^{\mathbf{m}^{-1}} f(\lambda) \right]. \tag{9.2}$$

Thus,

$$H_t^{\nu,m} = e^{-tA_{\nu,m}}$$

in the sense of the spectral theorem.

Proposition 9.2. *The family $\{H_t^{\nu,\mathbf{m}}\}_{t\geq 0}$ is a strongly continuous contraction semigroup on L^2_ν. If $f \in S_*(\mathbb{R})$, and*

$$u(t,x) = H_t^{\nu,\mathbf{m}} f(x),$$

then

$$\partial_t u(t,x) = \Delta_{\nu,\mathbf{m}} u(t,x), \qquad t > 0,$$

and

$$\lim_{t\downarrow 0} H_t^{\nu,\mathbf{m}} f = f$$

in $S_*(\mathbb{R})$ and in L^2_ν.

Proof. The multiplier $e^{-t\lambda^2/b^2}$ has modulus at most one, so Plancherel gives

$$\|H_t^{\nu,\mathbf{m}} f\|_{2,\nu} \leq \|f\|_{2,\nu}.$$

The semigroup law follows from multiplication of the spectral multipliers. Strong continuity at $t = 0$ follows by dominated convergence:

$$e^{-t\lambda^2/b^2} \mathcal{F}_\nu^{\mathbf{m}^{-1}} f(\lambda) \longrightarrow \mathcal{F}_\nu^{\mathbf{m}^{-1}} f(\lambda) \quad \text{in } L^2_\nu.$$

For $f \in S_*(\mathbb{R})$, differentiation under the transform is justified because the transformed function is rapidly decreasing. Hence,

$$\partial_t \mathcal{F}_\nu^{\mathbf{m}^{-1}} u(t,\lambda) = -\frac{\lambda^2}{b^2} e^{-t\lambda^2/b^2} \mathcal{F}_\nu^{\mathbf{m}^{-1}} f(\lambda),$$

which is precisely the transform-side expression of $\Delta_{\nu,\mathbf{m}} u(t,\cdot)$. □

The semigroup also has an explicit kernel representation. Let

$$\gamma = \frac{d}{b}, \qquad L_\gamma f(x) = e^{\frac{i\gamma}{2}x^2} f(x).$$

Because

$$A_{\nu,\mathbf{m}} = L_\gamma A_\nu L_{-\gamma}, \qquad A_\nu = -\mathcal{B}_\nu,$$

we have

$$H_t^{\nu,\mathbf{m}} = L_\gamma e^{-tA_\nu} L_{-\gamma} = L_\gamma e^{t\mathcal{B}_\nu} L_{-\gamma}.$$

The Bessel heat kernel with respect to $d\mu_\nu(y) = y^{2\nu+1} dy$ is

$$p_t^\nu(x,y) = \frac{2}{\Gamma(\nu+1)(4t)^{\nu+1}} \exp\left(-\frac{x^2+y^2}{4t}\right) j_\nu\left(\frac{ixy}{2t}\right), \qquad t > 0. \tag{9.3}$$

It satisfies

$$e^{-tA_\nu} g(x) = \int_0^\infty p_t^\nu(x,y) g(y) y^{2\nu+1}\, dy, \qquad \int_0^\infty p_t^\nu(x,y) y^{2\nu+1}\, dy = 1.$$

Theorem 9.3. *For $t > 0$ and suitable f, for instance $f \in S_*(\mathbb{R})$, one has*

$$H_t^{\nu,\mathbf{m}} f(x) = \int_0^\infty \mathcal{G}_t^{\nu,\mathbf{m}}(x,y) e^{-i\gamma y^2} f(y) y^{2\nu+1}\, dy, \tag{9.4}$$

where

$$\mathcal{G}_t^{\nu,\mathbf{m}}(x,y) = \frac{2}{\Gamma(\nu+1)(4t)^{\nu+1}} e^{\frac{i\gamma}{2}(x^2+y^2)} e^{-\frac{x^2+y^2}{4t}} j_\nu\left(\frac{ixy}{2t}\right). \tag{9.5}$$

Equivalently,

$$H_t^{\nu,\mathbf{m}} f(x) = \frac{2e^{\frac{i\gamma}{2}x^2}}{\Gamma(\nu+1)(4t)^{\nu+1}} \int_0^\infty e^{-\frac{i\gamma}{2}y^2} e^{-\frac{x^2+y^2}{4t}} j_\nu\left(\frac{ixy}{2t}\right) f(y) y^{2\nu+1}\, dy. \tag{9.6}$$

Proof. Using the conjugacy of the heat semigroups,

$$H_t^{\nu,m} f(x) = e^{\frac{i\gamma}{2}x^2} e^{-tA_\nu}(L_{-\gamma} f)(x).$$

By the Bessel heat kernel formula (9.3),

$$e^{-tA_\nu}(L_{-\gamma} f)(x) = \int_0^\infty p_t^\nu(x,y) e^{-\frac{i\gamma}{2}y^2} f(y) y^{2\nu+1}\, dy.$$

Therefore,

$$H_t^{\nu,m} f(x) = e^{\frac{i\gamma}{2}x^2} \int_0^\infty p_t^\nu(x,y) e^{-\frac{i\gamma}{2}y^2} f(y) y^{2\nu+1}\, dy,$$

which gives (9.6). Rewriting the phase as

$$e^{\frac{i\gamma}{2}x^2} e^{-\frac{i\gamma}{2}y^2} = e^{\frac{i\gamma}{2}(x^2+y^2)} e^{-i\gamma y^2}$$

gives (9.4) and (9.5). □

Remark 9.4. If the heat equation is written with conductivity $\sigma > 0$ as

$$\partial_t u = \sigma \Delta_\nu^m u,$$

then the semigroup is $e^{-t\sigma A_{\nu,m}}$, and the above heat kernel is obtained by replacing t with σt.

Theorem 9.5. *Let $0 < \alpha < 1$, and let $f \in S_*(\mathbb{R})$. Then,*

$$A_{\nu,\mathbf{m}}^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(x) - H_t^{\nu,\mathbf{m}} f(x)}{t^{1+\alpha}}\, dt. \tag{9.7}$$

Equivalently,

$$A_{\nu,\mathbf{m}}^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{1}{t^{1+\alpha}} \left[f(x) - \int_0^\infty \mathcal{G}_t^{\nu,\mathbf{m}}(x,y) e^{-i\gamma y^2} f(y) y^{2\nu+1}\, dy \right] dt. \tag{9.8}$$

Proof. For $r > 0$ and $0 < \alpha < 1$,

$$r^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{1 - e^{-tr}}{t^{1+\alpha}}\, dt.$$

Applying this scalar identity to the spectral multiplier

$$r = \frac{\lambda^2}{b^2}$$

gives

$$\left(\frac{\lambda^2}{b^2}\right)^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{1 - e^{-t\lambda^2/b^2}}{t^{1+\alpha}}\, dt.$$

Multiplying by $\mathcal{F}_\nu^{m^{-1}} f(\lambda)$ and applying $\mathcal{F}_\nu^m$ yields (9.7). Formula (9.8) follows from the heat kernel representation in Theorem 9.3. □

10. Lévy-type singular integral representation

The singular-integral formula gives the physical space form of the nonlocal operator. Its convergence is controlled by two complementary mechanisms. Near the singularity, the difference between f and its canonical generalized shift produces the same cancellation as in the classical fractional Laplacian. At infinity, the heat kernel decay and the rapid decay of test functions ensure integrability. These two mechanisms justify the principal value form of the Lévy-type representation below.

