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Research article

Stability of mean-field stochastic differential equations under sublinear expectation

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Abstract: This paper explores the properties of exponential decay and logarithmic decay in mean-field stochastic differential equations driven by G-Brownian motion. Using the distribution-dependent G-Lyapunov function approach, we derive explicit criteria for moment exponential stability and partial moment stability under general decay rates. By integrating moment estimation techniques, we also furnish analyzed formulas to compute the decay exponents. As an application, we present two examples to verify our results.

Keywords: mean-field stochastic differential equations; stability; G-Itô's formula
Mathematics Subject Classification: 34A12; 34A37; 34D05

1. Introduction

In this paper, we investigate the stability of mean-field stochastic differential equations driven by G-Brownian motion, described by

$$\begin{cases} \mathrm{d}Y_t = b_t(Y_t, \mathcal{L}_{Y_t})\mathrm{d}t + h_t(Y_t, \mathcal{L}_{Y_t})\mathrm{d}B_t + \sigma_t(Y_t, \mathcal{L}_{Y_t})\mathrm{d}\langle B\rangle_t, t > 0, \\ Y_t|_{t=0} = \xi, \\ \mathcal{L}_{Y_t} = \text{the probability distribution of } Y_t, \end{cases} \tag{1.1}$$

where
• ξ is an $\mathcal{F}_0$-measurable random variable;
• B_t is a one-dimensional G-Brownian motion, and $\langle B\rangle_t$ denotes its quadratic variation process on complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$;
• The functions

$$b_t(y,\mu),\ h_t(y,\mu),\ \sigma_t(y,\mu) : [0,T]\times\mathbb{R}^m\times\mathcal{M}_{\lambda^2}\to\mathbb{R} \tag{1.2}$$

are Borel measurable. Here, $\mathcal{M}_{\lambda^2}$ defined by (2.1) will be elaborated later.

For the one-dimensional G-Brownian motion, the associated sublinear function $G : \mathbb{R} \to \mathbb{R}$ is defined by

$$G(a) := \frac{1}{2}(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-) = \frac{1}{2} \sup_{v \in [\underline{\sigma}^2, \overline{\sigma}^2]} va, \quad a \in \mathbb{R},$$

where $0 \le \underline{\sigma}^2 \le \overline{\sigma}^2 < \infty$ are the lower and upper volatility bounds of B, respectively, and $a^+ := \max\{a, 0\}$ and $a^- := \max\{-a, 0\}$.

The model of mean-field equations driven by G-Brownian motion (1.1) holds significant application value in financial mathematics. This model follows the core idea proposed by Bachelier, namely, using stochastic processes to describe stock price movements [1], while its theoretical framework further incorporates the Black–Scholes model [2] and the extended model proposed by Merton [3]. Compared with traditional linear models, Model (1.1) can more effectively capture market uncertainty phenomena. Based on the theory of G-expectation and G-Brownian motion developed by Peng via backward stochastic differential equations [4], Model (1.1) generalizes the classical pricing framework to nonlinear expectation. Related studies on fractional stochastic differential equations can also be found in [5, 6].

We now review the stability results related to Eq (1.1) from the following two aspects. If the coefficients b, h, and σ are independent of the distribution $\mathcal{L}_{Y_t}$, Eq (1.1) reduces to a classical stochastic differential equation driven by G-Brownian motion,

$$\mathrm{d}Y(t) = b(Y(t))\mathrm{d}t + h(Y(t))\mathrm{d}B_t + \sigma(Y(t))\mathrm{d}\langle B\rangle_t, \quad t > 0. \tag{1.3}$$

Fei et al. [7] investigated the asymptotic stability in distribution of such equations under the Lipschitz condition. Yin et al. [8] researched average principle and second-moment ultimate boundedness and stability without general decay rate by using Lipschitz condition.

Conversely, when the coefficients b, h, and σ depend on the distribution $\mathcal{L}_{Y_t}$, Yuan and Zhu [9] established the exponential stability of Problem (1.1). Xu et al. [10] obtained existence results for the solution. Furthermore, Yuan and Zhu [11] explored moment exponential stability and global stability in probability under local Lipschitz continuity.

The foregoing stability theory generally assumes that the solution converges asymptotically to the origin. However, this condition is often difficult to satisfy in practical systems subject to disturbances. In fact, many systems can maintain operation within an acceptable range of oscillations despite failing to meet Lyapunov's criteria for asymptotic stability [12]. To meet such engineering demands, Lasalle and Lefeschetz first proposed the concept of practical stability in 1961. They emphasized the feasibility of maintaining bounded oscillations within finite ranges instead of mandating asymptotic convergence to the origin [13]. In this direction, Caraballo et al. [14] considered (1.3) with delay, i.e.,

$$\mathrm{d}Y_t = b(Y_t)\mathrm{d}t + h(Y_t)\mathrm{d}B_t + \sigma(Y_t)\mathrm{d}\langle B\rangle_t, \quad t > 0. \tag{1.4}$$

and investigated the practical stability of Eq (1.4) with a general decay rate. Indeed, Liu [15] pioneered the study of general decay rates for stochastic systems, moving beyond the conventional focus on purely exponential decay.

When the nonautonomous Eq (1.4) relies on a distribution-dependent term, we focus on the moment stability and partial moment stability of Eq (1.1) with a general decay rate. The main contributions of this work are summarized as follows.

(i) Yin et al. [8] used inequality techniques to investigate the exponential stability and boundedness of Equation (1.1) without general decay rates. Different from the previous work, we establish exponential stability under general decay rates by utilizing G-Lyapunov functions.
(ii) Note that when the system's dynamics and probability measure spaces exhibit evolutionary coupling due to distribution dependence, their stability analysis faces fundamental challenges. Compared with [16], where no distribution-dependent term is involved, the difficulty here lies in finding a suitable G-Lyapunov function that depends on the distribution term.
(iii) Although the work in [17] established the exponential stability of mean-field equations driven by Brownian motion with general decay rates, we extend these results to the sublinear expectation framework by overcoming the uncertainties caused by the second-order variation of G-Brownian motion and further investigate the practical exponential stability of Eq (1.1).

The remainder of this paper is organized as follows. Section 2 introduces the necessary preliminaries regarding the space under study, discusses the relevant stability concepts, and presents the main results. Section 3 provides detailed proofs of these results. Finally, Section 4 presents two examples to illustrate our theoretical findings.

2. Main results

In this section, we begin by introducing the function space associated with the distribution-dependent term. We then give the definitions and foundational properties of the moment exponential stability and boundedness. Finally, we outline the key assumptions and provide an overview of the main theorems discussed in this work. Throughout this exposition, we adopt the convention that $\mu = \mathcal{L}_Y$ denotes the law of the random variable.

2.1. Space related to distribution-dependent term

We establish the functional analytic framework for studying the distribution-dependent terms in Eq (1.1). We begin by defining the relevant function spaces and associated terms, then proceed to characterize the probability measure spaces and their properties.

Definition 2.1 (Continuous function space [18])**.** *Let $C_\rho(\mathbb{R}^m)$ denote the space of continuous functions equipped with the norm*

$$C_\rho(\mathbb{R}^m) := \left\{\varphi \in C(\mathbb{R}^m), \|\varphi\|_{C_\rho(\mathbb{R}^m)} := \sup_{y\in\mathbb{R}^m} \frac{\|\varphi(y)\|}{(1+\|y\|)^2} + \sup_{y\neq z} \frac{\|\varphi(y)-\varphi(z)\|}{\|y-z\|} < \infty\right\}.$$

This space consists of continuous functions with controlled growth at infinity and globally Lipschitz continuous behavior, where $\|\cdot\|$ denotes the matrix norm defined on $\mathbb{R}^m$.

Let $\mathcal{P}(\mathbb{R}^m)$ denote the collection of all probability measures on $\mathcal{B}(\mathbb{R}^m)$, where $\mathcal{B}(\mathbb{R}^m)$ is the Borel σ-algebra on $\mathbb{R}^m$.

