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Research article

Structural decomposition of arbitrary graded algebras

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Abstract: This paper is devoted to the study of the inner structure of graded algebras of arbitrary dimension over an arbitrary base field $\mathbb{F}$. By employing a framework that does not require additional identities such as associativity, commutativity, Lie, or Jordan, we investigate the decomposition of these algebras through connection techniques applied to the support of the grading. We show that, under certain mild conditions on the zero-component, any such algebra can be decomposed into a direct sum of orthogonal graded ideals. Furthermore, we provide conditions for the algebra to be represented as a direct sum of its simple graded ideals. Our results extend several well-known decomposition theorems from the commutative algebra case to the arbitrary algebra setting.

Keywords: arbitrary algebra; graded algebra; connectivity; ideal decomposition; structure theory
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1. Introduction

The study of gradings on different classes of algebras has garnered remarkable interest in recent years (for instance [1, 2]). This fascination is motivated, in part, because gradings provide a natural framework to decompose complicated algebraic structures into more manageable homogeneous components (for instance [3–5]). Indeed, often underlying symmetries that are not immediately apparent (for instance [6–8].)

In the present paper, we investigate the structural properties of graded algebras over an arbitrary base field $\mathbb{F}$ and of an arbitrary dimension. It is important to note that, within this framework, we do not assume further identities on the product (such as associativity, commutativity, Lie, or Jordan), allowing for a high degree of generality.

Definition 1.1. *By an* **algebra** $\mathcal{A}$*, we mean a vector space over* $\mathbb{F}$ *endowed with a bilinear map (called the product of the algebra):*

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad (x, y) \mapsto xy.$$

A **subalgebra** of an algebra $\mathcal{A}$ is a linear subspace B of $\mathcal{A}$ such that $BB \subset B$, while an **ideal** of $\mathcal{A}$ is a linear subspace I of $\mathcal{A}$ such that $I\mathcal{A} + \mathcal{A}I \subset I$.

Definition 1.2. *Let $\mathcal{A}$ be an algebra and $(G, +)$ a non-trivial Abelian group. We say that $\mathcal{A}$ is a **graded algebra** (by means of G) if it can be decomposed as a direct sum of linear subspaces:*

$$\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g, \tag{1.1}$$

*such that the grading follows the group operation: $\mathcal{A}_g\mathcal{A}_h \subset \mathcal{A}_{g+h}$ for all $g, h \in G$. The **support** of the grading is defined as the set:*

$$\Sigma := \{g \in G \setminus \{0\} : \mathcal{A}_g \neq 0\}. \tag{1.2}$$

Hence, we can write Eq (1.1) as

$$\mathcal{A} = \mathcal{A}_0 \oplus \left(\bigoplus_{g\in\Sigma} \mathcal{A}_g \right). \tag{1.3}$$

Here, we emphasize the special role that $\mathcal{A}_0$ plays in the theory of group-graded algebras, which stems directly from the unique properties of the identity element within the grading group. For instance, if we consider the Cartan decomposition of a finite-dimensional semisimple Lie algebra L with respect to a Cartan subalgebra H as a group grading, then the component L_0 coincides precisely with the Cartan subalgebra H. To underscore its significance, we shall refer to the homogeneous component $\mathcal{A}_0$ as the **core of the grading**. Indeed, a deep understanding of the core of a given grading naturally leads us to introduce the second equivalence relation in our study.

Definition 1.3. *A* **graded ideal** *I of $\mathcal{A}$ is an ideal that splits consistently with the grading, i.e.,*

$$I = \bigoplus_{g\in G} I_g,$$

where $I_g = I \cap \mathcal{A}_g$.

Definition 1.4. *We say that a graded algebra $\mathcal{A}$ is **graded simple** if its product is non-zero and its only graded ideals are $\{0\}$ and $\mathcal{A}$.*

In this work, we extend the results in [2] from the commutative case to the arbitrary setting. To do so, we investigate the connectivity of the support Σ of a G-graded algebra $\mathcal{A}$ in a very different way than in [2]. Indeed, we will need two different equivalence relations (on different sets), $\mathcal{R}_1$ and $\mathcal{R}_2$, to achieve our goals:

(1) The relation $\mathcal{R}_1$ will allow us to group homogeneous components based on their direct binary interactions. This construction will be carried out in Section 2.
(2) The relation $\mathcal{R}_2$ will take the coupling effects mediated by the core of the grading into account. Section 3 will be devoted to this goal.

