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Research article

Notes on convergence and stability of iteration processes

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Abstract: The theory developed in this paper defines a mathematical technology for automating residually stable convergence of iterative processes to common fixed points of semigroups of nonlinear operators. Our ultimate aim is to simplify future research on convergence and stability across a broad range of scenarios. By 'automation', we refer to the availability of a framework based on a single property of a given Banach space and a single theorem that enables conclusions to be drawn regarding stable convergence for a wide range of commonly used iterative processes. Such a novel framework is developed in this paper. This is exemplified by the direct application of our results to convergence in measure, convergence in Marcinkiewicz quasi-norm, and convergence in Lorentz quasi-norm within spaces such as Lebesgue spaces, Sobolev spaces, Orlicz spaces, Musielak-Orlicz spaces, variable exponent Lebesgue spaces, and other spaces constructed upon them. These specific outcomes, ready for use, are novel results in their own right.

Keywords: fixed point; semigroup of nonlinear operators; iteration process; process convergence; process stability
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1. Introduction

We describe the aim of this paper with the following problem statement. Though precise definitions of the concepts will be provided later, their intuitive meanings are quite clear.

Problem 1. *Let us consider the setting defined by the following assumptions:*

1) *X is a uniformly convex Banach space.*
2) *X is equipped with a convergence, defined later as Ψ-convergence, weaker than the norm convergence.*
3) *$D \subset X$ is a nonempty convex, bounded, closed in the norm sense, and sequentially compact with respect to the Ψ-convergence.*

4) $\mathcal{T}$ *is an asymptotic pointwise nonexpansive semigroup of operators acting within D.*
5) *Let* $\{w_n\}$ *be a sequence in D, generated by an iteration process. Assume that* $\{w_n\}$ *is an approximate fixed point sequence for the semigroup* $\mathcal{T}$*.*

Question: What single condition connecting the norm and the Ψ*-convergence will automatically ensure that* $\{w_n\}$ *is* Ψ*-convergent to a common fixed point of* $\mathcal{T}$ *in a stable manner?*

Hence, the theory developed in this paper defines a mathematical technology for automating residually stable convergence of iterative processes to common fixed points.

To make it absolutely clear, our goal is not to introduce yet another specialised set of techniques, nor is it merely to unify existing results. On the contrary, our ultimate aim is to simplify and automate future research on convergence and stability across a broad range of scenarios. By 'automation', we refer to the availability of a framework based on a single property of a given Banach space X and a single theorem that enables one to draw conclusions regarding stable convergence for a wide range of commonly used iterative processes. Such a framework is consequently developed throughout this paper, with Theorem 6.1 fulfilling the role of this main theorem.

In the recent book [1], the author considers the existence of common fixed points for semigroups of asymptotic pointwise nonexpansive mappings acting within uniformly convex Banach spaces. In this context, the book examines both the strong and weak convergence of iterative fixed point construction processes, including generalised forms of the Krasnosel'skii-Mann, Ishikawa, and implicit iteration algorithms, as well as the stability of these processes under summable errors. The original, weak topology, variant of the Opial property [2] is used to prove the convergence of iteration processes in the weak topology. However, because the weak Opial property is quite restrictive (failing even in simple cases such as L^p when $1 < p \neq 2$) alternative methods must be employed. Furthermore, from the perspective of applications, different types of convergence, such as convergence in probability, are often more meaningful.

In this paper, we present a novel solution to Problem 1 by introducing in Definition 3.1 a general concept of convergence called Ψ-convergence. The topology generated by Ψ-convergence is weaker than the topology defined by the norm, yet the norm remains lower semicontinuous with respect to this new convergence. Since both weak and strong convergence satisfy the conditions defining Ψ-convergence, all results related to weak and strong convergence remain valid in this new framework. More importantly, the advantage and novelty of the Ψ-convergence approach lie in its ability to guarantee the convergence and stability of iterative processes across a much broader class of scenarios than just weak and strong convergence, through a single property known as the Ψ-Opial property (Definition 4.6).

There are known examples of Opial-like properties that arise from types of convergence other than those based on the weak topology; refer, for example, to [3–9] and the literature cited therein. The authors of [10] discuss an interesting example of an Opial property for noncommutative modular function spaces of τ-measurable operators affiliated with a semifinite von Neumann algebra that has a faithful and normal trace τ. Such a space is defined using a 'normal' function modular ρ, as specified in [11, Definition 2.1.1]. This Opial property relates to the Luxemburg norm and the $\tilde{\rho}$-a.e. convergence, defined by $\tilde{\rho}(x_n - x) \to 0$ whenever $m_{x_n - x} \to 0$ a.e.; here, $m_{x_n - x}$ denotes a generalised rearrangement mapping. The authors demonstrate in their Theorem 4.3 that this noncommutative modular function space exhibits the uniform Opial property, provided the function modular ρ has the Δ_2-type property, similar to our Theorem 4.6.

In our case, this additional convergence in X is defined by Ψ, a family of mappings from X to $\mathbb{R}^+$. Our approach strikes an optimal balance between abstraction and practical usability. This is exemplified by the direct application of our results to convergence in measure, convergence in Marcinkiewicz quasi-norm, and convergence in Lorentz quasi-norm within spaces such as L^p spaces, Sobolev spaces, Orlicz spaces, Musielak-Orlicz spaces, variable exponent Lebesgue spaces, and other spaces constructed upon them.

For the fundamentals of the theory of asymptotic pointwise nonexpansive semigroups, we direct the reader to the monograph [1]. Specifically, we recommend familiarisation with the Introduction chapter, which contains examples of such semigroups along with justifications for their consideration. Generally, we do not recommend viewing such semigroups merely as a different type; rather, they should be considered in light of possibly insufficient information regarding their regularity. For instance, Examples 1.6 and 1.7 from [1] demonstrate that a semigroup can be nonexpansive within a set D_1 while being only asymptotically nonexpansive in another set D_2. The challenge lies in the fact that, in many application scenarios, selecting an appropriate set D is not straightforward. This underscores the importance of a more general theory. Moreover, there exist useful characterisations of asymptotic pointwise nonexpansive semigroups when their members are continuously Fréchet differentiable; refer to Proposition 2.1.

Our results are also novel for more regular types of semigroups, including asymptotic nonexpansive, eventually nonexpansive, and nonexpansive semigroups. Therefore, readers primarily interested in these cases will find the paper's results relevant as well. Note also that when the results are applied to a semigroup consisting of all iterations of a single operator T, namely $\mathcal{T} = \{T^k : k = 0, 1, \ldots\}$, the common fixed points of $\mathcal{T}$ coincide with the fixed points of T. Consequently, the iterative processes reduce to more straightforward methods for constructing fixed points for a single operator.

Section 2, 'Preliminaries and general results', summarises necessary background information from [1] related to fixed point theory for semigroups of asymptotic pointwise nonexpansive operators. It also reviews fundamental concepts of the theory of modular [12] and modular function spaces [13], which are essential for understanding how the results of this paper can be applied to Musielak-Orlicz spaces. The motivation is that, in these spaces, norm concepts interplay efficiently with their modular counterparts, producing results unattainable through norm and weak topology methods alone, as demonstrated in Theorem 4.5, along with the associated commentaries; see also Remark 6.1 regarding implications for L^p-spaces.

In Section 3, titled 'Banach Spaces with Convergence', we formally define Ψ-convergence along with its associated concepts, and present an extensive list of examples that encompass the cases mentioned earlier in this Introduction.

In the context of the Ψ-convergence, the following section, titled 'Convergence Results', introduces the key concepts of the Demiclosedness Principle and the Opial property. We also pose the main question of when a sequence $\{w_n\}$, generated by a well-defined iteration process and assumed to be an approximate fixed point sequence for an asymptotic pointwise nonexpansive semigroup $\mathcal{T}$, converges in the sense of Ψ to a common fixed point of $\mathcal{T}$. The answer is provided in Theorem 4.3, which underscores the critical importance of the Ψ-based variants of compactness and the Opial property. The remainder of this section is dedicated to a thorough examination of examples from Section 3, illustrating how this general convergence result applies in each of these important scenarios.

In Section 5, we demonstrate that the Ψ-convergence of iteration processes discussed in the previous section is residually stable. This stability question addresses a critical concern: whether a process remains convergent in the presence of algorithmical errors. We introduce and examine the residual stability and compare it with the stability under summable errors considered in [14–16].

In Section 6, titled 'Summary of Results', we summarise the paper's findings in Theorem 6.1, which provides a general and comprehensive answer to the question posed in Problem 1. Additionally, since the paper presents a series of new specific results for various Banach spaces and types of convergence, Theorems 6.2 through 6.6 summarise these novel contributions.

Section 7, titled 'Applications', is dedicated to the utilisation of the results from the preceding sections, in demonstrating the convergence and stability of the generalised Krasnosel'skii-Mann and generalised Ishikawa iteration processes for asymptotic pointwise nonexpansive semigroups. This includes all the sample scenarios presented in the 'Preliminaries'. In its last subsection, we discuss an example of the application of our results to the problem of residually stable construction of stationary points for dynamical systems represented by asymptotic pointwise nonexpansive semigroups.

Finally, in the 'Conclusions' section, we highlight the main achievements of the theory and its applications discussed in the article, examining them from the perspective of potential future developments and applications.

2. Preliminaries and general results

In this paper, X will always stand for a Banach space equipped with the norm $\|\cdot\|$, even if it is not directly mentioned. By D we will denote a nonempty convex subset of X, which is bounded and closed in the norm sense. Building on the work of [1], we will use the following definitions and notations.

2.1. Semigroups of operators

By J we will denote a subsemigroup of $[0, +\infty)$ with normal addition. We assume that $0 \in J$ and that there exists $t > 0$ such that $t \in J$. This assumption implies that $+\infty$ is a cluster point of J in the sense of the natural topology inherited from $[0, \infty)$. The notation $t \to \infty$ will mean that t tends to infinity over J. Common examples include $J = [0, \infty)$ and $J = \{n\alpha : n = 0, 1, 2, 3, ...\}$ for a given number $\alpha > 0$.

Definition 2.1. *[1, Definition 1.1] A one-parameter family $\mathcal{T} = \{T_t : t \in J\}$ of mappings from D into itself is called a continuous semigroup on D if it satisfies the conditions:*

1) $T_0(x) = x$, for $x \in D$;
2) $T_{t+s}(x) = T_t(T_s(x))$, for $x \in D$ and $t, s \in J$;
3) For each $x \in D$, the mapping $t \mapsto T_t(x)$ is continuous in the norm sense.

For every $t \in J$, define $F(T_t) = \{x \in D : T_t(x) = x\}$. We define then the set of all common set points for the mappings in $\mathcal{T}$ as the intersection:

$$F(\mathcal{T}) = \bigcap_{t \in J} F(T_t).$$

Definition 2.2. *[1, Definition 1.5] A one-parameter family $\mathcal{T} = \{T_t : t \in J\}$ of mappings from D into itself is called a pointwise Lipschitzian semigroup on D if it satisfies the following conditions:*

1) $\mathcal{T}$ is a continuous semigroup on D;

2) For each $t \in J$, T_t is a pointwise Lipschitzian mapping, which means that for every $x \in D$ and every $t \in J$ there exists $a_t(x) \geq 1$ such that,

$$\|T_t(x) - T_t(y)\| \leq a_t(x)\|x - y\| \; for \; all \; y \in D.$$

In this paper, we will always deal with the asymptotic pointwise nonexpansive semigroups, defined below.

Definition 2.3. *[1, Definition 1.6] A pointwise Lipschitzian semigroup $\mathcal{T}$ is called asymptotic pointwise nonexpansive if $\lim\limits_{t\to\infty} a_t(x) = 1$ for every $x \in C$. The class of all asymptotic pointwise nonexpansive semigroups on D will be denoted by $\mathcal{APNS}(D)$.*

Asymptotic pointwise nonexpansive semigroups, which generalise nonexpansive semigroups, represent an important class of semigroups with diverse applications, particularly in the theories of differential equations, dynamical systems, and stochastic evolutionary processes. The study of semigroups of iterations $\{T^k : k = 0, 1, 2, 3, ...\}$ of a Lipschitzian operator T was initially advanced through the asymptotic nonexpansive approach, without focusing on pointwise methods, by Goebel and Kirk in [17]. This framework was later broadened by Kirk and Xu in [18] to include pointwise nonexpansive mappings. Kozlowski further extended the approach in [19] to incorporate asymptotic pointwise nonexpansive semigroups. Since then, a substantial body of research has developed around these concepts, underscoring the importance of asymptotic pointwise nonexpansive semigroups. For more detailed information, readers can consult the recent book [1] and the references within.

Interestingly, in certain scenarios, we can characterise asymptotic pointwise nonexpansive semigroups by analysing the asymptotic behaviour of the Fréchet derivatives of the semigroup members, as illustrated in the following result.

Proposition 2.1. *[1, Proposition 1.4] Let $\mathcal{T}$ be a pointwise Lipschitzian semigroup on D, where D is a nonempty, bounded, closed and convex subset of a Banach space X. Assume that all mappings $T_t \in \mathcal{T}$ are continuously Fréchet differentiable on an open convex set A containing D then $\mathcal{T}$ is asymptotic pointwise nonexpansive on D if and only if for each $x \in D$, $\limsup\limits_{t\to\infty} \|(T_t)_x^{'}\| \leq 1$.*

The fundamental common fixed point existence result for asymptotic pointwise nonexpansive semigroups was originally proved in [19].

Theorem 2.1. *[19, Theorem 3.4] [1, Theorem 2.3] Assume that X is uniformly convex. Let $\mathcal{T}$ be an asymptotic pointwise nonexpansive semigroup on a nonempty, bounded, closed and convex set $D \subset X$. Under these conditions, $\mathcal{T}$ has a common fixed point, and the set of common fixed points is closed and convex.*

The technique of approximate fixed point sequences is essential in demonstrating convergence and stability of iteration processes.

Definition 2.4. *Given the semigroup $\mathcal{T}$, we say that a sequence $\{x_k\}$ of elements in D is an approximate fixed point sequence for $\mathcal{T}$ if $\lim_{k\to\infty} \|T_t(x_k) - x_k\| = 0$ holds for each $t \in J$.*

To enhance simplicity when dealing with fixed-point construction iteration processes, we recall the concept of asymptotic pointwise nonexpansive sequences.

Definition 2.5. *[1, Definition 3.6] Let D be a subset of a Banach space X. We say that a sequence $\{H_n\}$ of mappings that act within D is an asymptotic pointwise nonexpansive sequence if to every $x \in D$ there exists $A_n(x) \geq 1$ such that for every $x \in D$,*

$$\lim_{n\to\infty} A_n(x) = 1 \tag{2.1}$$

and

$$\|H_n(x) - H_n(y)\| \leq A_n(x)\|x - y\|$$

holds for all $y \in D$ and all $n \in \mathbb{N}$.

Define $B_n(x) = A_n(x) - 1$. In view of (2.1), we have

$$\lim_{n\to\infty} B_n(x) = 0.$$

By $\mathcal{A}(D)$ we will denote the class of all asymptotic pointwise nonexpansive sequences of mappings $\{H_n\}$ acting within D. Let $\{H_n\} \in \mathcal{A}(D)$. We say that a sequence $\{x_n\}$ of elements of D is generated by $\{H_n\}$ if $x_1 \in D$ and $x_{n+1} = H_n(x_n)$ for every $n \in \mathbb{N}$.

It is essential to recognise that asymptotic pointwise nonexpansive sequences serve as base for iteration processes for constructing common fixed points of semigroups of nonlinear operators. Thus, these sequences should not be confused with the semigroups themselves. An asymptotic pointwise nonexpansive sequence does not necessarily adhere to the semigroup conditions and may differ significantly from the semigroups to which it is related.

2.2. Modular spaces

When discussing examples related to Musielak-Orlicz spaces and their specialisations, such as Orlicz and variable Lebesgue spaces, we will draw upon fundamental concepts from the theory of modular spaces [12] and modular function spaces [11, 13]. This is because Musielak-Orlicz spaces, as defined in Example 3.5, belong to the class of modular function spaces. Our focus will be limited to convex modulars, as they are most pertinent for our setting.

Definition 2.6. *[12] Let X be a real vector space. A functional $\rho : X \to [0, \infty]$ is called a convex modular (or simply, a modular, when the context is clear) if*

1) $\rho(x) = 0$ *if and only if* $x = 0$
2) $\rho(|x|) = \rho(x)$
3) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ *for any* $x, y \in X$, *and* $\alpha, \beta \geq 0$ *with* $\alpha + \beta = 1$

The vector space $X_\rho = \{x \in X : \rho(\lambda x) \to 0,\ as\ \lambda \to 0\}$ is called a modular space .*

Definition 2.7. *[12, Def. 1.7] A modular ρ is called left-continuous if $\lim_{\lambda\to 1^-} \rho(\lambda x) = \rho(x)$ for all $x \in X_\rho$. Similarly, ρ is called right-continuous if $\lim_{\lambda\to 1^+} \rho(\lambda x) = \rho(x)$ for all $x \in X_\rho$.*

*A more general definition of a modular is obtained when the convexity condition 3 of DefInition 2.6 is replaced by a weaker condition $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for any $x, y \in X$, and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

It is well known that every convex modular space can be equipped with a norm, called the Luxemburg norm, defined as

$$\|x\|_\rho = \inf\Big\{\alpha > 0 : \rho\Big(\frac{x}{\alpha}\Big) \le 1\Big\}.$$

Note that, without any additional assumptions, one needs to prove that $\rho(\lambda(x_n - x)) \to 0$ for every $\lambda > 0$ to conclude that $\|x_n - x\|_\rho \to 0$, while the converse is always true. The following result illustrates the relationships between norm and modular behaviour, which will be utilised in our analysis.