Definition 2.2 (Weighted signed measure space [18]). *The Banach space $\mathcal{M}^s_{\lambda^2}(\mathbb{R}^m)$ comprises all signed and measures m on $\mathcal{B}(\mathbb{R}^m)$ with finite weighted variation*

$$\|m\|^2_{\lambda^2} := \int_{\mathbb{R}^m} (1 + \|y\|)^2 |m|(\mathrm{d}y) < \infty,$$

where $|m| = m^+ + m^-$ denotes the total variation measure; m^+ and m^-, respectively, represent positive and negative measures.

Definition 2.3 (Probability measures with finite moments [18]). *We define the subspace of probability measures with finite second moments*

$$\mathcal{M}_{\lambda^2}(\mathbb{R}^m) := \mathcal{M}^s_{\lambda^2}(\mathbb{R}^m) \cap \mathcal{P}(\mathbb{R}^m). \tag{2.1}$$

Definition 2.4 (Weierstrass distance [17]). *For $\mu_1, \mu_2 \in \mathcal{M}_{\lambda^2}(\mathbb{R}^m)$, we define the metric*

$$\rho(\mu_1, \mu_2) := \sup_{\|\varphi\|_{C_\rho(\mathbb{R}^m)} \le 1} \left| \int_{\mathbb{R}^m} \varphi(y)\mu_1(\mathrm{d}y) - \int_{\mathbb{R}^m} \varphi(y)\mu_2(\mathrm{d}y) \right|,$$

then $(\mathcal{M}_{\lambda^2}(\mathbb{R}^m), \rho)$ is a complete metric space.

Further, to investigate stability by the G-Lyapunov functional method, the related space is required.

Definition 2.5 (Differential with dependent distribution term [17]). *The function g is called in $C^1(\mu \in \mathcal{M}_{\lambda^2}(\mathbb{R}^m))$ if there exists a μ-modification of $\partial_\mu g(\mu)(\cdot)$ such that*

$$\partial_\mu g : \mathcal{M}_{\lambda^2}(\mathbb{R}^m) \times \mathbb{R}^m \to \mathbb{R}^m$$

is continuous.

Definition 2.6 (Regular function space [17]). *A function Φ is said to belong to the space*

$$C_b^{1,2,2;1}(\mathbb{R}_+ \times \mathbb{R}^m \times \mathcal{M}_{\lambda^2}(\mathbb{R}^m)).$$

If it satisfies the following conditions:

(i) *Φ is jointly continuous in (t, y, μ), where $t \in \mathbb{R}_+$, $y \in \mathbb{R}^m$ and $\mu \in \mathcal{M}_{\lambda^2}(\mathbb{R}^m)$.*
(ii) *For fixed t and y, the mapping*

$$\mu \mapsto \Phi(t, y, \mu) \in C_b^{2;2,1}(\mathcal{M}_{\lambda^2}(\mathbb{R}^m)),$$

meaning that it has bounded, twice-differentiable behavior with respect to μ; for a fixed y and μ, the function

$$t \mapsto \Phi(t, y, \mu) \in C^1(\mathbb{R}_+),$$

indicating that it is continuously differentiable in time; for a fixed t and μ, the function

$$y \mapsto \Phi(\cdot, y, \mu) \in C^2(\mathbb{R}^m),$$

indicating that it has continuous second-order derivatives in space.

Definition 2.7 (Positive regular function space [17]). *A function Φ is said to belong to the space*

$$C_{b,+}^{1,2,2;1}(\mathbb{R}_+ \times \mathbb{R}^m \times \mathcal{M}_{\lambda^2}(\mathbb{R}^m)). \tag{2.2}$$

If

$$\Phi \in C_b^{1,2,2;1}(\mathbb{R}_+ \times \mathbb{R}^m \times \mathcal{M}_{\lambda^2}(\mathbb{R}^m))$$

and $\Phi \geq 0$ hold simultaneously.

Definition 2.8 (Lipschitz differentiable function space [17]). *A function g belongs to $C_b^{1;1}(\mathcal{M}_{\lambda^2}(\mathbb{R}^m))$. If*

(i) $g \in C^1(\mathcal{M}_{\lambda^2}(\mathbb{R}^m))$.
(ii) *The measure derivative $\partial_\mu g$ is bounded and Lipschitz continuous, i.e., a constant $C > 0$ exists such that*

$$\text{for all } \mu, \widetilde{\mu} \in \mathcal{M}_{\lambda^2}(\mathbb{R}^m),\ y, \widetilde{y} \in \mathbb{R}^m \text{ satisfy}$$

- *boundedness:* $|\partial_\mu g(\mu)(y)| \leq C$,
- *Lipschitz continuity:*

$$|\partial_\mu g(\mu)(y) - \partial_\mu g(\widetilde{\mu})(\widetilde{y})| \leq C[\rho(\mu, \widetilde{\mu}) + |y - \widetilde{y}|],$$

where ρ is a metric on $\mathcal{M}_{\lambda^2}(\mathbb{R}^m)$.

Remark 2.1 ([18]). *The higher-order spaces $C^2(\mathcal{M}_{\lambda^2}(\mathbb{R}^m))$ and $C_b^{2;1}(\mathcal{M}_{\lambda^2}(\mathbb{R}^m))$ can be defined similarly.*

2.2. Moment exponential stability and boundedness

Liu [19] obtained the polynomial stability of highly nonlinear time-changed stochastic differential equations. Based on this work, we generalize the results of [11]. The definitions of the stability and second-moment ultimate boundedness are given below.

Definition 2.9 (p-th moment stability [14]). *Assume that*

$$\eta(t) \uparrow +\infty \text{ as } t \to +\infty, \tag{2.3}$$

and satisfies

$$\eta(t+s) \leq \eta(t)\eta(s) \text{ for } s, t \in \mathbb{R}_+ \text{ largely enough.}$$

Equation (1.1) *is defined as p-th moment stable, $p > 0$, with decay $\eta(t)$ of order α if there are positive constants α and C satisfying*

$$E\|Y_t\|^p \leq C(y_0)\eta(t)^{-\alpha}, \quad t \geq 0$$

holds for any

$$Y_0 = y_0 \in \mathbb{R}^m,$$

an $\mathcal{F}_0$-measurable random vector or, equivalently

$$\limsup_{t \to +\infty} \frac{\log(E\|Y_t\|^p)}{\log \eta(t)} \leq -\alpha.$$

Definition 2.10 (second-moment ultimate boundedness [14])**.** *If there are positive constants K, α, and M satisfying*

$$E\|Y_t\|^2 \leq K\eta(t)^{-\alpha}E\|\xi\|^2 + M,\ t \geq 0,$$

where $\eta(t)$ is defined by (2.3)*, then the solution Y_t of Problem* (1.1) *is said to be ultimately bounded in the second moment with decay $\eta(t)$ of order α.*

2.3. The definition on partial moment stability and the relevant lemma

To study the partial moment stability with a general decay rate, we introduce the following notations. Denote

$$Y := (X, Z)^T$$

with

$$X := (X_1, \cdots, X_{m_1})^T \in \mathbb{R}^{m_1}, Z := (Z_1, \cdots, Z_{m_2})^T \in \mathbb{R}^{m_2}$$

and

$$m_1 > 0, m_2 \geq 0, m_1 + m_2 = m,$$

where

$$\|X\| := \sqrt{X_1^2 + \cdots + X_{m_1}^2}, \quad \|Z\| := \sqrt{Z_1^2 + \cdots + Z_{m_2}^2}, \quad \|Y\| := \sqrt{\|X\|^2 + \|Z\|^2}.$$

Let

$$\mathcal{B}_r := \{Y \in \mathbb{R}^m : \|Y\| \leq r\}, r > 0.$$

Definition 2.11. *The solution to Problem* (1.1) *is defined as global uniform bounded with probability one, provided that for every $\varsigma > 0$, there is*

$$d = d(\varsigma) > 0 \ (\textit{independent of } t_0)$$

such that

$$\sup_{t \geq t_0} \|Y_t\| \leq d(\varsigma), \textit{almost surely for every } t_0 \geq 0 \textit{ and all } Y_0 \in \mathbb{R}^n \textit{ with } E\|Y_0\| \leq \varsigma.$$