We recall that a Bernstein function is a non-negative C^∞ function ϕ on $(0,\infty)$ whose derivative is completely monotone. Equivalently, it admits a Lévy–Khintchine representation of the form

$$\phi(r) = a + br + \int_0^\infty (1 - e^{-tr})\, d\eta(t), \qquad r > 0,$$

where $a, b \geq 0$, and η is a positive measure satisfying

$$\int_0^\infty \min\{1, t\}\, d\eta(t) < \infty;$$

see [5,36]. For $0 < \alpha < 1$, the function $\phi(r) = r^\alpha$ is a Bernstein function, and

$$r^\alpha = \int_0^\infty (1 - e^{-tr})\, d\eta_\alpha(t), \qquad d\eta_\alpha(t) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{dt}{t^{1+\alpha}}. \tag{10.1}$$

Using the Bessel heat kernel (9.3), define, for $x, y > 0$ and $x \neq y$,

$$\mathcal{K}_{\nu,\alpha}(x,y) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty p_t^\nu(x,y) \frac{dt}{t^{1+\alpha}}. \tag{10.2}$$

This is the jump kernel obtained by subordinating the Bessel heat semigroup by the Lévy measure in (10.1). Using

$$j_\nu(iz) = 2^\nu \Gamma(\nu+1) I_\nu(z) z^{-\nu}, \qquad z > 0,$$

where I_ν is the modified Bessel function of the first kind, and making the change of variables $r = 1/t$, one obtains

$$\mathcal{K}_{\nu,\alpha}(x,y) = \frac{\alpha}{2\Gamma(1-\alpha)}(xy)^{-\nu}\int_0^\infty r^\alpha \exp\left(-\frac{x^2+y^2}{4}r\right) I_\nu\left(\frac{xy}{2}r\right) dr. \tag{10.3}$$

The only singularity of this kernel occurs on the diagonal. Indeed, the standard asymptotic formula

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}}\Big(1 + O(z^{-1})\Big), \qquad z \to +\infty,$$

gives, locally uniformly for x and y in compact subintervals of $(0,\infty)$,

$$\mathcal{K}_{\nu,\alpha}(x,y) = c_\alpha (xy)^{-\nu-\frac{1}{2}}|x-y|^{-1-2\alpha}\left(1 + O\left(\frac{|x-y|^2}{xy}\right)\right), \qquad y \to x, \tag{10.4}$$

where

$$c_\alpha = \frac{\alpha 2^{2\alpha}\Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}\Gamma(1-\alpha)}. \tag{10.5}$$

Away from the diagonal, the kernel is locally integrable; hence, for every $x > 0$ and every $\varepsilon > 0$,

$$\int_{\{y>0:\,|x-y|>\varepsilon\}} \mathcal{K}_{\nu,\alpha}(x,y) y^{2\nu+1}\, dy < \infty.$$

Theorem 10.1. *Let $0 < \alpha < 1$, and let $f \in \mathscr{S}_{*,+}$. Then, for every $x > 0$,*

$$A^\alpha_{\nu,m} f(x) = e^{\frac{i\gamma}{2}x^2}\,\mathrm{p.\,v.}\int_0^\infty \Big(e^{-\frac{i\gamma}{2}x^2} f(x) - e^{-\frac{i\gamma}{2}y^2} f(y)\Big)\mathcal{K}_{\nu,\alpha}(x,y) y^{2\nu+1}\, dy. \tag{10.6}$$

Here,

$$\mathrm{p.\,v.}\int_0^\infty \Phi(x,y) y^{2\nu+1}\, dy = \lim_{\varepsilon\downarrow 0}\int_{\{y>0:\,|x-y|>\varepsilon\}} \Phi(x,y) y^{2\nu+1}\, dy,$$

whenever the limit exists.

Proof. Let

$$g = L_{-\gamma} f = e^{-\frac{i\gamma}{2}x^2} f.$$

Because

$$A_{\nu,\mathbf{m}} = L_\gamma A_\nu L_{-\gamma},$$

the spectral functional calculus gives

$$A^\alpha_{\nu,\mathbf{m}} = L_\gamma A^\alpha_\nu L_{-\gamma}.$$

It is therefore enough to establish the singular-integral formula for $A^\alpha_\nu g$. By the Balakrishnan formula for the non-negative- self-adjoint operator A_ν, one has

$$A^\alpha_\nu g(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{g(x) - e^{-tA_\nu} g(x)}{t^{1+\alpha}}\, dt.$$

The Bessel heat semigroup satisfies

$$e^{-tA_\nu} g(x) = \int_0^\infty p^\nu_t(x,y) g(y) y^{2\nu+1}\, dy$$

and

$$\int_0^\infty p_t^\nu(x,y)y^{2\nu+1}\,dy = 1.$$

Thus,

$$g(x) - e^{-tA_\nu}g(x) = \int_0^\infty (g(x)-g(y))p_t^\nu(x,y)y^{2\nu+1}\,dy.$$

For $\varepsilon > 0$, define the truncated integral

$$I_\varepsilon(x) = \int_{\{y>0:\,|x-y|>\varepsilon\}} (g(x)-g(y))\mathcal{K}_{\nu,\alpha}(x,y)y^{2\nu+1}\,dy.$$

On the truncated region $|x-y| > \varepsilon$, the kernel is locally integrable, and the rapid decay of g gives absolute convergence. Hence, Fubini's theorem yields

$$I_\varepsilon(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \int_{\{y>0:\,|x-y|>\varepsilon\}} (g(x)-g(y))p_t^\nu(x,y)y^{2\nu+1}\,dy\frac{dt}{t^{1+\alpha}}.$$

The limit as $\varepsilon \downarrow 0$ exists in the principal-value sense. Indeed, fix $x > 0$, and choose $\delta > 0$ such that $(x-\delta, x+\delta) \subset (0,\infty)$. Away from this interval, the integral is absolutely convergent. Near the diagonal, write $y = x+h$. Becaause g is smooth,

$$g(x) - g(x+h) = -g'(x)h - \frac{1}{2}g''(x)h^2 + O(h^3), \qquad h \to 0.$$

Using (10.4), we also have

$$\mathcal{K}_{\nu,\alpha}(x,x+h)(x+h)^{2\nu+1} = c_\alpha x^\nu |h|^{-1-2\alpha}(1+O(h)).$$

Therefore,

$$\begin{aligned}(g(x)-g(x+h))&\mathcal{K}_{\nu,\alpha}(x,x+h)(x+h)^{2\nu+1}\\ &= -c_\alpha x^\nu g'(x)h|h|^{-1-2\alpha} + O(|h|^{1-2\alpha}).\end{aligned}$$