Proposition 2.2. *[12] Let ρ be a convex modular. Then,*

1) *$\rho(x) \le \|x\|_\rho$ if $\|x\|_\rho < 1$.*
2) *If ρ is left-continuous, then the inequalities $\|x\|_\rho \le 1$ and $\rho(x) \le 1$ are equivalent.*
3) *If ρ is right-continuous, then the inequalities $\|x\|_\rho < 1$ and $\rho(x) < 1$ are equivalent.*

Let us define the relative growth function for a convex modular.

Definition 2.8. *Let ρ be a convex modular. For a $\delta \ge 0$, define:*

$$w_\delta(s) = \sup\left\{\frac{\rho(sx)}{\rho(x)} \,:\; \delta < \rho(x) < +\infty\right\}.$$

For convenience, if $\delta = 0$ we will simplify notation to $w = w_\delta$ and w will be referred to as the modular's growth function.

The following properties of the relative growth function are direct consequence of the definition.

Proposition 2.3. *For a convex modular ρ, the following statements are true:*

1) *w_δ is a nondecreasing function.*
2) *$w_\delta(0) = 0$ and $w_\delta(1) = 1$.*
3) *$\rho(sx) \le w_\delta(s)\rho(x), \forall s \ge 0, \forall x \in X_\rho$ such that $\delta < \rho(x) < +\infty$.*
4) *$w_\delta(s) \le t$ for $s \le 1$.*
5) *w_δ is a convex function.*
6) *w_δ is a continuous function at those $s \ge 0$ at which $w_\delta(s) < +\infty$.*

Definition 2.9. *[13, Definitions 3.5 and 3.6] The following properties of modulars will play important roles in the discussion presented throughout the following sections.*

1) *We say that a modular ρ has the Δ_2 property if $\rho(2x_n) \to 0$ whenever $\rho(x_n) \to 0$.*
2) *We say that a modular ρ has the Δ_2-type property if there exists a constant $0 < K < \infty$ such that for every $x \in X_\rho$ we have $\rho(2x) \le K\rho(x)$.*

Clearly, the Δ_2-type condition implies the Δ_2-condition. Furthermore, for modulars with the Δ_2 property, $\rho(x_n) \to 0$ if and only if $\|x_n\|_\rho \to 0$, [13, Proposition 3.12]. Let us recall another important property of modulars.

Definition 2.10. *[13, Definition 5.4] A modular ρ is called uniformly continuous if for every $\varepsilon > 0$ and every $0 < L < \infty$ there exists $\delta > 0$ such that,*

$$|\rho(x + y) - \rho(x)| < \varepsilon, \tag{2.2}$$

whenever $x \in X_\rho$, $y \in X_\rho$, $\rho(y) < \delta$, and $\rho(x) \le L$.

Proposition 2.4. *Let ρ be a uniformly continuous modular then for every $x \in X_\rho$,*

1) $\rho(x) < \infty$,
2) ρ is a continuous modular at x, that is, it is left and right-continuous at x
3) ρ has the Δ_2 property.

Proof. Let $x \in X_\rho$. Without loss of generality, we can assume that $x \neq 0$. From (2.2) it follows that $\rho(x)$ must be finite.

To prove part 2, fix $\varepsilon > 0$ and $L = \rho(x) < \infty$. It follows from the uniform continuity of ρ that there exists a $0 < \delta \leq L$ such that,

$$|\rho(x+y) - \rho(x)| < \varepsilon, \tag{2.3}$$

provided $\rho(y) < \delta$. Take $\lambda > 0$ such that $|\lambda - 1| < \min\left\{1, \dfrac{\delta}{\rho(x)}\right\}$. Let $y = \lambda x - x$. Then,

$$\rho(y) = \rho((\lambda - 1)x) \leq |\lambda - 1|\rho(x) < \delta.$$

By (2.3) we get,

$$|\rho(\lambda x) - \rho(x)| = |\rho(x+y) - \rho(x)| < \varepsilon.$$

Hence, ρ is a continuous modular at x.

To prove the Δ_2 property, assume that $\rho(x_n) \to 0$. Hence $\rho(x_n) < M < \infty$ for n sufficiently large. Let $\varepsilon > 0$ and take $0 < \delta < M$ corresponding to ε and M by the uniform continuity of ρ. Hence, for sufficiently large n such that $\rho(x_n) < \delta$, it holds that,

$$|\rho(2x_n) - \rho(x_n)| = |\rho(x_n + x_n) - \rho(x_n)| < \varepsilon.$$

Therefore, $\rho(2x_n) \to 0$, finishing the proof of the Δ_2 property. □

Remark 2.1. *Very useful property of uniform continuity holds for many important modulars. For instance, in Orlicz spaces over a finite atomless measure [20] or in sequence Orlicz spaces [21] the uniform continuity of the Orlicz modular is equivalent to the Δ_2 property. The next result provides conditions sufficient for the uniform continuity of convex modulars.*

Proposition 2.5. *[22, Theorem 3.3] If a modular ρ satisfies the Δ_2-type condition and its growth function w is right-continuous at $t = 1$ then ρ is uniformly continuous.*

We will need the following property of uniformly continuous convex modulars.

Theorem 2.2. *Let ρ be a uniformly continuous convex modular. Let $u \in X_\rho$ and $\{w_n\}$ be a sequence of elements of X_ρ be such that there exists $\beta > 1$ and $0 < L < \infty$, for which we have,*

$$\sup_{n\in\mathbb{N}} \rho(\beta(w_n - u)) \leq L. \tag{2.4}$$

Define function $\Lambda_u : [0,\beta] \to [0, L]$ by the following formula,

$$\Lambda_u(s) = \liminf_{n\to\infty} \rho(s(w_n - u)). \tag{2.5}$$

Then, Λ_u is a uniformly continuous function.

Proof. For each natural number n, define function $f_n(s) = \rho(s(w_n - u))$ and note that each f_n is a finite, convex, increasing, continuous function, and that $f_n(0) = 0$. We claim that $\{f_n\}$ is an equicontinuous sequence of functions. Indeed, let us fix an $\varepsilon > 0$. Let $n \in \mathbb{N}$. Let $s_1, s_2 \in [0,\beta]$ be such that $|s_2 - s_1| < 1$. Denoting $x = s_1(w_n - u)$ and $y = (s_2 - s_1)(w_n - u)$, and remembering the boundedness condition (2.4), use (2.2), with ε and L, to obtain a $0 < \delta < 1$ such that,

$$|f_n(s_2) - f_n(s_1)| = |\rho(y + x) - \rho(x)| < \varepsilon \tag{2.6}$$

holds, whenever $|s_2 - s_1| < \delta$. Since (2.6) holds for any $n \in \mathbb{N}$ and the choice of δ did not depend on n, the sequence $\{f_n\}$ is an equicontinuous. Assume that $|s_2 - s_1| < \delta$ and $s_1 < s_2$, then $f_n(s_1) \leq f_n(s_2)$. It follows from (2.6) that,

$$0 \leq f_n(s_2) < f_n(s_1) + \varepsilon$$

holds for all $n \in \mathbb{N}$, which implies that,

$$0 \leq \Lambda_x(s_2) = \liminf_{n\to\infty} f_n(s_2) \leq \liminf_{n\to\infty} f_n(s_1) + \varepsilon = \Lambda_x(s_1) + \varepsilon.$$

Finally,

$$|\Lambda_x(s_2) - \Lambda_x(s_1)| < \varepsilon,$$

completing the proof. □

Modular function spaces form a special subclass of the class of all modular spaces. We will use the following adaptation of the definition of modular function spaces. Let (Ω, μ) be a finite measure space. By $\mathcal{E}$ we denote the linear space of all simple measurable functions. By $\mathcal{M}_\infty$ we will denote the space of all extended measurable functions, that is, all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \mathcal{E}$, $|g_n| \leq |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$. By 1_A we denote the characteristic function of the set A.

Definition 2.11. *[13, Def. 3.1] Let $\rho : \mathcal{M}_\infty \to [0, \infty]$ be a nontrivial, convex and even function. We say that ρ is a regular convex function pseudomodular if*

(a) $\rho(0) = 0$*;*
(b) ρ *is* monotone*, that is,* $|f(\omega)| \leq |g(\omega)|$ *for all* $\omega \in \Omega$ *implies* $\rho(f) \leq \rho(g)$*, where* $f, g \in \mathcal{M}_\infty$*.*
(c) ρ *is orthogonally subadditive, that is,* $\rho(f 1_{A\cup B}) \leq \rho(f 1_A) + \rho(f 1_B)$ *for any* $A, B \in \Sigma$ *such that* $A \cap B \neq \emptyset$*, where* $f \in \mathcal{M}_\infty$*.*
(d) ρ *has the Fatou property:* $|f_n(\omega)| \uparrow |f(\omega)|$ *for all* $\omega \in \Omega$ *implies* $\rho(f_n) \uparrow \rho(f)$*, where* $f \in \mathcal{M}_\infty$*.*
(e) ρ *is order continuous in* $\mathcal{E}$*, that is,* $g_n \in \mathcal{E}$ *and* $|g_n(\omega)| \downarrow 0$ *implies* $\rho(g_n) \downarrow 0$*.*

Similarly as in the case of measure spaces, we say that a set $A \in \Sigma$ is ρ-null if $\rho(g 1_A) = 0$ for every $g \in \mathcal{E}$. We say that a property holds ρ-almost everywhere if the exceptional set is ρ-null. As usual we identify any pair of measurable sets whose symmetric difference is ρ-null as well as any pair of measurable functions differing only on a ρ-null set. With this in mind we define:

$$\mathcal{M} = \{f \in \mathcal{M}_\infty : |f(\omega)| < \infty \ \rho - a.e\},$$

where each $f \in \mathcal{M}$ is actually an equivalence class of functions equal ρ-a.e. rather than an individual function.

Definition 2.12. *[13, Def. 3.2] Let ρ be a regular convex function pseudomodular. We say that ρ is a regular convex function modular (or shortly, a function modular) if $\rho(f) = 0$ implies that $f = 0$ $\rho-a.e.$*

It is straightforward to demonstrate that every regular convex function modular is a left-continuous convex modular. The modular space defined by such modular is called a modular function space.

In the next proposition, following [13], we collect some fundamental properties of modular function spaces.

Proposition 2.6. *In modular function spaces the following statements are true:*

(a) $(X_\rho, \|\cdot\|_\rho)$ *is a complete Banach space.*
(b) *If $f_n \to f$ $\rho-a.e.$ then $\rho(f) \le \liminf_{n\to\infty} \rho(f_n)$. This property is a direct consequence of the Fatou property.*
(c) *If $f_n \to f$ $\rho-a.e.$ then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f$ ρ-a.e.*

We will be particularly interested in orthogonally additive function modulars, as defined below.

Definition 2.13. *We say that a function modular ρ is orthogonally additive if*

$$\rho(f 1_{A\cup B}) = \rho(f 1_A) + \rho(f 1_B),$$

whenever $A \cap B = \emptyset$.

We will need the following important property of orthogonally additive function modulars.

Theorem 2.3. *[13, Theorem 4.7] Let ρ be an orthogonally additive function modular. Let $\{f_n\} \subset X_\rho$ be ρ-a.e. convergent to zero. Assume that there exists $\beta > 1$ such that $\sup\limits_{n\in\mathbb{N}} \rho(\beta f_n) < \infty$. Let $g \in E_\rho$. Then,*

$$\liminf_{n\to\infty} \rho(f_n + g) = \liminf_{n\to\infty} \rho(f_n) + \rho(g).$$

Recall that $E_\rho = \{f \in X_\rho : \rho(\lambda f 1_{(\cdot)})$ is order continuous for every $\lambda > 0\}$, and that $E_\rho = X_\rho$ whenever ρ has the Δ_2 property ([13, Theorem 3.8]).

3. Banach spaces with convergence

Let X be a Banach space equipped with an additional convergence structure, as defined below.

Definition 3.1. *Let Ψ be a family of functions $\psi : X \to [0, +\infty)$ satisfying the following properties:*

(A1) *For each $\psi \in \Psi$ there exists $c_\psi > 0$ such that $\psi(x + y) \le c_\psi\Big(\psi(x) + \psi(y)\Big)$ for all $x, y \in X$.*
(A2) *$x = 0$ if and only if $\psi(x) = 0$ for every $\psi \in \Psi$.*
(A3) *Each $\psi \in \Psi$ is $\|\cdot\|$-continuous at zero.*
(A4) *If $\{x_n\}$ is a bounded sequence such that $\psi(x_n - x) \to 0$ for every $\psi \in \Psi$ then*

$$\|x\| \le \liminf_{n\to\infty} \|x_n\|. \tag{3.1}$$

In this context, we say that the triple $(X, \|\cdot\|, \Psi)$ forms a Banach space equipped with the sequential convergence Ψ. Furthermore, we will say that $\{x_n\}$ is Ψ-convergent to x, and write $x_n \overset{\Psi}{\to} x$, whenever $\psi(x_n - x) \to 0$ for every $\psi \in \Psi$. Equivalently, we will also write $x = \Psi - \lim_{n\to\infty} x_n$.

Remark 3.1. *Since $x_n \xrightarrow{\Psi} x$ means that $\psi(x_n - x) \to 0$ for every $\psi \in \Psi$, it follows that any subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is Ψ-convergent to x as well.*

It is easy to notice that the convergence defined by Ψ is the L^*-convergence in the sense of Kisyński [23]. Hence, accordingt to [23, Theorem 3], the Ψ-convergence is equivalent to convergence in the topology, where open sets are defined as all subsets V of X such that if $x \in V$ and $x_n \xrightarrow{\Psi} x$, then $x_n \in V$ for sufficiently large n. For more details see also [24] and [25]. Consequently, we have the following definitions of Ψ-closed sets and sequentially Ψ-compact sets.

Definition 3.2. *Given $(X, \|\cdot\|, \Psi)$, a set $K \subset X$ is called Ψ-closed if, whenever $x_n \in K$ for all $n \in \mathbb{N}$ and $x_{n_k} \xrightarrow{\Psi} x$, it follows that $x \in K$. Similarly, K is called sequentially Ψ-compact if from every sequence $\{x_n\}$ of elements of K we can choose a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \xrightarrow{\Psi} x$ for an $x \in X$.*

Note that condition (A4) of Definition 3.1 simply means that the norm $\|\cdot\|$ is lower semi-continuous with respect to the Ψ-convergence, while (A3) means that the Ψ-convergence is weaker than the convergence with respect to the norm.

Let us consider a few examples that effectively illustrate the generality of the concept. These examples will be extensively utilised in the following sections.

Example 3.1. *The norm convergence in X is itself a Ψ-convergence with $\Psi = \{\|\cdot\|\}$.*

Example 3.2. *The convergence in the weak topology is also a Ψ-convergence with $\Psi = \{|\psi| : \psi \in S_{X^*}\}$, where S_{X^*} is the unit sphere in the dual space X^*.*

Example 3.3. *Similarly, the convergence in the weak-$*$ topology becomes a Ψ-convergence in X^* for $\Psi = \{|\psi| : \psi \in S_X\}$, where S_X is the unit sphere in the X.*

In the subsequent examples, we will assume that (Ω, Σ, μ) represents a finite atomless measure space, unless explicitly specified otherwise. Consequently, we will interpret convergence in measure and convergence almost everywhere in relation to μ, even if this is not explicitly stated.

Example 3.4. *Consider L^p, where $1 < p < +\infty$. Let $\Psi = \{\psi_0\}$, where for every $x \in L^p$,*

$$\psi_0(x) = \int_\Omega \frac{|x(t)|}{1 + |x(t)|} d\mu(t). \tag{3.2}$$

It is well known that the Ψ-convergence defined by (3.2) *is equivalent to convergence in measure. It is clear that ψ_0 is sub-additive, thus satisfying condition (A1) of Definition* 3.1. *While (A2) is evident, it is also well established that L^p convergence implies convergence in measure, thereby proving (A3). Furthermore, the Fatou Lemma demonstrates that the L^p-norm is lower semi-continuous with respect to convergence in measure, which leads us to (A4).*

Example 3.5. *Consider the Musielak-Orlicz space [12], defined as*

$$L^\varphi = \{x \in L^0 : \rho_\varphi(\lambda x) \to 0 \text{ as } \lambda \to 0^+\},$$

where ρ_φ is a convex modular given by the formula:

$$\rho_\varphi(x) = \int_\Omega \varphi(t, |x(t)|) d\mu(t),$$

and $\varphi : \Omega \times [0, +\infty) \to [0, +\infty)$, called Musielak-Orlicz function, satisfies the following two conditions:

(i) for every $t \in \Omega$, $\varphi(t,\cdot)$ *is a convex function such that* $\varphi(t,0) = 0$, $\varphi(t,u) > 0$ *for* $u > 0$.

