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Research article

An $O(\nu \log \nu)$ upper bound for piercing discrete axis-parallel rectangles

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Abstract: Let $P \subset \mathbb{R}^2$ be a finite set, and let $\mathcal{R}$ be a nonempty finite family of axis-parallel rectangles, each containing at least one point of P. We study the relation between the piercing number $\tau = \tau(\mathcal{R}|_P)$ and the matching number $\nu = \nu(\mathcal{R}|_P)$ of the trace family $\mathcal{R}|_P = \{R \cap P : R \in \mathcal{R}\}$. We prove that $\tau \leq 24\nu \log_2(2\nu) = O(\nu \log \nu)$, improving the previously known upper bound $O((\nu \log \log \nu)^2)$.

Keywords: discrete axis-parallel rectangles; piercing number; matching number; (p, q)-property; geometric transversals; outerplanar graphs; treewidth
Mathematics Subject Classification: 05C10, 05C65, 05D15, 52A35, 52C15

1. Introduction

The intersection patterns of sets are widely studied in discrete geometry; see, for example, [1–3]. A classical result of this type is the (p, q)-theorem, which was formulated by Hadwiger and Debrunner in [4] and proved by Alon and Kleitman in [5].

Let $\mathcal{F}$ be a family of sets, and let $p \geq q$ be positive integers. We say that $\mathcal{F}$ satisfies the (p, q)-*property* if every subfamily of $\mathcal{F}$ of size p contains q members with a common point. A set S is called a *transversal*, or a *piercing set*, of $\mathcal{F}$ if it meets every member of $\mathcal{F}$. The minimum cardinality of such a set is the *piercing number*, denoted by $\tau(\mathcal{F})$. We write $\nu(\mathcal{F})$ for the maximum cardinality of a pairwise disjoint subfamily of $\mathcal{F}$; this is the *matching number*, or *packing number*, of $\mathcal{F}$. In this notation, the classical (p, q)-theorem can be stated as follows.

Theorem 1.1. *((p, q)-theorem) Let p, q, d be positive integers with $p \geq q \geq d + 1$. There exists a constant $c = c(p, q, d)$ such that every finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ satisfying the (p, q)-property has $\tau(\mathcal{F}) \leq c$.*

Even in the planar case, the best known quantitative bounds for $c(p, q, 2)$ are far from tight; see [6]. In this paper, we study the (p, q) problem for discrete axis-parallel rectangles and obtain an improved upper bound in the $(p, 2)$ case.

An *axis-parallel rectangle* (or simply a *rectangle*) is a nonempty compact set of the form $[a, b] \times [c, d] \subset \mathbb{R}^2$, where $a \leq b$ and $c \leq d$. Degenerate rectangles are allowed; either interval may consist of a single point. More generally, an *axis-parallel box* (or simply a *box*) in $\mathbb{R}^d$ is a Cartesian product of closed bounded intervals. In 2023, Tomon established the following upper bound on the piercing number of a finite family of boxes in terms of its matching number.

Theorem 1.2. *([7, Theorem 4.1]) Let $d \geq 2$ be an integer. There exists a constant c_d such that every finite family $\mathcal{B}$ of boxes in $\mathbb{R}^d$ satisfies*

$$\tau(\mathcal{B}) \leq c_d \nu (\log \nu)^{d-2} (\log\log \nu)^2,$$

where $\nu = \nu(\mathcal{B})$.

Conversely, lower-bound constructions show that

$$\tau = \Omega\left(\nu\left(\frac{\log \nu}{\log\log \nu}\right)^{d-2}\right);$$

see [7–9]. In particular, a finite family $\mathcal{R}$ of rectangles in $\mathbb{R}^2$ satisfies

$$\tau = O(\nu(\log\log \nu)^2),$$

while examples with $\tau = \Omega(\nu)$ are known.