Definition 2.12 (Partial moment exponential stability [20])**.** *Let $\eta(t)$ be a positive function defined for sufficiently large $t > 0$ such that*

$$\eta(t) \to +\infty \textit{ as } t \to +\infty.$$

A solution

$$Y(t) = (X(t), Z(t))$$

to Problem (1.1) *is said to tend to the ball $\mathcal{B}_r$ moment with respect to X with the decay function $\eta(t)$ and an order of at least $\alpha' > 0$ provided that there is a constant $\alpha' > 0$ satisfying*

$$\limsup_{t \to +\infty} \frac{\log\left(E(\|Y_t\|^p - r)\right)}{\log \eta(t)} \leq -\alpha',\ \textit{almost surely.}$$

If, in addition, 0 is a solution to Problem (1.1)*, the zero solution is defined as practical p-th moment stable with respect to X with the decay function $\eta(t)$ and an order of at least α', if every solution to Problem* (1.1) *tends to the boundary of the ball $\mathcal{B}_r$ with respect to X momently with decay function $\eta(t)$ and order at least α', for any $r > 0$ that is sufficiently small.*

Lemma 2.1 (Inequality estimation [21])**.** *Let $\delta_1(t)$ and $\delta_2(t)$ be two bounded Borel measurable non-negative functions. Suppose that $\varpi(t)$ is a continuous, non-negative, and non-decreasing function defined on $[0, T]$ and $0 \leq \mu < 1$. If*

$$\delta_1(t) \leq \varpi(t) + \int_0^t \delta_2(\tau)\delta_1(\tau)^\mu \mathrm{d}\tau, \ \ 0 \leq t \leq T,$$

then

$$\delta_1(t) \leq \left(\varpi(t)^{1-\mu} + (1-\mu)\int_0^t \delta_2(\tau)\mathrm{d}\tau\right)^{\frac{1}{1-\mu}}, \ 0 \leq t \leq T.$$

2.4. Some conditions and the contents of main theorems

We then introduce some suitable conditions to study the moment exponential stability and practical exponential stability. The assumptions (H_1)–(H_2) are given below. We further assume that the $\mathcal{F}_0$-measurable initial value ξ is square-integrable under the sublinear expectation, namely, $E\|\xi\|^2 < \infty$.

(H$_1$) The functions b, h, and σ satisfy Relation (1.2) and the following linear growth condition: There is a constant $L_1 > 0$ such that, for all $t \in [0, T]$, $y \in \mathbb{R}^m$, and $\mu \in \mathcal{M}_{\lambda^2}(\mathbb{R}^m)$, we have

$$\|b_t(y,\mu)\|^2 + \|h_t(y,\mu)\|^2 + \|\sigma_t(y,\mu)\|^2 \leq L_1(1 + \|y\|^2 + \|\mu\|_{\lambda^2}^2).$$

(H$_2$) The functions b, h, and σ satisfy Relation (1.2) and the following global Lipschitz condition: A constant $L_2 > 0$ exists such that, for all $t \in [0, T]$, $y_1, y_2 \in \mathbb{R}^m$, and $\mu_1, \mu_2 \in \mathcal{M}_{\lambda^2}(\mathbb{R}^m)$, we have

$$\begin{aligned}&\|b_t(y_1,\mu_1) - b_t(y_2,\mu_2)\|^2 + \|h_t(y_1,\mu_1) - h_t(y_2,\mu_2)\|^2 + \|\sigma_t(y_1,\mu_1) - \sigma_t(y_2,\mu_2)\|^2\\ &\leq L_2(\|y_1 - y_2\|^2 + \rho^2(\mu_1,\mu_2)).\end{aligned}$$

Lemma 2.2 (Well-posedness and moment estimate)**.** *Suppose that* (H$_1$)–(H$_2$) *hold and $E\|\xi\|^2 < \infty$. Then Eq* (1.1) *admits a unique adapted solution Y on every finite interval* $[0, T]$*. Moreover, there is a constant $C_T > 0$, independent of the initial value, such that*

$$E\Big(\sup_{0\leq s\leq T} \|Y_s\|^2\Big) \leq C_T(1 + E\|\xi\|^2).$$

Consequently, together with the regularity required of $\Phi \in C_b^{1,2,2;1}(\mathbb{R}_+ \times \mathbb{R}^m \times \mathcal{M}_{\lambda^2}(\mathbb{R}^m))$, the above estimate guarantees the integrability of the terms appearing in the distribution-dependent G-Itô formula used below.

Proof. The conclusion follows from the standard Picard iteration argument for distribution-dependent stochastic differential equations under G-expectation, together with the G-Burkholder–Davis–Gundy inequality and Gronwall's inequality; see [10, 11]. The linear growth condition (H$_1$) yields the required a priori second-moment bound, while the global Lipschitz condition (H$_2$) gives the uniqueness and convergence of the Picard sequence. This proves both the well-posedness assertion and the stated moment estimate. □

Finally, we consider the polynomial decay and logarithmic decay of the solutions to Problem (1.1). When the origin is an equilibrium point, the moment stability of stochastic mean-field differential equations with a general decay rate is characterized by Theorems 2.1 and 2.2. In contrast, when the origin is not an equilibrium point, the relevant results of partial moment stability with general decay rate are presented in Theorem 2.3–Theorem 2.5.

Theorem 2.1. *Let* (H_1) *and* (H_2) *hold. Suppose that*

$$\Phi \in C_{b,+}^{1,2,2;1}(\mathbb{R}_+ \times \mathbb{R}^m \times \mathcal{M}_{\lambda^2}(\mathbb{R}^m))$$

and $\phi_1(t)$ *and* $\phi_2(t)$ *are two continuous non-negative functions on* $\mathbb{R}_+$. *We see that* $\kappa, \vartheta \in \mathbb{R}$ *and* $p, n \in \mathbb{R}_+$ *exist such that the following conditions are satisfied for all* $t \in \mathbb{R}_+$, $Y_t \in \mathbb{R}^m$, *and* $\mathcal{L}_{Y_t} \in \mathcal{M}_{\lambda^2}(\mathbb{R}^m)$.

(i) $\eta(t)^n \|Y_t\|^p \leq \Phi(t, Y_t, \mathcal{L}_{Y_t})$;
(ii) $L\Phi(t, Y_t, \mathcal{L}_{Y_t}) \leq \phi_1(t) + \phi_2(t)\Phi(t, Y_t, \mathcal{L}_{Y_t})$;
(iii) $\limsup_{t\to\infty} \frac{\ln\left(\int_0^t \phi_1(s)\mathrm{d}s\right)}{\ln \eta(t)} \leq \kappa$ *and* $\limsup_{t\to\infty} \frac{\int_0^t \phi_2(s)\mathrm{d}s}{\ln \eta(t)} \leq \vartheta$.
(iv) Y_t *is globally uniformly bounded with probability one.*

Then the solution to Problem (1.1) *converges to zero in p-th moment with the asymptotic rate* $\eta(t)$. *Furthermore, we have*

$$\limsup_{t\to\infty} \frac{\log(E\|Y_t\|^p)}{\log \eta(t)} \leq -\alpha,$$

where the decay rate exponent $\alpha = n - \vartheta - \kappa > 0$.