(ii) for every $u \geq 0$, $\varphi(\cdot,u)$ *is an integrable function.*

It is well known (see, e.g., [12, Therem 7.7]) that L^φ *is a Banach space with the Luxemburg norm defined as*

$$\|x\|_\varphi = \inf\{\alpha > 0 : \rho_\varphi(x/\alpha) \leq 1\}. \tag{3.3}$$

Consider this space with convergence in measure, as discussed in Example 3.4. *Since* $\|x_n\|_\varphi \to 0$ *implies* $\rho_\varphi(\lambda x_n) \to 0$ *for any* $\lambda > 0$, *it easily follows that convergence in the Luxemburg norm implies convergence in measure (see [12, Remark 7.9]), thereby proving (A3). To demonstrate (A4), we first note that Musielak-Orlicz spaces are examples of modular function spaces, as discussed in the Preliminaries section of this paper. It is well known that in case of Musielak-Orlicz spaces, convergence* ρ*-a.e. is equivalent to convergence almost everywhere in the measure sense. In modular function spaces, modulars are lower semi-continuous with respect to convergence* ρ*-a.e. Hence,* $\rho_\phi(x) \leq \liminf_{n\to\infty} \rho_\phi(x_n)$ *provided* $x_n \to 0$ *almost everywhere. Assume that* $x_n \to x$ *in measure. By passing to a subsequence we can assume that* $x_{n_k} \to x$ *almost everywhere and that,*

$$\lim_{k\to\infty} \|x_{n_k}\|_\phi = \liminf_{n\to\infty} \|x_n\|_\phi.$$

Let $\lambda_k = \|x_{n_k}\|_\phi$ *and* $\lambda_0 = \lim_{k\to\infty} \lambda_k$. *Fix an arbitrary* $\varepsilon > 0$. *From* (3.3) *it follows that,*

$$\rho_\phi\Big(\frac{x_{n_k}}{\lambda_k + \varepsilon}\Big) \leq 1. \tag{3.4}$$

Since $\dfrac{x_{n_k}}{\lambda_k + \varepsilon} \to \dfrac{x}{\lambda_0 + \varepsilon}$ *almost everywhere, it follows from the lower semi-continuity of the modular, along with* (3.4)*, that,*

$$\rho_\phi\Big(\frac{x}{\lambda_0 + \varepsilon}\Big) \leq \liminf_{k\to\infty} \rho_\phi\Big(\frac{x_{n_k}}{\lambda_k + \varepsilon}\Big) \leq 1.$$

Thus, $\|x\|_\phi \leq \lambda_0 + \varepsilon$ *and, by the arbitrariness of* $\varepsilon > 0$, *it follows that,*

$$\|x\|_\phi \leq \lambda_0 = \liminf_{n\to\infty} \|x_n\|_\phi.$$

Note the special cases:

1) *If function* φ *does not depend on the second variable, then the corresponding Musielak-Orlicz space is referred to as Orlicz space. Orlicz space theory has been proved to be of major importance in mathematics and physics; see [20, 26–28].*
2) *When* $\varphi(u,t) = |u|^{p(t)}$ *for* $p(t) \geq 1$, *the space* L^φ *becomes the variable exponent Lebesgue space. The theory of such spaces has become highly significant due to its important applications in science and technology; see, for example, [29–31]. If* $p = p(t)$ *is a constant, we obtain the classical* L^p*-space.*

Example 3.6. *Sobolev spaces* $W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega)\ \forall |\alpha| \leqslant k\}$, *where* $\Omega \subset \mathbb{R}^d$ *is a bounded domain,* $1 \leq p < +\infty$, $k \in \mathbb{N}$ *and* $D^\alpha u$ *are weak derivatives, play an important in the theory of differential equations.* $W^{k,p}(\Omega)$ *becomes a Banach space with the norm defined by:*

$$\|u\|_{W^{k,p}(\Omega)} = \Bigg(\sum_{|\alpha|\leq k} \|D^\alpha u\|^p_{L^p(\Omega)}\Bigg)^{\frac{1}{p}}.$$

Consider this Banach space with Ψ defined as convergence in measure of measurable functions and their weak derivatives D^α for all $|\alpha| \le k$. Similarly, as we did in case of L^p in Example 3.4*, we can prove that $(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)}, \Psi)$ is a Banach space with convergence, as specified in Definition* 3.1*. This example can be easily extended to Orlicz-Sobolev and generalized Orlicz-Sobolev (Musielak-Orlicz-Sobolev) spaces discussed in [12, 32].*

Example 3.7. *Consider again L^p space as in Example* 3.4 *but this time the convergence Ψ is introduced by the Marcinkiewicz (weak L^p) quasi-norm defined as*

$$\|x\|_{L^{p,\infty}} = \sup_{\alpha>0} \alpha\, \mu\{u \in \Omega : |x(u)| > \alpha\}^{\frac{1}{p}}.$$

As a quasi-norm, $\|\cdot\|_{L^{p,\infty}}$ satisfies the generalised triangle inequality with a universal constant $c_\psi = 2^{\frac{1}{p}}$, thereby proving (A1) of Definition 3.1*. To prove (A3), note that the $L^{p,\infty}$-convergence is weaker than the L^p-convergence, which follows from the inequality $\|x\|_{L^{p,\infty}} \le \|x\|_{L^p}$. The latter inequality is a direct consequence of Markov's inequality:*

$$\mu\{u \in \Omega : |x(u)| > \alpha\} \le \frac{\|x\|_{L^p}^p}{\alpha^p}.$$

It is known that the L^p-norm is lower semicontinuous with respect to $\|x\|_{L^{p,\infty}}$. The standard proof relies on the Fatou Lemma and the fact that convergence in $\|x\|_{L^{p,\infty}}$ is stronger than convergence in measure, which allows to extract a subsequence converging almost everywhere, which is required by the Fatou Lemma. Therefore, condition (A4) is also satisfied.

Example 3.8. *The previous example can be easily extended to the more general case of Lorentz spaces $L^{p,q}$ for $q > p > 1$, where the Ψ-convergence is defined by the Lorentz quasi-norm:*

$$\|x\|_{p,q} = \left(\int_0^\infty \left(t^{\frac{1}{p}} x^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}. \tag{3.5}$$

In formula (3.5)*, x^* represents the decreasing rearrangement of the function x. The standard arguments show that conditions (A3) and (A4) hold in this case. This example can be further extended to Orlicz-Lorentz spaces, see, e.g., [33].*

In the following section, we will explore the concept of Ψ-convergence of sequences in the context of iterative fixed point construction processes. After presenting some general results, we will return to the examples discussed above to illustrate the effectiveness of the theory developed in this paper.

4. Convergence results

In this section, we will always assume that a pointwise Lipschitzian semigroup $\mathcal{T}$ is asymptotic pointwise nonexpansive on the set D, denoted symbolically as $\mathcal{T} \in \mathcal{APNS}(D)$. Recall that D is assumed to be a nonempty convex and bounded subset of X.

We will use the following definition to precisely describe convergence to a common fixed point of a pointwise Lipschitzian semigroup of operators acting within D.

Definition 4.1. *Given $\mathcal{T} \in \mathcal{APNS}(D)$, we say that a sequence $\{w_n\}$ of elements of D is Ψ-convergent to the set of common fixed point of $\mathcal{T}$, denoted by $F(\mathcal{T})$, if there exists a common fixed point $w \in F(\mathcal{T})$ such that $w = \Psi - \lim_{n\to\infty} w_n$.*

Consider a sequence of pointwise Lipschitzian operators $\{H_n\} \in \mathcal{A}(D)$. Such sequence introduces a generalised iterative procedure by setting a starting point, as specified by the following definitions.

Definition 4.2. *Let $\{H_n\} \in \mathcal{A}(D)$. An iteration process generated by $\{H_n\}$ starting at $u_1 \in D$, denoted by $\Gamma(\{H_n\}, u_1)$, is defined as the process producing a sequence $\{u_n\}$ by calculating $u_{n+1} = H_n(u_n)$ for $n \geq 1$. When we are not interested in a specific starting point, we will write $\Gamma(\{H_n\})$.*

We are interested in processes that are Ψ-convergent to some common fixed point of a given semigroup, as defined below. Let us introduce some convenient notation that will be used throughout thus paper: $S_k^n = H_n \circ H_{n-1} \circ ... \circ H_k$, where $n \geq k$ are natural numbers. Observe that $S_k^k = H_k$. By convention, we set $S_k^{k-1} = Id$, where Id stands for the identity operator.

Definition 4.3. *Let $\mathcal{T} \in \mathcal{APNS}(D)$ and $\{H_n\} \in \mathcal{A}(D)$. We say that an iteration process $\Gamma(\{H_k\})$ is Ψ-convergent to $F(\mathcal{T})$ if for every $x \in D$ and every $k \in \mathbb{N}$, the sequence $\{S_k^{k+n-2}(x)\}_{n\in\mathbb{N}}$ Ψ-converges to an element of $F(\mathcal{T})$.*

Note that the Ψ-convergence of an iteration process $\Gamma(H_n)$ to $F(\mathcal{T})$ simply means that, regardless of the initial starting point $x \in D$ selected, the iterates generated by $\{H_n(x)\}$ will Ψ-converge to a common fixed point of $\mathcal{T}$.

We will see that our convergence results, the general Theorem 4.3 and its specialisations in Section 7 for generalised Krasnosel'skii-Mann and Ishikawa processes, articulate precisely this phenomenon. However, it is important to recognise that, in general, asymptotic pointwise nonexpansive semigroups do not guarantee the uniqueness of common fixed points. Consequently, the result of the construction of a common fixed point, as outlined in Definition 4.3, may depend on the selection of the starting point.

In its classical form, Demiclosedness Principle provides an effective means for demonstrating convergence in the weak topology of fixed point construction processes. This method requires the specification of particular conditions on a Banach space X that guarantee an element of X serves as a fixed point if it is the weak limit of an approximate sequence. For more details, refer to the discussion in [1]. To adapt this idea to our setting, we introduce a concept of the Ψ-related Demiclosedness Property.

Definition 4.4. *We say that $(X, \|\cdot\|, \Psi)$ possesses the Demiclosedness Property, shortly the (DC)-property, if for every semigroup $\mathcal{T} \in \mathcal{APNS}(D)$ with nonempty $F(\mathcal{T})$ and every $\{w_n\}$, sequence of elements of D, Ψ-converging to $w \in D$, the following implication holds:*

$$\{w_n\} \textit{ is an asymptotic fixed point sequence} \Longrightarrow w \in F(\mathcal{T}). \tag{4.1}$$

The role of the Demiclosedness Property is elucidated in our next result. Although it may appear evident based on the definitions employed, it nonetheless provides clear guidance for addressing the convergence of iterative processes, which will be discussed later in this section.

Theorem 4.1. *Assume that $(X, \|\cdot\|, \Psi)$ has the Demiclosedness Property. Let, as usual, $D \subset X$ be closed, bounded and convex. In addition, assume that D is sequentially compact with respect to Ψ.*

Given $\mathcal{T} \in \mathcal{APNS}(D)$, assume that $\{w_n\}$ is an asymptotic fixed point sequence. If the sequence $\{w_n\}$ has at most one Ψ-cluster point in D, then this sequence Ψ-converges to $F(\mathcal{T})$.

Proof. Since D is sequentially compact with respect to Ψ and $\{w_n\}$ has at most one Ψ-cluster point in D, it follows that there exist $w \in D$ such that $w = \Psi - \lim_{n\to\infty} w_n$. By the (DC)-property, $w \in F(\mathcal{T})$. □

Since the (DC)-property plays the crucial role in the proof of Theorem 4.1, we need to establish some practical mechanisms that will guarantee this property for a large variety of cases including examples discussed in the previous section. In this paper, we will mainly deal with characteristics of Banach spaces that are related to the renowned Opial property [2].

Definition 4.5. *We say that $(X, \|\cdot\|, \Psi)$ has the non-strict Ψ-Opial property if for every $x \in X$,*

$$\liminf_{n\to\infty} \|w_n - w\| \leq \liminf_{n\to\infty} \|w_n - x\|, \tag{4.2}$$

or equivalently

$$\limsup_{n\to\infty} \|w_n - w\| \leq \limsup_{n\to\infty} \|w_n - x\|, \tag{4.3}$$

provided $\{w_n\}$ is a $\|\cdot\|$-bounded sequence such that $w_n \xrightarrow{\Psi} w$.

Definition 4.6. *We say that $(X, \|\cdot\|, \Psi)$ has the Ψ-Opial property if for every $x \neq w$,*

$$\liminf_{n\to\infty} \|w_n - w\| < \liminf_{n\to\infty} \|w_n - x\|, \tag{4.4}$$

or equivalently

$$\limsup_{n\to\infty} \|w_n - w\| < \limsup_{n\to\infty} \|w_n - x\|, \tag{4.5}$$

provided $\{w_n\}$ is a $\|\cdot\|$-bounded sequence such that $w_n \xrightarrow{\Psi} w$.

Note that Ψ-Opial property implies non-strict Ψ-Opial property.

The proof of equivalence between the lim inf and lim sup conditions uses the standard subsequence argument. For the sake of completeness we detail the proof for the strict case. The proof of the non-strict case follows the same approach.

Proposition 4.1. *Let $w_n \xrightarrow{\Psi} w$. Assume that $x \neq w$. Then, conditions* (4.4) *and* (4.5) *are equivalent.*

Proof. Part 1. The lim inf condition implies the lim sup condition.
Choose the subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that,

$$\lim_{k\to\infty} \|w_{n_k} - w\| = \limsup_{n\to\infty} \|w_n - w\|. \tag{4.6}$$

By Remark 3.1, $\psi(w_{n_k} - w) \to 0$. Therefore, we can apply (4.4) to it.

$$\liminf_{k\to\infty} \|w_{n_k} - w\| < \liminf_{k\to\infty} \|w_{n_k} - x\|.$$

Since $\lim\limits_{k\to\infty} \|w_{n_k} - w\|$ exists, it follows from (4.6) that,

$$\liminf_{k\to\infty} \|w_{n_k} - w\| = \lim_{k\to\infty} \|w_{n_k} - w\| = \limsup_{n\to\infty} \|w_n - w\|.$$

Hence,

$$\limsup_{n\to\infty} \|w_n - w\| = \liminf_{k\to\infty} \|w_{n_k} - w\| < \liminf_{k\to\infty} \|w_{n_k} - x\| \le \limsup_{k\to\infty} \|w_{n_k} - x\| \le \limsup_{n\to\infty} \|w_n - x\|.$$

Part 2. The lim sup condition implies the lim inf condition.
Choose the subsequence $\{w_{n_k}\}$ of $\{w_n\}$ so that,

$$\lim_{k\to\infty} \|w_{n_k} - x\| = \liminf_{n\to\infty} \|w_n - x\|. \tag{4.7}$$

By (4.5),

$$\limsup_{k\to\infty} \|w_{n_k} - w\| < \limsup_{k\to\infty} \|w_{n_k} - x\|.$$

Note that,

$$\liminf_{n\to\infty} \|w_n - w\| \le \liminf_{k\to\infty} \|w_{n_k} - w\| \le \limsup_{k\to\infty} \|w_{n_k} - w\|. \tag{4.8}$$

Because $\lim\limits_{k\to\infty} \|w_{n_k} - x\|$ exists, it follows from (4.7) that,

$$\limsup_{k\to\infty} \|w_{n_k} - x\| = \lim_{k\to\infty} \|w_{n_k} - x\| = \liminf_{n\to\infty} \|w_n - x\|. \tag{4.9}$$

Finally, using (4.8) and (4.9), we conclude that,

$$\liminf_{n\to\infty} \|w_n - w\| \le \limsup_{k\to\infty} \|w_{n_k} - w\| < \limsup_{k\to\infty} \|w_{n_k} - x\| = \liminf_{n\to\infty} \|w_n - x\|.$$

□

We will now demonstrate that the non-strict Ψ-Opial property provides a universal mechanism for proving the (DC)-property in uniformly convex Banach spaces.