A family $\mathcal{F}$ with the $(p, 2)$-property has matching number at most $p - 1$, whereas a family with matching number ν has the $(\nu + 1, 2)$-property. Let $M(p, q, d)$ denote the analogue of $c(p, q, d)$ for axis-parallel boxes. Theorem 1.2 implies that

$$M(p, 2, d) = O(p(\log p)^{d-2}(\log\log p)^2),$$

and the corresponding lower-bound constructions imply that

$$M(p, 2, d) = \Omega\left(p\left(\frac{\log p}{\log\log p}\right)^{d-2}\right).$$

We henceforth focus on the relation between τ and ν.

The discrete setting differs from the continuous piercing problem for ordinary rectangles, in which piercing points may be chosen anywhere in $\mathbb{R}^2$. Let $P \subset \mathbb{R}^d$ be finite, and let $\mathcal{B}$ be a finite family of boxes in $\mathbb{R}^d$ such that each box contains a point of P. The *trace family* of $\mathcal{B}$ on P is

$$\mathcal{B}|_P = \{B \cap P : B \in \mathcal{B}\}.$$

Thus, $\mathcal{B}|_P$ is a hypergraph with vertex set P, and its transversals are required to be subsets of P. This setting has recently received considerable attention.

Halman [10] proved a Helly-type theorem for the above setting. Edwards and Soberón [11] established colorful, fractional, and quantitative extensions of this theorem. Eom et al. [12] proved a fractional discrete Helly theorem for families of boxes in which a sufficiently high proportion of pairs intersect in the prescribed point set. Frankl and Jung [13] developed a unified approach to Helly-type theorems for monotone properties of boxes, including the property of containing a point of a prescribed set. Gangopadhyay et al. [14] obtained new quantitative colorful and (p, q)-type results for discrete boxes. The most closely related result, proved in [15], is the following.

Theorem 1.3. *([15, Theorem 1.5]) Let $P \subset \mathbb{R}^2$ be finite, and let $\mathcal{R}$ be a finite family of rectangles in $\mathbb{R}^2$ such that each rectangle contains a point of P. Then,*

$$\tau(\mathcal{R}|_P) = O((\nu \log\log \nu)^2),$$

where $\nu = \nu(\mathcal{R}|_P)$.

Moreover, there exist finite sets $P \subset \mathbb{R}^2$ and finite families $\mathcal{R}$ of rectangles in $\mathbb{R}^2$ such that each rectangle contains a point of P and

$$\tau(\mathcal{R}|_P) = \Omega(\nu \log \nu),$$

where $\nu = \nu(\mathcal{R}|_P)$.

The upper bound was obtained by combining a continuous piercing theorem with a separate discrete argument. If $\mathcal{R}|_P$ has the $(p, 2)$-property, then $\nu(\mathcal{R}) \leq p - 1$. By Theorem 1.2, $\mathcal{R}$ can be partitioned into $O(p(\log\log p)^2)$ pairwise-intersecting subfamilies. Theorem 3.1 in [15], proved via a reduction to 4-intervals and the polytopal Knaster-Kuratowski-Mazurkiewitz-Shapley theorem, pierces each such subfamily with at most $4(p - 1)$ points of P. Hence, the total bound is $O((p \log\log p)^2)$, and taking $p = \nu + 1$ gives $O((\nu \log\log \nu)^2)$.

The contribution of the present paper is the following improved upper bound.

Theorem 1.4. *Let $P \subset \mathbb{R}^2$ be finite, and let $\mathcal{R}$ be a nonempty finite family of rectangles such that $R \cap P \neq \emptyset$ for every $R \in \mathcal{R}$. Then,*

$$\tau(\mathcal{R}|_P) \leq 24\nu \log_2(2\nu),$$

where $\nu = \nu(\mathcal{R}|_P)$.