The ultimate goal of this decomposition is to prove that any arbitrary graded algebra (indexed through an arbitrary Abelian grading group) can be represented as a direct sum of mutually orthogonal ideals, effectively reducing the study of complicated graded structures to the analysis of these ideals. Section 4 will be centered on this objective.

Finally, Section 5 focuses on the study of arbitrary graded algebras of maximal length $\mathcal{A}$. A theorem will be established by providing a decomposition of $\mathcal{A}$ as the direct sum of its graded simple ideals.

2. The first relation

Let

$$\mathcal{A} = \mathcal{A}_0 \oplus \left(\bigoplus_{g\in\Sigma} \mathcal{A}_g \right)$$

be a G-graded algebra defined over a base field $\mathbb{F}$, where G is a non-trivial Abelian group. We impose no cardinality restrictions on the dimension of $\mathcal{A}$ or the support Σ.

We define an auxiliary index set $\bar{\Sigma} = \{\bar{g} : g \in \Sigma\}$, which serves as a formal copy of the support. Let the interaction domain be defined by the union:

$$\Omega = \Sigma \cup \bar{\Sigma}.$$

Denote $\overline{(\bar{g})} := g$ for any $g \in \Sigma$.

To characterize the non-trivial algebraic interactions between homogeneous components, we introduce the following **transition map**

$$\nu : \Sigma \times \Omega \to \Sigma \cup \{0\}. \tag{2.1}$$

For any $g, h \in \Sigma$, the map is defined as follows:

$$\begin{aligned} &\nu(g,h) = g+h \quad \text{if} \quad \mathcal{A}_g\mathcal{A}_h + \mathcal{A}_h\mathcal{A}_g \neq 0 \text{ and } g+h \neq 0;\\ &\nu(g,\bar{h}) = g-h \quad \text{if} \quad \mathcal{A}_{g-h}\mathcal{A}_h + \mathcal{A}_h\mathcal{A}_{g-h} \neq 0 \text{ and } g-h \neq 0;\\ &\nu(g,h) = \nu(g,\bar{h}) = 0 \quad \text{in the remaining cases.} \end{aligned}$$

Lemma 2.1. *For any $g, h \in \Sigma$ and $\alpha \in \Omega$, the transition map satisfies $\nu(g,\alpha) = h$ if and only if $\nu(h,\bar{\alpha}) = g$.*

Proof. To establish the result, we consider the following two cases based on the nature of the element α:

Case 1: $\alpha \in \Sigma$. Suppose $\alpha = k$ for some $k \in \Sigma$. The condition $\nu(g,\alpha) = h$ implies the non-vanishing of either the product $\mathcal{A}_g\mathcal{A}_k \neq 0$ or $\mathcal{A}_k\mathcal{A}_g \neq 0$, which necessitates the relation $g + k = k + g = h$ within the Abelian grading group. Consequently, we observe that either $\mathcal{A}_{h-k}\mathcal{A}_k \neq 0$ or $\mathcal{A}_k\mathcal{A}_{h-k} \neq 0$, which yields

$$\nu(h,\bar{k}) = h - k = g$$

in either case. From this, we conclude that $\nu(h,\bar{\alpha}) = g$.

Case 2: $\alpha \in \bar{\Sigma}$. Suppose $\alpha = \bar{k}$ for some $k \in \Sigma$. In this scenario, the hypothesis

$$\nu(g,\alpha) = \nu(g,\bar{k}) = h$$

implies that either $\mathcal{A}_{g-k}\mathcal{A}_k \neq 0$ or $\mathcal{A}_k\mathcal{A}_{g-k} \neq 0$, where $h = g - k$. From this, $\nu(h,k) = h + k = g$ holds in either case. Since

$$\overline{\alpha} = \overline{(\bar{k})} = k,$$

we obtain the desired identity $\nu(h,\overline{\alpha}) = g$. □

We define an equivalence relation $\mathcal{R}_1$ on the support Σ.

