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Research article

Geometric and analytic properties of a q-Lune-type subclass of univalent functions defined by q-calculus

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Abstract: In this paper, we introduced and explored a new subclass $\mathcal{R}_L^q$ of univalent functions in the open unit disk $\mathbb{E}$ related to the q-difference operator and newly define q-Lune-type domain. The class is defined by the subordination condition

$$D_q f(z) \prec z + \sqrt{1 + \beta(q) z^2}, \qquad 0 < q < 1,$$

where $\beta(q) = \frac{\log(1+q)}{\log 2}$. First, we showed that this class is non-empty for both identity and non-identity functions. Using techniques from subordination theory and Carathéodory functions, we derived sharp coefficient estimates and obtained bounds of the second and third Hankel determinant. Furthermore, we investigated logarithmic coefficients γ_1, γ_2, and γ_3. In addition, graphical illustrations were provided to demonstrate the geometric behavior of the class and to verify the subordination condition. The results presented here extend several results when $q \to 1^-$ and contribute to the growing theory of analytic and univalent function classes.

Keywords: q-calculus; q-difference operator; subordination; Hankel determinant; logarithmic coefficients; analytic and univalent functions; q-Lune-type domain
Mathematics Subject Classification: Primary 05A30, 30C45; Secondary 11B65, 47B38

1. Introduction

Geometric function theory (GFT) is a substantial field of complex analysis that concentrates on the geometric behavior of analytic and univalent functions. Classical subclasses, for example, starlike, convex, and close-to-convex functions have been widely studied due to their structural significance and applications in different fields (see [1, 2]). A central topic in GFT is the study of coefficient problems and functional inequalities connected with the theory of analytic functions and among these, Hankel determinants and logarithmic coefficients have drawn notable focus in recent years. The Hankel determinant gives important facts regarding the growth and geometric properties of analytic functions (AFs), while logarithmic coefficients play a crucial role in studying univalence and distortion properties (see [3–5]). In the theory of AFs, coefficient problems have taken a central position, as they offer key tools for estimating and understanding the behavior of analytic functions. These problems cover a broad range of topics, including coefficient bounds, covering theorems, Hankel and Toeplitz determinants, necessary and sufficient conditions, along with multiple coefficient results and conjectures. The Hankel matrix and its determinant play a useful role in this field [6]. It is a square matrix characterized by constant entries along each skew-diagonal. In geometric function theory, this matrix is constructed using the coefficients of power series expansions of analytic functions.

Let f be an analytic function in $E = \{z \in \mathbb{C} : |z| < 1\}$ satisfying the normalized conditions $f(0) = 0$ and $f'(0) = 1$. A function f belongs to the set of analytic functions $\mathcal{A}$ which admits the Taylor series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in E. \tag{1.1}$$

A function f defined in E will be univalent if it is injective on E, that is, whenever

$$f(z_1) = f(z_2), \quad z_1, z_2 \in E, \tag{1.2}$$

it necessarily follows that $z_1 = z_2$. The class $\mathcal{S}$ is defined by:

$$\mathcal{S} = \{f \in \mathcal{A} \text{ and } f \text{ is univalent in } E\}.$$

A domain is called starlike if each point in the region can be connected to w_0 without leaving the domain. We say a function is starlike if the image of E through the function f is starlike with respect to the origin.

Analytically, let $f \in \mathcal{S}^*$ if the following property holds:

$$0 < \Re\left(\frac{zf'(z)}{f(z)}\right), \quad z \in E.$$

The next very important family is the class $\mathcal{P}$, made up of analytic functions p in E that are normalized by $h(0) = 1$ and having a positive real part in E, which is,

$$\mathcal{P} = \{h : h(0) = 1, \text{ also } \Re(h(z)) > 0, \ z \in E\}.$$

Such functions admit the Taylor series form defined below:

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{1.3}$$

Let f_1 and f_2 be analytic in $\mathbb{E}$. Then f_1 is subordinate to f_2, represented as $f_1(z) \prec f_2(z)$, for some Schwarz function $v(z)$ that satisfies $v(0) = 0$, $|v(z)| < 1$, and $f_1(z) = f_2(v(z))$. This idea is broadly used to investigate the new subclasses of univalent functions with suggested properties related to geometry.

Logarithmic coefficients, defined by the series

$$\log \frac{f(z)}{z} = 2\sum_{k=1}^{\infty} \gamma_k z^k, \qquad z \in \mathbb{U}, \tag{1.4}$$

play a useful role in the study of the geometric characteristics of univalent functions. These coefficients, investigated in the context of univalent function theory by Duren and Leung [7], have been widely studied for various subclasses of analytic and univalent functions. Although the Koebe function $k(z) = z(1-z)^{-2}$ satisfies $\gamma_k = 1/k$, sharp estimates for logarithmic coefficients of a general function in the set of univalent functions remain known only for the first few coefficients. Especially, considerable attention has been given to obtaining sharp bounds for the second and third logarithmic coefficients and their associated Hankel determinants [8,9]. More recently, Kowalczyk and Lecko [10] investigated Hankel determinants of logarithmic coefficients for starlike and convex classes. Further inestigations include sharp estimates for strongly starlike and strongly convex functions [11], as well as results for subclasses of univalent and close-to-convex functions [12]. These studies establised the continued importance of logarithmic coefficients and their Hankel determinants in GFT and provide motivation for investigating analogous problems for the class of AFs considered in the present work.

The jth Hankel determinant (HD) of $f \in \mathcal{A}$ is defined as:

$$H_{j,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+j-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+j-1} & a_{n+j-2} & \cdots & a_{n+2j-2} \end{vmatrix}, \quad j, n \in \mathbb{N}.$$

A primary objective in the study of Hankel determinants is to investigate their sharp upper bounds. For particular choices of j and n, one obtains well-known special cases:

$$H_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_1a_3 - a_2^2, \quad H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$

The functional $H_{2,1}$ corresponds to the classical Fekete–Szegö problem, while $H_{2,2}$ is refereed to a second Hankel determinant (SHD). Furthermore, the third-order Hankel determinant (THD) is defined as:

$$H_{3,1}(f) = a_3(a_2a_4 - a_3^2) + a_5(a_3 - a_2^2) - a_4(a_4 - a_2a_3). \tag{1.5}$$

Consequently, the following estimate holds:

$$|H_{3,1}(f)| \leq |a_5||a_3 - a_2^2| + |a_4||a_4 - a_2a_3| + |a_3||H_{2,2}(f)|. \tag{1.6}$$

In parallel, the theory of q-calculus, also refereed as quantum calculus, has become an important technique in modern analysis. The q-difference operator generalizes the classical derivative and allows the development of q-analogues of many classical results. This operator has been widely applied in GFT to define new subclasses of univalent functions and to investigate their properties (see [13–15]).

Recent years have been significant progress in the study of Hankel determinant and logarithmic coefficients for various classes of $\mathcal{A}$. In this direction, Panigrahi et al. [16] investigated $H_{2,1}$ and the inverse of $H_{2,1}$ for certain subclasses of univalent functions. Mamon et al. [17] also obtained sharp estimates of $H_{2,1}$ and $H_{3,1}$ as well as logarithmic coefficients for a well-defined family of analytic functions connected with a domain bounded by lung-shaped curve, which highlights the geometric influence of such domains. Furthermore, Sabir et al. [18] studied $H_{2,1}$ and $H_{3,1}$ for a new set of symmetric bi-univalent functions (BUFs) investigated via Ruscheweyh operators, extending several earlier results in the literature. In another recent contribution, Nawaz et al. [19] further refined sharpness estimates of $H_{2,1}$ and logarithmic coefficients for analytic functions related to specific geometric domains. In addition, Sokół et al. [20] introduced and investigated λ-pseudo starlike functions connected with vertical strip domains and derived important coefficient bounds. Alsoboh et al. [21] established Fekete-Szegö results for a newly defined subfamily of bi-univalent functions classes linked with q-ultra-spherical functions. Moreover, using the cosine function, Marimuthu et al. [22] defined subclasses of S^* and C, and obtained coefficient estimates for a new set of starlike and convex functions connected with q-calculus. These studies demonstrate the growing interest in obtaining sharp coefficient estimates, Hankel determinants, and logarithmic coefficient bounds for various function classes associated with different geometric domains and operators.