Theorem 1.4 improves the previous upper bound $O((\nu \log\log \nu)^2)$ to

$$24\nu \log_2(2\nu) = O(\nu \log \nu),$$

a near-linear bound in ν, without relying on topological arguments or on known results for the continuous piercing problem. The proof is based on two new ingredients. One is a linear bound for families in which every rectangle meets a fixed vertical line. In this setting, every nonempty trace on either side of the line is connected in an appropriate outerplanar graph. Since outerplanar graphs have treewidth at most 2, Alon's theorem on families of subgraphs of bounded-treewidth graphs yields the linear estimate. The other ingredient is a balanced vertical separator, which partitions the family into two side subfamilies, each with matching number at most half that of the original family, and a line-pierced middle subfamily. Recursion on the two side subfamilies then proves Theorem 1.4.

The rest of the paper is organized as follows. In Section 2, we prove the line-pierced estimate by constructing the two outerplanar graphs and applying Alon's bounded-treewidth theorem. In Section 3, we prove the separator lemma and complete the proof of Theorem 1.4. Section 4 contains concluding remarks and open questions.

2. The line-pierced case

The main step is the following linear estimate for families of rectangles that meet a fixed vertical line.

Lemma 2.1. *Let $P \subset \mathbb{R}^2$ be finite, and let $\mathcal{R}$ be a finite family of axis-parallel rectangles such that every rectangle contains a point of P. Assume that every rectangle in $\mathcal{R}$ intersects a fixed vertical line ℓ. Then,*

$$\tau(\mathcal{R}|_P) \leq 24\nu(\mathcal{R}|_P).$$

We use the following terminology for families of subgraphs. If $\mathcal{H}$ is a family of subgraphs of a graph G, a transversal of $\mathcal{H}$ is a set $S \subseteq V(G)$ such that $S \cap V(H) \neq \emptyset$ for every $H \in \mathcal{H}$. Its minimum cardinality is denoted by $\tau(\mathcal{H})$. The matching number $\nu(\mathcal{H})$ is the maximum cardinality of a subfamily of $\mathcal{H}$ whose members are pairwise vertex-disjoint. The following theorem will be applied to the rectangle traces.

Theorem 2.2. *([16, Theorem 1.2]) Let G be a graph of treewidth at most b, and let $\mathcal{H}$ be a finite family of subgraphs of G, each having at most r connected components. Then,*

$$\tau(\mathcal{H}) \leq 2(b+1)r^2\nu(\mathcal{H}).$$

We now construct two auxiliary graphs on P. Without loss of generality, assume that every rectangle intersects

$$\ell = \{(x, y) \in \mathbb{R}^2 : x = 0\}.$$

Let

$$P^+ = P \cap \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$$

be the set of points of P on or to the right of ℓ. Order these points by nondecreasing y-coordinate, breaking ties arbitrarily, and denote the resulting order by $p_1, p_2, \ldots, p_n$, where $p_i = (x_i, y_i)$.

Define a graph G^+ on the vertex set P^+ as follows. For $i < j$, join p_i to p_j if $x_t > \max\{x_i, x_j\}$ for every $i < t < j$. In particular, consecutive vertices are adjacent.

Claim 2.3. *The graph G^+ is outerplanar. In particular,* $\mathrm{tw}(G^+) \leq 2$.

Proof. Place $p_1, \ldots, p_n$ on a horizontal line in this order, and draw every edge p_ip_j as a semicircle above the line with the segment p_ip_j as its diameter. Two such semicircles with no common endpoint can cross only if their endpoints alternate.

Suppose that two edges cross. Then, there are indices $i < j < k < l$ such that p_ip_k and p_jp_l are edges. Since p_ip_k is an edge, $x_j > \max\{x_i, x_k\}$, and hence $x_j > x_k$. Similarly, since p_jp_l is an edge, $x_k > \max\{x_j, x_l\}$, and hence $x_k > x_j$, a contradiction. Thus, the arcs do not cross, so G^+ is outerplanar. It is known that every outerplanar graph has treewidth at most 2. □

Claim 2.4. *For every $R \in \mathcal{R}$, the induced subgraph $G^+[R \cap P^+]$ is empty or connected.*

Proof. Write $R = [a, b] \times [c, d]$. Since R intersects ℓ, we have $a \leq 0 \leq b$. Hence,

$$R \cap P^+ = \{p_i : c \leq y_i \leq d \text{ and } x_i \leq b\}.$$

Let $p_{i_1}, p_{i_2}, \ldots, p_{i_m}$, where $i_1 < i_2 < \cdots < i_m$, be the points of $R \cap P^+$. If $m \leq 1$, the assertion is immediate.