Definition 2.1. *Two elements $g, h \in \Sigma$ are* **connected** *($g\mathcal{R}_1 h$) if $h \in \{g, -g\}$ or if there exists a finite sequence $(\alpha_1, \dots, \alpha_n) \subset \Omega$, called a* **connection** *from g to h of* **length** *n, such that:*

- $\nu(\epsilon_0 g, \alpha_1) = g_1$, *for some* $\epsilon_0 \in \{1, -1\}$;
- $\nu(\epsilon_1 g_1, \alpha_2) = g_2$, *for some* $\epsilon_1 \in \{1, -1\}$;

  $\vdots$
- $\nu(\epsilon_i g_i, \alpha_{i+1}) = g_{i+1}$, *for some* $\epsilon_i \in \{1, -1\}$;

  $\vdots$
- $\nu(\epsilon_{n-2} g_{n-2}, \alpha_{n-1}) = g_{n-1}$, *for some* $\epsilon_{n-2} \in \{1, -1\}$;
- $\nu(\epsilon_{n-1} g_{n-1}, \alpha_n) = \epsilon_n h$, *for some* $\epsilon_{n-1}, \epsilon_n \in \{1, -1\}$.

Lemma 2.2. *The connection relation $\mathcal{R}_1$ defined on the support Σ is symmetric.*

Proof. Let $g, h \in \Sigma$. In the case where $h \in \{g, -g\}$, it is clear that $g\mathcal{R}_1 h$ implies $h\mathcal{R}_1 g$. Then, suppose that $g\mathcal{R}_1 h$ with $h \notin \{g, -g\}$. In accordance with Definition 2.1, there exists a connection $(\alpha_1, \alpha_2, \dots, \alpha_n)$ in Ω of length $n \geq 1$. Let us verify that $h\mathcal{R}_1 g$.

By applying Lemma 2.1 to the chain of conditions in Definition 2.1, we derive the following identities:

- $\nu(g_1, \bar{\alpha}_1) = \epsilon_0 g \in \Sigma$, for some $\epsilon_0 \in \{1, -1\}$;
- $\nu(g_2, \bar{\alpha}_2) = \epsilon_1 g_1 \in \Sigma$, for some $\epsilon_1 \in \{1, -1\}$;

  $\vdots$
- $\nu(g_{i+1}, \bar{\alpha}_{i+1}) = \epsilon_i g_i \in \Sigma$, for some $\epsilon_i \in \{1, -1\}$;

  $\vdots$
- $\nu(g_{n-1}, \bar{\alpha}_{n-1}) = \epsilon_{n-2} g_{n-2} \in \Sigma$, for some $\epsilon_{n-2} \in \{1, -1\}$;
- $\nu(\epsilon_n h, \bar{\alpha}_n) = \epsilon_{n-1} g_{n-1} \in \Sigma$, for some $\epsilon_{n-1}, \epsilon_n \in \{1, -1\}$.

The existence of the above relations implies that the sequence $(\bar{\alpha}_n, \bar{\alpha}_{n-1}, \dots, \bar{\alpha}_1)$ in Ω provides a connection from h to g as required by Definition 2.1. Thus, $h\mathcal{R}_1 g$, confirming that the relation is symmetric. □

Lemma 2.3. *The connection relation $\mathcal{R}_1$ on the support Σ is transitive.*

Proof. Let $g, h, k \in \Sigma$ such that $g\mathcal{R}_1 h$ and $h\mathcal{R}_1 k$. In the cases where either $h \in \{g, -g\}$ or $k \in \{h, -h\}$, it is clear that $g\mathcal{R}_1 k$. Thus, suppose $h \notin \{g, -g\}$ and $k \notin \{h, -h\}$.