The leading term is odd in h and cancels in the symmetric principal value, and the remainder is integrable because $0 < \alpha < 1$. Consequently,

$$\lim_{\varepsilon\downarrow 0} I_\varepsilon(x) = \text{p. v.}\int_0^\infty (g(x)-g(y))\mathcal{K}_{\nu,\alpha}(x,y)y^{2\nu+1}\,dy$$

exists. Passing to the limit in the truncated identity gives

$$\text{p. v.}\int_0^\infty (g(x)-g(y))\mathcal{K}_{\nu,\alpha}(x,y)y^{2\nu+1}\,dy = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{g(x)-e^{-tA_\nu}g(x)}{t^{1+\alpha}}\,dt,$$

and the right-hand side is $A_\nu^\alpha g(x)$. Hence,

$$A_\nu^\alpha g(x) = \text{p. v.}\int_0^\infty (g(x)-g(y))\mathcal{K}_{\nu,\alpha}(x,y)y^{2\nu+1}\,dy.$$

Multiplying by $e^{i\gamma x^2/2}$ and substituting

$$g(x) = e^{-\frac{i\gamma}{2}x^2}f(x), \qquad g(y) = e^{-\frac{i\gamma}{2}y^2}f(y),$$

gives (10.6). □

The preceding proof makes explicit why the principal-value prescription is necessary in the Lévy representation. The subordinated kernel has the diagonal behavior $\mathcal{K}_{\nu,\alpha}(x,y) \asymp |x-y|^{-1-2\alpha}$, so the singularity cannot in general be handled by absolute integration. Expanding the compensated difference near $y = x$ shows that its leading contribution is odd and therefore cancels under symmetric truncation, and the remaining term is locally integrable for every $0 < \alpha < 1$. Thus, the principal value is not merely a notation. It is the analytic mechanism that realizes the cancellation required by the fractional spectral multiplier. The proof also displays the construction route from the Balakrishnan semigroup formula to the physical-space Lévy kernel through heat-kernel subordination.

Remark 10.2. The principal value is needed precisely because of the diagonal singularity. Near $y = x$, the kernel behaves as $|x-y|^{-1-2\alpha}$. because

$$g(x) - g(y) = O(|x-y|),$$

the absolute value of the integrand behaves as $|x-y|^{-2\alpha}$, which is not locally integrable when $\alpha \geq 1/2$. The principal value removes the odd leading term, and the remaining part is of order $|x-y|^{1-2\alpha}$, which is integrable for every $0 < \alpha < 1$.

11. Fractional heat equation

Let $0 < \alpha \leq 1$, and let $A_{\nu,m} = -\Delta_\nu^m$ be the non-negative, self-adjoint realization of the canonical Fourier–Bessel operator. The space-fractional heat equation is

$$\begin{cases} \partial_t u(t,x) + A_{\nu,m}^\alpha u(t,x) = 0, & t > 0,\ x > 0, \\ u(0,x) = f(x). \end{cases} \tag{11.1}$$

For $\alpha = 1$, this becomes the ordinary canonical Fourier–Bessel heat equation $\partial_t u = \Delta_{\nu,\mathbf{m}} u$.

Theorem 11.1. *Let $0 < \alpha \leq 1$ and $f \in \mathcal{H}_\nu$. The unique mild solution of* (11.1) *is*

$$u(t,x) = H_{t,\alpha}^{\nu,\mathbf{m}} f(x),$$

where

$$H_{t,\alpha}^{\nu,\mathbf{m}} f(x) = \mathcal{F}_\nu^{\mathbf{m}} \left[\exp\left(-t \left(\frac{\lambda^2}{b^2} \right)^\alpha \right) \mathcal{F}_\nu^{\mathbf{m}^{-1}} f(\lambda) \right](x). \tag{11.2}$$

Moreover, $\{H_{t,\alpha}^{\nu,\mathbf{m}}\}_{t\geq 0}$ is a strongly continuous contraction semigroup on $\mathcal{H}_\nu$. If $f \in \mathcal{D}(A_{\nu,\mathbf{m}}^\alpha)$, then the solution is strong and satisfies the equation in $\mathcal{H}_\nu$.

Proof. Applying $\mathcal{F}_\nu^{\mathbf{m}^{-1}}$ gives the scalar Cauchy problem

$$\partial_t \widehat{u}(t,\lambda) + \left(\frac{\lambda^2}{b^2} \right)^\alpha \widehat{u}(t,\lambda) = 0, \qquad \widehat{u}(0,\lambda) = \mathcal{F}_\nu^{\mathbf{m}^{-1}} f(\lambda).$$

Thus,

$$\widehat{u}(t,\lambda) = \exp\left(-t \left(\frac{\lambda^2}{b^2} \right)^\alpha \right) \mathcal{F}_\nu^{\mathbf{m}^{-1}} f(\lambda),$$

and applying $\mathcal{F}_\nu^{\mathbf{m}}$ gives (11.2). The contraction and semigroup properties follow from Plancherel and the multiplier bound. $\square$

The fractional heat kernel is defined by

$$G^{\nu,\mathbf{m}}_{t,\alpha}(x,y) = \frac{c_\nu}{(ib)^{\nu+1}} \int_0^\infty K^m_\nu(x,\lambda) e^{-t(\lambda^2/b^2)^\alpha} K^{m^{-1}}_\nu(\lambda,y)\lambda^{2\nu+1} d\lambda.$$

For suitable f,

$$H^{\nu,\mathbf{m}}_{t,\alpha} f(x) = \int_0^\infty G^{\nu,\mathbf{m}}_{t,\alpha}(x,y) f(y) y^{2\nu+1} dy.$$

When $\alpha = 1$, this kernel is the ordinary canonical heat kernel. Because $r \mapsto r^\alpha$ is a Bernstein function, the fractional heat semigroup is subordinated to the ordinary heat semigroup [3, 5]:

$$H^{\nu,\mathbf{m}}_{t,\alpha} f = \int_0^\infty H^{\nu,\mathbf{m}}_s f\, \eta_{t,\alpha}(s) ds,$$

where $\eta_{t,\alpha}$ is the positive stable density determined by

$$e^{-tr^\alpha} = \int_0^\infty e^{-sr} \eta_{t,\alpha}(s) ds.$$

Consequently,

$$G^{\nu,\mathbf{m}}_{t,\alpha}(x,y) = \int_0^\infty \mathcal{G}^{\nu,\mathbf{m}}_s(x,y) \eta_{t,\alpha}(s) ds.$$

Proposition 11.2. *Let $u(t) = H^{\nu,\mathbf{m}}_{t,\alpha} f$. Then,*

$$\|u(t)\|_{2,\nu} \leq \|f\|_{2,\nu}, \qquad t \geq 0.$$

If $f \in \mathcal{D}(A^{\alpha/2}_{\nu,\mathbf{m}})$, then,

$$\frac{d}{dt}\|u(t)\|^2_{2,\nu} = -2\|A^{\alpha/2}_{\nu,\mathbf{m}} u(t)\|^2_{2,\nu}.$$

Proof. Both statements follow by applying Plancherel to the multiplier representation (11.2) and differentiating under the spectral integral. □

For $0 < \beta < 1$, the Caputo time-fractional problem

$$\partial^\beta_t u(t,x) + A^\alpha_{\nu,m} u(t,x) = 0, \qquad u(0,x) = f(x),$$

has the spectral solution

$$u(t,x) = \mathcal{F}^{\mathbf{m}}_\nu \left[E_\beta \left(-t^\beta \left(\frac{\lambda^2}{b^2} \right)^\alpha \right) \mathcal{F}^{\mathbf{m}^{-1}}_\nu f(\lambda) \right](x),$$

where $E_\beta(z) = \sum_{k=0}^\infty z^k/\Gamma(\beta k + 1)$ is the Mittag-Leffler function.