Consider consecutive trace vertices p_{i_s} and $p_{i_{s+1}}$. For every t with $i_s < t < i_{s+1}$, the nondecreasing order gives $y_{i_s} \leq y_t \leq y_{i_{s+1}}$. Both endpoint y-coordinates lie in $[c, d]$, so $y_t \in [c, d]$. This conclusion remains valid when some y-coordinates are equal, because arbitrary tie-breaking preserves the displayed inequalities. Since $p_t \notin R \cap P^+$, it follows that $x_t > b$.

Since $x_{i_s}, x_{i_{s+1}} \leq b$, we obtain $x_t > \max\{x_{i_s}, x_{i_{s+1}}\}$ for every $i_s < t < i_{s+1}$. Thus, $p_{i_s}p_{i_{s+1}}$ is an edge of G^+. The trace vertices form a path in $G^+[R \cap P^+]$, proving the claim. □

On the other side of the line, let

$$P^- = P \cap \{(x, y) \in \mathbb{R}^2 : x < 0\}.$$

Order the points of P^- by nondecreasing y-coordinate, again breaking ties arbitrarily, and denote the resulting order by $q_1, q_2, \ldots, q_m$, where $q_i = (u_i, v_i)$. Define a graph G^- on P^- by joining q_i to q_j, for $i < j$, if $u_t < \min\{u_i, u_j\}$ for every $i < t < j$.

Claim 2.5. *The graph G^- is outerplanar. In particular,* $\mathrm{tw}(G^-) \leq 2$.

Proof. Place $q_1, \ldots, q_m$ on a horizontal line in this order, and draw the edges as semicircles above the line. If two edges crossed, there would be indices $i < j < k < l$ such that q_iq_k and q_jq_l are edges. The edge q_iq_k would give $u_j < \min\{u_i, u_k\}$ and hence $u_j < u_k$, whereas q_jq_l would give $u_k < \min\{u_j, u_l\}$ and hence $u_k < u_j$, a contradiction. Thus, the drawing is noncrossing, so G^- is outerplanar and has treewidth at most 2. □

Claim 2.6. *For every $R \in \mathcal{R}$, the induced subgraph $G^-[R \cap P^-]$ is empty or connected.*

Proof. Write $R = [a, b] \times [c, d]$. Since $a \leq 0 \leq b$, a point $q_i = (u_i, v_i) \in P^-$ belongs to R precisely when $c \leq v_i \leq d$ and $u_i \geq a$. Let $q_{i_1}, \ldots, q_{i_s}$ be the trace vertices in increasing index order. For two consecutive trace vertices q_{i_j} and $q_{i_{j+1}}$, every intermediate index t satisfies $v_{i_j} \leq v_t \leq v_{i_{j+1}}$, and hence $v_t \in [c, d]$. As above, arbitrary tie-breaking among equal v-coordinates does not affect this conclusion.

Since $q_t \notin R \cap P^-$, it follows that $u_t < a$. The endpoints satisfy $u_{i_j}, u_{i_{j+1}} \geq a$, so $u_t < \min\{u_{i_j}, u_{i_{j+1}}\}$. Consequently, consecutive trace vertices are adjacent in G^- and form a path. This proves the claim. □

Proof of Lemma 2.1. Let

$$G = G^+ \sqcup G^-.$$

By Claims 2.3 and 2.5, both G^+ and G^- are outerplanar and therefore have treewidth at most 2. Since the treewidth of a disjoint union is the maximum of the treewidths of its components (see, e.g., [17, Corollary 8]) and G^+, G^- are connected, it follows that $\mathrm{tw}(G) \leq 2$.