By Definition 2.1, there exist two finite sequences $(\alpha_1, \dots, \alpha_n)$ and $(\beta_1, \dots, \beta_m)$ in Ω, with $n, m \geq 1$, which establish the connections from g to h and from h to k, respectively. Specifically, the first sequence implies:

- $\nu(\epsilon_0 g, \alpha_1) = g_1 \in \Sigma$, for some $\epsilon_0 \in \{1, -1\}$;
- $\nu(\epsilon_1 g_1, \alpha_2) = g_2 \in \Sigma$, for some $\epsilon_1 \in \{1, -1\}$;

  $\vdots$
- $\nu(\epsilon_{n-1} g_{n-1}, \alpha_n) = \epsilon_n h$, for some $\epsilon_{n-1}, \epsilon_n \in \{1, -1\}$.

Similarly, the second sequence yields:

- $\nu(\tau_0 h, \beta_1) = h_1 \in \Sigma$, for some $\tau_0 \in \{1, -1\}$;
- $\nu(\tau_1 h_1, \beta_2) = h_2 \in \Sigma$, for some $\tau_1 \in \{1, -1\}$;

  $\vdots$
- $\nu(\tau_{m-1} h_{m-1}, \beta_m) = \tau_m k$, for some $\tau_{m-1}, \tau_m \in \{1, -1\}$.

By concatenating these sequences, we obtain the sequence $(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m)$ in Ω, which satisfies the requirements of Definition 2.1 for a connection between g and k. Thus, $g\mathcal{R}_1 k$, confirming the transitivity of the relation. □

Proposition 2.1. *The connection relation $\mathcal{R}_1$ on Σ constitutes an equivalence relation.*

Proof. Reflexivity directly follows from the definition of a connection. The symmetry and transitivity of $\mathcal{R}_1$ are guaranteed by Lemmas 2.2 and 2.3, respectively. □

In view of Proposition 2.1, the connection relation $\mathcal{R}_1$ induces a partition on the support Σ. We denote the quotient set by

$$\Sigma/\mathcal{R}_1 := \{[g] : g \in \Sigma\},$$

where each equivalence class $[g]$ consists of all elements in Σ connected to g. Our subsequent goal is to associate a suitable graded linear subspace of the algebra $\mathcal{A}$ with each class $[g]$.

For any equivalence class $[g] \in \Sigma/\mathcal{R}_1$, we define the graded subspace:

$$\mathcal{A}_{[g]} := \bigoplus_{h \in [g]} \mathcal{A}_h. \tag{2.2}$$

Consequently, the non-core part of the algebra, (see Eq (1.3)), admits the following decomposition:

$$\bigoplus_{g \in \Sigma} \mathcal{A}_g = \bigoplus_{[g] \in \Sigma/\mathcal{R}_1} \mathcal{A}_{[g]}. \tag{2.3}$$

Lemma 2.4. *For distinct classes $[g] \neq [h]$, the orthogonality condition*

$$\mathcal{A}_{[g]}\mathcal{A}_{[h]} = 0$$

holds.

Proof. Suppose there exist $g' \in [g]$ and $h' \in [h]$ such that $\mathcal{A}_{g'}\mathcal{A}_{h'} \neq 0$. We aim to show that this condition implies $g'\mathcal{R}_1 h'$, which would contradict the assumption that $[g]$ and $[h]$ are distinct equivalence classes.

We begin by observing that $g' + h' \neq 0$. Indeed, in case $g' + h' = 0$, then $h' = -g'$ and so

$$[g] = [g'] = [h'] = [h],$$

a contradiction.

Now, the non-vanishing of the product $\mathcal{A}_{g'}\mathcal{A}_{h'}$, coupled with the fact that $g' + h' \neq 0$, implies by definition that $\nu(g', h') = g' + h'$. Now, we examine the pair $(h', \overline{g'})$ as a potential connection chain. First, we have $\nu(g', h') = g' + h'$. Second, we have:

$$\mathcal{A}_{g'}\mathcal{A}_{(g'+h')-g'} = \mathcal{A}_{g'}\mathcal{A}_{h'} \neq 0.$$

Since $(g' + h') - g' = h' \neq 0$, it follows that $\nu(g' + h', \overline{g'}) = h'$.