Theorem 2.2. *Let* $(H_1) - (H_2)$ *hold. Suppose there is a function*

$$\Phi \in C_{b,+}^{1,2,2;1}(\mathbb{R}_+ \times \mathbb{R}^m \times \mathcal{M}_{\lambda^2}(\mathbb{R}^m))$$

defined by (2.2), *such that*

$$L\Phi(t, Y_t, \mathcal{L}_{Y_t}) + \beta_1(t)\Phi(t, Y_t, \mathcal{L}_{Y_t}) \leq \beta_2(t) \tag{2.4}$$

and

$$c_1 \int_{\mathbb{R}^m} \|Y\|^2 \mu(\mathrm{d}Y) - K_1 \leq \int_{\mathbb{R}^m} \Phi(t, Y_t, \mathcal{L}_{Y_t})\mu(\mathrm{d}Y) \leq c_2 \int_{\mathbb{R}^m} \|Y\|^2 \mu(\mathrm{d}Y) - K_2, \tag{2.5}$$

where $\beta_i(t) : \mathbb{R}_+ \to \mathbb{R}_+$ *are positive functions and* K_i *are positive constants,* $i = 1, 2$. *If the following asymptotic conditions are satisfied*

$$\limsup_{t\to\infty} \frac{\int_0^t \left(\frac{n\eta'(\tau)}{\eta(\tau)} - \beta_1(\tau)\right)\mathrm{d}\tau}{\ln \eta(t)} \leq \vartheta \text{ and } \lim_{t\to\infty} \frac{\ln \int_0^t \eta(\tau)^n \beta_2(\tau)\mathrm{d}\tau}{\ln \eta(t)} \leq \kappa,$$

then the solution to Problem (1.1) *is ultimately bounded in the second moment with decay rate* $\eta(t)$. *Specially, the constants* $C > 0$ *and* $M_3 > 0$ *exist such that*

$$E\|Y_t\|^2 \leq C\eta(t)^{-\alpha} E\|\xi\|^2 + M_3, t \geq 0.$$

Theorem 2.3. *Let the function*

$$\Phi \in C_{b,+}^{1,2,2;1}(\mathbb{R}_+ \times \mathbb{R}^m \times \mathcal{M}_{\lambda^2}(\mathbb{R}^m)).$$

Assume that $\eta_i(t) > 0$ $(i = 1, 2, 3)$ *are continuous functions. Suppose further that for all* $Y = (X, Z) \in \mathbb{R}^m$ *and* $t \in \mathbb{R}_+$, *Conditions* (i) *and* (iv) *in Theorem* 2.1 *hold, and a positive constant* $p \in N^+$, $n \geq 0$, *and the real numbers* ω, ϱ, θ, *and* $0 \leq \mu < 1$ *exist such that*

(i) $L\Phi(t, Y_t, \mathcal{L}_{Y_t}) \le \eta_1(t)\Phi(t, Y_t, \mathcal{L}_{Y_t}) + \eta_2(t)\Phi(t, Y_t, \mathcal{L}_{Y_t})^\mu + \eta_3(t)$;

(ii) $\limsup_{t\to+\infty} \frac{\int_0^t \eta_1(\tau)\mathrm{d}\tau}{\ln \eta(t)} \le \varrho(1-\mu)$, $\lim_{t\to+\infty} \frac{\ln(\int_0^t \eta_2(\tau)\mathrm{d}\tau)}{\ln \eta(t)} \le \theta$, $\lim_{t\to+\infty} \frac{\ln(\int_0^t \eta_3(\tau)\mathrm{d}\tau)}{\ln \eta(t)} \le \omega$.

The following estimate holds

$$\limsup_{t\to+\infty} \frac{\ln(E\|X_t\|^p)}{\ln \eta(t)} \le -(n - (\varrho + \omega \vee \theta)).$$

Theorem 2.4. *Let the function*

$$\Phi \in C_{b,+}^{1,2,2;1}(\mathbb{R}_+ \times \mathbb{R}^m \times \mathcal{M}_{\lambda^2}(\mathbb{R}^m)).$$

Assume that the continuous functions $\psi_1(t) \ge 0$ *and* $\psi_2(t) > 0$*; the constants* $p \in N^+$*,* $n \ge 0$*,* $\overline{\psi} \ge 0$*, and* $\widetilde{\psi} \ge 0$*; and a small constant* $\sigma \ge 0$ *exist such that for all* $t \ge 0$ *and all* $Y_t = (X_t, Z_t) \in \mathbb{R}^m$*; the Conditions* (i) *and* (iv) *in Theorem* 2.1 *hold; and*

(i) $L\Phi(t, Y_t, \mathcal{L}_{Y_t}) \le \psi_1(t)\Phi(t, Y_t, \mathcal{L}_{Y_t}) + \psi_2(t)$;

(ii) $\limsup_{t\to+\infty} \frac{\int_0^t \psi_1(\tau)\mathrm{d}\tau}{\ln \eta(t)} \le \overline{\psi}$, $\limsup_{t\to+\infty} \frac{\ln(t)}{\ln \eta(t)} \le \widetilde{\psi}$;

(iii) $\lim_{t\to+\infty} \frac{\psi_2(t)}{\eta(t)^n} = \sigma$, $\psi_2(t) \le \widehat{\psi}$,

then we obtain

$$\limsup_{\substack{t\to+\infty \\ E\left(\|X_t\|^p - \frac{\psi_2(t)}{\eta(t)^n}\right)>0}} \frac{\ln\left(E\left(\|X_t\|^p - \frac{\psi_2(t)}{\eta(t)^n}\right)\right)}{\ln \eta(t)} \le -(n - \overline{\psi} - \widetilde{\psi}).$$

Theorem 2.5. *Let the function*

$$\Phi \in C_{b,+}^{1,2,2;1}(\mathbb{R}_+ \times \mathbb{R}^m \times \mathcal{M}_{\lambda^2}(\mathbb{R}^m)).$$

Suppose that we have the continuous functions $\psi_1(t) \ge 0$*,* $\psi_2(t) \ge 0$*, and* $\psi_3(t) > 0$*; the constants* $p \in N^+$ *and* $n \ge 0$*; and the nonnegative real numbers* $\overline{\psi}_1$*,* $\overline{\psi}_2$*, and* $\overline{\psi}_3$ *such that for all* $t \ge 0$ *and all* $Y_t = (X_t, Z_t) \in \mathbb{R}^m$*; the following conditions hold:*

(i) *Conditions* (i) *and* (iv) *in Theorem* 2.1 *and condition* (ii) *in Theorem* 2.4 *hold;*

(ii) $L\Phi(t, Y_t, \mathcal{L}_{Y_t}) \le \psi_1(t)\Phi(t, Y_t, \mathcal{L}_{Y_t}) + \psi_2(t)\Phi(t, Y_t, \mathcal{L}_{Y_t})^\mu + \psi_3(t)$;

(iii) $\limsup_{t\to+\infty} \frac{\int_0^t \psi_1(\tau)\mathrm{d}\tau}{\ln \eta(t)} \le \overline{\psi}_1(1-\mu)$, $\limsup_{t\to+\infty} \frac{\ln(\int_0^t \psi_2(\tau)\mathrm{d}\tau)}{\ln \eta(t)} \le \overline{\psi}_2$, $\limsup_{t\to+\infty} \frac{\ln(t)}{\ln \eta(t)} \le \overline{\psi}_3$.

The following inequality holds:

$$\limsup_{\substack{t\to+\infty \\ E\left(\|X_t\|^p - \frac{\psi_3(t)}{\eta(t)^n}\right)>0}} \frac{\ln\left(E\left(\|X_t\|^p - \frac{\psi_3(t)}{\eta(t)^n}\right)\right)}{\ln \eta(t)} \le -(n - (\overline{\psi}_1 + \overline{\psi}_2 \vee \overline{\psi}_3)).$$