Theorem 4.2 (Demiclosedness Principle)**.** *Let X be a uniformly convex Banach space. Assume that $(X, \|\cdot\|, \Psi)$ has the non-strict Ψ-Opial property. Then, $(X, \|\cdot\|, \Psi)$ has the (DC)-property.*

Proof. Assume that $\mathcal{T} \in \mathcal{APNS}(D)$. Since X is uniformly convex, it follows that $F(\mathcal{T}) \neq \emptyset$. Let $\{w_n\}$ be an asymptotic fixed point sequence Ψ-converging to $w \in D$. We need to show that $w \in F(\mathcal{T})$. Let φ be a function defined for every $x \in D$ by

$$\varphi(x) = \limsup_{n\to\infty} \|w_n - x\|.$$

Let us arbitrarily fix an $s \in J$. We claim that,

$$\lim_{k\to\infty} \varphi(T_{ks}(w)) = \varphi(w). \tag{4.10}$$

From the non-strict Opial condition (4.3) it follows that for every $x \in D$,

$$\varphi(w) = \limsup_{n\to\infty} \|w_n - w\| \le \limsup_{n\to\infty} \|w_n - x\| = \varphi(x).$$

This implies that φ attains its minimum at $w \in D$. Thus, to prove (4.10) it is enough to demonstrate that,

$$\lim_{k\to\infty} \varphi(T_{ks}(w)) \le \varphi(w). \tag{4.11}$$

Since $\{x_n\}$ is an asymptotic fixed point sequence, we observe that for any $x \in D$,

$$\varphi(x) \leq \limsup_{n\to\infty} \|w_n - T_{ks}(w_n)\| + \limsup_{n\to\infty} \|T_{ks}(w_n) - x\| = \limsup_{n\to\infty} \|T_{ks}(w_n) - x\|.$$

By applying this to $x = T_{ks}(w)$, we derive the following inequality:

$$\varphi(T_{ks}(w)) \leq \limsup_{n\to\infty} \|T_{ks}(w_n) - T_{ks}(w)\| \leq a_{ks}(w) \limsup_{n\to\infty} \|w_n - w\| = a_{ks}(w)\varphi(w).$$

Remembering that $\lim_{k\to\infty} a_{ks}(w) = 1$, take the limits of both sides as $k \to \infty$ to obtain inequality (4.11), proving our claim (4.10). Choose $d > 0$ such that $\|u\| \leq \frac{d}{2}$ for every $u \in D$. Because X is uniformly convex, if follows from [34, Theorem 2] that there exists a continuous, strictly increasing and convex function $\lambda_d : [0, \infty) \to [0, \infty)$ such that $\lambda_d(t) = 0$ if and only $t = 0$, and the following parallelogram inequality holds for any $\alpha \in [0, 1]$ and all x, y such that $\|x\| \leq d$ and $\|y\| \leq d$,

$$\|\alpha x + (1 - \alpha)y\|^2 \leq \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\lambda_d(\|x - y\|).$$

By applying this inequality to $x = w_n - w$, $y = w_n - T_{ks}(w)$ and $\alpha = \frac{1}{2}$ we have,

$$\left\|w_n - \frac{1}{2}\Big(w + T_{ks}(w)\Big)\right\|^2 \leq \frac{1}{2}\|w_n - w\|^2 + \frac{1}{2}\|w_n - T_{ks}(w)\|^2 - \frac{1}{4}\lambda_d(\|T_{ks}(w) - w\|),$$

which implies that,

$$\varphi(w)^2 \leq \frac{1}{2}\varphi(w)^2 + \frac{1}{2}\varphi(T_{ks}(w))^2 - \frac{1}{4}\lambda_d(\|T_{ks}(w) - w\|), \tag{4.12}$$

where we used the fact that φ attains its minimum at w. By taking the limits in (4.12) as $k \to \infty$ and using the equality (4.10), we conclude that,

$$\lim_{k\to\infty} \lambda_d(\|T_{ks}(w) - w\|) = 0. \tag{4.13}$$

Since λ_d is strictly increasing and continuous, equal zero only at zero, it follows from (4.13) that $T_{ks}(w) \to w$ as $k \to \infty$. Using the continuity of T_s, we infer from this that,

$$T_s(T_{ks}(w)) \to T_s(w), \tag{4.14}$$

when $k \to \infty$. By the same argument as above, we can conclude that $T_{(1+k)s}(w) \to w$ as $k \to \infty$. Thus,

$$T_s(T_{ks}(w)) = T_{(1+k)s}(w) \to w. \tag{4.15}$$

Given the uniqueness of the limit, equations (4.14) and (4.15) establish that $T_s(w) = w$. By arbitrariness of $s \in J$, we get $w \in F(\mathcal{T})$. Therefore, $(X, \|\cdot\|, \Psi)$ has the (DC)-property, as claimed. □

In the following discussion, we will need to assume some degree of control over the iteration processes. As will be clarified in the next sections, typical iteration processes conform to these restrictions.

Definition 4.7. *Let $\{x_k\}$ be a sequence of elements from D. Let $\{H_k\} \in \mathcal{A}(D)$. Given $1 < L < +\infty$, a sequence $\{H_k\}$ will be called well-controlled by L if there exists a natural number p_L such that,*

$$\prod_{p=p_L}^{\infty} L_p \leq L < +\infty, \tag{4.16}$$

where

$$L_p = \sup\{A_p(z) : z \in D\}, \tag{4.17}$$

where functions $\{A_k(\cdot)\}$ are associated with $\{H_k\}$ by the definition of $\mathcal{A}(D)$. We say that $\{H_k\}$ is well-controlled if it is controlled by some L. In this situation, we will also say that the process $\Gamma(\{H_k\})$ is well-controlled.

Recall that for each $\{H_k\} \in \mathcal{A}(D)$ we assume that to every $x \in D$ there exists $A_k(x) \geq 1$ such that for each $y \in D$,

$$\|H_k(x) - H_k(y)\| \leq A_k(x)\|x - y\|, \tag{4.18}$$

and $\lim_{k\to\infty} A_k(x) = 1$. We also use notation: $B_k(x) = A_k(x) - 1$.

We will need the following technical result.

Lemma 4.1. *[1, Lemma 3.5] Let $\{H_k\} \in \mathcal{A}(D)$ and let $\{w_k\}$ be a sequence generated by $\{H_k\}$. Assume that $w \in \bigcap_{k=1}^{\infty} F(H_k)$ is such that,*

$$\sum_{i=1}^{\infty} B_i(w) < \infty. \tag{4.19}$$

Then, there exists an $r \in \mathbb{R}$ such that $\lim_{k\to\infty} \|x_k - w\| = r$.

When discussing convergence of iterative processes to common fixed points of semigroups, some structural connection between them is necessary. This necessary compatibility connection is captured by the following definition.

Definition 4.8. *We say that the sequence $\{H_k\} \in \mathcal{A}$ (alternatively, that the process $\Gamma\{H_k\})$) is $\mathcal{T}$-compatible if*

$$F(\mathcal{T}) \subset \bigcap_{n=1}^{\infty} F(H_n). \tag{4.20}$$

We are now ready to formulate our main convergence problem.

Problem 2. *Let us consider the setting defined by the following assumptions:*

1) *X is a uniformly convex Banach space.*
2) *$D \subset X$ is a nonempty convex, bounded and closed in the norm sense.*
3) *D is sequentially Ψ-compact.*
4) *$\mathcal{T} \in \mathcal{APNS}(D)$.*
5) *$\{w_n\} = \Gamma(\{H_n\})$ is an approximate fixed point sequence, where $\{H_n\} \in \mathcal{A}(D)$ is well-controlled and $\mathcal{T}$-compatible.*

Question: What do we need to assume about $(X, \|\cdot\|, \Psi)$ to ensure that the process $\Gamma(\{H_n\})$ is Ψ-convergent to $F(\mathcal{T})$?

In our next result, we resolve Problem 2.

Theorem 4.3. *In the settings of Problem 2, assume that $(X, \|\cdot\|, \Psi)$ has the Ψ-Opial property. Then, the sequence $\{w_n\} = \Gamma(\{H_n\})$ is Ψ-convergent to $F(\mathcal{T})$, which means that there exists $w \in D$ such that $w_n \xrightarrow{\Psi} w$.*

Proof. Since X is uniformly convex and has the Ψ-Opial property, it follows from the Demiclosedness Principle (Theorem 4.2) that $(X, \|\cdot\|, \Psi)$ posses the (DC)-property. Because of this and the Ψ-compactness of D, by employing Theorem 4.1, the only thing needed to show that $\{w_n\}$ Ψ-converges to an element of $F(\mathcal{T})$ is to prove is that the sequence $\{w_n\}$ has at most one Ψ-cluster point. Assume to the contrary that $u \neq v$ are two different Ψ-cluster points of $\{w_n\}$. There exist then $\{u_n\}$ and $\{v_n\}$, subsequences of $\{w_n\}$, such that $u_n \xrightarrow{\Psi} u$ and $v_n \xrightarrow{\Psi} v$. By the Demiclosedness Property, both u and v belong to $F(\mathcal{T}) \subset \bigcap_{n=1}^{\infty} F(H_n)$. Note that condition 4.19 is satisfied because $\{H_n\} \in \mathcal{A}(D)$ is well-controlled, which means by (4.16) that,

$$\sum_{i=1}^{\infty} B_i(u) = \sum_{i=1}^{\infty}(A_i(u) - 1) \leq \sum_{i=1}^{\infty}(L_i - 1) \leq \prod_{i=1}^{\infty} L_i < \infty.$$

Therefore, it follows from Lemma 4.1 that there exist $r_1, r_2 \in \mathbb{R}$ such that

$$r_1 = \lim_{n\to\infty} \|w_n - u\| = \lim_{n\to\infty} \|w_n - u\|, \; r_2 = \lim_{n\to\infty} \|w_n - v\|. \tag{4.21}$$

Because $\{v_n\}$ and $\{u_n\}$ are both subsequences of $\{w_n\}$, the Ψ-Opial property and (4.21) imply that,

$$\begin{aligned} r_1 &= \lim_{n\to\infty} \|w_n - u\| = \liminf_{n\to\infty} \|u_n - u\| < \liminf_{n\to\infty} \|u_n - v\| = \lim_{n\to\infty} \|w_n - v\| = r_2 \\ &= \liminf_{n\to\infty} \|v_n - v\| < \liminf_{n\to\infty} \|v_n - u\| = \lim_{n\to\infty} \|w_n - u\| = r_1. \end{aligned}$$

The contradiction implies that $\{w_n\}$ has at most one Ψ-cluster point in D. Since D is sequentially compact with respect to Ψ, it follows that there exist $w \in D$ such that $w_n \xrightarrow{\Psi} w$. By the (DC)-property, $w \in F(\mathcal{T})$. The proof is complete. □

The above result is new. The closest examples, would be a combination of results from [1] spread across Theorem 3.4 (Demiclosedness Principle), weak convergence results for Krasnosel'skii-Mann, Ishikawa and implicit iteration processes (Theorems 4.1, 5.1, 6.1, respectively), strong convergence results for these processes (Theorems 4.11, 5.7, 6.3, respectively). These convergence results can be traced back to [35, Theorems 22.23, 22.35, 22.39].

Now, let us revisit Examples 3.1 to 3.8 in light of the application of Theorem 4.3. In the scenarios discussed below, we assume that all assumptions on X, D, $\mathcal{T}$, $\{H_n\}$, $\{w_n\}$, as stated in Problem 2, are satisfied.

Proposition 4.2. *Strong (norm) convergence (Example 3.1). When Ψ represents convergence with respect to the original norm in the Banach space X, the Ψ-Opial inequality given in (4.4) is trivially fulfilled. By Theorem 4, the sequence $\{w_n\} = \Gamma(\{H_n\})$ converges in norm to $F(\mathcal{T})$ provided D is norm-compact. The difficulty of this case lies in the fact that the norm-compactness in infinite-dimensional Banach spaces is a very strong assumption, which is usually unrealistic in practice.*

Proposition 4.3. *Weak convergence (Example 3.2). When X is a uniformly convex Banach space with the weak-Opial property, Theorem 4.3 guarantees the weak convergence of* $\{w_n\} = \Gamma(\{H_n\})$ *to* $F(\mathcal{T})$*. Weak-compactness of D follows from its convexity, boundedness and norm-closedness. This is the case, for example for* l^p *for* $p > 1$*; however it does not apply to* L^p *when* $1 < p \neq 2$ *because such spaces fail the weak-Opial condition. For the latter case, convergence in measure (Example 3.4), convergence in the sense of the Marcinkiewicz-quasi-norm (Example 3.7), or Lorentz-quasi-norm (Example 3.8) may be more suitable.*

Remark 4.1. *We would like to highlight the existence of alternative methods for spaces* L^p *when* $1 < p \neq 2$ *and for other Banach spaces that are both uniformly convex and uniformly smooth, as discussed in [1, 36]. However, the techniques utilised in those references are more complex, with stricter assumptions compared to the current framework that relies on the* Ψ*-Opial property. Additionally, the approach presented in the current paper enables a unified treatment of all* L^p *and other related spaces, as indicated by the examples provided.*

Proposition 4.4. *Weak-$*$ convergence (Example 3.3). If X is uniformly smooth, then* X^* *is uniformly convex and Theorem 4.3 guarantees the weak-$*$-convergence of* $\{w_n\} = \Gamma(\{H_n\})$ *to* $F(\mathcal{T})$ *provided* X^* *has the weak-$*$ Opial property and the convex set D is bounded and closed in* X^**.*

The following result will be utilised when considering other examples.

Proposition 4.5. *Consider* $(L^p, \|\cdot\|_p, \Psi)$*, where* $p > 1$*. If from every bounded sequence* $\{w_n\}$*, which* Ψ*-converges to* $w \in L^p$*, we can extract a subsequence almost everywhere converging to w, then the Banach space with convergence* $(L^p, \|\cdot\|_p, \Psi)$ *possesses the* Ψ*-Opial property.*

Proof. Assume to the contrary that there exists $x \neq w$ such that,

$$\liminf_{n\to\infty} \|w_n - x\|_p \leq \liminf_{n\to\infty} \|w_n - w\|_p,$$

where $\{w_n\}$ is a bounded sequence, which Ψ-converges to $w \in L^p$. By passing (possibly twice) to a subsequence, we can select a subsequence $\{w_{n_k}\}$ such that $w_{n_k} \to w$ almost everywhere and

$$\lim_{k\to\infty} \|w_{n_k} - x\|_p = \liminf_{n\to\infty} \|w_n - x\|_p \leq \liminf_{n\to\infty} \|w_n - w\|_p \leq \liminf_{k\to\infty} \|w_{n_k} - w\|_p.$$

Thus, from now on, we can assume that $w_n \to w$ almost everywhere and

$$\lim_{n\to\infty} \|w_n - x\|_p \leq \liminf_{n\to\infty} \|w_n - w\|_p. \tag{4.22}$$

Since the function $t \mapsto t^p$ is for $t > 0$ continuous and increasing, it follows from (4.22) that,

$$\lim_{n\to\infty} \|w_n - x\|_p^p \leq \liminf_{n\to\infty} \|w_n - w\|_p^p. \tag{4.23}$$

From the Brezis-Lieb Theorem ([37, Theorem 1]) it follows that,

$$\liminf_{n\to\infty} \|w_n - x\|_p^p = \liminf_{n\to\infty} \|w_n - w\|_p^p + \|w - x\|_p^p.$$

This implies that,

$$\liminf_{n\to\infty} \|w_n - w\|_p^p < \liminf_{n\to\infty} \|w_n - x\|_p^p,$$

which contradicts (4.23), finishing the proof. □

Corollary 4.1. *It follows directly from Proposition* 4.5 *that the following spaces possess the respective* Ψ*-Opial property:*

1) L^p *for* $p > 1$ *(Example* 3.4*), where* Ψ *is the convergence in measure: using Proposition* 4.5*, we conclude from Theorem* 4.3 *that* $\{w_n\} = \Gamma(\{H_n\})$ *converges to* $F(\mathcal{T})$*, provided the convex set* D *is sequentially compact with respect to the topology of convergence in measure. Note that the convergence in measure of* $\{w_n\}$ *implies that the sequence* $\{w_n\}$ *converges to* $F(\mathcal{T})$ *in the weak topology as well, provided the sequence* $\{w_n\}$ *is bounded.*
2) *Sobolev space with* Ψ *being the convergence in measure of functions and all their relevant weak derivatives (Example* 3.6*): apply Proposition* 4.5 *to functions and each of their derivatives.*
3) L^p *(where* $p > 1$*) with the convergence with respect to the Marcinkiewicz quasi-norm and Lorentz-quasi-norm (Examples* 3.7 *and* 3.8*, respectively): the* Ψ*-Opial property is ensured by Proposition* 4.5*, as both of these convergences imply convergence in measure, which in turn guarantees the existence of a subsequence that converges almost everywhere (since the measure is finite).*

The case of Musielak-Orlicz spaces requires additional preparatory work.

Next, let us recall the following definition of the Δ_2-condition for Musielak-Orlicz functions φ (see [12, Definition 8.13]).