For each $R \in \mathcal{R}$, let

$$Q_R = G[R \cap P].$$

Since

$$P = P^+ \sqcup P^-,$$

we have

$$Q_R = G^+[R \cap P^+] \sqcup G^-[R \cap P^-].$$

Claims 2.4 and 2.6 imply that each subgraph on the righthand side is either empty or connected. Consequently, Q_R has at most two connected components. Moreover, by definition,

$$V(Q_R) = R \cap P.$$

Let

$$\mathcal{Q} = \{Q_R : R \in \mathcal{R}\}.$$

Since

$$V(G) = P \quad \text{and} \quad V(Q_R) = R \cap P,$$

a subset of P meets every member of $\mathcal{Q}$ if and only if it meets every trace in $\mathcal{R}|_P$. Hence,

$$\tau(\mathcal{Q}) = \tau(\mathcal{R}|_P).$$

Similarly, two members $Q_R, Q_{R'}$ are vertex-disjoint if and only if the corresponding traces $R \cap P$ and $R' \cap P$ are disjoint, so

$$\nu(\mathcal{Q}) = \nu(\mathcal{R}|_P).$$

The preceding observations verify the hypotheses of Theorem 2.2 with $b = 2$ and $r = 2$. Therefore,

$$\tau(\mathcal{R}|_P) = \tau(\mathcal{Q}) \leq 2(b+1)r^2\nu(\mathcal{Q}) = 2(2+1)2^2\nu(\mathcal{Q}) = 24\nu(\mathcal{R}|_P).$$

This completes the proof. □

3. Proof of Theorem 1.4

For $x \in \mathbb{R}$, let

$$\ell_x = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x\}.$$

For a rectangle $R = [a,b] \times [c,d]$, we say that R lies *strictly to the left* of ℓ_x if $b < x$ and *strictly to the right* of ℓ_x if $a > x$. For a finite family $\mathcal{R}$, define

$$\begin{aligned}
\mathcal{R}^-_{\ell_x} &= \{R \in \mathcal{R} : R \text{ lies strictly to the left of } \ell_x\},\\
\mathcal{R}^+_{\ell_x} &= \{R \in \mathcal{R} : R \text{ lies strictly to the right of } \ell_x\},\\
\mathcal{R}^0_{\ell_x} &= \{R \in \mathcal{R} : R \cap \ell_x \neq \emptyset\}.
\end{aligned}$$

These three families form a partition of $\mathcal{R}$. In particular, a rectangle of zero width contained in ℓ_x belongs to $\mathcal{R}^0_{\ell_x}$.

Lemma 3.1. *Let $P \subset \mathbb{R}^2$ be finite, and let $\mathcal{R}$ be a finite family of axis-parallel rectangles such that $R \cap P \neq \emptyset$ for every $R \in \mathcal{R}$. Let $k = \nu(\mathcal{R}|_P)$. Then, there exists a vertical line ℓ such that*

$$\nu(\mathcal{R}^-_\ell|_P) \leq \frac{k}{2}, \quad \nu(\mathcal{R}^+_\ell|_P) \leq \frac{k}{2}, \quad \textit{and} \quad \nu(\mathcal{R}^-_\ell|_P) + \nu(\mathcal{R}^+_\ell|_P) \leq k.$$

Proof. If $k = 0$, then $\mathcal{R}$ is empty and any vertical line satisfies the conclusion. Assume henceforth that $k \geq 1$. Define the integer-valued functions

$$a(x) = \nu(\mathcal{R}^-_{\ell_x}|_P) \quad \text{and} \quad b(x) = \nu(\mathcal{R}^+_{\ell_x}|_P).$$

If $x \leq y$, then $\mathcal{R}^-_{\ell_x} \subseteq \mathcal{R}^-_{\ell_y}$ and $\mathcal{R}^+_{\ell_y} \subseteq \mathcal{R}^+_{\ell_x}$. Hence, a is nondecreasing and b is nonincreasing.