3. Proofs of Theorems 2.1–2.5

The proof of Theorem 2.1. Following the methodology of Theorem 2.1 in [17], we provide a proof sketch. Using G-Itô's formula, we have

$$
\begin{aligned}
&\Phi(t, Y_t, \mathcal{L}_{Y_t}) - \Phi(s, Y_s, \mathcal{L}_{Y_s}) \\
&= \int_s^t (\partial_t \Phi(\tau, Y_\tau, \mathcal{L}_{Y_\tau}) + (\partial_Y \Phi)(\tau, Y_\tau, \mathcal{L}_{Y_\tau}) \cdot b) \mathrm{d}\tau + \int_s^t (\partial_Y \Phi)(\tau, Y_\tau, \mathcal{L}_{Y_\tau}) \cdot h \mathrm{d}B_\tau \\
&\quad + \int_s^t (\partial_Y \Phi)(\tau, Y_\tau, \mu_\tau) \cdot \sigma \mathrm{d}\langle B \rangle_\tau + \frac{1}{2} (\partial_{YY}^2 \Phi)(\tau, Y_\tau, \mathcal{L}_{Y_\tau}) \cdot h^2 \mathrm{d}\langle B \rangle_\tau \\
&\quad + \int_s^t \int_0^1 (\partial_Y \Phi)(\tau, Y_\tau, \mathcal{L}_{Y_{\tau(\lambda)}}) \mathcal{L}_{Y_{\tau(\lambda)}} (b \mathrm{d}\tau + h \mathrm{d}B_\tau + \sigma \mathrm{d}\langle B \rangle_\tau) \mathrm{d}\lambda,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\Phi(t, Y_t, \mathcal{L}_{Y_t}) =& \Phi(0, \xi, \mathcal{L}_\xi) + \int_0^t (\partial_t \Phi(\tau, Y_\tau, \mathcal{L}_{Y_\tau}) + \partial_Y \Phi(\tau, Y_\tau, \mathcal{L}_{Y_\tau}) \cdot b) \mathrm{d}\tau \\
&+ \int_0^t (\partial_Y \Phi)(\tau, Y_\tau, \mathcal{L}_{Y_\tau}) \cdot h \mathrm{d}B_\tau + \int_0^t (\partial_Y \Phi)(\tau, Y_\tau, \mathcal{L}_{Y_\tau}) \cdot \sigma \mathrm{d}\langle B \rangle_\tau \\
&+ \frac{1}{2} \int_0^t (\partial_{YY}^2 \Phi)(\tau, Y_\tau, \mathcal{L}_{Y_\tau}) \cdot h^2 \mathrm{d}\langle B \rangle_\tau \\
&+ \int_0^t \int_0^1 (\partial_Y \Phi)(\tau, Y_\tau, \mathcal{L}_{Y_{\tau(\lambda)}}) \mathcal{L}_{Y_{\tau(\lambda)}} (b \mathrm{d}\tau + h \mathrm{d}B_\tau + \sigma \mathrm{d}\langle B \rangle_\tau) \mathrm{d}\lambda \\
\leq& \Phi(0, \xi, \mathcal{L}_\xi) + \int_0^t L\Phi(\tau, Y_\tau, \mathcal{L}_{Y_\tau}) \mathrm{d}\tau + \int_0^t (\partial_Y \Phi)(\tau, Y_\tau, \mathcal{L}_{Y_\tau}) \cdot h \mathrm{d}B_\tau \\
&+ \frac{1}{2} \int_0^t \zeta_\tau(\Phi, Y_\tau, \mathcal{L}_{Y_\tau}) \mathrm{d}\langle B \rangle_\tau - \int_0^t G(\zeta_\tau(\Phi, Y_\tau, \mathcal{L}_{Y_\tau})) \mathrm{d}\tau,
\end{aligned}
$$

where

$$
Y_{\tau(\lambda)} := Y_\tau + \lambda (b \mathrm{d}\tau + h \mathrm{d}B_\tau + \sigma \mathrm{d}\langle B \rangle_\tau), \lambda \in [0, 1] \text{ and } \tau \in [0, t],
$$

Define

$$
\begin{aligned}
H_\tau &:= \partial_Y \Phi(\tau, Y_\tau, \mathcal{L}_{Y_\tau}) \cdot h_\tau(Y_\tau, \mathcal{L}_{Y_\tau}), \\
\zeta_\tau(\Phi, Y_\tau, \mathcal{L}_{Y_\tau}) &:= 2 \partial_Y \Phi(\tau, Y_\tau, \mathcal{L}_{Y_\tau}) \cdot \sigma_\tau(Y_\tau, \mathcal{L}_{Y_\tau}) \\
&\quad + \partial_{YY}^2 \Phi(\tau, Y_\tau, \mathcal{L}_{Y_\tau}) \cdot h_\tau(Y_\tau, \mathcal{L}_{Y_\tau})^2.
\end{aligned}
$$

and

$$
\begin{aligned}
L\Phi =& \partial_t \Phi + (\partial_Y \Phi)(\tau, Y_\tau, \mathcal{L}_{Y_\tau}) b + \int_0^1 \partial_Y \Phi(t, Y, z_1(\lambda)) z_2(\lambda, \varkappa) \mathrm{d}\lambda + \int_0^1 \partial_Y \Phi(t, Y, z_1(\lambda)) \overline{\sigma}^2 z_2(\lambda, |\sigma|) \mathrm{d}\lambda \\
&+ 2G\Big(\sigma \cdot \partial_Y \Phi + \frac{1}{2} h^2 \cdot \partial_{YY}^2 \Phi\Big),
\end{aligned} \tag{3.1}
$$

in which $z_1(\lambda)$ is a function depending on Y and λ, and $z_2(\lambda, \varkappa)$ is a function depending on Y, λ and $\varkappa$. By Lemma 2.2 and the regularity imposed on Φ, the processes H and ζ defined above belong to the standard G-Itô integrability spaces $H_G^2(0,T)$ and $M_G^1(0,T)$, respectively. Hence

$$M_t := \int_0^t H_\tau \mathrm{d}B_\tau$$

is a symmetric G-martingale. Therefore, we have

$$E[M_t] = E[-M_t] = 0.$$

Furthermore, by the quadratic variation compensation property of G-Brownian motion (see Theorem 4.5 in [22]), we have

$$K_t := \frac{1}{2}\int_0^t \zeta_\tau(\Phi, Y_\tau, \mathcal{L}_{Y_\tau})\mathrm{d}\langle B\rangle_\tau - \int_0^t G(\zeta_\tau(\Phi, Y_\tau, \mathcal{L}_{Y_\tau}))\mathrm{d}\tau$$

is a nonincreasing G-martingale and, consequently, $E[K_t] \leq 0$. Notice that this conclusion does not require the pointwise condition $G(\zeta_\tau) \geq 0$. Indeed, under each probability measure in the representation of the sublinear expectation, $\mathrm{d}\langle B\rangle_\tau = v_\tau \mathrm{d}\tau$ with $v_\tau \in [\underline{\sigma}^2, \overline{\sigma}^2]$, and the definition of G gives

$$\frac{1}{2}v_\tau\zeta_\tau - G(\zeta_\tau) \leq 0.$$

Taking the sublinear expectation in the preceding G-Itô inequality therefore yields

$$E\Phi(t, Y_t, \mathcal{L}_{Y_t}) - E\Phi(0, \xi, \mathcal{L}_\xi) \leq E\int_0^t L\Phi(\tau, Y_\tau, \mathcal{L}_{Y_\tau})\mathrm{d}\tau.$$

Combining the known Condition (ii) that

$$E\Phi(t, Y_t, \mathcal{L}_{Y_t}) \leq E\Phi(0, \xi, \mathcal{L}_\xi) + \int_0^t [\phi_1(\tau) + \phi_2(\tau)E\Phi(\tau, Y_\tau, \mathcal{L}_{Y_\tau})]\mathrm{d}\tau.$$

Applying the Grönwall inequality yields

$$E\Phi(t, Y_t, \mathcal{L}_{Y_t}) \leq \Big[E\Phi(0, \xi, \mathcal{L}_\xi) + \int_0^t \phi_1(\tau)\mathrm{d}\tau\Big]\exp\Big(\int_0^t \phi_2(\tau)\mathrm{d}\tau\Big),$$

i.e.,

$$\ln\left(E\Phi(t, Y_t, \mathcal{L}_{Y_t})\right) \leq \ln\Big[E\Phi(0, \xi, \mathcal{L}_\xi) + \int_0^t \phi_1(\tau)\mathrm{d}\tau\Big] + \int_0^t \phi_2(\tau)\mathrm{d}\tau.$$