For $0 < q < 1$, the q-difference operator D_q can be defined as follows:

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad z \neq 0,$$

with $D_q f(0) = f'(0)$. Utilizing this operator, many authors have investigated the subclasses q-starlike ($q - S^*$) and q-convex ($q - C$), and examined their geometric properties (see [23–25]). These investigations determined that q-calculus provides an adjustable framework for extending classical problems in geometric function theory.

In current years, q-calculus (quantum calculus) has gained significant attention as a generalization of classical calculus without limits. The q-difference operator plays a central role in this theory and has been widely used to define a new subfamily of univalent functions and to study their geometric characteristics (see [13, 26]). The incorporation of q-operators into GFT has led to the development of various q-analogues of classical function classes. Recently, there has been increasing interest in the study of Hankel determinant and logarithmic coefficients for a subfamily of starlike functions defined via q-operators. In particular, sharp bounds for $H_{2,1}$ and $H_{3,1}$ have been obtained for various subclasses of $q - S^*$ and $q - C$ using subordination techniques and Carathéodory function methods [27–29]. Recent studies have investigated Hankel determinants for various subclasses of analytic functions and established corresponding coefficient estimates using suitable linear operators [30] and subclasses of analytic functions associated with Mathieu-type series have been studied, yielding new coefficient inequalities and related geometric properties [31]. Furthermore, several new subclasses defined via special functions and q-operators have been investigated, where sharp coefficient estimates and Fekete–Szegö type results were established. Recent works have explored subclasses associated with symmetric domains and special mappings, leading to improved bounds for analytic function coefficients, logarithmic coefficients, and higher-order Hankel determinants [32–34]. These developments demonstrate that the combination of q-calculus and subordination theory provides a powerful framework for studying geometric characteristics of analytic and univalent functions.

Motivated by the above developments, and inspired by recent works on Hankel determinants in analytic function theory (see [27, 28, 32]), we study a new subfamily of AFs related to the q-Lune-type mapping.

Definition 1.1. The function $\Phi_q : E \to \mathbb{C}$ is defined by

$$\Phi_q(z) = z + \sqrt{1 + \beta(q)z^2}, \qquad \beta(q) = \frac{\log(1+q)}{\log 2}, \quad 0 < q < 1. \tag{1.7}$$

This function takes E onto a symmetric q-Lune-shaped region and it is fundamental to defining our class.

Remark 1.1. *The mapping*

$$\Phi_q(z) = z + \sqrt{1 + \beta(q)z^2}$$

transforms E onto a symmetric q-Lune-shaped domain. The geometry of the image region depends on the parameter q, where increasing q widens the lune, (see Figure 1).

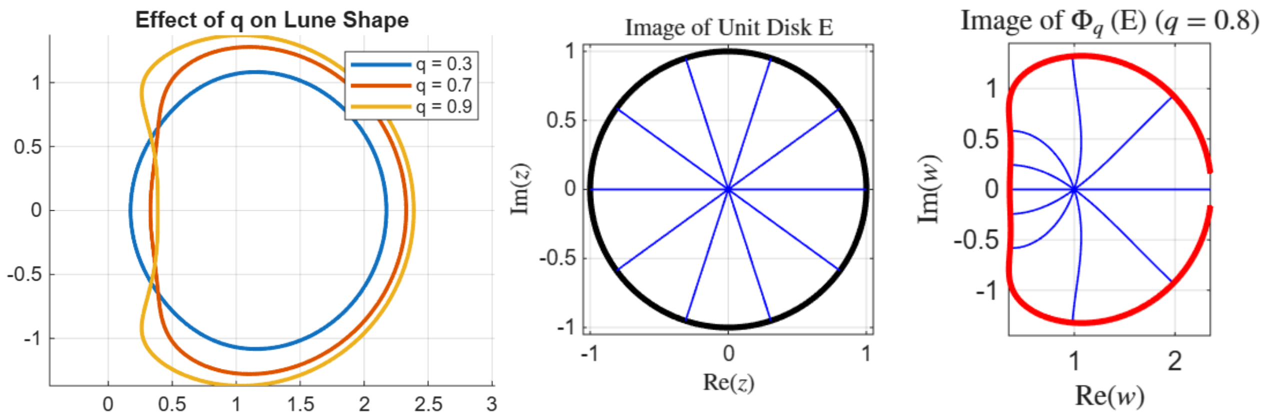

Figure 1. Left: Effect of q on the lune-shaped domain. Right: Mapping of the unit disk $\mathbb{E}$ under $\Phi_q(z)$ for $q = 0.8$.

Here, we determine a new subfamily $\mathcal{R}_L^q$ of analytic functions defined by using the subordination condition.

Definition 1.2. A univalent function $f \in \mathcal{R}_L^q$ if

$$D_q f(z) \prec \Phi_q(z), \quad z \in E, \tag{1.8}$$

where the function $\Phi_q(z)$ is defined in (1.7).

1.1. Non-emptiness of the class $\mathcal{R}_L^q$

We first establish rigorously that the class $\mathcal{R}_L^q$ is non-empty. Recall that, for $0 < q < 1$,

$$\Phi_q(z) = z + \sqrt{1 + \beta(q)z^2}, \qquad \beta(q) = \frac{\log(1+q)}{\log 2},$$

and that

$$\mathcal{R}_L^q = \left\{ f \in \mathcal{A} : D_q f(z) \prec \Phi_q(z) \right\}.$$

We have

$$0 < q < 1 \quad \Longrightarrow \quad 1 < 1 + q < 2,$$

and therefore

$$0 < \log(1+q) < \log 2.$$

Consequently,

$$0 < \beta(q) = \frac{\log(1+q)}{\log 2} < 1. \tag{1.9}$$

Moreover,

$$\Phi_q(0) = 1.$$

Since $0 < \beta(q) < 1$, the function

$$1 + \beta(q) z^2$$

does not vanish in E, because for $|z| < 1$,

$$\left|\beta(q) z^2\right| < \beta(q) < 1.$$

Hence the analytic branch of

$$\sqrt{1 + \beta(q) z^2},$$

which satisfies

$$\sqrt{1 + \beta(q) 0^2} = 1$$

is well-defined and analytic in E. Thus Φ_q is analytic in E.

Construction of an explicit member of $\mathcal{R}_L^q$:

Consider the identity function

$$f_0(z) = z.$$

Clearly,

$$f_0 \in \mathcal{A}.$$

Indeed, f_0 is analytic in E and satisfies $f_0(0) = 0$, and

$$f_0'(0) = 1.$$

For the q-difference operator

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \qquad z \neq 0,$$

we obtain

$$D_q f_0(z) = \frac{z - qz}{(1-q)z} = 1, \qquad z \neq 0.$$

At $z = 0$, the value is understood by analytic continuation, and hence

$$D_q f_0(0) = f_0'(0) = 1.$$

Therefore,

$$D_q f_0(z) \equiv 1, \quad z \in E. \tag{1.10}$$

Now we verify the subordination condition. Define

$$v_0(z) \equiv 0, \quad z \in E.$$

Then v_0 is a Schwarz function, since

$$v_0(0) = 0, \qquad |v_0(z)| = 0 < 1 \quad (z \in E).$$

Furthermore, using $\Phi_q(0) = 1$, we have

$$\Phi_q(v_0(z)) = \Phi_q(0) = 1.$$

Combining this identity with (1.10), we obtain

$$D_q f_0(z) = \Phi_q(v_0(z)).$$

Thus,

$$D_q f_0(z) \prec \Phi_q(z).$$

Consequently,

$$f_0 \in \mathcal{R}_L^q.$$

Therefore,

$$\mathcal{R}_L^q \neq \emptyset, \qquad 0 < q < 1. \tag{1.11}$$

Now, we also construct a non-trivial member of $\mathcal{R}_L^q$. Let $v(z) = z^2$. Then v is a Schwarz function in E. Since

$$v(0) = 0, \quad |v(z)| = |z|^2 < 1 \quad (z \in E).$$

Define f_1 by prescribing its q-derivative as

$$D_q f_1(z) = \Phi_q(v(z)) = z^2 + \sqrt{1 + \beta(q) z^4}. \tag{1.12}$$

Since $|\beta(q) z^4| < 1$ for $z \in E$, the function

$$\sqrt{1 + \beta(q) z^4}$$

is analytic in E. Its Taylor expansion is

$$\sqrt{1 + \beta(q) z^4} = 1 + \frac{\beta(q)}{2} z^4 - \frac{\beta(q)^2}{8} z^8 + \cdots .$$

Consequently,

$$D_q f_1(z) = 1 + z^2 + \frac{\beta(q)}{2} z^4 - \frac{\beta(q)^2}{8} z^8 + \cdots . \tag{1.13}$$

Now let

$$f_1(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Since

$$D_q f_1(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},$$

the comparison of coefficients in (1.13) gives

$$a_2 = 0, \qquad a_3 = \frac{1}{[3]_q}, \qquad a_4 = 0, \qquad a_5 = \frac{\beta(q)}{2[5]_q},$$

and, more generally, all coefficients are obtained uniquely by division by the non-zero numbers $[n]_q$, $n \geq 2$.