Definition 4.9. *We say that the Musielak-Orlicz function* φ *satisfies the* Δ_2 *condition if there exists a nonnegative function* $h \in L^1$ *and a constant* $0 < M(2) < +\infty$*, for which the following inequality*

$$\varphi(t, 2u) \leq M(2)\varphi(t, u) + h(t),$$

holds for all $u \geq 0$ *and almost all* $t \in \Omega$*. Hence, for every* $x \in L^\varphi$*,*

$$\rho_\varphi(2x) \leq M(2)\rho_\varphi(x) + H, \tag{4.24}$$

where $H = \int_\Omega h(t)d\mu(t) < +\infty$.

If the constant $H = 0$*, we will say that* φ *satisfies the* Δ_2*-type condition.*

Remark 4.2. *Clearly, the* Δ_2*-type condition for* φ *implies the* Δ_2*-condition for* φ*.*

Proposition 4.6. *Assume that the Musielak-Orlicz function* φ *satisfies the* Δ_2 *condition. Then, for any* $\beta > 1$ *there exist* $M(\beta) > 0$ *and a nonnegative function* $g \in L^1$ *such that,*

$$\varphi(t, \beta u) \leq M(\beta)\varphi(t, u) + g(t), \tag{4.25}$$

holds for all $u \geq 0$ *and almost all* $t \in \Omega$*. Hence, for every* $x \in L^\varphi$*,*

$$\rho_\varphi(\beta x) \leq M(\beta)\rho_\varphi(x) + G, \tag{4.26}$$

where $G = \int_\Omega g(t)d\mu(t) < +\infty$

Proof. Assume that $\beta \leq 2^n$ for an $n \in \mathbb{N}$. Then, the rest of the proof follows from the following inequality:

$$\varphi(t, \beta u) \leq M(2)^n\varphi(t, u) + \Big(\sum_{i=0}^{n-1} M(2)^i\Big)h(t).$$

□

The following estimate shows, in particular, that the norm bounded sets are also ρ_φ-bounded, provided φ satisfies the Δ_2 condition.

Remark 4.3. *Assume that φ satisfies the Δ_2 condition. Let $\|x\|_\rho \le \alpha$ and $\beta \ge 1$, then*

$$\rho_\varphi(\beta x) \le M(\alpha\beta)\rho_\varphi\Big(\frac{x}{\alpha}\Big) + H \le M(\alpha\beta) + H < +\infty, \tag{4.27}$$

where we used the fact that for every left-continuous ρ, $\|w\|_\rho \le 1$ if only if $\rho \le 1$.

Proposition 4.7. *The Musielak-Orlicz function φ satisfies the Δ_2 condition if and only if the modular ρ_φ has the Δ_2 property (as defined in Definition 2.9).*

Proof. It follows from the Egoroff Theorem and the fact that $f_n \rightrightarrows 0$ implies $\rho(f_n) \to 0$, that if φ satisfies the Δ_2 condition, then the modular ρ_φ has the Δ_2 property. For the other direction, let us assume that ρ_φ has the Δ_2 property but φ fails the Δ_2 condition. Take $M = 2$ and $h = 1_\Omega$. Hence there exists a measurable set $\Omega' \subset \Omega$ with $\mu(\Omega') > 0$ and a number $u > 0$ such that,

$$\varphi(t, 2u) > 2\varphi(t, u) + 1_\Omega(t), \tag{4.28}$$

for every $t \in \Omega'$. Let $\{\Omega'_n\}$ be a decreasing to an empty set sequence of measurable subsets of Ω'. Define $x_n = 1_{\Omega'_n}$. Since ρ_φ has the Δ_2 property, it follows that $\rho_\varphi(x_n) \to 0$, which implies (also by the the Δ_2 property of ρ_φ) that,

$$\rho_\varphi(2x_n) \to 0. \tag{4.29}$$

On the other hand, by using (4.28), we have,

$$\rho_\varphi(2x_n) = \int_{\Omega'_n} \varphi(t, 2u)d\mu(t) \ge 2\int_{\Omega'_n} \varphi(t, u)d\mu(t) + \mu(\Omega) > \mu(\Omega) > 0,$$

contradicting (4.29). □

Remark 4.4. *Using the Lebesgue Dominated Convergence Theorem, it is easy to demonstrate that the Δ_2-property implies that ρ_φ is a right-continuous modular. Since $\rho_\varphi(\lambda x) \to 0$ as $\lambda \to 0^+$, it follows from (4.26) that, in the Δ_2 case, $\rho_\varphi(x) < +\infty$ for every $x \in L^\varphi$. Moreover, according to (4.26), for any $\delta > 0$, we have $w_\delta(s) < +\infty$ for all $s \ge 0$. This leads us to conclude, based on Proposition 2.3, that w_δ is a continuous function, whenever the Musielak-Orlicz function φ satisfies the Δ_2 condition.*

Note that every Musielak-Orlicz modular is orthogonally additive. Therefore, remembering that for Musielak-Orlicz spaces, ρ-a.e. convergence is equivalent to convergence almost everywhere with respect to measure μ, we can adapt Theorem 2.3 to our Musielak-Orlicz setting, as stated below.

Theorem 4.4. *Let L^φ be Musielak-Orlicz space with the Δ_2-property. Let $w_n \to w$ m-a.e., where $w \in L^\varphi$ and $\sup_{n\in\mathbb{N}} \rho_\varphi(w_n) < +\infty$, then for every $x \in L^\varphi$ such that $x \ne w$, we have,*

$$\liminf_{n\to\infty} \rho_\varphi(w_n - x) = \liminf_{n\to\infty} \rho_\varphi(w_n - w) + \rho_\varphi(x - w) > \liminf_{n\to\infty} \rho_\varphi(w_n - w).$$

Our next result shows that, if ρ_φ is uniformly continuous, then L^φ possesses the Ψ-Opial property, where Ψ denotes convergence in measure m.

Theorem 4.5. *Let L^φ be Musielak-Orlicz space such that ρ_φ is uniformly continuous. Let $w_n \to w$ in measure, where $w \in L^\varphi$ and $\sup_{n\in\mathbb{N}} \|w_n\|_\varphi < +\infty$, then for every $x \in L^\varphi$ such that $x \neq w$, we have,*

$$\liminf_{n\to\infty} \|w_n - w\|_\varphi < \liminf_{n\to\infty} \|w_n - x\|_\varphi. \tag{4.30}$$

Therefore, L^φ possesses the Opial property with respect to convergence in measure.

Proof. Note first that, by part 3 of Proposition 2.4, the modular ρ_φ has the Δ_2 property. This implies, by Proposition 4.7, that φ satisfies the Δ_2 condition. Next, let us assume to the contrary that (4.30) does not hold, that is that,

$$\liminf_{n\to\infty} \|w_n - x\|_\varphi \leq \liminf_{n\to\infty} \|w_n - w\|_\varphi.$$

Using our assumptions and the same subsequence argument as the one used in the proof of Proposition 4.5, we can assume that $w_n \to w$ almost everywhere and

$$\lim_{n\to\infty} \|w_n - x\|_\varphi \leq \liminf_{n\to\infty} \|w_n - w\|_\varphi. \tag{4.31}$$

Without any loss of generality, we can assume that,

$$\liminf_{n\to\infty} \|w_n - w\|_\varphi = \nu > 0, \tag{4.32}$$

as otherwise, by (4.31), $\lim_{n\to\infty} \|w_n - x\|_\varphi$ would equal zero, implying $w_n \to x$ in measure, which contradicts our assumption $x \neq w$. Observe that, by Theorem 4.4, for every $\alpha > 0$,

$$\liminf_{n\to\infty} \rho_\varphi\left(\frac{w_n - x}{\alpha}\right) - \liminf_{n\to\infty} \rho_\varphi\left(\frac{w_n - w}{\alpha}\right) = \rho_\varphi\left(\frac{x - w}{\alpha}\right) > 0, \tag{4.33}$$

because $\varphi_\alpha(t, u) := \varphi\left(t, \frac{u}{\alpha}\right)$ is also a Musielak-Orlicz function satisfying the Δ_2-condition, sequence $\left\{\frac{w_n}{\alpha}\right\}$ converges to $\frac{w}{\alpha}$ almost everywhere, and $\sup_{n\in\mathbb{N}} \rho_\varphi\left(\frac{w_n}{\alpha}\right) < +\infty$ due to the norm boundedness of $\{\|w_n\|_\varphi\}$; refer to Remark 4.3.

Define two functions, Λ_w and Λ_x, on the interval $[0, \infty)$ by the formula:

$$\Lambda_w(s) = \liminf_{n\to\infty} \rho_\varphi(s(w_n - w)), \ \Lambda_x(s) = \liminf_{n\to\infty} \rho_\varphi(s(w_n - x)).$$

We claim that there exists $s_0 > \frac{1}{\nu}$ such that $\Lambda_w(s_0) \geq 1$. Indeed, assume that this is not the case, that is, for every $s > \frac{1}{\nu}$ it holds,

$$\Lambda_w(s) = \liminf_{n\to\infty} \rho_\varphi(s(w_n - w)) < 1.$$

This implies that for some subsequence $\{w_{a_n}\}$ of $\{w_n\}$, we have $\rho_\varphi(s(w_{a_n} - w)) < 1$, and finally that $\|w_{a_n} - w\|_\varphi \leq \frac{1}{s} < \nu$. Thus,

$$\nu = \liminf_{n\to\infty} \|w_n - w\|_\varphi \leq \liminf_{n\to\infty} \|w_{a_n} - w\|_\varphi \leq \frac{1}{s} < \nu,$$

which is impossible. Since $\{\|w_n\|_\varphi\}$ is a bounded sequence, there exists $K > 0$ such that $\|w_n - w\|_\varphi \leq K$ and $\|w_n - x\|_\varphi \leq K$ hold for all $n \in \mathbb{N}$. Let $\beta = s_0 + 2$ and note that by Remark 4.3,

$$\sup_{n\in\mathbb{N}} \rho_\varphi(\beta(w_n - u)) \leq M(\beta K) + H < \infty, \tag{4.34}$$

holds for both $u = w$ and $u = x$. It follows from Theorem 2.2 that both Λ_w and Λ_x functions are continuous and nondecreasing on $[0,\beta]$. Since $\Lambda_w(0) = 0$ and $\Lambda_w(s_0) \geq 1$, there must exist $s_1 \leq s_0 \leq \beta$ such that $\Lambda_w(s_1) = 1$. Denote $\alpha = \dfrac{1}{s_1}$. By utilising (4.33), we conclude that there exists $\gamma > 1$ such that,

$$\liminf_{n\to\infty} \rho_\varphi\Big(\frac{w_n - w}{\alpha}\Big) = 1 < \gamma < \liminf_{n\to\infty} \rho_\varphi\Big(\frac{w_n - x}{\alpha}\Big). \tag{4.35}$$

Select a subsequence of $\{w_n\}$ so that,

$$\lim_{k\to\infty} \rho_\varphi\Big(\frac{w_{n_k} - w}{\alpha}\Big) = \liminf_{n\to\infty} \rho_\varphi\Big(\frac{w_n - w}{\alpha}\Big). \tag{4.36}$$

Thus,

$$\lim_{k\to\infty} \rho_\varphi\Big(\frac{w_{n_k} - w}{\alpha}\Big) = 1 < \gamma < \liminf_{n\to\infty} \rho_\varphi\Big(\frac{w_n - x}{\alpha}\Big) \leq \liminf_{k\to\infty} \rho_\varphi\Big(\frac{w_{n_k} - x}{\alpha}\Big). \tag{4.37}$$

Hence, for every $0 < \varepsilon' < \gamma - 1$ there exists $k_{\varepsilon'} \in \mathbb{N}$ such that for every $k \geq k_{\varepsilon'}$,

$$\rho_\varphi\Big(\frac{w_{n_k} - w}{\alpha}\Big) \leq 1 + \varepsilon' < \gamma,$$

which implies that for such k,

$$\rho_\varphi\Big(\frac{w_{n_k} - w}{\alpha(1 + \varepsilon')}\Big) \leq \frac{1}{1 + \varepsilon'}\, \rho_\varphi\Big(\frac{w_{n_k} - w}{\alpha}\Big) \leq 1. \tag{4.38}$$

By (4.37), there exists also k_0, which is independent of ε', such that,

$$\rho_\varphi\Big(\frac{w_{n_k} - x}{\alpha}\Big) > \gamma > 1, \tag{4.39}$$

holds for $k \geq k_0$. By the left-continuity of ρ_φ, we conclude from (4.38) that $\|w_{n_k} - w\|_\rho \leq \alpha(1 + \varepsilon')$ for $k \geq k_{\varepsilon'}$. Utilising this and (4.31), we have,

$$\lim_{k\to\infty} \|w_{n_k} - x\|_\varphi = \lim_{n\to\infty} \|w_n - x\|_\varphi \leq \liminf_{n\to\infty} \|w_n - w\|_\varphi \leq \liminf_{k\to\infty} \|w_{n_k} - w\|_\varphi \leq \alpha(1 + \varepsilon').$$

Since $\varepsilon' > 0$ can be arbitrarily small, we conclude that $\lim_{k\to\infty} \|w_{n_k} - x\|_\varphi \leq \alpha$. Therefore, for every $\varepsilon > 0$ there exists a $k_\varepsilon > k_0$ such that for every $k \geq k_\varepsilon$, $\|w_{n_k} - x\|_\varphi \leq \alpha + \varepsilon$. This implies that,

$$\rho_\varphi\Big(\frac{w_{n_k} - x}{\alpha + \varepsilon}\Big) \leq 1, \tag{4.40}$$

for $k \geq k_\varepsilon$. Assume now that $\varepsilon < 1$. Thus,

$$0 < \delta := \liminf_{k\to\infty} \rho_\varphi\Big(\frac{w_{n_k} - w}{\alpha + 1}\Big) < \liminf_{k\to\infty} \rho_\varphi\Big(\frac{w_{n_k} - x}{\alpha + 1}\Big) \leq \liminf_{k\to\infty} \rho_\varphi\Big(\frac{w_{n_k} - x}{\alpha + \varepsilon}\Big). \tag{4.41}$$

Note that $\delta > 0$ because if it did not, then by Δ_2 it would imply existence of a subsequence $\{w_{n_{k_l}}\}$ such that $\lim_{l\to\infty} \|w_{n_{k_l}} - w\|_\varphi = 0$, contradicting (4.32). It follows from (4.41) that there exists $k_\delta \geq k_\varepsilon$ such that,

$$\rho_\varphi\left(\frac{w_{n_k} - x}{\alpha + \varepsilon}\right) > \delta \tag{4.42}$$

holds for all $k \geq k_\delta$.

In view of Remark 4.4, the growth function w_δ, is right-continuous. Hence, we can take $0 < \varepsilon < 1$ such that $w_\delta\left(1 + \dfrac{\varepsilon}{\alpha}\right) < \gamma$. Using this together with (4.39) and (4.40), for any $k \geq k_\delta$ we have,

$$1 < \gamma < \rho_\varphi\left(\frac{w_{n_k} - x}{\alpha}\right) \leq w_\delta\left(1 + \frac{\varepsilon}{\alpha}\right)\rho_\varphi\left(\frac{w_{n_k} - x}{\alpha + \varepsilon}\right) \leq w_\delta\left(1 + \frac{\varepsilon}{\alpha}\right) < \gamma.$$

The contradiction completes the proof. □

Remark 4.5. *It is known that the Δ_2-property, along with the uniform convexity of φ, implies the uniform convexity of the Musielak-Orlicz space equipped with the Luxemburg norm [12, Theorem 11.6].*

Corollary 4.2. *From Theorems* 4.5 *along with Remark* 4.5*, it follows immediately that any Musielak-Orlicz space L^φ, where φ is uniformly convex and ρ_φ is uniformly continuous, possesses the Ψ-Opial property, with Ψ representing convergence in measure. This guarantees the convergence in measure of $\{w_n\} = \Gamma(H_n)$ to $F(\mathcal{T})$, provided $\{w_n\}$ is an approximate fixed point sequence and D is sequentially compact with respect to the topology of convergence in measure.*

Our next result shows that if ρ_φ has the Δ_2-type property, then the Musielak-Orlicz space L^φ possesses the Ψ-Opial property, where Ψ denotes convergence in measure m.

Theorem 4.6. *Let L^φ be Musielak-Orlicz space, where ρ_φ has the Δ_2-type property. Let $w_n \to w$ in measure, where $w \in L^\varphi$ and $\sup_{n\in\mathbb{N}} \|w_n\|_\varphi < +\infty$, then for every $x \in L^\varphi$ such that $x \neq w$, we have,*

$$\liminf_{n\to\infty} \|w_n - w\|_\varphi < \liminf_{n\to\infty} \|w_n - x\|_\varphi.$$

Therefore, L^φ possesses the Opial property with respect to convergence in measure.

Proof. The Δ_2-type condition of ρ_φ implies that the growth function $w(s) < \infty$ for every $s \geq 0$. Hence, by Proposition 2.3, the growth function w is continuous at every $s \geq 0$, and in particular at $s = 1$. It then follows from Proposition 2.5 that ρ_φ is uniformly continuous. Therefore, we can apply Theorem 4.5, which completes the proof. □

Note that, by utilising Japon's Theorem 4.1 from [4], we can strengthen our results for Musielak-Orlicz spaces to obtain a uniform variant of Ψ-Opial property, provided ρ_φ has the Δ_2-type property. Since our main results do not rely on such uniform conditions, we will not explore this topic in the current paper. However, future research may reveal its significance.