Combining the known Conditions (ii) and (iii) with the limit property, it follows that for any $\varepsilon > 0$, there is a large enough $t > 0$ satisfying

$$\ln(E\Phi(t, Y_t, \mathcal{L}_{Y_t})) \leq \ln\Big[E\Phi(0, \xi, \mathcal{L}_\xi) + \eta(t)^{\kappa+\varepsilon}\Big] + \ln\Big[\eta(t)^{\vartheta+\varepsilon}\Big],$$

which indicates that

$$\limsup_{t\to+\infty} \frac{\ln\left(E\Phi(t, Y_t, \mathcal{L}_{Y_t})\right)}{\ln \eta(t)} \leq \kappa + \varepsilon + \vartheta.$$

Taking $\varepsilon \to 0$, Condition (i) tells

$$\limsup_{t\to+\infty} \frac{\ln(E\|Y_t\|^p)}{\ln \eta(t)} \le \limsup_{t\to+\infty} \frac{\ln\left(\eta(t)^{-n} E\Phi(t, Y_t, \mathcal{L}_{Y_t})\right)}{\ln \eta(t)} \le -(n-\kappa-\vartheta).$$

The proof of Theorem 2.2. G-Itô's formula says

$$\begin{aligned}\eta(t)^n E\Phi(t, Y_t, \mathcal{L}_{Y_t}) \le& \eta(0)^n \Phi(0, \xi, \mathcal{L}_\xi) + n\int_0^t \eta(\tau)^{n-1}\eta'(\tau) E\Phi(\tau, Y_\tau, \mathcal{L}_{Y_\tau})\mathrm{d}\tau \\ &+ E\int_0^t \eta(\tau)^n L\Phi(\tau, Y_\tau, \mathcal{L}_{Y_\tau})\mathrm{d}\tau.\end{aligned}$$

Condition (2.4) gives

$$\begin{aligned}\eta(t)^n E\Phi(t, Y_t, \mathcal{L}_{Y_t}) \le& \eta(0)^n \Phi(0, \xi, \mathcal{L}_\xi) + \int_0^t \beta_2(\tau)\eta(\tau)^n \mathrm{d}\tau \\ &+ \int_0^t \Big(\frac{n\eta'(\tau)}{\eta(\tau)} - \beta_1(s)\Big)\eta(\tau)^n E\Phi(\tau, Y_\tau, \mathcal{L}_{Y_\tau})\mathrm{d}\tau.\end{aligned}$$

The Gronwall's inequality tells

$$\eta(t)^n E\Phi(t, Y_t, \mathcal{L}_{Y_t}) \le \Big(\eta(0)^n \Phi(0, \xi, \mathcal{L}_\xi) + \int_0^t \beta_2(\tau)\eta(\tau)^n \mathrm{d}\tau\Big)\cdot \exp\Big(\int_0^t \Big[\frac{n\eta'(\tau)}{\eta(\tau)} - \beta_1(\tau)\Big]\mathrm{d}\tau\Big).$$

Taking the logarithm on both sides, we obtain

$$\ln\left(\eta(t)^n E\Phi(t, Y_t, \mu_t)\right) \le \ln\Big[\eta(0)^n \Phi(0, \xi, \mathcal{L}_\xi) + \int_0^t \beta_2(\tau)\eta(\tau)^n \mathrm{d}\tau\Big] + \int_0^t \Big[\frac{n\eta'(\tau)}{\eta(\tau)} - \beta_1(\tau)\Big]\mathrm{d}\tau.$$

Under the stated assumptions and by the fundamental properties of limits, for any $\varepsilon > 0$, a sufficiently large $t > 0$ exists such that

$$\ln\left(\eta(t)^n E\Phi(t, Y_t, \mathcal{L}_{Y_t})\right) \le \ln\left(\eta(0)^n \Phi(0, \xi, \mathcal{L}_\xi) + \eta(t)^{\kappa+\varepsilon}\right) + \ln\left(\eta(t)^{\vartheta+\varepsilon}\right),$$

which means that

$$\limsup_{t\to+\infty} \frac{\ln\left(\eta(t)^n E\Phi(t, Y_t, \mathcal{L}_{Y_t})\right)}{\ln \eta(t)} \le \kappa + \varepsilon + \vartheta + \varepsilon.$$

Letting $\varepsilon \to 0$, it follows that

$$n + \limsup_{t\to+\infty} \frac{\ln\left(E\Phi(t, Y_t, \mathcal{L}_{Y_t})\right)}{\ln \eta(t)} \le \limsup_{t\to+\infty} \frac{\ln\left(\eta(t)^n E\Phi(t, Y_t, \mathcal{L}_{Y_t})\right)}{\ln \eta(t)} \le \kappa + \vartheta.$$

From Condition (2.5)

$$c_1 E\|Y_t\|^2 - K_1 \le E\Phi(t, Y_t, \mathcal{L}_{Y_t})$$

and applying the monotonicity of the function $\log Y$, we get the desired result.

The proof of Theorem 2.3. Similar to Theorem 2.1, we know

$$E\Phi(t, Y_t, \mathcal{L}_{Y_t}) - E\Phi(0, \xi, \mathcal{L}_\xi) \le E\int_0^t L\Phi(\tau, Y_\tau, \mathcal{L}_{Y_\tau})\mathrm{d}\tau.$$

By Condition (i), we have

$$E\Phi(t,Y_t,\mathcal{L}_{Y_t}) \le E\Phi(0,\xi,\mathcal{L}_\xi) + \int_0^t \Big(\eta_1(\tau)E\Phi(\tau,Y_\tau,\mathcal{L}_{Y_\tau}) + \eta_2(\tau)E(\Phi^\mu(\tau,Y_\tau,\mathcal{L}_{Y_\tau})) + \eta_3(\tau)\Big)\mathrm{d}\tau.$$

The Grönwall–Bellman lemma [23] gives

$$E\Phi(t,Y_t,\mathcal{L}_{Y_t}) \le \Big(\Phi(0,\xi,\mathcal{L}_\xi) + \int_0^t \eta_3(\tau)\mathrm{d}\tau + \int_0^t \eta_2(\tau)E(\Phi^\mu(\tau,Y_\tau,\mathcal{L}_{Y_\tau}))\mathrm{d}\tau\Big)\cdot\exp\Big(\int_0^t \eta_1(\tau)\mathrm{d}\tau\Big).$$

From Lemma 2.1, we obtain

$$\begin{aligned} E\Phi(t,Y_t,\mathcal{L}_{Y_t}) \le & \bigg(\Big(\Phi(0,\xi,\mathcal{L}_\xi) + \int_0^t \eta_3(\tau)\mathrm{d}\tau\Big)^{1-\mu}\cdot\exp\Big((1-\mu)\int_0^t \eta_1(\tau)\mathrm{d}\tau\Big) \\ & + (1-\mu)\int_0^t \eta_2(\tau)\mathrm{d}\tau\cdot\exp\Big(\int_0^t \eta_2(\tau)\mathrm{d}\tau\Big)\bigg)^{\frac{1}{1-\mu}}, \end{aligned}$$

which implies that

$$E\Phi(t,Y_t,\mathcal{L}_{Y_t}) \le \bigg(\exp\Big(\int_0^t \eta_1(\tau)\mathrm{d}\tau\Big)\Big(\Big(\Phi(0,\xi,\mathcal{L}_\xi) + \int_0^t \eta_3(\tau)\mathrm{d}\tau\Big)^{1-\mu} + (1-\mu)\int_0^t \eta_2(\tau)\mathrm{d}\tau\Big)\bigg)^{\frac{1}{1-\mu}}.$$

Taking logarithms on both sides, we have

$$\ln E\Phi(t,Y_t,\mathcal{L}_{Y_t}) \le \frac{1}{1-\mu}\int_0^t \eta_1(\tau)\mathrm{d}\tau + \frac{1}{1-\mu}\ln\Big(\Big(\Phi(0,\xi,\mathcal{L}_\xi) + \int_0^t \eta_3(\tau)\mathrm{d}\tau\Big)^{1-\mu} + (1-\mu)\int_0^t \eta_2(\tau)\mathrm{d}\tau\Big).$$