Since $0 < q < 1$, we have

$$[n]_q = \frac{1 - q^n}{1 - q} > 0, \qquad n \geq 1.$$

Thus the coefficient construction is well-defined. Moreover, the analytic right-hand side of (1.12) has a Taylor series with a radius of convergence of at least 1. Hence the corresponding power series for f_1 is analytic in E.

By construction,

$$D_q f_1(z) = \Phi_q(z^2) = \Phi_q(v(z)),$$

where $v(z) = z^2$ is a Schwarz function. Therefore,

$$D_q f_1(z) \prec \Phi_q(z),$$

and hence $f_1 \in \mathcal{R}_L^q$. Since

$$a_3 = \frac{1}{[3]_q} \neq 0,$$

the function f_1 is non-trivial and is different from the identity function. This proves rigorously that

$$\mathcal{R}_L^q \neq \emptyset \quad \text{for every } 0 < q < 1.$$

The paper consists of five sections. In Section 1, we discussed the required background and define the function $\Phi_q(z)$. We establish that this function maps E onto a q-Lune-type domain for various values of q. Motivated by this mapping, we define a new class of analytic functions, denoted by $\mathcal{R}_L^q$, associated with $\Phi_q(z)$. In Section 2, we investigate the geometric interpretation of the class $\mathcal{R}_L^q$. In particular, we demonstrate that the class is non-empty for both trivial and non-trivial cases, supported by graphical illustrations using the Maple program. Section 3 provides the auxiliary results that are crucial for the subsequent analysis. In Section 4, we investigate coefficient estimates, Hankel determinants, and bounds for logarithmic coefficients. In Section 5 is devoted to the conclusion.

2. Geometric interpretation of the class $\mathcal{R}_L^q$

In this section, we study the geometric effect of the class $\mathcal{R}_L^q$ by using MATLAB-generated figures for different values of q. These figures illustrate the image of E under the q-difference operator and confirm the subordination condition geometrically. The effect of the parameter q on the deformation of the image domain is also highlighted. In addition to the analytical results, we provide graphical illustrations to demonstrate the geometric behavior of the class $\mathcal{R}_L^q$. These figures confirm that the image of E under the q-difference operator remains inside the region determined by $\Phi_q(z)$, thereby validating the subordination condition. Further, we investigate several important characteristics of this class, including coefficient estimates, and second and third Hankel determinants. In particular, we obtain some sharp bounds and establish new inequalities for logarithmic coefficients.

Figure 2 represent the image of init disk and Figure 3 determined the image of E by using concentric circles and radial lines. This geometric framework helps as the base domain for the theory of AFs in the class $\mathcal{R}_L^q$ and gives insight into how mappings act on the unit disk.

Image of the unit disk

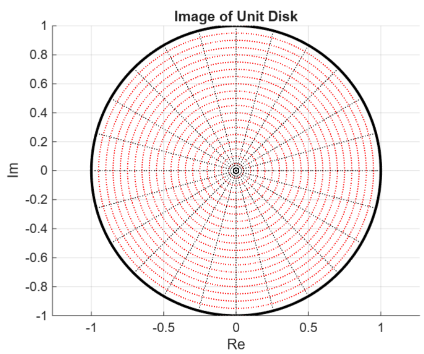

Figure 2. Geometry of the unit disk.

Identity case

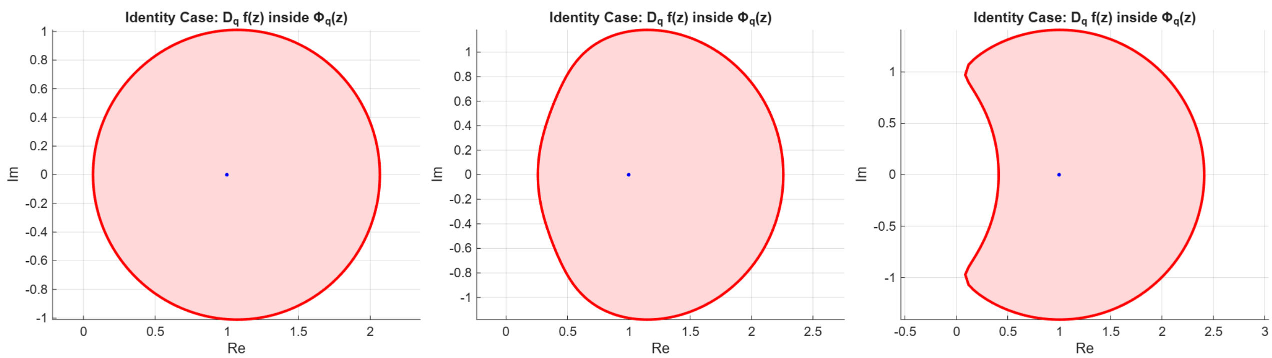

Figure 3. $D_qf(z)$ lies inside $\Phi_q(z)$ for different $q = 0.1, 0.5, 0.9$.

Here, we let the identity function $f(z) = z$. The q-difference operator creates a constant-type image, shown like a point inside the region bounded by $\Phi_q(z) = z + \sqrt{1+\beta(q)z^2}$. The shaded region signifies the image domain of Φ_q. These figures confirm that the subordination condition

$$D_qf(z) \prec \Phi_q(z)$$

holds. Thus, the class $\mathcal{R}_L^q$ is non-empty.

Figure 4 determine the non-trivial case of the class $\mathcal{R}_L^q$. The color gradient represents the radial parameter $|z|$, displaying how the unit disk E is transformed under this mapping. The image stays entirely inside the boundary defined by $\Phi_q(z)$, illustrating that the subordination condition holds for the non-trivial function. As q increases, the deformation of the image becomes more distinct.

Non-trivial case

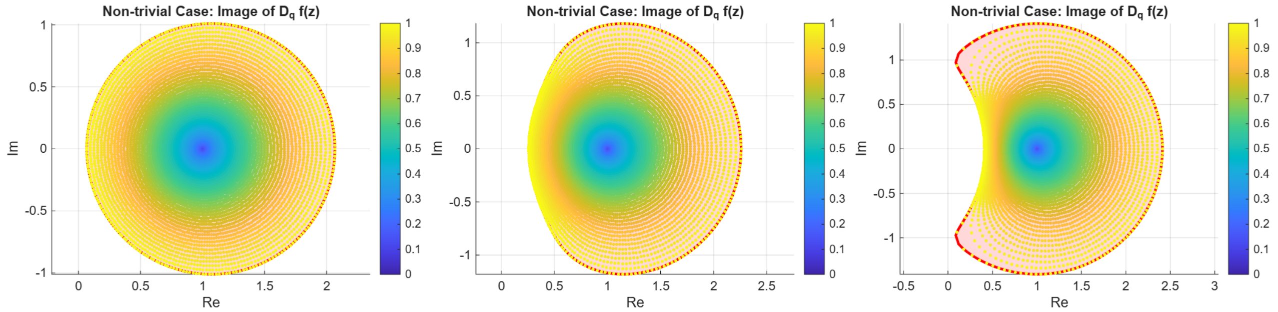

Figure 4. Non-trivial case: $D_qf(z)$ lies inside the class $\mathcal{R}_L^q$ for different values of $q = 0.1, 0.5, 0.9$.

Figure 5 determine the direct representation of the subordination condition. The blue points indicate the image of $D_qf(z)$, while the red boundary corresponds to the region defined by $\Phi_q(z)$. It is obviously seen that the whole image of $D_qf(z)$ lies inside the target domain for all values of q. This verifies that the geometric validity of the class $\mathcal{R}_L^q$ and highlights the role of the parameter q in controlling the shape of the image domain.