Remark 4.6. *Since every L^p space, with $p > 1$, is also the Musielak-Orlicz space L^φ, where φ is uniformly convex and satisfies the Δ_2-type condition, the process convergence in L^p follows from the more general scenario summarised in Corollary* 4.2*. However, we treated the L^p-case separately due to its inherent simplicity.*

5. Stability of Ψ-convergent processes

In the preceding section, we presented a general method for demonstrating the Ψ-convergence of iterative processes, summarised in Theorem 4.3 and illustrated by the subsequent examples. This section will focus on the stability of such convergent processes.

Consider $\{H_k\}$, a sequence of pointwise Lipschitzian operators. We define a generalised iteration process as follows: starting from an arbitrarily chosen element $w_1 \in D$, we generate a sequence $\{w_k\}$ by iteratively computing $w_{k+1} = H_k(w_k)$ for every $k \in \mathbb{N}$. If this process Ψ-converges to a common fixed point of a semigroup $\mathcal{T}$, irrespective of the starting point, it serves as a method for approximating such common fixed points. In practical applications, each iteration may introduce algorithmical errors, making it vital to be able to evaluate the stability of this process—specifically, whether it continues to Ψ-converge to $F(\mathcal{T})$ when each iteration k yields a point x_{k+1} that is close to, but not precisely equal to $H_k(x_k)$. This section aims to present a general method for establishing a particular type of stability, so called residual stability, for iteration processes that Ψ-converge to $F(\mathcal{T})$. In Section 7, we will illustrate this method with examples from the preceding section, applied to generalised Krasnosel'skii-Mann and Ishikawa iteration processes. These examples may serve as templates that can be adapted to various other contexts.

Similar concepts of stability can be traced back to Ostrowski's 1967 paper [38]. Following this trajectory, Butnariu, Reich, and Zaslavski explored the stability of the iterates of a nonexpansive mapping [39, 40]. Additionally, noteworthy contributions have been made by Reich, Taiwo, and Zaslavski [14, 41–43], along with references to pertinent literature. The books authored by Zaslavski [44–47] provide an excellent exposition of the current state of stability theory for iterative processes related to fixed point problems. In the context of weakly and strongly convergent iteration processes for pointwise Lipschitzian semigroups, recent works [1, 15, 16, 48, 49] address stability under summable errors.

Definition 5.1. *Assume that $\{H_k\} \in \mathcal{A}(D)$. Let $\Gamma(\{H_k\})$ be the iteration process generated by $\{H_k\}$. Assume that the process $\Gamma(\{H_k\})$ is Ψ-convergent to $F(\mathcal{T})$. We say that $\Gamma(\{H_k\})$ is residually stable if for any sequence $\{x_n\}$ of elements from D such that its k-th maximal residual error,*

$$\delta_k(\{x_n\}) = \sup_{m \geq k} \|x_{m+1} - H_m \circ H_{m-1} \circ \ldots \circ H_k(x_k)\|, \tag{5.1}$$

tends to zero as $k \to +\infty$, there exists a common fixed point $w \in F(\mathcal{T})$, which is the Ψ-limit of $\{x_n\}$.

In this case, we will say that the Ψ-convergence of this process is residually stable.

For the sake of brevity, we will use δ_k to denote $\delta_k(\{x_n\})$ when there is no risk of confusion. From a practical numerical implementation perspective, residual stability implies that if we are confident that, after a sufficient number of steps, the algorithmical errors are and will remain sufficiently small, the results of subsequent iterations will converge progressively closer to a common fixed point of the semigroup in question. However, unless we know in advance that such a common fixed point is unique, the procedure does not provide any indication of which fixed point is being approximated.

Stability under summable errors, defined below, is less demanding than residual stability; see Proposition 5.1 and the associated commentary.

Definition 5.2. *Let $\{H_k\} \in \mathcal{A}(D)$. We define an iteration process $\Gamma(\{H_k\})$, which Ψ-converges to $F(\mathcal{T})$, as being stable under summable errors if for every sequence $\{x_k\}$ satisfying the estimation:*

$$\sum_{k=1}^{\infty} \|x_{k+1} - H_k(x_k)\| < \infty, \tag{5.2}$$

there exists an $w \in F(\mathcal{T})$ such that $x_k \overset{\Psi}{\to} w$.

We will need the following technical lemma.

Lemma 5.1. *[1, Lemma 7.1] Assume that $\{H_k\} \in \mathcal{A}(D)$. Let $\{x_k\}$ be a sequence of elements from D. Then, the inequality:*

$$\begin{aligned} \| \quad & S_{k+q+1}^{k+q+i}(x_{k+q+1}) - S_k^{k+q+i}(x_k)\| \\ \le \quad & \prod_{j=1}^{i} A_{k+q+j}(S_{k+q+1}^{k+q+j-1}(x_{k+q+1}))\|x_{k+q+1} - S_k^{k+q}(x_k)\| \end{aligned} \tag{5.3}$$

holds for any three integers $k \ge 1, q \ge 0, i \ge 1$.

We are now ready to prove our stability result.

Theorem 5.1. *Let $\mathcal{T} \in \mathcal{APNS}(D)$. Assume that the sequence $\{H_k\} \in \mathcal{A}(D)$ is well-controlled by L. If the iteration process $\Gamma(\{H_k\})$ is Ψ-convergent to $F(\mathcal{T})$ then $\Gamma(\{H_k\})$ is residually stable.*

Proof. Let $\{x_n\}$ be a sequence of elements of D such that its maximal residual error $\delta_k \to 0$ as $k \to \infty$. Let $k \ge k_L$. Take any two integers $q \ge 0$ and $i \ge 1$. Using (5.3) along with the assumption that $\{H_k\}$ is well-controlled by L, we have,

$$\|S_{k+q+1}^{k+q+i}(x_{k+q+1}) - S_k^{k+q+i}(x_k)\| \le L\|x_{k+q+1} - S_k^{k+q}(x_k)\| \le L\,\delta_k, \tag{5.4}$$

which tends to zero as $k \to \infty$. Since the process $\Gamma(\{H_k\})$ is Ψ-convergent to $F(\mathcal{T})$ it follows that for every $k \ge 1$ and $q \ge 0$ there exists $y_k \in F(\mathcal{T})$ and $y_{k+q+1} \in F(\mathcal{T})$ such that for every $\psi \in \Psi$,

$$\psi\Big(S_{k+q+1}^{k+q+i}(x_{k+q+1}) - S_k^{k+q+i}(x_k) - y_{k+q+1} - y_k\Big) \to 0,$$

as $i \to \infty$. Therefore, by condition (A4) of Definition 3.1,

$$\|y_{k+q+1} - y_k\| \le \liminf_{i\to\infty} \Big\|S_{k+q+1}^{k+q+i}(x_{k+q+1}) - S_k^{k+q+i}(x_k)\Big\| \le L\,\delta_k \to 0,$$

as $k \to \infty$. Hence, $\{y_n\}$ is a Cauchy sequence. Since D is assumed in this paper to be norm closed, there exists then a $w \in D$ such that,

$$\lim_{k\to\infty} \|y_k - w\| = 0.$$

We claim that $w \in F(\mathcal{T})$. Indeed, fix any $t \in J$ and $k \in \mathbb{N}$ and observe that,

$$\|T_t(w) - w\| \le \|T_t(w) - T_t(y_k)\| + \|T_t(y_k) - y_k\| + \|y_k - w\| \le (a_t(w) + 1)\|y_k - w\| \to 0.$$

It remains to prove that $x_n \xrightarrow{\Psi} w$. Let us fix any $\psi \in \Psi$ and $\varepsilon > 0$. Since $\lim_{k\to\infty} \|y_k - w\| = 0$, it follows from (A3) of Definition 3.1 that there exists $k_0 \in \mathbb{N}$ such that,

$$\psi(w - y_k) < \frac{\varepsilon}{2c_\psi}. \tag{5.5}$$

Similarly, it follows from (5.4), used with $q = 0$, and (A3) of Definition 3.1 that there exists $k_1 \geq k_0$ such that for every $k \geq k_1$,

$$\psi\Big(S^{k+i}(x_k) - x_{k+i+1}\Big) < \frac{\varepsilon}{2c_\psi^2} \tag{5.6}$$

holds for all $i \in \mathbb{N}$. Fix temporarily $i \in \mathbb{N}$. For such i and any $k \geq k_1$, using (A1) of Definition 3.1 along inequalities (5.5) and (5.6), we obtain the following:

$$\begin{aligned}
\psi(w - x_{k+i+1}) &\leq c_\psi \psi(w - y_k) + c_\psi^2 \psi\Big(y_k - S_k^{k+i}(x_k)\Big) + c_\psi^2 \psi\Big(S_k^{k+i}(x_k) - x_{k+i+1}\Big) \\
&\leq \frac{\varepsilon}{2} + c_\psi^2 \psi\Big(y_k - S_k^{k+i}(x_k)\Big) + \frac{\varepsilon}{2} \\
&= \varepsilon + c_\psi^2 \psi\Big(y_k - S_k^{k+i}(x_k)\Big).
\end{aligned} \tag{5.7}$$

Using the definition of y_k, it follows from (5.7) that,

$$\limsup_{m\to\infty} \psi(w - x_m) = \limsup_{i\to\infty} \psi(w - x_{k+i+1}) < \varepsilon + \limsup_{i\to\infty} c_\psi^2 \psi\Big(y_k - S_k^{k+i}(x_k)\Big) = \varepsilon.$$

By arbitrariness of $\varepsilon > 0$, we conclude that $\lim_{m\to\infty} \psi(w - x_m) = 0$. Since this is true for any $\psi \in \Psi$, it follows that $x_n \xrightarrow{\Psi} w$, completing the proof. □

Theorem 5.1 recovers the stability results [1, Theorem 7.3, 7.5] (albeit under less demanding regime of stability under summable errors).

Theorem 5.1 is significant because it elucidates the relationship between convergence and residual stability in iterative processes. In light of the scenarios explored in the previous sections of this paper, we can encapsulate this phenomenon in the following result.

Corollary 5.1. *From Theorem 5.1 it follows that, in the settings of Problem 2, every well-controlled process $\Gamma(\{H_k\})$ is residually stable provided assumed conditions on the set D and the semigroup $\mathcal{T}$ guarantee that this process is Ψ-convergent to $F(\mathcal{T})$. Therefore, the residual stability will be guaranteed in all scenarios discussed in Propositions 4.3 and 4.4, as well as Corollaries 4.1 and 4.2.*

The next result demonstrates that residual stability is a more demanding type of stability comparing to the stability under summable errors.

Proposition 5.1. *For well-controlled by L sequences $\{H_k\} \in \mathcal{A}(D)$, residual stability implies stability under summable errors.*

Proof. Assume that $\Gamma(\{H_k\})$ is residually stable. We want to prove that it is also stable under summable errors. To this end, assume that for a sequence $\{x_k\}$ the inequality (5.2) holds. Let us introduce the notation: $r_k = \|x_{k+1} - H_k(x_k)\|$. Then, the following inequality holds for every $k \geq k_L$ and every integer $i \geq 0$ (c.f., [1, Lemma 7.2]):

$$\|x_{k+i+1} - S_k^{k+i}(x_k)\| \leq \Big(\prod_{j=k}^{k+i} A_j(x_j)\Big)\Big(\sum_{j=k}^{k+i} r_j\Big) \leq \Big(\prod_{j=k}^{\infty} A_j(x_j)\Big)\Big(\sum_{j=k}^{\infty} r_j\Big) \leq L\Big(\sum_{j=k}^{\infty} r_j\Big),$$

which tends to zero as $k \to \infty$, because of (5.2). By residual stability of $\Gamma(\{H_k\})$, there exists an $w \in F(\mathcal{T})$ such that $x_k \overset{\Psi}{\to} w$. Hence, the process is stable under summable errors. The proof is complete. □

To interpret the residual stability and its comparison with the stability under summable errors, let us consider $H_k(x_k)$ as an expected outcome of application of the procedure to x_k, while x_{k+1} should be understood as the actual reading of this iterative computational step. From this perspective δ_k, defined by the formula (5.1), represents the difference between the expected value after $m - k + 1$ iterations starting from x_k and the actual reading after these iterations. Consequently, the residual stability means that, given any stopping condition of the form $\delta_k < c_0$ for all $k \geq k_0$, this stopping condition will be met after a sufficiently large number of iterations. Assume that the process is residually stable and that $\delta_k = \dfrac{1}{k}$. Hence, there exists a common fixed point w such that $x_k \overset{\Psi}{\to} w$. However, knowing only that the process is stable under summable errors, the condition $\|x_{k+1} - H_k(x_k)\| = \dfrac{1}{k}$ does not allow us to deduce such convergence.

From a practical, numerical standpoint, one can use the quadratic convergence approach, which is frequently employed for estimating how fast numerical algorithms reach the solution for problems like impulsive differential systems, [50, 51]. Assume that $\{x_n\}$ is quadratically convergent to the solution x, which means that there exists a constant $M > 0$ such that $\|x_{n+1} - x\| \leq M\|x_{n+1} - x\|^2$ at each step n. Thus,

$$\|x_{m+1} - H_m \circ H_{m-1} \circ ... \circ H_k(x_k)\| \leq M \sum_{i=k}^{m} \|x_i - x\|^2 \leq M \sum_{i=k}^{\infty} \|x_i - x\|^2.$$

Consequently, if we knew that $\sum_{i=1}^{\infty} \|x_i - x\|^2 < \infty$, then we would conclude that the process is residually stable. The finiteness of this infinite sum may follow from conditions like,

$$\|x_{n+1} - x\| \leq \frac{1}{n}. \tag{5.8}$$

Since, in our context, the solution x is typically unknown, condition (5.8) is in practical implementations replaced by a stopping condition of the form $\|x_{n+1} - x_n\| < c_0$. For instance, for accuracy to 6 decimal places, one could use $c_0 = 10^{-6}$.

The above commentary indicates that the residual stability of iterative processes, arising from our automation technology under an appropriate Ψ-Opial condition, can be practically assessed using routine numerical techniques, such as quadratic convergence.

6. Summary of results

Let us recall the overarching settings of this paper:

1) X is a uniformly convex Banach space equipped with Ψ-convergence,
2) $D \subset X$ is a nonempty convex, bounded, closed in the norm sense, and sequentially Ψ-compact,
3) $\mathcal{T} \in \mathcal{APNS}(D)$, that is, $\mathcal{T}$ is an asymptotic pointwise nonexpansive semigroup of operators acting within D.

Using the notation and overall assumptions presented above, the following theorem summarises the operation of the mathematical technology introduced in the preceding sections.

Theorem 6.1. *Let sequence $\{w_n\}$ be generated by a well-controlled, $\mathcal{T}$-compatible, iterative process. Assume that X has the Ψ-Opial property. If $\{w_n\}$ is an approximate fixed point sequence for the semigroup $\mathcal{T}$, then there exists a common fixed point $w \in F(\mathcal{T})$ such that $w_n \overset{\Psi}{\to} w$. Moreover, this convergence is residually stable.*

Proof. The proof uses results obtained in Sections 4 and 5, and can be divided into the three steps:

1) Since X is uniformly convex it follows from Theorem 2.1 that $F(\mathcal{T})$, the set of all common fixed point of $\mathcal{T}$, is nonempty.
2) Since X has the Ψ-Opial property, Theorem 4.3 guarantees that the process Ψ-converges to an element of $F(\mathcal{T})$.
3) Since the process is Ψ-convergent to an element of $F(\mathcal{T})$, it follows from Theorem 5.1, that this convergence is residually stable.

□

Theorem 6.1 is a new and original result based on a novel mathematical technology. It has no equivalents, even in special cases like nonexpansive semigroups. It recovers [16, Theorem 4.1], which is a special case when Ψ is a weak convergence and stability is considered as a less demanding stability under summable errors. Theorem 3.1 in [15] is a somewhat related result in metric spaces.

Let us emphasise that Theorem 6.1 provides the sought-after single property of space X (the Ψ-Opial property), which guarantees that a sequence generated by a well-controlled process compatible with the semigroup is Ψ-convergent to a common fixed point of this semigroup, given that the sequence is an approximate fixed point sequence. Importantly, this convergence is automatically residually stable.

As indicated in the Introduction, the aim of this paper is not to introduce yet another specialised set of techniques, nor is it merely a unification of existing results. The objective is focused on the simplification and automation of future work. Given a scheme with a Banach space X together with a suitably chosen Ψ-convergence, to prove residually stable Ψ-convergence of iterative common fixed point processes for asymptotic pointwise non-expansive semigroups on a closed, bounded, convex, and Ψ-sequentially compact set D, one needs to verify that only these three conditions are satisfied:

1) X is uniformly convex (to ensure that such common fixed points exist);
2) X has the Ψ-Opial property;
3) The iterative process under consideration is well-controlled and $\mathcal{T}$-compatible, and the sequence generated by this process is an approximate fixed point sequence.