Applying Condition (ii), we obtain

$$\ln E\Phi(t,Y_t,\mathcal{L}_{Y_t}) \le \ln\big(\eta(t)^{\varrho+\varepsilon}\big) + \frac{1}{1-\mu}\ln\Big(\big(\Phi(0,\xi,\mathcal{L}_\xi) + \eta(t)^{\omega+\varepsilon}\big)^{1-\mu} + (1-\mu)\eta(t)^{\theta+\varepsilon}\Big),\ \forall\varepsilon > 0,$$

which means that

$$\lim_{t\to+\infty}\frac{\ln E\Phi(t,Y_t,\mathcal{L}_{Y_t})}{\ln\eta(t)} \le (\varrho+\varepsilon) + (\omega+\varepsilon)\vee(\theta+\varepsilon),\ \text{almost surely.} \tag{3.2}$$

Letting $\varepsilon \to 0$ in (3.2), we have

$$\lim_{t\to+\infty}\frac{\ln E\Phi(t,Y_t,\mathcal{L}_{Y_t})}{\ln\eta(t)} \le \varrho + \omega\vee\theta,\ \text{almost surely.}$$

Theorem 2.1 (i) indicates that

$$\lim_{t\to+\infty}\frac{\ln\big(E\|Y_t\|^p\big)}{\ln\eta(t)} \le \lim_{t\to+\infty}\frac{\ln\Big(\eta(t)^{-n}E\Phi(t,Y_t,\mathcal{L}_{Y_t})\Big)}{\ln\eta(t)} \le -(n-(\varrho+\omega\vee\theta)),\ \text{almost surely.}$$

Therefore, the solution to Problem (1.1) is p-th moment stable with respect to Y_t, with a decay function $\eta(t)$ and an order of at least $\alpha = n - (\varrho + \omega\vee\theta)$ provided that $n > \varrho + \omega\vee\theta$.

The proof of Theorem 2.4. Since

$$\eta(t)^n\|Y_t\|^p - \psi_2(t) = \eta(t)^n\Big(\|Y_t\|^p - \frac{\psi_2(t)}{\eta(t)^n}\Big),$$

then from Theorem 2.1 (i), we have

$$\Phi(t, Y_t, \mathcal{L}_{Y_t}) \geq \eta(t)^n\|Y_t\|^p \geq \eta(t)^n\|Y_t\|^p - \psi_2(t) \geq \eta(t)^n\Big(\|Y_t\|^p - \frac{\psi_2(t)}{\eta(t)^n}\Big),$$

For those values of t satisfying

$$E\Big(\|Y_t\|^p - \frac{\psi_2(t)}{\eta(t)^n}\Big) > 0,$$

taking logarithms in the inequality above gives

$$\ln\big(E\Phi(t, Y_t, \mathcal{L}_{Y_t})\big) \geq n\ln\big(\eta(t)\big) + \ln\bigg(E\Big(\|Y_t\|^p - \frac{\psi_2(t)}{\eta(t)^n}\Big)\bigg).$$

We then obtain

$$\ln\bigg(E\Big(\|Y_t\|^p - \frac{\psi_2(t)}{\eta(t)^n}\Big)\bigg) \leq \ln\big(E\Phi(t, Y_t, \mathcal{L}_{Y_t})\big) - n\ln\big(\eta(t)\big). \tag{3.3}$$

On the other hand, G-Itô's formula yields

$$E\Phi(t, Y_t, \mathcal{L}_{Y_t}) - E\Phi(0, \xi, \mathcal{L}_{\xi}) \leq E\int_0^t L\Phi(\tau, Y_\tau, \mathcal{L}_{Y_\tau})\mathrm{d}\tau, \forall\, t \geq 0.$$

Condition (i) yields

$$E\Phi(t, Y_t, \mathcal{L}_{Y_t}) \leq \Phi(0, \xi, \mathcal{L}_{\xi}) + \int_0^t \Big(\psi_2(\tau) + \psi_1(\tau)E\Phi(\tau, Y_\tau, \mathcal{L}_{Y_\tau})\Big)\mathrm{d}\tau.$$

Applying Gronwall's lemma [23] together with Condition (i), we have

$$\begin{aligned} E\Phi(t, Y_t, \mathcal{L}_{Y_t}) \leq& \Big(\Phi(0, \xi, \mathcal{L}_{\xi}) + \int_0^t \psi_2(\tau)\mathrm{d}\tau\Big)\exp\Big(\int_0^t \psi_1(\tau)\mathrm{d}\tau\Big) \\ \leq& (\Phi(0, \xi, \mathcal{L}_{\xi}) + \widehat{\psi} t)\exp\Big(\int_0^t \psi_1(\tau)\mathrm{d}\tau\Big). \end{aligned} \tag{3.4}$$

According to Condition (ii), for $\varepsilon > 0$, whenever $t > 0$ is large,

$$\int_0^t \psi_1(\tau)\mathrm{d}\tau \leq (\overline{\psi} + \varepsilon)\ln\eta(t) \text{ and } \log(t) \leq (\widetilde{\psi} + \varepsilon)\ln\eta(t).$$

Then Inequality (3.4) becomes

$$\ln\big(E\Phi(t, Y_t, \mathcal{L}_{Y_t})\big) \leq \ln\Big(\eta(t)^{\widetilde{\psi}+\varepsilon}\Big) + \ln\Big(\eta(t)^{\overline{\psi}+\varepsilon}\Big).$$

That is, we have

$$\limsup_{t\to+\infty}\frac{\ln E\Phi(t,Y_t,\mathcal{L}_{Y_t})}{\ln\eta(t)}\le(\overline{\psi}+\varepsilon)+(\widetilde{\psi}+\varepsilon).$$

Letting $\varepsilon\to 0$, we have

$$\limsup_{t\to+\infty}\frac{\ln E\Phi(t,Y_t,\mathcal{L}_{Y_t})}{\ln\eta(t)}\le(\overline{\psi}+\widetilde{\psi}).$$

From (3.3), we deduce that

$$\limsup_{\substack{t\to+\infty\\ E\left(\|Y_t\|^p-\frac{\psi_2(t)}{\eta(t)^n}\right)>0}}\frac{\ln\left(E\left(\|Y_t\|^p-\frac{\psi_2(t)}{\eta(t)^n}\right)\right)}{\ln\eta(t)}\le-(n-(\overline{\psi}+\widetilde{\psi})).$$

The proof of Theorem 2.5. Similar to Theorem 2.4, for those t satisfying

$$E\left(\|Y_t\|^p-\frac{\psi_3(t)}{\eta(t)^n}\right)>0,$$

we have

$$\ln\left(E\left(\|Y_t\|^p-\frac{\psi_3(t)}{\eta(t)^n}\right)\right)\le\ln\left(E\Phi(t,Y_t,\mathcal{L}_{Y_t})\right)-n\ln\eta(t). \tag{3.5}$$

By utilization the G-Itô's formula, we obtain

$$E\Phi(t,Y_t,\mathcal{L}_{Y_t})-E\Phi(0,\xi,\mathcal{L}_\xi)\le E\int_0^t L\Phi(\tau,Y_\tau,\mathcal{L}_{Y_\tau})\mathrm{d}\tau\ \forall\ t\ge 0.$$

Condition (ii) yields

$$E\Phi(t,Y_t,\mathcal{L}_{Y_t})\le\Phi(0,\xi,\mathcal{L}_\xi)+\int_0^t\Big(\psi_1(\tau)E\Phi(\tau,Y_\tau,\mathcal{L}_{Y_\tau})+\psi_2(\tau)E(\Phi^\mu(\tau,Y_\tau,\mathcal{L}_{Y_\tau}))+\psi_3(\tau)\Big)\mathrm{d}\tau.$$

The Gronwall–Bellman lemma in [23] deduces that

$$E\Phi(t,Y_t,\mathcal{L}_{Y_t})\le\left(\Phi(0,\xi,\mathcal{L}_\xi)+\int_0^t\psi_3(\tau)\mathrm{d}\tau+\int_0^t\psi_2(\tau)E(\Phi^\mu(\tau,Y_\tau,\mathcal{L}_{Y_\tau}))\mathrm{d}\tau\right)\cdot\exp\left(\int_0^t\psi_1(\tau)\mathrm{d}\tau\right).$$