Subordination visualization

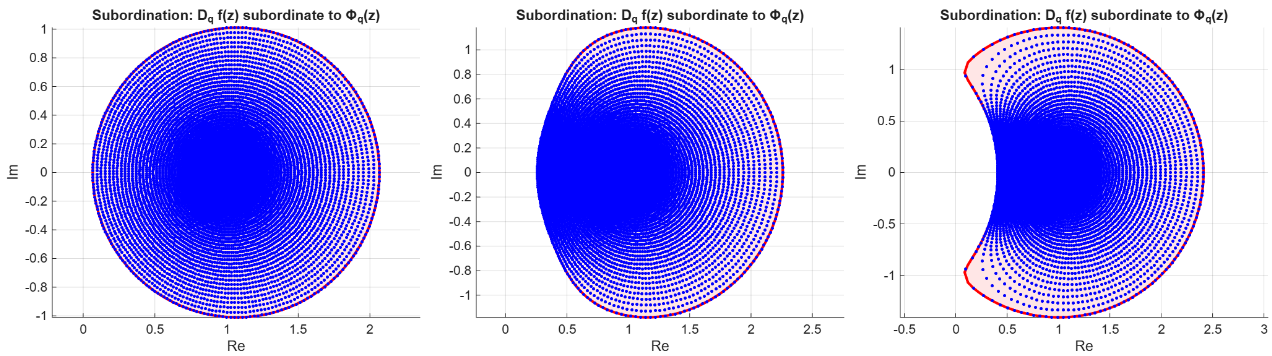

Figure 5. Subordination visualization: $D_qf(z) \prec \Phi_q(z)$.

3. Set of lemmas

Lemma 3.1. *[1, 35] If $h(z)$ is defined by* (1.3)*, then*

$$c_3 = \frac{1}{4}\left[2(4-c_1^2)c_1x - (4-c_1^2)c_1x^2 + 2(4-c_1^2)(1-|x|^2)z + c_1^3\right]$$

for some z with $|z| \leq 1$ and

$$c_2 = \frac{1}{2}\left[x(4-c_1^2) + c_1^2\right].$$

Lemma 3.2. *Let h be the function defined in* (1.3)*. Then*

$$\left|Qc_1^3 - 2Pc_1c_2 + c_3\right| \leq 2,$$

where $0 \leq P \leq 1$ *and* $P(2P-1) \leq Q \leq P$*.*

Lemma 3.3. *[1] Let h be the function defined in* (1.3)*. Then*

$$|c_n| \leq 2 \quad (n \in \mathbb{N})$$

and

$$|c_n - \mu c_i c_{n-i}| \leq 2, \quad n > i, \ \mu \in [0,1]. \tag{3.1}$$

Lemma 3.4. *[36] Let h be the function defined in* (1.3)*, where* $0 < b < 1$ *and* $0 < d < 1$*. If*

$$8b(1-b)\left\{(dl - 2\Lambda)^2 + (d(b+d) - l)^2\right\} + d(1-d)(l - 2bd)^2 \leq 4d^2b(1-d)^2(1-b),$$

then

$$\left|bc_2^2 + 2dc_1c_3 - \frac{3}{2}lc_1^2c_2 - c_4 + \Lambda c_1^4\right| \leq 2.$$

Lemma 3.5. *[37] Let* $N, M, O \in R$ *and*

$$Y(N, M, O) := \max\left\{|N + Mx + Ox^2| + 1 - |x|^2 : x \in E\right\}.$$

I. If $NO \geq 0$*, then*

$$Y(N,M,O) = \begin{cases} |N| + |M| + |O|, & \text{if } |M| \geq 2(1 - |O|), \\ 1 + |N| + \dfrac{M^2}{4(1-|O|)}, & \text{if } |M| < 2(1 - |O|). \end{cases}$$

II. If $NO < 0$*, then*

$$Y(N,M,O) = \begin{cases} 1 - |N| + \dfrac{M^2}{4(1-|O|)}, & \text{if } -4NO\left(\frac{1}{C^2} - 1\right) \leq M^2 \ \wedge \ |M| < 2(1-|O|), \\ 1 + |N| + \dfrac{M^2}{4(1+|O|)}, & \text{if } M^2 < \min\left\{4(1+|O|)^2, -4NO\left(\frac{1}{O^2} - 1\right)\right\}, \\ R(N,M,O), & \text{otherwise}, \end{cases}$$

$$\text{where} \quad R(N,M,O) := \begin{cases} |N| + |M| - |O|, & \text{if } |O|(|M| + 4|N|) \leq |NM|, \\ -|N| + |M| + |O|, & \text{if } |NM| \leq |O|(|M| - 4|N|), \\ (|N| + |O|)\sqrt{1 - \dfrac{M^2}{4NO}}, & \text{otherwise}. \end{cases}$$

Lemma 3.6. *[38] Let h be the function defined in* (1.3)*. Then for real constants* T, U, V*, we have*

$$\left|Tc_1^3 - Uc_1c_2 + Vc_3\right| \leq 2|T| + 2|U - 2T| + 2|T - U + V|. \tag{3.2}$$

4. Main results

Let $f \in \mathcal{R}_L^q$. Then, we have a Schwarz function $v(z)$ satisfying

$$v(0) = 0 \quad \text{and} \quad |v(z)| \leq 1,$$

such that

$$D_q f(z) = v(z) + \sqrt{1 + \beta(q) v^2(z)}. \tag{4.1}$$

Define the function

$$h(z) = \frac{1 + v(z)}{1 - v(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots .$$

We see that $h(z) \in \mathcal{P}$ and

$$\begin{aligned} v(z) &= \frac{h(z) - 1}{h(z) + 1} \\ &= \frac{1}{2}c_1 z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right) z^2 + \left(\frac{1}{8}c_1^3 - \frac{1}{2}c_1 c_2 + \frac{1}{2}c_3\right) z^3 \\ &+ \left(-\frac{1}{16}c_1^4 + \frac{3}{8}c_1^2 c_2 - \frac{1}{4}c_2^2 - \frac{1}{2}c_1 c_3 + \frac{1}{2}c_4\right) z^4 \\ &+ \left(\frac{1}{32}c_1^5 - \frac{1}{4}c_1^3 c_2 + \frac{3}{8}c_1 c_2^2 + \frac{3}{8}c_1^2 c_3 - \frac{1}{2}c_2 c_3 - \frac{1}{2}c_1 c_4 + \frac{1}{2}c_5\right) z^5 \\ &+ O(z^6). \end{aligned} \tag{4.2}$$

The series form of $D_q f(z)$ is given as:

$$D_q f(z) = 1 + [2]_q a_2 z + [3]_q a_3 z^2 + [4]_q a_4 z^3 + [5]_q a_5 z^4 + O(z^5) \tag{4.3}$$

and

$$\begin{aligned} v(z) + \sqrt{1 + \beta(q)(v(z))^2} = 1 + \frac{1}{2}c_1 z &+ \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2 + \frac{\beta}{8}c_1^2\right) z^2 \\ &+ \left(\frac{1}{2}c_3 - \frac{1}{2}c_1 c_2 + \frac{1}{8}c_1^3 + \beta\left(\frac{1}{4}c_1 c_2 - \frac{1}{8}c_1^3\right)\right) z^3 \\ &+ \left(\frac{1}{2}c_4 - \frac{1}{2}c_1 c_3 + \frac{3}{8}c_1^2 c_2 - \frac{1}{4}c_2^2 - \frac{1}{16}c_1^4 \right. \\ &\left. + \beta\left(\frac{1}{8}c_2^2 - \frac{1}{2}c_1^2 c_2 + \frac{12 - \beta}{128}c_1^4 + \frac{1}{4}c_1 c_3\right)\right) z^4 + \ldots, \end{aligned} \tag{4.4}$$

where

$$\beta = \frac{\log(1 + q)}{\log 2}.$$

Comparing (4.4) and (4.3), we achieve

$$a_2 = \frac{c_1}{2[2]_q}, \tag{4.5}$$

$$a_3 = \frac{1}{[3]_q}\left(\frac{1}{2}c_2 + \left(\frac{\beta}{8} - \frac{1}{4}\right)c_1^2\right), \tag{4.6}$$

$$a_4 = \frac{1}{[4]_q}\left[\frac{1}{2}c_3 + \left(\frac{\beta}{4} - \frac{1}{2}\right)c_1c_2 + \left(\frac{1}{8} - \frac{\beta}{8}\right)c_1^3\right], \tag{4.7}$$

$$a_5 = \frac{1}{[5]_q}\left[\frac{1}{2}c_4 + \left(\frac{\beta}{4} - \frac{1}{2}\right)c_1c_3 + \left(\frac{3}{8} - \frac{\beta}{2}\right)c_1^2c_2 + \left(\frac{\beta}{8} - \frac{1}{4}\right)c_2^2 + \left(\frac{\beta(12-\beta)}{128} - \frac{1}{16}\right)c_1^4\right]. \tag{4.8}$$