Considering that in a large number of interesting cases, uniform convexity is well studied, and that for many classical Ψ-convergences, known compactness criteria exist, this leaves only two conditions needing verification. It is quite likely that the relevant Ψ-Opial condition is already known or can be relatively easily derived from other established results. Hence, in many situations, the only task will be to verify the last of the aforementioned conditions.

However, even the latter task may be substantially simplified or completely automated. This is because many standard iterative processes are inherently compatible with the considered semigroup (a natural assumption for common fixed-point construction processes) and well-controlled. Notably, in the nonexpansive, eventually nonexpansive and asymptotic nonexpansive cases, this property can usually be easily obtained by a suitable choice of parameters. Even in the more complex scenario of asymptotic pointwise nonexpansive semigroups, relatively straightforward guidelines exist to achieve this. Furthermore, many standard processes generate approximate fixed-point sequences, as this is a natural approach to approximating fixed points (as the name itself suggests). In the next section, we will illustrate the above commentary by discussing two popular iterative schemes: the Krasnosel'skii-Mann and Ishikawa processes.

The above discussion is not purely theoretical. This paper applies the theory summarised in Theorem 6.1 to prove residual stability across a range of significant scenarios. These results are novel; indeed, each could warrant a dedicated paper. However, our methods enable us to achieve these results with considerably less effort, primarily focusing on the identification of relevant Ψ-Opial conditions. In accordance with the general philosophy outlined in this study, we believe it is critically important that the application of the introduced technology allows us to minimise the otherwise necessary repetition of argumentation. These new results are summarised in the following theorems.

Theorem 6.2. *Let $L^\varphi = L^\varphi(\Omega, \Sigma, \mu)$ be a Musielak-Orlicz space, where μ is a finite, atomless measure, φ is uniformly convex and ρ_φ is uniformly continuous. Assume that Ψ is the convergence in measure μ, and $D \subset L^\varphi$ is sequentially compact with respect to convergence in measure. Assume that $\mathcal{T}$ is an asymptotic pointwise nonexpansive semigroup of operators acting within D. Let sequence $\{w_n\}$ be an approximate fixed point sequence, generated by a well-controlled, $\mathcal{T}$-compatible, iterative process. Then, there exists a common fixed point $w \in F(\mathcal{T})$ such that $w_n \xrightarrow{\mu} w$. Moreover, this convergence is residually stable.*

Proof. The proof is divided into the following steps:

1) From Remark 4.5, it follows that L^φ is uniformly convex with respect to the Luxemburg norm $\|\cdot\|_\varphi$, which guarantees that the set of all common fixed point of $\mathcal{T}$, is nonempty.
2) By Theorem 4.5, L^φ has the Ψ-Opial property.
3) From Theorem 6.1 it follows that the sequence $\{w_n\}$ is convergent in measure to a common fixed point $w \in F(\mathcal{T})$, and that this convergence is residually stable.

□

By applying Theorem 6.2 to specific instances of Musielak-Orlicz spaces, we can directly derive the following results on residually stable convergence. We are not aware of any comparable findings.

Theorem 6.3. *Let $L^\varphi = L^\varphi(\Omega, \Sigma, \mu)$ be an Orlicz space, where μ is a finite, atomless measure, φ is uniformly convex and and satisfies the Δ_2-condition. Assume that Ψ is the convergence in measure μ, and $D \subset X$ is sequentially compact with respect to convergence in measure. Assume that $\mathcal{T}$ is an asymptotic pointwise nonexpansive semigroup of operators acting within D. Let sequence $\{w_n\}$ be generated by a well-controlled, $\mathcal{T}$-compatible, iterative process. Then, there exists a common fixed point $w \in F(\mathcal{T})$ such that $w_n \xrightarrow{\mu} w$. Moreover, this convergence is residually stable.*

Proof. Remark 2.1 indicates that the modular ρ_φ is uniformly continuous. Since every Orlicz space is also a Musielak-Orlicz space, the desired conclusions are derived directly from Theorem 6.2. □

Theorem 6.4. *Let $L^{p(\cdot)}$ be a variable Lebesgue space, where μ is a finite, atomless measure on Ω, and there exists $a > 1$ and $b < \infty$ such that $a \le p(t) \le b$ for all $t \in \Omega$. Assume that Ψ is the convergence in measure μ, and $D \subset L^{p(\cdot)}$ is sequentially compact with respect to convergence in measure. Assume that $\mathcal{T}$ is an asymptotic pointwise nonexpansive semigroup of operators acting within D. Let sequence $\{w_n\}$ be an approximate fixed point sequence, generated by a well-controlled, $\mathcal{T}$-compatible, iterative process. Then, there exists a common fixed point $w \in F(\mathcal{T})$ such that $w_n \xrightarrow{\mu} w$. Moreover, this convergence is residually stable.*

Proof. Since $p(t) \ge a > 1$ for all $t \in \Omega$, it follows that $L^{p(\cdot)}$ is uniformly convex with respect to the Luxemburg norm induced by the convex modular:

$$\rho(x) = \int_\Omega |x(t)|^{p(t)} d\mu(t). \tag{6.1}$$

Modular ρ defined by (6.1) has the Δ_2-type property and its growth function $w(t)$ is right-continuous at $t = 1$. Hence, by Proposition 2.5, ρ is uniformly continuous. Since $L^{p(\cdot)}$ is also a Musielak-Orlicz space with $\varphi(t, u) = u^t$, the desired conclusions are derived directly from Theorem 6.2. □

Remark 6.1. *Note that, in particular, if $p(t) = p$ for all $t \in \Omega$, then Theorem 6.4 gives us residually stable convergence in measure of processes in L^p spaces for $1 < p < \infty$. Since convergence in measure on bounded subsets of such L^p space implies weak convergence, it follows that Theorem 6.4 gives us also convergence and residual stability with respect to the weak convergence. For $p \ne 2$, this result cannot be directly obtained by using the weak Opial property, as such spaces do not posses this classical version of Opial property. This phenomenon shows the advantage of our approach.*

Let us consider now how our theory works in Sobolev spaces. Recall from Example 3.6 the definition of Sobolev space: $W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega)\ \forall |\alpha| \leqslant k\}$, where $\Omega \subset \mathbb{R}^d$ is a bounded domain, $1 < p < +\infty$, $k \in \mathbb{N}$ and $D^\alpha u$ are weak derivatives. The convergence Ψ is defined by the family $\Psi = \{\psi_\alpha(x) = \psi_0(D^\alpha(x)) : |\alpha| \le k\}$, where ψ_0 relates to the convergence in measure and is defined by formula (3.2).

Theorem 6.5. *Let $W^{k,p}(\Omega)$ be a Sobolev space, as defined above. Assume that $D \subset W^{k,p}(\Omega)$ is sequentially compact with respect to the Ψ-convergence. Assume that $\mathcal{T}$ is an asymptotic pointwise nonexpansive semigroup of operators acting within D. Let sequence $\{w_n\}$ be an approximate fixed point sequence, generated by a well-controlled, $\mathcal{T}$-compatible, iterative process. Then, there exists a common fixed point $w \in F(\mathcal{T})$ such that $w_n \xrightarrow{\mu} w$. Moreover, this convergence is residually stable.*

Proof. The proof is divided into the three usual steps:

1) Because $p > 1$, $W^{k,p}(\Omega)$ is uniformly convex, which guarantess that the set of all common fixed point of $\mathcal{T}$, is nonempty.
2) As observed in part 3 of Corollary 4.1, space $W^{k,p}(\Omega)$ possesses the Ψ-Opial property.
3) From Theorem 6.1 it follows that the sequence $\{w_n\}$ is Ψ-convergent in measure to a common fixed point $w \in F(\mathcal{T})$, and that this convergence is residually stable.

□

Let us finish this summary section with the following novel result.

Theorem 6.6. *Let D be a subset of L^p, where $p > 1$. Let Ψ be the convergence defined by the Marcinkiewicz quasi-norm or Lorentz-quasi-norm (Examples* 3.7 *and* 3.8*). Assume that D is sequentially compact with respect to this Ψ-convergence. Assume that $\mathcal{T}$ is an asymptotic pointwise nonexpansive semigroup of operators acting within D. Let sequence $\{w_n\}$ be an approximate fixed point sequence, generated by a well-controlled, $\mathcal{T}$-compatible, iterative process. Then, there exists a common fixed point $w \in F(\mathcal{T})$ such that $w_n \overset{\Psi}{\to} w$. Moreover, this convergence is residually stable.*

Proof. In this case, the usual steps of the proof are:

1) L^p is uniformly convex since $p > 1$.
2) The Ψ-Opial property is ensured by Proposition 4.5, as both of these convergences imply convergence in measure, which in turn guarantees the existence of a subsequence that converges almost everywhere.
3) Hence, from Theorem 6.1 it follows that the sequence $\{w_n\}$ is Ψ-convergent in measure to a common fixed point $w \in F(\mathcal{T})$, and that this convergence is residually stable.

□

7. Applications

In this section, we will illustrate how our technology, applies to some commonly used iterative processes. We will also outline an application to differential equations and dynamical systems.

In first two subsections, we will utilise the framework and results discussed in the preceding sections to the generalised Krasnosel'skii - Mann and generalised Ishikawa iteration processes. Both algorithms are commonly used for the approximation of fixed points. In our exposition, we will loosely base our discussion on [1, 16], where convergence in the weak topology and stability under summable errors were considered. However, in this section, we will seek to answer the question of when these processes are Ψ-convergent in a residually stable manner.

Next, in the subsection 'Application to differential equations and dynamical systems', we discuss residually stable construction of stationary points for dynamical systems defined by asymptotic pointwise nonexpansive semigroups related to differential equations in Banach spaces.

7.1. Generalised Krasnosel'skii-Mann process

Definition 7.1. *[1, Definition 4.1] Let $\mathcal{T} \in \mathcal{APNS}(\mathcal{D})$. Let $\{t_n\}$ be a sequence of elements in J and $\{c_n\}$ be a sequence of numbers from* $(0, 1)$*. The generalised Krasnosel'skii-Mann iteration process $gKM(\mathcal{T}, \{c_n\}, \{t_n\})$ generated by the semigroup $\mathcal{T}$, the sequences $\{c_n\}$ and $\{t_n\}$, is defined by the iterative formula:*

$$x_{n+1} = c_n T_{t_n}(x_n) + (1 - c_n)x_n, \tag{7.1}$$

where $x_1 \in D$ is chosen arbitrarily, and

(i) $\{c_n\}$ *is bounded away from* 0 *and* 1*,*

(ii) $\{t_n\}$ *is chosen so there exists* $n_0 \in \mathbb{N}$ *such that* $\sum_{n=n_0}^{\infty} M_{t_n} < \infty$*, where* $M_{t_n} = \sup\{b_{t_n}(x) : x \in D\}$.

Recall that $b_t(x) = a_t(x) - 1$*, where* $a_t(x) \geq 1$ *is chosen so that the inequality:*

$$\|T_t(x) - T_t(y)\| \leq a_t(x)\|x - y\|$$

holds for any $y \in D$.

Remark 7.1. *It is worthwhile noting that assumption (ii) of Definition* 7.1 *implies that,*

$$\|T_{t_n}(x) - T_{t_n}(y)\| \leq (1 + b_{t_n}(x))\|x - y\| \leq (1 + M_{t_n})\|x - y\|, \tag{7.2}$$

where $M_{t_n} \to 0$*. However, this does not imply that the semigroup* $\mathcal{T}$ *must be asymptotic nonexpansive, where all* $a_t(\cdot)$ *are constants. The only correct implication is that we can select a sequence* $\{t_n\}$ *such that* (7.2) *holds. This assumption is quite reasonable, as for every* $x \in D$*,* $b_t(x) \to 0$ *as* $t \to +\infty$*. However, the selected sequence* $\{t_n\}$ *does not need to tend to infinity; any sequence satisfying (ii) will do. Note that there exist less restrictive, yet more intricate, assumptions; see [1,16]. However, we limit our considerations to a much simpler, yet quite general, condition (ii). Readers whose requirements demand more general conditions may modify the considerations of this section accordingly.*

Proposition 7.1. *Let* $gKM(\mathcal{T}, \{c_n\}, \{t_n\})$ *be a generalised Krasnosel'skii-Mann iteration process. For every* $n \in \mathbb{N}$*, define:*

$$H_n(x) = c_n T_{t_n}(x) + (1 - c_n)x. \tag{7.3}$$

Then, $\{H_n\}$ *is a well-controlled asymptotic pointwise nonexpansive sequence, which is* $\mathcal{T}$*-compatible, i.e., such that,*

$$F(\mathcal{T}) \subset \bigcap_{n=1}^{\infty} F(H_n). \tag{7.4}$$

Proof. Observe that,

$$\|H_n(x) - H_n(y)\| \leq c_n a_{t_n}(x)\|x - y\| + (1 - c_n)\|x - y\| = (1 + c_n b_{t_n}(x))\|x - y\|.$$

which proves that $\{H_n\} \in \mathcal{A}(D)$ because $b_{t_n}(x) \to 0$ by condition (ii) of Definition 7.1. To prove that $\{H_n\}$ is well-controlled, let us denote:

$$A_n(x) = (1 + c_n b_{t_n})(x),$$

and observe that,

$$L_n := \sup\{A_n(x) : x \in D\} \leq 1 + M_{t_n},$$

and, therefore, that,

$$L := \prod_{n=n_0}^{\infty} L_n \leq e^M,$$

where $M = \sum_{n=n_0}^{\infty} M_{t_n} < +\infty$, in accordance with point (ii) of Definition 7.1. This proves that $\{H_n\}$ is well-controlled. Inclusion (7.4) immediately follows from (7.3). □

Corollary 7.1. *Let semigroup $\mathcal{T} \in \mathcal{APNS}(\mathcal{D})$. From Proposition* 7.1*, Corollary* 5.1 *and Theorem* 4.3 *it follows that for all scenarios discussed in Propositions* 4.3 *and* 4.4*, as well as Corollaries* 4.1 *and* 4.2*, the generalised Krasnosel'skii-Mann iteration process $gKM(\mathcal{T}, \{c_n\}, \{t_n\})$ is a residually stable Ψ-convergent to $F(\mathcal{T})$ iteration process, provided the sequence $\{x_n\}$ generated by $gKM(\mathcal{T}, \{c_n\}, \{t_n\})$ is an approximate fixed-point sequence.*

In Corollary 7.1, we concluded that for a very large class of spaces $(X, \|\cdot\|, \Psi)$ with the relevant Ψ-Opial property (including all examples from Section 3), the Ψ-convergence and residual stability of generalised Krasnosel'skii-Mann iteration processes are assured when the sequence generated by this process is an approximate fixed-point sequence. In applications, this result simplifies proceedings considerably, as verification of only a single condition is required. This demonstrates the power of our approach.

The literature provides several criteria for the sequence generated by the generalised Krasnosel'skii-Mann iteration process to be an approximate fixed-point sequence. As an example, we below adapt Lemma 4.3 from [1] to our context. The proof closely follows that of the cited result and is therefore omitted.

Theorem 7.1. *Let $\mathcal{T} \in \mathcal{APNS}(\mathcal{D})$, where $J = [0, +\infty)$, be such that every $a_t(\cdot)$ is a bounded function. Let $\{x_k\} = gKM(\mathcal{T}, \{c_k\}, \{t_k\})$ be the generalised Krasnosel'skii-Mann process. Let us assume that to every $s \in J$ there exists a strictly increasing sequence of natural numbers $\{j_k\}$ satisfying the following conditions:*

$$\lim_{k\to\infty} \|x_k - x_{j_k}\| = 0,$$

and

$$\lim_{k\to\infty} \|T_{e_k}(x_{j_k}) - x_{j_k}\| = 0,$$

where $e_k = |t_{j_{k+1}} - t_{j_k} - s| \in J$. Then, $\{x_k\}$ is an approximate fixed point sequence for $\mathcal{T}$.

For practical reasons, assumptions of Theorem 7.1 may be restricted to a much smaller subset of J, see [1] for more details. Observe that boundedness of each $a_t(\cdot)$ implies that each operator T_t is Lipschitzian with some Lipschitzian constant $1 \leq L_t < +\infty$. This however is not very restrictive assumption since these constants might be extremely large. For alternative assumptions consult [1,16].

As another example, let us adapt Theorem 4.2 from [1] and note that we can always construct $\{t_n\}$ with the properties specified in the assumptions of Theorem 7.2.