Furthermore, the extended Gronwall Lemma 2.1 follows:

$$\begin{aligned}E\Phi(t,Y_t,\mathcal{L}_{Y_t})\le\Bigg(&\left(\Phi(0,\xi,\mathcal{L}_\xi)+\int_0^t\psi_3(\tau)\mathrm{d}\tau\right)^{1-\mu}\cdot\exp\left(\int_0^t\psi_1(\tau)\mathrm{d}\tau\right)\\&+(1-\mu)\int_0^t\psi_2(\tau)\mathrm{d}\tau\cdot\exp\left(\int_0^t\psi_1(\tau)\mathrm{d}\tau\right)\Bigg)^{\frac{1}{1-\mu}},\end{aligned}$$

which means that

$$E\Phi(t,Y_t,\mathcal{L}_{Y_t}) \le \left(\exp\Big(\int_0^t \psi_1(\tau)\mathrm{d}\tau\Big)\bigg(\Big(\Phi(0,\xi,\mathcal{L}_\xi)+\int_0^t \psi_3(\tau)\mathrm{d}\tau\Big)^{1-\mu} + (1-\mu)\int_0^t \psi_2(\tau)\mathrm{d}\tau\bigg)\right)^{\frac{1}{1-\mu}}.$$

Taking the logarithm on both sides yields

$$\ln E\Phi(t,Y_t,\mathcal{L}_{Y_t}) \le \frac{1}{1-\mu}\int_0^t \psi_1(\tau)\mathrm{d}\tau + \frac{1}{1-\mu}\ln\bigg(\Big(\Phi(0,\xi,\mathcal{L}_\xi)+\int_0^t \psi_3(\tau)\mathrm{d}\tau\Big)^{1-\mu} + (1-\mu)\int_0^t \psi_2(\tau)\mathrm{d}\tau\bigg).$$

Condition (iii) says

$$\ln E\Phi(t,Y_t,\mathcal{L}_{Y_t}) \le \ln\Big(\eta(t)^{\overline{\psi}_1+\varepsilon}\Big) + \frac{1}{1-\mu}\ln\bigg(\Big(\Phi(0,\xi,\mathcal{L}_\xi)+\eta(t)^{\overline{\psi}_3+\varepsilon}\Big)^{1-\mu} + (1-\mu)\eta(t)^{\overline{\psi}_2+\varepsilon}\bigg), \forall\varepsilon>0.$$

Letting $\varepsilon \to 0$, we derive

$$\lim_{t\to+\infty}\frac{\ln E\Phi(t,Y_t,\mathcal{L}_{Y_t})}{\ln\eta(t)} \le \overline{\psi}_1 + \overline{\psi}_2 \vee \overline{\psi}_3.$$

Inequality (3.5) tells

$$\limsup_{\substack{t\to+\infty \\ E\left(\|Y_t\|^p - \frac{\psi_3(t)}{\eta(t)^n}\right)>0}} \frac{\ln E\Big(\|Y_t\|^p - \frac{\psi_3(t)}{\eta(t)^n}\Big)}{\ln\eta(t)} \le -(n-(\overline{\psi}_1+\overline{\psi}_2\vee\overline{\psi}_3)).$$

4. Applications

We now present a specific application of Eq (1.1). Consider a system comprising a finite number of agents, where individuals interact locally and influence each other, with their behaviors governed by ordinary differential equations. As the population size within the system tends to infinity, the collective behavior and stability characteristics emerging at the macroscopic level can be characterized by the limiting dynamical behavior of Model (1.1). To analyze the limiting behavior of such systems, the following two examples are presented to verify moment exponential stability and practical exponential stability with a general decay rate, respectively.

Example 4.1. *Consider the following mean-field stochastic differential equations driven by G-Brownian motion:*

$$\begin{cases} \mathrm{d}Y_t = -\dfrac{Y_t - EY_t}{1+t}\mathrm{d}t + \dfrac{1}{(1+t)^2}\mathrm{d}B_t + 1\mathrm{d}\langle B\rangle_t, \quad t>0, \\ Y_t|_{t=0} = \xi. \end{cases} \tag{4.1}$$

Then the solution to Problem (4.1) *is exponentially stable.*

It can be verified that Problem (4.1) has a mild solution immediately. Let

$$\Phi(t,Y_t,\mathcal{L}_{Y_t}) = (1+t)^3(Y_t - EY_t)^2.$$

Then $\Phi(t,Y_t,\mathcal{L}_{Y_t})$ satisfies

$$L\Phi(t,Y_t,\mathcal{L}_{Y_t}) \le 1 + \frac{1}{1+t}\Phi(t,Y_t,\mathcal{L}_{Y_t}).$$

Applying Theorem 2.1 and Definition 2.9, we have

$$\limsup_{t\to+\infty} \frac{\log(E|Y(t)|^p)}{\log \eta(t)} \leq -2.$$

Example 4.2. *Consider the following system:*

$$\mathrm{d}Y_t = \frac{-2\mu(Y_t - EY_t)}{1+t}\mathrm{d}t + (Y_t - EY_t)\mathrm{d}B_t - 2(Y_t - EY_t)\mathrm{d}\langle B\rangle_t, \quad t \geq 0, \tag{4.2}$$

where $\mu > 1$. *Then System* (4.2) *is practical exponential stability.*

Let

$$\Phi(t, Y_t, \mathcal{L}_{Y_t}) = (1+t)^{2\mu}(Y_t - EY_t)^2.$$

Equation (3.1) indicates

$$\begin{aligned} L\Phi(t, Y_t, \mathcal{L}_{Y_t}) =& 2\mu(1+t)^{2\mu-1}(Y_t - EY_t)^2 - 4\mu(1+t)^{2\mu-1}(Y_t - EY_t)^2 \\ & - 2\int_0^1 (Y_t - EY_t)\mathcal{L}_{Y_{t(\lambda)}}(-Y_t + EY_t)\mathrm{d}\lambda \\ & + 2G\Big(- 4(1+t)^{2\mu}(Y_t - EY_t)^2 + (1+t)^{2\mu}(Y_t - EY_t)^2\Big) \\ & - 2\int_0^1 (Y_t - EY_t)\mathcal{L}_{Y_{t(\lambda)}}(-Y_t + EY_t)\mathrm{d}\lambda \\ \leq & - 2\mu(1+t)^{2\mu-1}(Y_t - EY_t)^2. \end{aligned}$$

Setting $\eta(t) = t$, $n = 2\mu$, p=2, $\overline{\psi}_1 = 0$, $\overline{\psi}_2 = 1$, and $\overline{\psi}_3 = 1$, Theorem 2.5 shows that System (4.2) is practically stable.

Remark 4.1 (Open problem)**.** *This paper establishes analytical stability criteria under general decay rates. The examples above illustrate these criteria for specific parameter values. However, the dependence of the decay exponents on the drift and diffusion coefficients has not been fully characterized. An open problem is to determine how perturbations of these coefficients affect the corresponding decay exponents. In particular, the continuity and sensitivity of the mapping from the coefficients to the decay exponents deserve further investigation. Such results may provide a quantitative characterization of the robustness of the stability criteria over a broader parameter space.*

5. Conclusions

This paper investigated the stability of mean-field stochastic differential equations driven by G-Brownian motion under general decay rates. By using distribution-dependent G-Lyapunov functions, sufficient conditions for moment exponential stability, second-moment ultimate boundedness, and partial moment stability were established, and two examples were provided to illustrate the theoretical results. Future work will focus on the sensitivity of the decay exponents to perturbations in the drift and diffusion coefficients.

Use of Generative-AI tools declaration

The author used OpenAI's ChatGPT only for language polishing, grammatical checking, and improving the clarity of expression. All scientific content, theoretical derivations, results, analyses, and conclusions were independently completed, checked, and verified by the author.

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Conflict of interest

The author declares no conflicts of interest.

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