The logarithmic coefficients of $f(z)$ are given as:

$$\gamma_1 = \frac{1}{2}a_2, \tag{4.9}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \tag{4.10}$$

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right), \tag{4.11}$$

$$\gamma_4 = \frac{1}{2}\left(a_5 - a_2a_4 + a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4\right). \tag{4.12}$$

From Eq (4.5), we obtain

$$\gamma_1 = \frac{1}{2}a_2 = \frac{c_1}{4[2]_q}. \tag{4.13}$$

From Eq (4.6), we get

$$\gamma_2 = \frac{c_2}{4[3]_q} + \left(\frac{\beta}{16[3]_q} - \frac{1}{8[3]_q} - \frac{1}{16[2]_q^2}\right)c_1^2. \tag{4.14}$$

From Eq (4.7), we obtain

$$\gamma_3 = \frac{c_3}{4[4]_q} + \left(\frac{\beta}{8[4]_q} - \frac{1}{4[4]_q} - \frac{1}{4[2]_q[3]_q}\right)c_1c_2 + \left(\frac{1-\beta}{16[4]_q} - \frac{\beta-2}{16[2]_q[3]_q} + \frac{1}{48[2]_q^3}\right)c_1^3. \tag{4.15}$$

Using (4.8), we compute

$$\begin{aligned}\gamma_4 = {} & \frac{c_4}{4[5]_q} + \left(\frac{\beta}{8[5]_q} - \frac{1}{4[5]_q} - \frac{1}{4[2]_q[4]_q}\right)c_1c_3 \\ & + \left(\frac{\beta}{16[5]_q} - \frac{1}{8[5]_q} - \frac{1}{8[3]_q^2}\right)c_2^2 \\ & + \left(\frac{3-4\beta}{16[5]_q} + \frac{2-\beta}{8}\left(\frac{1}{[2]_q[4]_q} + \frac{1}{2[3]_q^2}\right) + \frac{1}{8[2]_q^2[3]_q}\right)c_1^2c_2 \\ & + \left(\frac{12\beta-\beta^2-8}{256[5]_q} + \frac{\beta-1}{15[2]_q[4]_q} + \frac{\beta-2}{[2]_q^2[3]_q} + \frac{4\beta-\beta^2-4}{128[3]_q^2} - \frac{1}{64[2]_q^4}\right)c_1^4,\end{aligned} \tag{4.16}$$

where

$$\begin{gathered} {}_q = 1, [2]_q = 1+q, [3]_q = 1+q+q^2, \\ [4]_q = 1+q+q^2+q^3, [5]_q = 1+q+q^2+q^3+q^4. \end{gathered}$$

Theorem 4.1. *For $0 < q < 1$, let $f \in \mathcal{R}_L^q$, where f is defined in* (1.1) *and*

$$\beta(q) = \frac{\log(1+q)}{\log 2} \in (0,1).$$

Then the coefficients a_2, a_3, and a_4 satisfy

$$|a_2| \leq \frac{1}{[2]_q}, \qquad |a_3| \leq \frac{1}{[3]_q}, \qquad |a_4| \leq \frac{1}{[4]_q}.$$

Proof. From Eq (4.5), we have

$$a_2 = \frac{c_1}{2[2]_q}.$$

Applying the first inequality of Lemma 3.3 yields

$$|a_2| \leq \frac{1}{[2]_q}.$$

Next, from Eq (4.6), we have

$$a_3 = \frac{1}{2[3]_q}\left(c_2 - \mu c_1^2\right), \qquad \mu = \frac{1}{2} - \frac{\beta}{4}.$$

Since $0 < \beta < 1$, we have $0 < \mu < 1$. Using the second inequality of Lemma 3.3, we get

$$|a_3| \leq \frac{1}{[3]_q}.$$

Now, from Eq (4.7), we can write

$$a_4 = \frac{1}{[4]_q}\left[\frac{1}{2}c_3 + \left(\frac{\beta}{4} - \frac{1}{2}\right)c_1c_2 + \left(\frac{1}{8} - \frac{\beta}{8}\right)c_1^3\right].$$

Factoring $\frac{1}{2[4]_q}$, we get

$$a_4 = \frac{1}{2[4]_q}\left(c_3 - 2P\,c_1c_2 + Q\,c_1^3\right),$$

where

$$P = \frac{1}{2} - \frac{\beta}{4}, \qquad Q = \frac{1}{4} - \frac{\beta}{4}.$$

Since $0 < \beta < 1$, it follows that $0 < P < 1$. Moreover, a direct computation shows that

$$P(2P-1) \leq Q \leq P.$$

Thus, all the required conditions of Lemma 3.2 are satisfied, so we obtain

$$|a_4| \leq \frac{1}{[4]_q}.$$

Hence, we have reached our required result. □

Theorem 4.2. *Let $f \in \mathcal{R}_L^q$, where f is defined by* (1.1)*, where* $0 < q < 1$ *and*

$$\beta = \frac{\log(1+q)}{\log 2} \in (0,1).$$

Then the fifth Taylor coefficient a_5 satisfies

$$|a_5| \leq \frac{1}{[5]_q}.$$

Proof. From Eq (4.8), we have

$$2[5]_q a_5 = \left[c_4 + \left(\frac{\beta}{2} - 1\right)c_1c_3 + \left(\frac{3}{4} - \beta\right)c_1^2c_2 + \left(\frac{\beta}{4} - \frac{1}{2}\right)c_2^2 + \left(\frac{12\beta - \beta^2}{64} - \frac{1}{8}\right)c_1^4\right]. \tag{4.17}$$

Equation (4.17) can be rewritten as

$$2[5]_q a_5 = -\Big(\Lambda c_1^4 + bc_2^2 + 2dc_1c_3 - \frac{3}{2}\ell c_1^2c_2 - c_4\Big), \tag{4.18}$$

where the parameters are identified by coefficient comparison as

$$d = \frac{1}{2} - \frac{\beta}{4}, \qquad b = \frac{1}{2} - \frac{\beta}{4}, \qquad \ell = \frac{1}{2} - \frac{2\beta}{3}, \qquad \Lambda = \frac{1}{8} - \frac{(12\beta - \beta^2)}{64}. \tag{4.19}$$

Since $0 < \beta < 1$, it follows immediately that

$$0 < d < 1, \qquad 0 < b < 1.$$

We now verify the technical condition of Lemma 3.4. First, compute

$$d\ell - 2\Lambda = \frac{\beta(13\beta - 92)}{96}.$$

Next, since $b = d$, we have

$$d(b+d) - \ell = 2d^2 - \ell = \frac{\beta}{2}\left(\frac{\beta}{4} + \frac{1}{3}\right)$$

and

$$\ell - 2d^2 = -\frac{\beta}{2}\left(\frac{\beta}{4} + \frac{1}{3}\right).$$

Moreover,

$$b(1-b) = d(1-d) = \Big(\frac{1}{2} - \frac{\beta}{4}\Big)\Big(\frac{1}{2} + \frac{\beta}{4}\Big) = \frac{1}{4} - \frac{\beta^2}{16}.$$

Substituting the above into the left-hand side of Lemma 3.4 yields

$$\begin{aligned} &8b(1-b)\Big[(d\ell - 2\Lambda)^2 + (d(b+d) - \ell)^2\Big] + d(1-d)(\ell - 2bd)^2 \\ &= d(1-d)\left[\frac{13\beta^2 - 92\beta}{12} + \Big(\frac{\beta}{4} + \frac{1}{3}\Big)\Big(\frac{\beta}{4} + \frac{1}{2}\Big)\right]. \end{aligned} \tag{4.20}$$