Theorem 7.2. *Assume that the semigroup $\mathcal{T} \in \mathcal{APNS}(D)$ is parametrised by J that is generated by a finite set $A = \{\alpha_1, \alpha_2, \alpha_3..., \alpha_M\}$ where each α_i is a natural number. Assume that every $a_t(\cdot)$ is a bounded function. Let $\{x_k\} = gKM(\mathcal{T}, \{c_k\}, \{t_k\})$ be the generalised Krasnosel'skii-Mann process. Assume that for every natural number $m \leq M$, there exists a strictly increasing, quasi-periodic sequence of natural numbers $\{j_k(m)\}$, with a quasi-period p_m, such that the equality:*

$$t_{j_{k+1}(m)} = \alpha_m + t_{j_k(m)}$$

holds for every $k \in \mathbb{N}$. Then, the sequence $\{x_k\}$, generated by $gKM(\mathcal{T}, \{c_k\}, \{t_k\})$, is an approximate fixed point sequence for $F(\mathcal{T})$.

In practice, Theorems 7.1 and 7.2 provide guidance on how to construct the sequence $\{t_n\}$ when implementing concrete algorithms based on the Krasnosel'skii-Mann method.

7.2. Generalised Ishikawa process

Let us examine the two-step Ishikawa iteration process, a generalisation of the one-step Krasnosel'skii-Mann method. The Ishikawa iteration process offers increased flexibility in defining algorithm parameters, which is crucial for improving performance of numerical implementations. Additionally, this section illustrates how the theoretical results presented earlier can be leveraged to tackle increasingly complex processes, building on the foundation of the one-step generalised Krasnosel'skii-Mann process. Let us start with the following definition.

Definition 7.2. *Let $\mathcal{T} \in \mathcal{APNS}(D)$, $\{t_k\} \subset J$. Let $\{t_k\}$ be a sequence of elements in J and let $\{c_k\}$ $\{d_k\}$ be two sequences of numbers from $(0, 1)$. The generalised Ishikawa iteration process $gI(\mathcal{T}, \{c_k\}, \{d_k\}, \{t_k\})$ generated by the semigroup $\mathcal{T}$, the sequences $\{c_k\}$, $\{d_k\}$ and $\{t_k\}$, is defined by the iterative formula:*

$$x_{k+1} = c_k T_{t_k}(d_k T_{t_k}(x_k) + (1 - d_k)x_k) + (1 - c_k)x_k,$$

where $x_1 \in D$ is chosen arbitrarily, and

(i) $\{c_k\}$ is bounded away from 0 and 1, and $\{d_k\}$ is bounded away from 1,

(ii) $\{t_n\}$ is chosen so there exists $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^{\infty} M_{t_n} < \infty$, where $M_{t_n} = \sup\{b_{t_n}(x) : x \in D\}$.

Similarly, as in the previous subsection, we will need the following result.

Proposition 7.2. *Let $gI(\mathcal{T}, \{c_k\}, \{d_k\}, \{t_k\})$ be a generalised Ishikawan iteration process. For every $n \in \mathbb{N}$ and all $u \in D$, define:*

$$H_k(u) = c_k T_{t_k}(d_k T_{t_k}(u) + (1 - d_k)u) + (1 - c_k)u. \tag{7.5}$$

Then, $\{H_n\}$ is a well-controlled asymptotic pointwise nonexpansive sequence, which is $\mathcal{T}$-compatible, i.e., such that,

$$F(\mathcal{T}) \subset \bigcap_{n=1}^{\infty} F(H_n). \tag{7.6}$$

Proof. For any $u, v \in D$, denote $u_k = d_k T_{t_k}(u) + (1-d_k)u$ and $v_k = d_k T_{t_k}(v) + (1-d_k)v$. The straightforward calculation shows that,

$$\|H_k(u) - H_k(v)\| \le a_{t_k}(u_k)a_{t_k}(u)\|u - v\|.$$

Denote $A_k(u) = a_{t_k}(u_k)a_{t_k}(u)$. It follows from assumption (ii) that $A_k(u) \to 1$. Hence,

$$A_k(u) = a_{t_k}(u_k)a_{t_k}(u) \le 1 + b_{t_k}(u) + b_{t_k}(u_k) + b_{t_k}(u)b_{t_k}(u_k) \le 1 + 2M_{t_k} + M_{t_k}^2 \to 1,$$

and, therefore, $\{H_k\} \in \mathcal{A}(D)$. To prove that it is well-controlled, observe that,

$$L_n := \sup\{A_n(x) : x \in D\} \le 1 + 2M_{t_n} + M_{t_n}^2,$$

and, therefore, that

$$L := \prod_{n=n_0}^{\infty} L_n \le e^M,$$

where $M = 1 + \sum_{n=n_0}^{\infty} M_{t_n} + \sum_{n=n_0}^{\infty} M_{t_n}^2 < +\infty$, in accordance with point (ii) of Definition 7.2. Hence, $\{H_n\}$ is well-controlled. Inclusion (7.6) immediately follows from (7.5). □

Corollary 7.2. *Let semigroup $\mathcal{T} \in \mathcal{APNS}(\mathcal{D})$. From Proposition* 7.2*, Corollary* 5.1 *and Theorem* 4.3 *it follows that for all scenarios discussed in Propositions* 4.3 *and* 4.4*, as well as Corollaries* 4.1 *and* 4.2*, the generalised Ishikawa iteration process is a residually stable Ψ-convergent to $F(\mathcal{T})$ iteration process, provided the sequence $\{x_n\}$, generated by it, is an approximate fixed-point sequence.*

Corollary 7.2 implies that for a very large class of spaces $(X, \|\cdot\|, \Psi)$ with the relevant Ψ-Opial property (including all examples from Section 3), the Ψ-convergence and consequential residual stability of generalised Ishikawa iteration processes are assured when the sequence generated by this process is an approximate fixed-point sequence. Results analogous to Theorems 7.1 and 7.2 can be also obtained for the Ishikawa process case; see [1, Theorems 5.4 and 5.5].

7.3. Application to differential equations and dynamical systems

In this section, we will discuss an example of the application of our results to the problem of residually stable construction of stationary points for dynamical systems defined by asymptotic pointwise nonexpansive semigroups. In numerous application, dynamical systems relate to initial value problems for differential equations. Let us consider the following initial value problem for an unknown function $u(x, \cdot) : [0, \infty) \to D \subset X$,

$$\begin{cases} u(x, 0) = x \\ \dfrac{\partial u}{\partial t}(x, t) + (I - H)(u(x, t)) = 0, \end{cases} \tag{7.7}$$

where I stands for the identity operator, and $H : D \to D$ is pointwise Lipschitzian mapping, which means that for every $x \in D$ there exists $a(x) \geq 1$ such that $\|H(x) - H(y)\| \leq a(x)\|x - y\|$ for all $y \in D$. Recall that, for H to be pointwise Lipschitzian, it is sufficient (but not necessary) for H to be Lipschitzian (e.g., nonexpansive) or continuously Fréchet differentiable on a bounded, open, convex set A containing D; see [1, Corollary 1.1]. Hence, the procedure described below is valid for a large class of seed operators H. The set D is assumed to be, as always in this paper, a nonempty, closed, bounded, and convex subset of a uniformly convex Banach space X. We assume that $(X, \|\cdot\|, \Psi)$ is a Banach space equipped with a sequential convergence Ψ. We will later show in this section how this scenario works in some concrete cases of X and Ψ.

Define the sequence of functions $u_n : D \times [0, \infty) \to D$ by:

$$\begin{cases} u_0(x, t) = x \\ u_{n+1}(x, t) = e^{-t}x + \int_0^t e^{s-t} H(u_n(x, s))ds. \end{cases} \tag{7.8}$$

Define the following two quantities: $M_n(t) = \sup\limits_{s \in [0,t]} a(u_n(x, s))$, and $M(t) = \sup\limits_{n \in \mathbb{N}_0} M_n(t)$. Observe that for all n and t, both $M_n(t)$ and $M(t)$ are equal to 1 when H is nonexpansive. By employing Theorems 9.1, 9.2 and 9.3 from [1], we obtain our next result.

Theorem 7.3. *Assume that $(1 - e^{-t})M(t) < 1$ for every $t \geq 0$, and that $\prod_{n=0}^{\infty} M_n(t) \to 1$ as $t \to \infty$. Then,*

1) *The sequence $\{u_n\}$ converges almost uniformly to the solution of the initial value problem* (7.7)*, denoted by u.*

2) *Denoting $T_t(x) = u(x,t)$, where $t \geq 0$, $x \in D$, the family of mappings $\mathcal{T} = \{T_t\}_{t\geq 0}$ forms an asymptotic pointwise nonexpansive semigroup (nonexpansive semigroup if H was nonexpansive).*
3) *The set of common fixed points of this semigroup, $F(\mathcal{T})$, is the set of all stationary points of the dynamical system defined by* (7.7).

The next result follows immediately from Theorems 6.1 and 7.3.

Theorem 7.4. *Assume that X possesses Ψ-Opial property and D is sequentially Ψ-compact. Under assumptions of Theorem* 7.3, *let $T_t(x) = u(x,t)$, where u is the solution of the initial value problem* (7.7). *Denote $\mathcal{T} = \{T_t\}_{t\geq 0}$. Let $\{w_k\} = \Gamma(\{H_k\})$ be a well-defined iterative process. If $\{w_k\}$ is an approximate fixed point sequence for the semigroup $\mathcal{T}$, then there exists a stationary point $w \in F(\mathcal{T})$ such that $w_n \overset{\Psi}{\to} w$ and its convergence is residually stable.*

This result well demonstrates the workings of our automated technology: in the presence of the Ψ-Opial property, the only action required is to use the iterative process, which generates an approximate fixed point sequence. The rest is taken care of automatically.

If the seed operator H is nonexpansive and $J = [0,\infty)$, then the assumptions of Theorem 7.3 are automatically met, the semigroup $\mathcal{T}$ is nonexpansive and the sequence $\{w_k\}$ generated by the generalised Krasnosel'skii-Mann iteration process is automatically an approximate fixed point sequence, giving us the following very useful result.

Theorem 7.5. *Assume that X possesses Ψ-Opial property and D is sequentially Ψ-compact. Consider the initial value problem* (7.7). *Assume that the seed operator H is nonexpansive and that the dynamical system deals with continuous time, that is that $J = [0,\infty)$. Let $T_t(x) = u(x,t)$, where u is the solution of* (7.7), *and denote $\mathcal{T} = \{T_t\}_{t\geq 0}$. Let $\{w_k\} = \Gamma(\{H_k\})$ be a well-defined iterative process. Then there exists a stationary point $w \in F(\mathcal{T})$ such that $w_n \overset{\Psi}{\to} w$ and its convergence is residually stable.*

Let us consider a special case when X is a Musielak-Orlicz space L^φ, where φ is uniformly convex and ρ_φ is uniformly continuous. Then, by Corollary 4.2, L^φ has the Ψ-Opial property, where Ψ denotes convergence in measure. Let $\{w_k\} = \Gamma(\{H_k\})$ be the generalised Krasnosel'skii-Mann iteration process generated by the semigroup $\mathcal{T}$. If $\{w_k\}$ is an approximate fixed point sequence, then it follows from Theorem 7.4 that there exists a stationary point $w \in F(\mathcal{T})$ such that $w_n \to w$ in measure, and this convergence is residually stable. This result is new, even for nonexpansive semigroups in Musielak-Orlicz spaces, and of particular interest because convergence in measure is in this case much easier to deal with than the weak convergence, and is stronger than the weak convergence. Also, and importantly, no Opial condition needs to be directly verified. If $\varphi(t,u) = u^{p(t)}$, then the Musielak-Orlicz space becomes a variable exponent Lebesgue space. If $1 < a < p(t) < b < \infty$, then $\varphi(t,u) = u^{p(t)}$ is uniformly convex and ρ_φ satisfies the Δ_2-type condition and, therefore, the space is uniformly convex, guaranteeing the stable convergence in measure to a stationary point.

Residually stable convergence in measure to a stationary point of a dynamical system can also be interpreted as stable convergence in probability, providing additional probabilistic insights into the long-term behavior of semigroups associated with nonlinear stochastic processes. A classical linear Markov process is defined as a stochastic process where future states depend solely on the current state, independent of past states. In contrast, nonlinear Markov processes consider future states that rely not only on the present state but also on its current distribution. This concept was first introduced by McKean in [52] within the framework of mechanical transport problems and associated nonlinear parabolic equations. Nonlinear Markov processes have garnered substantial attention over recent decades. For more comprehensive information, refer to the monograph by Kolokoltsov [53], as well as recent contributions by Kolokoltsov [54], Carmona and Delarue [55], Pham and Wei [56], and Neumann [57], along with the literature cited therein. Nonlinear Markov chains and related Markov semigroups emerge naturally in various fields of applied science, including evolutionary biology, epidemiology, and game theory.

8. Conclusions

The aim of this paper, as outlined in Problem 1 from the Introduction, is to develop a theory that defines a mathematical technology for automating residually stable convergence of iterative processes to common fixed points for semigroups of nonlinear operators. This is achieved by establishing a new convergence structure for Banach spaces that can encompass a wide range of known and useful convergences, as well as be general enough to accommodate future scenarios. This structure was designed to ensure convergence and stability for iterative processes used in constructing common fixed points for asymptotic pointwise nonexpansive semigroups. An important goal was to achieve this using a single property. The workings of our theory, which we used to address this problem, can be summarised as follows:

1) We defined Ψ-convergence by means of a family of functions, $\Psi = \{\psi : X \to [0, +\infty)\}$, satisfying certain conditions, see Definition 3.1.
2) We demonstrated that the Ψ-convergence encompasses commonly used convergence cases:
   (a) strong, weak, weak-$*$ convergences (Examples 3.1–3.3),
   (b) L^p with convergence in measure and with convergence in the $L^{(p,q)}$-quasi-norms (Examples 3.4, 3.7, 3.8),
   (c) Sobolev spaces with convergence in measure of all weak derivatives (Example 3.6),
   (d) convergence in measure in Musielak-Orlicz spaces (Example 3.5).
3) We defined the Ψ-Opial property (Definitions 4.5 and 4.6), which generalises several known variants of the Opial property, including the classic weak Opial property and its modular counterpart.
4) In Theorem 4.3, we demonstrated that an approximate fixed point sequence for an asymptotic pointwise semigroup $\mathcal{T}$, generated by a well-controlled iteration process, is Ψ-convergent to a common fixed point of $\mathcal{T}$, provided that the Banach space X possesses the Ψ-Opial property.
5) In Theorem 5.1, we proved that every well-controlled iteration process is residually stable, provided it is Ψ-convergent to a common fixed point of $\mathcal{T}$.
6) We summarised the paper's findings in Theorem 6.1, which provides a general and comprehensive answer to the question posed in Problem 1. Additionally, since the paper presents a series of new specific results for various Banach spaces and types of convergence, Theorems 6.2 through 6.6 summarise these novel contributions.
7) In Section 7, we applied our theory to generalised Krasnosel'skii-Mann and Ishikawa iteration processes. We proved there that these processes are Ψ-convergent to a common fixed point of $\mathcal{T}$ provided that the sequences generated by them are approximate fixed point sequences for $\mathcal{T}$. Therefore, these standard processes are also residually stable for all our scenarios.

8) In the same section, we discussed an example of the application of our results to the problem of residually stable construction of stationary points for dynamical systems defined by asymptotic pointwise nonexpansive semigroups.

The reader may have observed that convergence in measure played a crucial role in the sample scenarios illustrating the power of our theory. Also known as convergence in probability within probability theory, convergence in measure is often favoured for its intuitive nature, as it allows for 'convergence while ignoring small sets.' This characteristic makes it particularly well-suited for addressing statistics-based problems in applied scientific fields. In L^p and Musielak-Orlicz spaces over sets of finite measure, convergence in measure is weaker than norm convergence but stronger than weak convergence (on bounded sets). Given this, along with the relative ease of working with sequences converging in measure in these spaces, and the presence of well-established criteria for sequential compactness with respect to this convergence, our results provide a convenient toolkit for tackling fixed point construction problems for semigroups of nonlinear operators acting in these spaces.

An important open question is whether the results of this paper related to Musielak-Orlicz spaces can be generalised to the non-Δ_2 situations. This is particularly urgent for the case of Orlicz spaces because of the critical role played by L^{cosh} space used as the space of observables in classical and quantum statistical physics, see [58]. It is worthwhile noting that the convergence in measure is central to these applications. The need for a thorough study of semigroups and their common fixed points in such spaces equipped with variations of convergence in measure relates to recent investigations into the nature of dynamics in these spaces (see [59, 60]).

A similar observation can be made regarding the application of Marcinkiewicz and Lorentz quasi-norms as Ψ-convergence. These types of convergence are also weaker than L^p norms (but stronger than convergence in measure), have well-understood properties, and are extensively utilised in solving variational problems, as well as in harmonic analysis, PDE theory, and the theory of interpolation spaces, with significant applications in physics and various branches of science, see for example [28].

If X is an ordered Banach space, wet must also consider when the semigroup associated with the system (7.7) becomes a semigroup of monotone operators, provided the seed operator H is itself monotone and asymptotically pointwise nonexpansive only on elements compatible in the partial order sense. An answer to this question can be found in [1, Theorem 9.4]. Since conditions such as the positivity of initial values and solutions are critical for many applications, it will be important to consider the automated residual stability technologies applicable to these scenarios.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

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