12. Caffarelli–Silvestre type extension

The extension problem below is designed so that the nonlocal operator $A^\alpha_{\nu,m}$ appears as a Dirichlet-to-Neumann map. The degenerate weight $y^{1-2\alpha}$ is the same one that appears in the classical Caffarelli–Silvestre extension, and the tangential operator is replaced here by the canonical Fourier–Bessel operator $A_{\nu,m}$. Thus, the extension is not an independent construction; it is the local realization of

the same spectral multiplier $(\lambda^2/b^2)^\alpha$.
Let $0 < \alpha < 1$. The extension method used below is the spectral Caffarelli–Silvestre extension for non-negative self-adjoint operators, adapted to the canonical Fourier–Bessel calculus [37, 38]. In the present setting it is completely explicit because $A_{\nu,\mathbf{m}}$ is diagonalized by the canonical Fourier–Bessel transform:

$$\mathcal{F}_\nu^{\mathbf{m}^{-1}}(A_{\nu,\mathbf{m}}f)(\lambda) = \frac{\lambda^2}{b^2}\mathcal{F}_\nu^{\mathbf{m}^{-1}}f(\lambda). \tag{12.1}$$

Consequently, $A_{\nu,\mathbf{m}}^{1/2}$ is represented by the multiplier $\lambda/|b|$, and $A_{\nu,\mathbf{m}}^{\alpha}$ is represented by $(\lambda^2/b^2)^\alpha$. We first recall the special function appearing in the extension. For $\beta \in \mathbb{R}$, the modified Bessel function of the second kind, also called the Macdonald function, is denoted by K_β. It is the solution, decaying at infinity, of the modified Bessel equation

$$r^2w''(r) + rw'(r) - (r^2 + \beta^2)w(r) = 0, \qquad r > 0. \tag{12.2}$$

Moreover, $K_{-\beta} = K_\beta$, and, for $\beta > 0$,

$$K_\beta(r) \sim 2^{\beta-1}\Gamma(\beta)r^{-\beta}, \qquad r \downarrow 0, \tag{12.3}$$

and $K_\beta(r)$ decays exponentially as $r \to +\infty$; see [18, 39]. For $0 < \alpha < 1$, define

$$\varphi_\alpha(r) = \frac{2^{1-\alpha}}{\Gamma(\alpha)}r^\alpha K_\alpha(r), \qquad r > 0. \tag{12.4}$$

By (12.3),

$$\lim_{r\downarrow 0}\varphi_\alpha(r) = 1. \tag{12.5}$$

We shall also use the derivative identity

$$\frac{d}{dr}(r^\alpha K_\alpha(r)) = -r^\alpha K_{\alpha-1}(r) = -r^\alpha K_{1-\alpha}(r), \tag{12.6}$$

which gives

$$\varphi'_\alpha(r) = -\frac{2^{1-\alpha}}{\Gamma(\alpha)}r^\alpha K_{1-\alpha}(r). \tag{12.7}$$

Because $0 < 1-\alpha < 1$, (12.3) applied to $K_{1-\alpha}$ yields

$$\lim_{r\downarrow 0} r^{1-2\alpha}\varphi'_\alpha(r) = -\frac{2^{1-2\alpha}\Gamma(1-\alpha)}{\Gamma(\alpha)}. \tag{12.8}$$

Consider the degenerate elliptic extension problem

$$\begin{cases} \partial_y^2 U(y,x) + \dfrac{1-2\alpha}{y}\partial_y U(y,x) - A_{\nu,m}U(y,x) = 0, \quad y > 0,\ x > 0, \\ U(0,x) = f(x). \end{cases} \tag{12.9}$$

Because $A_{\nu,m} = -\Delta_\nu^m$, the differential equation can also be written as

$$\partial_y^2 U + \frac{1-2\alpha}{y}\partial_y U + \Delta_\nu^m U = 0.$$

Theorem 12.1. *Let* $0 < \alpha < 1$, *and let* $f \in \mathcal{D}(A^{\alpha}_{\nu,m})$. *Define*

$$U(y,\cdot) = \varphi_\alpha(yA^{1/2}_{\nu,m})f, \qquad y > 0, \tag{12.10}$$

by the spectral functional calculus. Equivalently,

$$U(y,x) = \mathcal{F}^m_\nu\left[\varphi_\alpha\left(\frac{y\lambda}{|b|}\right)\mathcal{F}^{m^{-1}}_\nu f(\lambda)\right](x). \tag{12.11}$$

Then, U solves (12.9) *in $\mathcal{H}_\nu$ for every $y > 0$, and*

$$\lim_{y\downarrow 0} U(y,\cdot) = f \qquad \text{in } \mathcal{H}_\nu. \tag{12.12}$$

Moreover,

$$-\lim_{y\downarrow 0} y^{1-2\alpha}\partial_y U(y,\cdot) = \frac{2^{1-2\alpha}\Gamma(1-\alpha)}{\Gamma(\alpha)}A^{\alpha}_{\nu,m}f \tag{12.13}$$

in $\mathcal{H}_\nu$. Equivalently,

$$A^{\alpha}_{\nu,m}f = -2^{2\alpha-1}\frac{\Gamma(\alpha)}{\Gamma(1-\alpha)}\lim_{y\downarrow 0} y^{1-2\alpha}\partial_y U(y,\cdot). \tag{12.14}$$

Proof. Put

$$\widehat{f}(\lambda) = \mathcal{F}^{m^{-1}}_\nu f(\lambda), \qquad \mu(\lambda) = \frac{\lambda^2}{b^2}.$$

By (12.10),

$$\widehat{U}(y,\lambda) := \mathcal{F}^{m^{-1}}_\nu U(y,\cdot)(\lambda) = \varphi_\alpha(y\sqrt{\mu(\lambda)})\widehat{f}(\lambda) = \varphi_\alpha\left(\frac{y\lambda}{|b|}\right)\widehat{f}(\lambda).$$

We first verify the equation. Set $r = y\sqrt{\mu(\lambda)}$. From (12.2) and (12.4), a direct calculation shows that φ_α satisfies

$$\varphi''_\alpha(r) + \frac{1-2\alpha}{r}\varphi'_\alpha(r) - \varphi_\alpha(r) = 0, \qquad r > 0. \tag{12.15}$$