Now the right-hand side of Lemma 3.4 becomes

$$4d^2b(1-d)^2(1-b) = 4[d(1-d)]^3. \tag{4.21}$$

Dividing the inequality in Lemma 3.4 by $d(1-d) > 0$, and after some simple calculations, we arrive at

$$4(55\beta^2 - 179\beta + 8) \leq 3(4-\beta)^2. \tag{4.22}$$

Since $0 < \beta < 1$, we verify the required inequality

$$4(55\beta^2 - 179\beta + 8) \leq 3(4-\beta)^2.$$

Indeed, after expanding and rearranging, the above inequality is equivalent to

$$16 + 692\beta - 217\beta^2 \geq 0.$$

Define

$$g(\beta) = 16 + 692\beta - 217\beta^2.$$

Then

$$g'(\beta) = 692 - 434\beta.$$

For $0 < \beta < 1$, we have

$$g'(\beta) > 692 - 434 = 258 > 0.$$

Thus, g is strictly increasing on $(0, 1)$. Consequently,

$$g(\beta) > g(0) = 16 > 0, \qquad 0 < \beta < 1.$$

Therefore,

$$4(55\beta^2 - 179\beta + 8) < 3(4-\beta)^2, \qquad 0 < \beta < 1.$$

Thus, all the required hypotheses of Lemma 3.4 hold, and hence from (4.18), we have

$$|a_5| \leq \frac{1}{[5]_q}.$$

This completes the proof. □

Sharpness: Let $f_n : \mathbb{E} \to \mathbb{C}$ defined by

$$f_n(z) = z^{n-1} + \sqrt{1 + \beta(q)\, z^{2(n-1)}}, \qquad n \geq 2,$$

where

$$\beta(q) = \frac{\log(1+q)}{\log 2}.$$

Expanding the square root, we obtain

$$z^{n-1} + \sqrt{1 + \beta(q)\, z^{2(n-1)}} = 1 + z^{n-1} + \frac{\beta(q)}{2} z^{2(n-1)} + O\Big(z^{4(n-1)}\Big)$$

$$= 1 + z^{n-1} + O\Big(z^{2(n-1)}\Big)$$

and

$$D_q f_n(z) = z + \frac{1}{[n]_q} z^n + O\Big(z^{2n-1}\Big).$$

In particular, we have

$$\begin{aligned}
&(i) \quad f_2(z) = z + \frac{1}{[2]_q} z^2 + \cdots, \\
&(ii) \quad f_3(z) = z + \frac{1}{[3]_q} z^3 + \cdots, \\
&(iii) \quad f_4(z) = z + \frac{1}{[4]_q} z^4 + \cdots, \\
&(iv) \quad f_5(z) = z + \frac{1}{[5]_q} z^5 + \cdots.
\end{aligned}$$

Thus, the coefficient bounds obtained in the preceding theorems are sharp.

Theorem 4.3. *Let $f \in \mathcal{R}_L^q$, where f is defined by* (1.1). *Then the following Fekete-Szegö–type inequality holds:*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{[3]_q}.$$

The result is sharp for the function $f_3(z)$.

Proof. From straightforward calculation of (4.5) and (4.6), we have

$$a_3 - a_2^2 = \frac{1}{2[3]_q}\left(c_2 - \mu c_1^2\right),$$

where

$$\mu = \frac{1}{2} - \frac{\beta}{4} + \frac{[3]_q}{2[2]_q^2}.$$

Since $0 < \beta < 1$ and $0 < q < 1$, it follows that $0 \leq \mu \leq 1$. Hence, by inequality (3.1) of Lemma 3.3, we get

$$\left|a_3 - a_2^2\right| \leq \frac{1}{[3]_q}.$$

Its completes the proof of the theorem. □

Theorem 4.4. *Let $f \in \mathcal{R}_L^q$, where f is defined by* (1.1). *Then*

$$|a_4 - a_3 a_2| \leq \frac{1}{[4]_q}.$$

The result is sharp for the function $f_4(z)$.

Proof. First, from (4.5)–(4.7), we compute

$$a_4 - a_3a_2 = \frac{1}{2[4]_q}\left[c_3 - \left(1 - \frac{\beta}{2} + \frac{[4]_q}{[2]_q[3]_q}\right)c_1c_2 + \frac{1}{4}\left(1 - \beta - \frac{[4_q]}{2[2]_q[3]_q}(1 - 2\beta)\right)c_1^3\right], \tag{4.23}$$

which asserts that

$$\left|c_3 - 2P\,c_1c_2 + Q\,c_1^3\right| \leq 2,$$

where

$$\begin{aligned} P &= \frac{1}{2}\left(1 - \frac{\beta}{2} + \frac{[4_q]}{[2]_q[3]_q}\right), \\ Q &= \frac{1}{4}\left(1 - \beta - \frac{[4_q]}{2[2]_q[3]_q}(1 - 2\beta)\right). \end{aligned} \tag{4.24}$$

Since

$$[2]_q = 1 + q, \qquad [3]_q = 1 + q + q^2, \qquad 0 < q < 1,$$

then

$$0 < \frac{1}{[2]_q[3]_q} < \frac{1}{2} \quad \text{and} \quad 0 < \beta < 1.$$

Hence,

$$0 < P < 1.$$

From (4.24), we have

$$P - Q = \frac{1}{4} + \frac{[4]_q(3 + 2\beta)}{8[2]_q[3]_q}.$$

Since $\beta > 0$ and $[2]_q[3]_q > 0$, we conclude that

$$P - Q > 0, \quad \text{that is,} \quad Q < P.$$

To verify $P(2P - 1) < Q$, we first compute

$$2P - 1 = -\frac{\beta}{2} + \frac{[4]_q}{[2]_q[3]_q}.$$

Since $0 < \frac{1}{[2]_q[3]_q} < \frac{1}{4}$ and $-\beta/2 < 0$, we have

$$2P - 1 < 0.$$

Because $P > 0$, it follows that

$$P(2P - 1) < 0. \tag{4.25}$$

Next, from (4.24), we see that each term inside the parentheses is positive for $0 < \beta < 1$, thus we have

$$Q = \frac{1}{4}\left(1 - \beta - \frac{[4_q]}{2[2]_q[3]_q}(1 - 2\beta)\right) > 0. \tag{4.26}$$

Combining (4.25) and (4.26), we obtain

$$P(2P-1) < Q.$$

All conditions of Lemma 3.2 are satisfied, so we finally obtain

$$|a_4 - a_3a_2| \le \frac{1}{[4]_q}.$$

Thus this completes the proof. □

Remark 4.1. *Suppose $q \to 1^-$, we have $[4]_q \to 4$, and the above inequality reduces to*

$$|a_4 - a_3a_2| \le \frac{1}{4},$$

which coincides with the corresponding classical result.

Theorem 4.5. *Let $f \in \mathcal{R}_L^q$, where f is of the form (1.1), where $0 < q < 1$. Then*

$$\left|H_{2,2}(f)\right| \le \frac{1}{[3]_q^2}.$$

The result is sharp.

Proof. First, from (4.5)–(4.7), we compute

$$a_2a_4 - a_3^2 = \frac{1}{4[2]_q[4]_q}c_1c_3 - \frac{1}{4[3]_q^2}c_2^2 + \left(\frac{\beta}{8} - \frac{1}{4}\right)\left(\frac{1}{[2]_q[4]_q} - \frac{1}{[3]_q^2}\right)c_1^2c_2 + \left(\frac{1-\beta}{16[2]_q[4]_q} - \frac{(2-\beta)^2}{64[3]_q^2}\right)c_1^4. \tag{4.27}$$

Since $H_{2,2}(f)$ is rotationally invariant, we may assume

$$c_1 = c, \qquad 0 \le c \le 2. \tag{4.28}$$

Hence

$$a_2a_4 - a_3^2 = \frac{c}{4[2]_q[4]_q}c_3 - \frac{1}{4[3]_q^2}c_2^2 + \left(\frac{\beta}{8} - \frac{1}{4}\right)\left(\frac{1}{[2]_q[4]_q} - \frac{1}{[3]_q^2}\right)c^2c_2 + \left(\frac{1-\beta}{16[2]_q[4]_q} - \frac{(2-\beta)^2}{64[3]_q^2}\right)c^4. \tag{4.29}$$