For every x, we also have $a(x) + b(x) \leq k$. Indeed, choose a matching of size $a(x)$ in $\mathcal{R}^-_{\ell_x}|_P$ and a matching of size $b(x)$ in $\mathcal{R}^+_{\ell_x}|_P$. Every rectangle strictly to the left of ℓ_x is disjoint from every rectangle strictly to its right. Consequently, the union of these two matchings is a matching in $\mathcal{R}|_P$.

Let $A = \{x \in \mathbb{R} : b(x) \le k/2\}$. Since $\mathcal{R}$ is finite and its rectangles are bounded, every rectangle lies strictly to the right of ℓ_x for all sufficiently small x, so $b(x) = k > k/2$. For all sufficiently large x, no rectangle lies strictly to the right of ℓ_x, so $b(x) = 0$. Thus, A is nonempty and bounded below. Let $x_0 = \inf A$.

We claim that $b(x_0) \le k/2$. Suppose instead that $b(x_0) > k/2$. Then, there exists a subfamily $\mathcal{M} \subseteq \mathcal{R}^+_{\ell_{x_0}}$ of more than $k/2$ rectangles whose traces are pairwise disjoint. Write each $R \in \mathcal{M}$ as

$$R = [\alpha_R, \beta_R] \times [c_R, d_R].$$

Since $\alpha_R > x_0$ for every $R \in \mathcal{M}$ and $\mathcal{M}$ is finite, there exists $\varepsilon > 0$ such that $x_0 + \varepsilon < \alpha_R$ for every $R \in \mathcal{M}$.

Thus, $\mathcal{M} \subseteq \mathcal{R}^+_{\ell_y}$ for every $y \in [x_0, x_0 + \varepsilon]$, and hence $b(y) > k/2$ throughout this interval. On the other hand, $x_0 = \inf A$ and, under the present supposition, $x_0 \notin A$. Therefore, there exists $z \in A$ such that $x_0 < z < x_0 + \varepsilon$, contradicting $b(z) \le k/2$. Hence, $b(x_0) \le k/2$.

We also claim that $a(x_0) \le k/2$. Suppose instead that $a(x_0) > k/2$. Choose a subfamily $\mathcal{N} \subseteq \mathcal{R}^-_{\ell_{x_0}}$ of more than $k/2$ rectangles whose traces are pairwise disjoint. Writing each $R \in \mathcal{N}$ as

$$R = [\alpha_R, \beta_R] \times [c_R, d_R],$$

we have $\beta_R < x_0$ for every $R \in \mathcal{N}$. Since $\mathcal{N}$ is finite, we may choose y such that $\max_{R \in \mathcal{N}} \beta_R < y < x_0$.

Then, $\mathcal{N} \subseteq \mathcal{R}^-_{\ell_y}$, so $a(y) > k/2$. Since $y < x_0 = \inf A$, we have $y \notin A$, and hence $b(y) > k/2$. This contradicts the inequality $a(y) + b(y) \le k$. Therefore, $a(x_0) \le k/2$. The remaining inequality follows by evaluating $a(x) + b(x) \le k$ at $x = x_0$. Taking $\ell = \ell_{x_0}$ proves the lemma. □

Proof of Theorem 1.4. We argue by induction on $k = \nu(\mathcal{R}|_P)$. Since $\mathcal{R}$ is nonempty and every trace is nonempty, $k \ge 1$. If $k = 1$, Lemma 3.1 yields a vertical line ℓ such that the matching numbers of $\mathcal{R}^-_\ell|_P$ and $\mathcal{R}^+_\ell|_P$ are at most $1/2$. Since matching numbers are integers and every rectangle has a nonempty trace, both side families are empty. Thus, every rectangle meets ℓ, and Lemma 2.1 gives

$$\tau(\mathcal{R}|_P) \le 24 = 24 \cdot 1 \cdot \log_2 2.$$

Now let $k \ge 2$, and assume that the result holds for every nonempty rectangle family satisfying the hypotheses and having trace matching number smaller than k. Choose a vertical line ℓ as in Lemma 3.1, and consider the partition

$$\mathcal{R} = \mathcal{R}^-_\ell \sqcup \mathcal{R}^0_\ell \sqcup \mathcal{R}^+_\ell.$$

Set

$$k_- = \nu(\mathcal{R}^-_\ell|_P) \quad \text{and} \quad k_+ = \nu(\mathcal{R}^+_\ell|_P).$$

Lemma 3.1 gives $k_-, k_+ \le k/2$ and $k_- + k_+ \le k$. In particular, each positive value among k_- and k_+ is smaller than k.