Indeed, inserting $\varphi_\alpha(r) = c_\alpha r^\alpha K_\alpha(r)$ into the left-hand side and using (12.2) gives zero. Therefore,

$$\begin{aligned}\partial_y^2\widehat{U}(y,\lambda) &+ \frac{1-2\alpha}{y}\partial_y\widehat{U}(y,\lambda) - \mu(\lambda)\widehat{U}(y,\lambda)\\ &= \mu(\lambda)\left[\varphi''_\alpha(r) + \frac{1-2\alpha}{r}\varphi'_\alpha(r) - \varphi_\alpha(r)\right]\widehat{f}(\lambda) = 0.\end{aligned}$$

By the diagonalization (12.1), this is exactly the canonical transform of

$$\partial_y^2 U + \frac{1-2\alpha}{y}\partial_y U - A_{\nu,m}U = 0.$$

Thus, the equation holds in $\mathcal{H}_\nu$ for every fixed $y > 0$. We next prove the trace condition. By (12.5),

$$\varphi_\alpha\left(\frac{y\lambda}{|b|}\right) \to 1 \qquad \text{for every } \lambda > 0$$

as $y \downarrow 0$. The function φ_α is bounded on $[0,\infty)$ because it has a finite limit at the origin and decays at infinity. Hence, by Plancherel and the dominated convergence theorem,

$$\begin{aligned}\|U(y,\cdot)-f\|^2_{\mathcal{H}_\nu} &= \int_0^\infty \left|\varphi_\alpha\left(\frac{y\lambda}{|b|}\right)-1\right|^2 |\widehat{f}(\lambda)|^2\lambda^{2\nu+1}\,d\lambda \\ &\longrightarrow 0.\end{aligned}$$

This proves (12.12). It remains to justify the Dirichlet-to-Neumann limit. Differentiating the spectral formula gives

$$\partial_y\widehat{U}(y,\lambda) = \sqrt{\mu(\lambda)}\,\varphi'_\alpha(y\sqrt{\mu(\lambda)})\widehat{f}(\lambda).$$

Using (12.8), for every $\mu > 0$,

$$\begin{aligned}\lim_{y\downarrow 0} y^{1-2\alpha}\partial_y\varphi_\alpha(y\sqrt{\mu}) &= \lim_{y\downarrow 0} y^{1-2\alpha}\sqrt{\mu}\,\varphi'_\alpha(y\sqrt{\mu}) \\ &= \mu^\alpha \lim_{r\downarrow 0} r^{1-2\alpha}\varphi'_\alpha(r) \\ &= -\frac{2^{1-2\alpha}\Gamma(1-\alpha)}{\Gamma(\alpha)}\mu^\alpha.\end{aligned}$$

Therefore,

$$-y^{1-2\alpha}\partial_y\widehat{U}(y,\lambda) \to \frac{2^{1-2\alpha}\Gamma(1-\alpha)}{\Gamma(\alpha)}\left(\frac{\lambda^2}{b^2}\right)^\alpha \widehat{f}(\lambda)$$

pointwise in λ. To pass from pointwise convergence to convergence in $\mathcal{H}_\nu$, we need a uniform domination. From (12.7), the function $r^{1-2\alpha}\varphi'_\alpha(r)$ is bounded on $(0,\infty)$. Near zero, this follows from (12.8), and near infinity, it follows from the exponential decay of $K_{1-\alpha}$. Hence, there is a constant $M_\alpha > 0$ such that

$$\left|r^{1-2\alpha}\varphi'_\alpha(r)\right| \le M_\alpha, \qquad r > 0.$$

With $r = y\sqrt{\mu(\lambda)}$, this gives

$$\begin{aligned}\left|y^{1-2\alpha}\partial_y\varphi_\alpha(y\sqrt{\mu(\lambda)})\right| &= \left|y^{1-2\alpha}\sqrt{\mu(\lambda)}\varphi'_\alpha(y\sqrt{\mu(\lambda)})\right| \\ &= \mu(\lambda)^\alpha\left|(y\sqrt{\mu(\lambda)})^{1-2\alpha}\varphi'_\alpha(y\sqrt{\mu(\lambda)})\right| \\ &\le M_\alpha\mu(\lambda)^\alpha.\end{aligned}$$

Because $f \in \mathcal{D}(A^\alpha_{\nu,m})$,

$$\int_0^\infty \mu(\lambda)^{2\alpha}|\widehat{f}(\lambda)|^2\lambda^{2\nu+1}\,d\lambda < \infty.$$

Another application of dominated convergence gives (12.13). The formula (12.14) is just a rearrangement of (12.13). □

The preceding proof also explains the construction principle behind the elliptic extension equation. The auxiliary variable $y > 0$ is introduced so that the nonlocal operator $A^\alpha_{\nu,m}$ becomes a local Dirichlet-to-Neumann map. After applying the canonical Fourier–Bessel transform in the tangential variable, $A_{\nu,m}$ is replaced by the scalar multiplier λ^2/b^2, and the extension equation reduces, for each fixed λ, to a modified Bessel ordinary differential equation in y. The decaying solution is normalized to have boundary value one at $y = 0$, producing the spectral Poisson factor $\varphi_\alpha(y\lambda/|b|)$. Inverting the transform gives the degenerate elliptic extension, and the weighted normal derivative at the boundary recovers the fractional power. This spectral reduction is the essential reason for the particular form of the extension equation.

Remark 12.2. The proof is purely spectral. It does not require a preliminary fractional Bessel singular-integral formula. The only ingredients are the self-adjointness of $A_{\nu,m}$, its canonical Fourier–Bessel diagonalization, and the elementary properties of the Macdonald function K_α.

The same extension admits a heat-semigroup representation. For $\mu > 0$,

$$\varphi_\alpha(y\sqrt{\mu}) = \frac{y^{2\alpha}}{4^\alpha\Gamma(\alpha)} \int_0^\infty e^{-y^2/(4t)} e^{-t\mu} \frac{dt}{t^{1+\alpha}}. \tag{12.16}$$

Applying the spectral theorem to (12.16) gives

$$U(y, x) = \frac{y^{2\alpha}}{4^\alpha\Gamma(\alpha)} \int_0^\infty e^{-y^2/(4t)} H_t^{\nu,m} f(x) \frac{dt}{t^{1+\alpha}}, \tag{12.17}$$

where $H_t^{\nu,m} = e^{-tA_{\nu,m}}$ is the canonical heat semigroup. If $\mathcal{G}_t^{\nu,m}$ denotes the canonical heat kernel, then

$$U(y, x) = \int_0^\infty P_y^{\alpha,\mathbf{m}}(x, z) e^{-i\gamma z^2} f(z) z^{2\nu+1}\, dz, \tag{12.18}$$

where

$$P_y^{\alpha,\mathbf{m}}(x, z) = \frac{y^{2\alpha}}{4^\alpha\Gamma(\alpha)} \int_0^\infty e^{-y^2/(4t)} \mathcal{G}_t^{\nu,m}(x, z) \frac{dt}{t^{1+\alpha}}. \tag{12.19}$$

13. Hardy inequalities

We record two Hardy inequalities adapted to the canonical Fourier–Bessel operator. The quadratic inequality follows by an elementary integration-by-parts argument after chirp conjugacy. The fractional inequality follows from the sharp Pitt inequality for the Fourier–Bessel transform and therefore has the sharp constant [40, 41].