From Lemma 3.1, we reach

$$\begin{aligned} H_{2,2}(f) = a_2a_4 - a_3^2 &= C_{c^4}\,c^4 + C_{(4-c^2)c^2x}\,(4-c^2)c^2x \\ &\quad + C_{c(4-c^2)x^2}\,(4-c^2)x^2 + C_{(1-|x|^2)z}\,c(4-c^2)(1-|x|^2)z, \end{aligned} \tag{4.30}$$

where

$$\begin{aligned} C_{c^4} = &\frac{1}{16[2]_q[4]_q} - \frac{1}{16[3]_q^2} + \frac{1}{2}\left(\frac{\beta}{8} - \frac{1}{4}\right)\left(\frac{1}{[2]_q[4]_q} - \frac{1}{[3]_q^2}\right) \\ &+ \left(\frac{1-\beta}{16[2]_q[4]_q} - \frac{(2-\beta)^2}{64[3]_q^2}\right), \end{aligned}$$

$$C_{(4-c^2)c^2x} = \frac{1}{8[2]_q[4]_q} - \frac{1}{8[3]_q^2} + \frac{1}{2}\left(\frac{\beta}{8} - \frac{1}{4}\right)\left(\frac{1}{[2]_q[4]_q} - \frac{1}{[3]_q^2}\right),$$

$$C_{c(4-c^2)x^2} = -\frac{c}{16[2]_q[4]_q} - \frac{(4-c^2)}{16[3]_q^2},$$

$$C_{(1-|x|^2)z} = \frac{1}{8[2]_q[4]_q},$$

where $x, z \in \mathbb{C}$ along with $|x| \leq 1$ and $|z| \leq 1$.
First, we let $c = 2$, and then, we get

$$H_{2,2}(f) = C_{c^4}\,(2)^4 = 16\,C_{c^4}.$$

Using the expression for C_{c^4}, we obtain

$$H_{2,2}(f) = 16\left[\frac{1}{16[2]_q[4]_q} - \frac{1}{16[3]_q^2} + \frac{1}{2}\left(\frac{\beta}{8} - \frac{1}{4}\right)\left(\frac{1}{[2]_q[4]_q} - \frac{1}{[3]_q^2}\right) + \left(\frac{1-\beta}{16[2]_q[4]_q} - \frac{(2-\beta)^2}{64[3]_q^2}\right)\right].$$

Therefore,

$$\left|H_{2,2}(f)\right| = 16\,|C_{c^4}|\,.$$

From the expression of C_{c^4}, we set

$$C_{c^4} = \frac{1}{[2]_q[4]_q}\left[\frac{1}{16} + \frac{1}{2}\left(\frac{\beta}{8} - \frac{1}{4}\right) + \frac{1-\beta}{16}\right] - \frac{1}{[3]_q^2}\left[\frac{1}{16} + \frac{1}{2}\left(\frac{\beta}{8} - \frac{1}{4}\right) + \frac{(2-\beta)^2}{64}\right]. \tag{4.31}$$

Therefore,

$$C_{c^4} = -\frac{\beta^2}{64[3]_q^2}. \tag{4.32}$$

It follows immediately that

$$|C_{c^4}| = \frac{\beta^2}{64[3]_q^2}. \tag{4.33}$$

We obtain

$$\left|H_{2,2}(f)\right| = 16 \cdot \frac{\beta^2}{64[3]_q^2} = \frac{\beta^2}{4[3]_q^2}. \tag{4.34}$$

Using the fact that $0 < \beta < 1$ for $0 < q < 1$, we finally conclude that

$$\left|H_{2,2}(f)\right| = \frac{\beta^2}{4[3]_q^2} \leq \frac{1}{4[3]_q^2}. \tag{4.35}$$

For $c = 0$, we obtain

$$H_{2,2}(f) = -\frac{x^2}{[3]_q^2}.$$

Taking absolute values and using $|x| \leq 1$, we have

$$\left|H_{2,2}(f)\right| = \frac{|x|^2}{[3]_q^2} \leq \frac{1}{[3]_q^2}.$$

Next, let $c \in (0,2)$, and we get

$$H_{2,2}(f) = C_{(1-|x|^2)z}\, c(4-c^2)\left[\frac{C_{c^4}}{C_{(1-|x|^2)z}}\frac{c^3}{4-c^2} + \frac{C_{(4-c^2)c^2x}}{C_{(1-|x|^2)z}}\, cx + \frac{C_{c(4-c^2)x^2}}{C_{(1-|x|^2)z}}\frac{x^2}{c} + (1-|x|^2)z\right]. \tag{4.36}$$

After taking the modulus, we have

$$|H_{2,2}(f)| \leq C_{(1-|x|^2)z}\, c(4-c^2)\left(|N + Mx + Ox^2| + 1 - |x|^2\right), \tag{4.37}$$

where

$$N = 8[2]_q[4]_q\, C_{c^4}\, \frac{c^3}{4-c^2},$$

$$M = 8[2]_q[4]_q\, C_{(4-c^2)c^2x}\, c,$$

$$O = 8[2]_q[4]_q\, C_{c(4-c^2)x^2}\, \frac{1}{c}.$$

From (4.32), we can see that

$$C_{c^4} = -\frac{\beta^2}{64[3]_q^2} < 0$$

and

$$C_{c(4-c^2)x^2} = -\frac{c}{16[2]_q[4]_q} - \frac{(4-c^2)}{16[3]_q^2} < 0.$$

Since $8[2]_q[4]_q > 0$, $\frac{c^3}{4-c^2} > 0$ for $0 < c < 2$, and $\frac{1}{c} > 0$ for $c > 0$. It follows that

$$N < 0 \quad \text{and} \quad O < 0.$$

Consequently,

$$NO > 0.$$

Using the value of $C_{(4-c^2)c^2x}$ in M, and after some simple calculations, we get

$$M = \frac{\beta c}{2}\left(1 - \frac{[2]_q[4]_q}{[3]_q^2}\right).$$

Since all quantities are real, thus

$$|M| = \frac{\beta c}{2}\left|1 - \frac{[2]_q[4]_q}{[3]_q^2}\right|. \tag{4.38}$$

Similarly

$$O = -\frac{1}{2} - \frac{[2]_q[4]_q}{2[3]_q^2}\frac{4-c^2}{c}.$$

Since the right-hand side is negative, we have

$$|O| = \frac{1}{2} + \frac{[2]_q[4]_q}{2[3]_q^2}\frac{4-c^2}{c} \tag{4.39}$$

and

$$2(1-|O|) = 1 - \frac{[2]_q[4]_q}{[3]_q^2}\frac{4-c^2}{c}. \tag{4.40}$$

Let

$$R(q) = \frac{[2]_q[4]_q}{[3]_q^2} = \frac{(1+q)(1+q+q^2+q^3)}{(1+q+q^2)^2}.$$

Then

$$|M| = \frac{\beta c}{2}|1-R|, \qquad 2(1-|O|) = 1 - R\frac{4-c^2}{c},$$

and hence

$$0 < R < 1.$$

Since $0 < R < 1$, we have

$$1 - R > 0,$$

and therefore

$$|1-R| = 1-R.$$

Thus

$$|M| = \frac{\beta c}{2}(1-R). \tag{4.41}$$

To determine the sign of $|M| - 2(1-|O|)$, define

$$\phi(c) := |M| - 2(1-|O|), \qquad 0 < c < 2.$$

Then

$$\phi(c) = \frac{\beta c}{2} - 1 + R\left(\frac{4-c^2}{c} - \frac{\beta c}{2}\right).$$

Multiply by the positive factor c and define

$$\psi(c) := c\,\phi(c).$$

Then

$$\psi(c) = \frac{\beta c^2}{2} - c + R\left(4 - c^2 - \frac{\beta c^2}{2}\right).$$

Differentiating with respect to (w.r.t.) c, we obtain

$$\psi'(c) = c[\beta - R(2+\beta)] - 1. \tag{4.42}$$

Since $0 < R < 1$ and $0 < \beta < 1$, we have

$$\psi'(c) < 0.$$

Therefore $\psi(c)$ is strictly decreasing on $(0,2)$. In particular,

$$\psi'(c) < 0 \quad \text{for all } c \in (0,2).$$

Hence all conditions are satisfied, so the second branch of Lemma 3.5 applies. Therefore, we obtain

$$|H_{2,2}(f)| \leq C_{(1-|x|^2)z}\, c(4-c^2)\left(1+|N|+\frac{M^2}{4(1-|O|)}\right). \tag{4.43}$$

Using

$$C_{(1-|x|^2)z} = \frac{1}{8[2]_q[4]_q}$$

in (4.43), we have

$$|H_{2,2}(f)| \leq \frac{c(4-c^2)}{8[2]_q[4]_q}\left(1+|N|+\frac{M^2}{4(1-|O|)}\right). \tag{4.44}$$

Since

$$N, M, O \in \mathbb{R}, \quad 0<c<2, \quad 0<q<1, \quad 0<\beta<1,$$

hence, the R.H.S (right-hand side) of (4.44) is based on the parameter c. Define

$$\Phi(c) = \frac{c(4-c^2)}{8[2]_q[4]_q}\left(1+|N|+\frac{M^2}{4(1-|O|)}\right). \tag{4.45}$$

Then inequality (4.44) can be written as

$$|H_{2,2}(f)| \leq \Phi(c), \qquad 0<c<2. \tag{4.46}$$

We already verified that

$$|M| < 2(1-|O|).$$

Hence,

$$\frac{M^2}{4(1-|O|)} < 1-|O| < 1.$$

Therefore,

$$1+|N|+\frac{M^2}{4(1-|O|)} < 2+|N|.$$

Hence (4.45) becomes

$$\Phi(c) < \frac{c(4-c^2)}{8[2]_q[4]_q}\left(2+|N|\right). \tag{4.47}$$

Using the expressions for N and C_{c^4} in (4.47), we have

$$\Phi(c) < \frac{c(4-c^2)}{8[2]_q[4]_q}\left(2+\frac{\beta^2[2]_q[4]_q}{8[3]_q^2}\cdot\frac{c^3}{4-c^2}\right).$$

Simplifying this gives

$$\Phi(c) < \frac{c(4-c^2)}{4[2]_q[4]_q}+\frac{\beta^2}{64[3]_q^2}c^4. \tag{4.48}$$

Define

$$\Psi(c) = \frac{c(4-c^2)}{4[2]_q[4]_q}+\frac{\beta^2}{64[3]_q^2}c^4.$$

Then $\Phi(c) \leq \Psi(c)$ for $0 < c < 2$.