Every rectangle in the middle family meets ℓ. Moreover, $\mathcal{R}^0_\ell \subseteq \mathcal{R}$, so the monotonicity of the matching number under passage to a subfamily gives $\nu(\mathcal{R}^0_\ell|_P) \le k$. Lemma 2.1 therefore yields

$$\tau(\mathcal{R}^0_\ell|_P) \le 24\nu(\mathcal{R}^0_\ell|_P) \le 24k.$$

If $k_\sigma = 0$ for some $\sigma \in \{-,+\}$, then $\mathcal{R}_\ell^\sigma$ is empty, and requires no piercing points. For each $\sigma \in \{-,+\}$ with $k_\sigma > 0$, the induction hypothesis gives

$$\tau(\mathcal{R}_\ell^\sigma|_P) \leq 24k_\sigma \log_2(2k_\sigma).$$

Taking the union of transversals for the three subfamilies gives

$$\tau(\mathcal{R}|_P) \leq 24 \sum_{\substack{\sigma\in\{-,+\} \\ k_\sigma>0}} k_\sigma \log_2(2k_\sigma) + 24k.$$

For every positive k_σ, the inequality $k_\sigma \leq k/2$ implies $\log_2(2k_\sigma) \leq \log_2 k$. Consequently,

$$\begin{aligned}\tau(\mathcal{R}|_P) &\leq 24(k_- + k_+) \log_2 k + 24k \\ &\leq 24k \log_2 k + 24k \\ &= 24k \log_2(2k),\end{aligned}$$

which completes the induction. □

4. Concluding remarks

The proof of the $O(\nu \log \nu)$ upper bound has two structural ingredients. In the line-pierced case, every nonempty one-sided trace is connected in an auxiliary graph. The disjoint union of the two auxiliary graphs has treewidth at most 2, which yields a linear estimate. The balanced vertical separator then reduces a general family to two side subfamilies with smaller matching numbers and a line-pierced middle subfamily, giving the logarithmic recursion.

The numerical factor 24 in the line-pierced estimate follows directly from Alon's general theorem: the auxiliary graph has treewidth parameter $b = 2$, and each trace subgraph has at most $r = 2$ connected components, so $2(b+1)r^2 = 24$. We do not claim that this constant is optimal; the best constant remains unknown. It is natural to ask whether the special outerplanar structure of the auxiliary graphs yields a smaller factor than the general bounded-treewidth theorem. This numerical question is distinct from the asymptotic $O(\nu \log \nu)$ form of the main bound.

From an algorithmic perspective, the discrete formulation may also be relevant to geometric hitting-set problems in which the admissible piercing points are restricted to a finite candidate set. Related algorithmic questions concerning independent and hitting sets of rectangles are studied in [8]. The present paper is structural rather than algorithmic, and we do not pursue these computational aspects here.

Further directions include finding a useful analogue of the line-pierced lemma for discrete axis-parallel boxes in higher dimensions and determining whether additional assumptions on the point set P or the rectangle family yield stronger bounds. A direct higher-dimensional extension is not immediate: after a cut by a hyperplane, the one-sided traces already contain lower-dimensional box-trace problems and no longer reduce to a single ordered coordinate and a single threshold.

Use of Generative-AI tools declaration

The author declares that the use of AI tools is limited to language editing and checking the logical consistency of certain mathematical arguments. All scientific content, proofs, results, and conclusions were developed and independently verified by the author, who takes full responsibility for the work.

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Conflict of interest

The author declares no conflict of interest.

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