The constants in these inequalities are tied to the weighted Bessel geometry. The measure $x^{2\nu+1}\,dx$ corresponds to the effective homogeneous dimension $2\nu + 2$, and the chirp factor has modulus one. Hence, the phase changes the covariant derivative f' into $f' - i\gamma x f$, but it does not change the underlying Hardy weight. This explains why the sharp constants are inherited from the Bessel Hardy, and Pitt inequalities, and the explicit canonical form of the energy contains the matrix parameter through $\gamma = d/b$.

Theorem 13.1. *Assume $\nu > 0$, and let $f \in \mathscr{S}_{*,+}$. Then,*

$$\nu^2 \int_0^\infty |f(x)|^2 x^{2\nu-1}\, dx \le \langle A_{\nu,\mathbf{m}} f, f\rangle_{\mathcal{H}_\nu}. \tag{13.1}$$

Equivalently,

$$\nu^2 \int_0^\infty |f(x)|^2 x^{2\nu-1}\, dx \le \int_0^\infty |f'(x) - i\gamma x f(x)|^2 x^{2\nu+1}\, dx. \tag{13.2}$$

The constant ν^2 is sharp.

Proof. Let

$$g = L_{-\gamma} f = e^{-i\gamma x^2/2} f.$$

Because $|g| = |f|$, and $A_{\nu,m} = L_\gamma(-\mathcal{B}_\nu)L_{-\gamma}$, we have

$$\begin{aligned}\langle A_{\nu,m} f, f\rangle_{\mathcal{H}_\nu} &= \langle(-\mathcal{B}_\nu)g, g\rangle_{\mathcal{H}_\nu}\\ &= \int_0^\infty |g'(x)|^2 x^{2\nu+1}\, dx.\end{aligned}$$

The last equality follows by integration by parts on $\mathscr{S}_{*,+}$:

$$\langle -\mathcal{B}_\nu g, g\rangle_{\mathcal{H}_\nu} = \int_0^\infty |g'(x)|^2 x^{2\nu+1}\, dx,$$

because the boundary terms vanish at 0 by even regularity and at infinity by rapid decay. The classical weighted Hardy inequality says that, for $\nu > 0$,

$$\nu^2 \int_0^\infty |g(x)|^2 x^{2\nu-1}\, dx \le \int_0^\infty |g'(x)|^2 x^{2\nu+1}\, dx.$$

Substituting $g = e^{-i\gamma x^2/2} f$ gives (13.1). Because

$$g'(x) = e^{-i\gamma x^2/2}(f'(x) - i\gamma x f(x)),$$

we also obtain (13.2). Sharpness is inherited from the sharpness of the classical weighted Hardy inequality. □

Theorem 13.2. *Let $0 < \alpha < \nu + 1$, and let $f \in \mathcal{D}(A^{\alpha/2}_{\nu,\mathbf{m}})$. Then,*

$$C_{\nu,\alpha} \int_0^\infty |f(x)|^2 x^{2\nu+1-2\alpha}\, dx \le \|A^{\alpha/2}_{\nu,\mathbf{m}} f\|^2_{2,\nu}, \tag{13.3}$$

where

$$C_{\nu,\alpha} = 2^{2\alpha}\left[\frac{\Gamma\left(\frac{\nu+1+\alpha}{2}\right)}{\Gamma\left(\frac{\nu+1-\alpha}{2}\right)}\right]^2. \tag{13.4}$$

Equivalently, if $f \in \mathcal{D}(A^{\alpha}_{\nu,\mathbf{m}})$, then

$$C_{\nu,\alpha} \int_0^\infty |f(x)|^2 x^{2\nu+1-2\alpha}\, dx \le \langle A^{\alpha}_{\nu,\mathbf{m}} f, f\rangle_{\mathcal{H}_\nu}. \tag{13.5}$$

The constant $C_{\nu,\alpha}$ is sharp.

Proof. We give the details. Let

$$A_\nu = -\mathcal{B}_\nu, \qquad \mathcal{B}_\nu = \frac{d^2}{dx^2} + \frac{2\nu+1}{x}\frac{d}{dx}.$$

The ordinary Fourier–Bessel transform $\mathcal{F}_\nu$ diagonalizes A_ν:

$$\mathcal{F}_\nu(A_\nu h)(\lambda) = \lambda^2 \mathcal{F}_\nu h(\lambda).$$

Hence, by spectral calculus,

$$\|A_\nu^{\alpha/2}h\|_{2,\nu}^2 = \int_0^\infty \lambda^{2\alpha}|\mathcal{F}_\nu h(\lambda)|^2\lambda^{2\nu+1}\,d\lambda. \tag{13.6}$$

We use the following sharp Pitt inequality for the Fourier–Bessel transform. For $0 < \alpha < \nu + 1$ and suitable ϕ,

$$\int_0^\infty |\mathcal{F}_\nu\phi(x)|^2x^{2\nu+1-2\alpha}\,dx \le D_{\nu,\alpha}\int_0^\infty |\phi(\lambda)|^2\lambda^{2\nu+1+2\alpha}\,d\lambda, \tag{13.7}$$

where

$$D_{\nu,\alpha} = 2^{-2\alpha}\left[\frac{\Gamma\left(\frac{\nu+1-\alpha}{2}\right)}{\Gamma\left(\frac{\nu+1+\alpha}{2}\right)}\right]^2. \tag{13.8}$$

Thus, $D_{\nu,\alpha}^{-1} = C_{\nu,\alpha}$. Let $h \in \mathcal{D}(A_\nu^{\alpha/2})$, and put

$$\phi = \mathcal{F}_\nu h.$$

Since $\mathcal{F}_\nu$ is a unitary involution on $\mathcal{H}_\nu$, $h = \mathcal{F}_\nu\phi$. Applying (13.7) to ϕ gives

$$\begin{aligned}\int_0^\infty |h(x)|^2x^{2\nu+1-2\alpha}\,dx &= \int_0^\infty |\mathcal{F}_\nu\phi(x)|^2x^{2\nu+1-2\alpha}\,dx\\ &\le D_{\nu,\alpha}\int_0^\infty |\phi(\lambda)|^2\lambda^{2\nu+1+2\alpha}\,d\lambda\\ &= D_{\nu,\alpha}\int_0^\infty \lambda^{2\alpha}|\mathcal{F}_\nu h(\lambda)|^2\lambda^{2\nu+1}\,d\lambda\\ &= D_{\nu,\alpha}\|A_\nu^{\alpha/2}h\|_{2,\nu}^2.\end{aligned}$$