Differentiating, we obtain

$$\Psi'(c) = \frac{4-3c^2}{4[2]_q[4]_q} + \frac{\beta^2}{16[3]_q^2}c^3.$$

For $0 < c < \frac{2}{\sqrt{3}}$, both terms are positive, so $\Psi'(c) > 0$. For $c > \frac{2}{\sqrt{3}}$, the first term will be negative, while the second term remains bounded since $0 < \beta < 1$.

Moreover, evaluating at $c = 2$, we have

$$\Psi'(2) = -\frac{2}{4[2]_q[4]_q} + \frac{\beta^2}{2[3]_q^2} < 0.$$

Thus, $\Psi'(c) < 0$ near $c = 2$, and hence $\Psi(c)$ increases on $(0, \frac{2}{\sqrt{3}})$ and decreases on $(\frac{2}{\sqrt{3}}, 2)$.

Therefore, $\Psi(c)$ attains its maximum at

$$c = \frac{2}{\sqrt{3}}.$$

Substituting this value in (4.48), we obtain

$$\Psi(c) \leq \frac{2}{3\sqrt{3}[2]_q[4]_q} + \frac{\beta^2}{12\sqrt{3}[3]_q^2}.$$

Combining this estimate with the boundary cases $c = 0$ and $c = 2$, we conclude that

$$|H_{2,2}(f)| \leq \max\left\{\frac{1}{[3]_q^2},\ \frac{2}{3\sqrt{3}[2]_q[4]_q} + \frac{\beta^2}{12\sqrt{3}[3]_q^2}\right\}. \tag{4.49}$$

Since $0 < q < 1$ and $0 < \beta < 1$. It follows that

$$\frac{1}{[3]_q^2} > \frac{2}{3\sqrt{3}[2]_q[4]_q} + \frac{\beta^2}{12\sqrt{3}[3]_q^2}.$$

Hence, (4.49) becomes

$$|H_{2,2}(f)| \leq \frac{1}{[3]_q^2}.$$

Thus the theorem is complete.

□

Theorem 4.6. *Let $f \in \mathcal{R}_L^q$, where f is defined by* (1.1). *Then*

$$|H_{3,1}(f)| \leq \frac{1}{[3]_q^3} + \frac{1}{[4]_q^2} + \frac{1}{[5]_q}\frac{1}{[3]_q}. \tag{4.50}$$

Proof. From (1.6), we have

$$|H_{3,1}(f)| \leq |a_5||a_3 - a_2^2| + |a_4||a_4 - a_2a_3| + |a_3||H_{2,2}(f)|. \tag{4.51}$$

Substituting the estimates of Theorems 4.1–4.5 in (4.51), we have

$$|H_{3,1}(f)| \leq \frac{1}{[3]_q^3} + \frac{1}{[4]_q^2} + \frac{1}{[5]_q}\frac{1}{[3]_q}. \tag{4.52}$$

This completes the proof. □

Logarithmic problems

Theorem 4.7. *Let $f \in \mathcal{R}_L^q$, where f is defined in* (1.1). *Then the logarithmic coefficients satisfy*

$$|\gamma_1| \leq \frac{1}{2[2]_q}, \tag{4.53}$$

$$|\gamma_2| \leq \frac{1}{2[3]_q}, \tag{4.54}$$

$$\begin{aligned} |\gamma_3| \leq \frac{1}{2[4]_q}\Bigg[& \left(\frac{1-\beta}{4} + \frac{(2-\beta)}{4}\frac{[4]_q}{[2]_q[3]_q} + \frac{1}{12}\frac{[4]_q}{[2]_q^3}\right) \\ & + \left|-\frac{1}{2} - \frac{\beta}{2}\frac{[4]_q}{[2]_q[3]_q} + \frac{1}{6}\frac{[4]_q}{[2]_q^3}\right| \\ & + \left|\frac{\beta+1}{4} - \frac{2+\beta}{4}\frac{[4]_q}{[2]_q[3]_q} + \frac{1}{12}\frac{[4]_q}{[2]_q^3}\right|\Bigg]. \end{aligned} \tag{4.55}$$

Proof. From (4.13), and applying Lemma 3.3, we get

$$|\gamma_1| \leq \frac{1}{2[2]_q}.$$

From the expression of γ_2 in (4.14), we write

$$\gamma_2 = \frac{1}{4[3]_q}\left(c_2 - \mu c_1^2\right),$$

where

$$\mu = \frac{1}{2} - \frac{\beta}{4} + \frac{[3]_q}{4[2]_q^2}.$$

Since $0 < \beta < 1$, it follows that $0 < \mu < 1$. Applying the inequality of Lemma 3.3, we obtain

$$|\gamma_2| \leq \frac{1}{2[3]_q}.$$

From the expression of γ_3 in (4.15), we write

$$\gamma_3 = \frac{1}{4[4]_q}\left(c_3 + Tc_1c_2 + Uc_1^3\right),$$

where

$$T = 4[4]_q\left(\frac{\beta}{8[4]_q} - \frac{1}{4[4]_q} - \frac{1}{4[2]_q[3]_q}\right),$$

$$U = 4[4]_q\left(\frac{1-\beta}{16[4]_q} - \frac{\beta-2}{16[2]_q[3]_q} + \frac{1}{48[2]_q^3}\right).$$

$$V = 1.$$

Applying Lemma 3.6, we obtain the required result.

□

Remark 4.2. *In the limiting case $q \to 1^-$, our results reduce to the following known estimates previously established in [39]:*

$$|\gamma_1| \leq \frac{1}{4}, \quad |\gamma_2| \leq \frac{1}{6}.$$

5. Conclusions

In this research work, we introduced and studied a new subclass $\mathcal{R}_L^q$ of analytic functions defined in E via the q-difference operator and a q-Lune-type mapping. The class was characterized through a subordination condition involving the function

$$\Phi_q(z) = z + \sqrt{1 + \beta(q)z^2}, \qquad \beta(q) = \frac{\log(1+q)}{\log 2}, \quad 0 < q < 1,$$

which generates a symmetric region on the right-hand side of plane.

Using techniques of subordination theory and the theory of Carathéodory functions, we derived several fundamental properties of the class $\mathcal{R}_L^q$. In particular, we obtained coefficient estimates and established sharp bounds of $H_{2,2}(f)$. Furthermore, we derived a new estimates of the third Hankel determinant $H_{3,1}(f)$, investigated logarithmic coefficients, and obtained explicit bounds for γ_1, γ_2, and γ_3 using suitable lemmas and functional inequalities. Furthermore, we studied graphical images to provide geometric understanding of the behavior of the class $\mathcal{R}_L^q$. These figures confirm that the image of E under the q-difference operator stays inside the region defined by $\Phi_q(z)$. The impact of the parameter q on the deformation of the image domain was also emphasized. The results investigated in this work illustrate that the combination of q-calculus and subordination theory provide a powerful framework for studying geometric properties of analytic functions. The investigated bounds not only generalize the classical results but also add to the ongoing development of q-analysis in GFT.

For future research, we will study and define new subclasses of q-starlike and q-convex functions related to $\Phi_q(z)$ and investigate the higher-order Hankel determinants and Fekete-Szegö results. Additionally, extending these results to harmonic and meromorphic function classes may provide further interesting developments.

Author contributions

All authors contributed equally to this work and have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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