Because $D_{\nu,\alpha}^{-1} = C_{\nu,\alpha}$, we obtain the sharp Bessel-Hardy inequality

$$C_{\nu,\alpha}\int_0^\infty |h(x)|^2x^{2\nu+1-2\alpha}\,dx \le \|A_\nu^{\alpha/2}h\|_{2,\nu}^2. \tag{13.9}$$

We now transfer (13.9) to the canonical operator. Recall that

$$A_{\nu,\mathbf{m}} = L_\gamma A_\nu L_{-\gamma}, \qquad L_\gamma h(x) = e^{i\gamma x^2/2}h(x), \qquad \gamma = \frac{d}{b}.$$

Because L_γ is unitary on $\mathcal{H}_\nu$, the spectral theorem gives

$$A_{\nu,\mathbf{m}}^{\alpha/2} = L_\gamma A_\nu^{\alpha/2}L_{-\gamma}, \qquad \mathcal{D}(A_{\nu,\mathbf{m}}^{\alpha/2}) = L_\gamma\mathcal{D}(A_\nu^{\alpha/2}).$$

Let

$$g = L_{-\gamma}f = e^{-i\gamma x^2/2}f.$$

Then, $g \in \mathcal{D}(A_\nu^{\alpha/2})$, $|g(x)| = |f(x)|$, and

$$\begin{aligned}\|A_{\nu,m}^{\alpha/2}f\|_{2,\nu}^2 &= \|L_\gamma A_\nu^{\alpha/2}L_{-\gamma}f\|_{2,\nu}^2\\ &= \|A_\nu^{\alpha/2}g\|_{2,\nu}^2.\end{aligned}$$

Applying (13.9) to g gives

$$C_{\nu,\alpha}\int_0^\infty |g(x)|^2 x^{2\nu+1-2\alpha}\,dx \le \|A_\nu^{\alpha/2}g\|_{2,\nu}^2.$$

Using $|g| = |f|$ and the preceding norm identity gives (13.3). If $f \in \mathcal{D}(A_{\nu,m}^\alpha)$, then, because $A_{\nu,m}$ is non-negative and self-adjoint,

$$\|A_{\nu,\mathbf{m}}^{\alpha/2}f\|_{2,\nu}^2 = \langle A_{\nu,\mathbf{m}}^\alpha f, f\rangle_{\mathcal{H}_\nu},$$

which proves (13.5). Finally, the sharpness of $C_{\nu,\alpha}$ follows from the sharpness of the Pitt constant (13.8). Indeed, if a larger constant were valid in the canonical inequality, then choosing $f = L_\gamma h$ would give the same larger constant in the ordinary Bessel inequality (13.9), contradicting sharpness. □

Remark 13.3. For $\alpha = 1$ and $\nu > 0$, the fractional constant becomes

$$C_{\nu,1} = 4\left[\frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\right]^2 = \nu^2,$$

because $\Gamma(\nu/2 + 1) = (\nu/2)\Gamma(\nu/2)$. Hence, the fractional Hardy inequality is consistent with the quadratic Hardy inequality in Theorem 13.1.

14. Conclusions

The canonical Fourier–Bessel transform diagonalizes the differential operator Δ_ν^m and therefore provides a natural functional calculus for $A_{\nu,m}$. Fractional powers are obtained as spectral multipliers with the symbol $(\lambda^2/b^2)^\alpha$. The comparison with the ordinary fractional Bessel operator shows that the canonical construction is a chirp-twisted and canonically scaled version of the Bessel fractional calculus. The generalized shift and convolution associated with the transform supply the correct physical space structure, and the heat semigroup yields Balakrishnan, Lévy, and resolvent representations. The special cases above show that the same framework contains the ordinary Fourier–Bessel transform, inverse Fourier–Bessel transform, fractional Fourier–Bessel transform, Fresnel–Bessel transform, chirped and scaled Fourier–Bessel transforms, and the limiting canonical chirp–dilation operators. These formulas provide a foundation for further work on nonlocal canonical Fourier–Bessel equations, regularity theory, and boundary value problems.

The revised presentation also clarifies the difference between the present canonical fractional operator and the usual fractional Bessel operator. The new ingredients are the matrix-dependent coupling, phase modulation, and canonical spectral scaling. Together, they yield a unified framework that can be specialized to several classical transforms and extended to further canonical or radial nonlocal models.

From an application viewpoint, the proposed operator has advantages that are absent from a classical fractional Bessel operator with fixed geometry. The matrix m permits the same fractional model to encode Fourier–Bessel analysis, fractional rotations, Fresnel propagation, dilations, and input/output chirps without changing the underlying functional calculus. The parameter b controls the canonical spectral scale, and a and d control quadratic phase modulation. This additional flexibility is useful for radially symmetric signal and image processing, optical propagation, phase-sensitive filtering, and nonlocal diffusion in media where both Bessel geometry and canonical phase-space effects are present. By contrast, the classical fractional Bessel operator captures radial nonlocality but does not by itself provide this matrix-controlled interpolation among optical and time–frequency regimes.

From the numerical viewpoint, the chirp–dilation–Fourier–Bessel factorization is especially useful because it separates the canonical operator into components for which mature algorithms already exist. The chirp factors are pointwise multipliers, the dilation is implemented by resampling, and the remaining radial transform can be treated by fast or quasi-discrete Hankel methods. Efficient fractional Fourier algorithms and comparative implementations are described in [12,13], and the quasi-discrete Hankel method of Guizar-Sicairos and Gutiérrez-Vega [14] and the fast stable discrete Hankel algorithm of Townsend [15] provide representative numerical tools for the Bessel component. This factorized structure gives a concrete route toward stable discretization of the canonical heat semigroup, spectral multipliers, and the extension operator without discretizing the full oscillatory kernel in a single step.

The framework developed here is applicable to several further problems. A first direction is the perturbation theory of $A^{\alpha}_{\nu,m}$ under lower-order potentials, boundary conditions, or matrix perturbations. A second direction is the numerical approximation of the canonical heat semigroup, the singular integral, and the extension problem, especially for different values of ν, b, and α. A third direction is the construction of multidimensional and product-type canonical Fourier–Bessel models, where the radial Bessel structure interacts with higher-dimensional canonical transforms. These problems will be considered in future work.

Author contributions

S. Mesloub and F. Bouzeffour contributed to the mathematical development, analysis, and preparation of the manuscript. All authors have read and approved the final version of the manuscript for publication.

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The authors used ChatGPT (OpenAI) for language editing and LaTeX formatting during manuscript preparation. The authors are responsible for the accuracy, originality, and integrity of the final manuscript.

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Conflict of interest

The authors declare that they have no conflict of interest.

References

1. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, 1970.
2. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: Theory and applications*, Gordon and Breach, 1993.
3. A. V. Balakrishnan, Fractional powers of closed operators and the semigroups generated by them, *Pacific J. Math.*, **10** (1960), 419–437. https://doi.org/10.2140/pjm.1960.10.419
4. E. L. Shishkina, S. M. Sitnik, On fractional powers of Bessel operators, *J. Inequal. Spec. Funct.*, **8** (2017), 49–67.
5. D. Applebaum, *Lévy Processes and Stochastic Calculus*, 2nd ed., Cambridge University Press, Cambridge, 2009.
6. S. A. Collins, Lens-system diffraction integral written in terms of matrix optics, *J. Opt. Soc. Amer.*, **60** (1970), 1168–1177. https://doi.org/10.1364/JOSA.60.001168
7. M. Moshinsky, C. Quesne, Linear canonical transformations and their unitary representations, *J. Math. Phys.*, **12** (1971), 1772–1780. https://doi.org/10.1063/1.1665805
8. K. B. Wolf, Development of linear canonical transforms: A historical sketch, in *Linear Canonical Transforms*, Springer, 2016.
9. V. Namias, The fractional order Fourier transform and its application to quantum mechanics, *J. Inst. Math. Appl.*, **25** (1980), 241–265. https://doi.org/10.1093/imamat/25.3.241
10. H. M. Ozaktas, Z. Zalevsky, M. A. Kutay, *The fractional Fourier transform with applications in optics and signal processing*, Wiley, Chichester, 2001.
11. J. J. Healy, M. A. Kutay, H. M. Ozaktas, J. T. Sheridan, *Linear canonical transforms: Theory and applications*, Springer, New York, 2016.
12. H. M. Ozaktas, O. Arikan, M. A. Kutay, G. Bozdagi, Digital computation of the fractional Fourier transform, *IEEE Trans. Signal Process.*, **44** (1996), 2141–2150. https://doi.org/10.1109/78.536672
13. A. Bultheel, H. Martínez-Sulbaran, Computation of the fractional Fourier transform, *Appl. Comput. Harmon. Anal.*, **16** (2004), 182–202.
14. M. Guizar-Sicairos, J. C. Gutiérrez-Vega, Computation of quasi-discrete Hankel transforms of integer order for propagating optical wave fields, *J. Opt. Soc. Amer. A*, **21** (2004), 53–58.
15. A. Townsend, A fast analysis-based discrete Hankel transform using asymptotic expansions, *SIAM J. Numer. Anal.*, **53** (2015), 1897–1917. https://doi.org/10.1137/151003106
16. V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform. Part I, *Comm. Pure Appl. Math.*, **14** (1961), 187–214. https://doi.org/10.1002/cpa.3160140303
17. G. B. Folland, *Harmonic analysis in phase space*, Princeton University Press, Princeton, 1989.
18. G. N. Watson, *A treatise on the theory of Bessel functions*, 2nd ed., Cambridge University Press, Cambridge, 1944.

19. B. M. Levitan, *Generalized translation operators and some of their applications*, Israel Program for Scientific Translations, Jerusalem, 1964.
20. F. Bouzeffour, M. Garayev, On the fractional Bessel operator, *Integr. Transf. Spec. F.*, **33** (2022), 230–246.
21. L. Dhaouadi, J. Sahbani, A. Fitouhi, Harmonic analysis associated to the canonical Fourier–Bessel transform, *Integr. Transf. Spec. F.*, **32** (2021), 290–315.
22. S. Ghazouani, J. Sahbani, Canonical Fourier–Bessel transform and their applications, *J. Pseudo-Differ. Oper. Appl.*, **14** (2023), 3.
23. J. Sahbani, Quantitative uncertainty principles for the canonical Fourier–Bessel transform, *Acta Math. Sin. Engl. Ser.*, **38** (2022), 331–346. https://doi.org/10.1007/s10114-022-1008-7
24. F. M. Cholewinski, Generalized Fock spaces and associated operators, *SIAM J. Math. Anal.*, **15** (1984), 177–202. https://doi.org/10.1137/0515015
25. J. Schmeelk, Hankel transforms in generalized Fock spaces, *Int. J. Math. Math. Sci.*, **17** (1994), 269–276.
26. J. J. Betancor, X. T. Duong, M. Y. Lee, J. Li, B. D. Wick, The Riesz transform and fractional integral operators in the Bessel setting, *Integr. Equat. Oper. Th.*, **97** (2025), 15. https://doi.org/10.1007/s00020-025-02801-1
27. D. Pasawan, Pseudo-differential operators associated with the fractional Hankel–Bessel transform, *arXiv:2601.03091*, 2026.
28. Y. Q. Xu, B. Z. Li, Linear canonical space-time transform and convolution theorems, *arXiv:2405.10990*, 2024.
29. F. Bouzeffour, Fractional Bessel derivative within the Mellin transform framework, *J. Nonlinear Math. Phys.*, **31** (2024), 3. https://doi.org/10.1007/s44198-024-00170-8
30. B. Halouani, F. Bouzeffour, On the fractional Laplace–Bessel operator, *AIMS Math.*, **9** (2024), 21524–21537. https://doi.org/10.3934/math.20241045
31. H. B. Mohamed, A. Saoudi, Linear canonical Fourier–Bessel wavelet transform: properties and inequalities, *Integral Transforms Spec. Funct.*, **35** (2024), 270–290. https://doi.org/10.1080/10652469.2024.2317724.
32. H. Benlaajine, M. El Hamma, R. Daher, Generalization of Titchmarsh's theorem for the linear canonical Fourier–Bessel transform, *Integral Transforms Spec. Funct.*, **36** (2025), 911–920. https://doi.org/10.1080/10652469.2025.2472349.
33. F. Bouzeffour, On the generalized fractional Laplace–Bessel operator, *Georgian Math. J.*, **32** (2025), 371–379. https://doi.org/10.1515/gmj-2024-2074
34. M. El Hamma, M. El Bouazizi, H. Benlaajine, Estimates of the K-functional and Jackson-type theorems for the linear canonical Fourier–Bessel transform, *Annali dell'Università di Ferrara*, 2026. https://doi.org/10.1007/s11565-026-00687-4
35. H. Mejjaoli and A. Poria, The linear canonical transforms associated with the Dunkl-type operator and its applications, *Integral Transforms Spec. Funct.*, **37** (2026), 7. https://doi.org/10.1080/10652469.2025.2527167.

36. R. L. Schilling, R. Song, Z. Vondraček, *Bernstein functions: Theory and applications*, 2nd ed., De Gruyter, Berlin, 2012.

37. L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, *Commun. Part. Diff. Eq.*, **32** (2007), 1245–1260. https://doi.org/10.1080/03605300600987306

38. P. R. Stinga, J. L. Torrea, Extension problem and Harnack's inequality for some fractional operators, *Commun. Part. Diff. Eq.*, **35** (2010), 2092–2122. https://doi.org/10.1080/03605301003735680

39. I. S. Gradshteyn, I. M. Ryzhik, *Table of integrals, series, and products*, 7th ed., Academic Press, Amsterdam, 2007.

40. D. V. Gorbachev, V. I. Ivanov, S. Yu. Tikhonov, Sharp Pitt inequality and logarithmic uncertainty principle for Dunkl transform in L^2, *J. Approx. Theory*, **202** (2016), 109–118.

41. R. L. Frank, R. Seiringer, Non-linear ground state representations and sharp Hardy inequalities, *J. Funct. Anal.*, **255** (2008), 3407–3430. https://doi.org/10.1016/j.jfa 2008.05.015