Consequently, the sequence $(h', \overline{g'})$ constitutes a valid connection from g' to h' in the sense of Definition 2.1. This ensures that g' and h' belong to the same equivalence class, i.e., $[g] = [h]$. By contraposition, if $[g] \neq [h]$, then, we must have $\mathcal{A}_{g'}\mathcal{A}_{h'} = 0$ for all $g' \in [g]$ and $h' \in [h]$, which yields the desired orthogonality condition $\mathcal{A}_{[g]}\mathcal{A}_{[h]} = 0$. □

Lemma 2.5. *For any $g, h \in \Sigma$ such that $\mathcal{A}_g\mathcal{A}_h \neq 0$ and $g + h \neq 0$, we have $[g + h] = [g] = [h]$.*

Proof. First, observe that the connection (h) gives us $g\mathcal{R}_1(g + h)$ and so $[g] = [g + h]$. In a similar way, the connection (g) gives us $h\mathcal{R}_1(g + h)$ and so $[h] = [g + h]$. Finally, the transitivity of $\mathcal{R}_1$ implies that $[g] = [h]$. □

3. The second relation

In this section, we relate the elements in the core of the grading $\mathcal{A}_0$. In the main results we will assume that

$$\mathcal{A}_0 = \sum_{g\in\Sigma} \mathcal{A}_g\mathcal{A}_{-g}.$$

This is a standard hypothesis in the study of group-graded algebras, (see, for instance, [2, 7]), in order to avoid superfluous elements to the grading in $\mathcal{A}_0$.

We begin as follows:

First, recall that $\mathcal{A}_0$ is a subalgebra of $\mathcal{A}$ since $\mathcal{A}_0\mathcal{A}_0 \subset \mathcal{A}_0$.

Second, for each class $[g] \in \Sigma/\mathcal{R}_1$, we can associate a **core subalgebra**

$$\mathcal{A}_{0,[g]} \subset \mathcal{A}_0$$

defined as the ideal of $\mathcal{A}_0$ generated by the set

$$\{\mathcal{A}_h\mathcal{A}_{-h} : h \in [g]\} \subset \mathcal{A}_0.$$

That is, the smallest ideal of $\mathcal{A}_0$ containing the aforementioned set.

Observe that Eq (2.2) and Lemma 2.5 imply that

$$\mathcal{A}_{0,[g]} \oplus \mathcal{A}_{[g]} \tag{3.1}$$

is a graded subalgebra of $\mathcal{A}$ for any $[g] \in \Sigma/\mathcal{R}_1$.

Now, we define a secondary equivalence relation $\mathcal{R}_2$ on

$$\Sigma/\mathcal{R}_1 = \{[g] : g \in \Sigma\}.$$

Definition 3.1. *Two classes $[g], [h]$ in $\Sigma/\mathcal{R}_1$ are* **related** *($[g]\mathcal{R}_2[h]$) if either $[g] = [h]$ or there exists a chain*

$$([k_1], \ldots, [k_n]) \subset \Sigma/\mathcal{R}_1,$$

with $n \geq 2$ (called a **connection sequence** *from $[g]$ to $[h]$), such that $[k_1] = [g]$, $[k_n] = [h]$, and the following coupling condition holds for each $1 \leq i \leq n - 1$*

$$\mathcal{A}_{0,[k_i]}\mathcal{A}_{[k_{i+1}]} + \mathcal{A}_{[k_{i+1}]}\mathcal{A}_{0,[k_i]} + \mathcal{A}_{0,[k_{i+1}]}\mathcal{A}_{[k_i]} + \mathcal{A}_{[k_i]}\mathcal{A}_{0,[k_{i+1}]} + \mathcal{A}_{0,[k_i]}\mathcal{A}_{0,[k_{i+1}]} + \mathcal{A}_{0,[k_{i+1}]}\mathcal{A}_{0,[k_i]} \neq 0.$$

Lemma 3.1. *The relation $\mathcal{R}_2$ is an equivalence relation on the set $\Sigma/\mathcal{R}_1$.*

Proof. The reflexivity of the relation $\mathcal{R}_2$ immediately follows from its definition.

To establish the symmetry of $\mathcal{R}_2$, suppose that $[g]\mathcal{R}_2[h]$ is satisfied by the existence of a connection sequence $([k_1],[k_2],\dots,[k_n])$. It is then evident that the reversed sequence $([k_n],[k_{n-1}],\dots,[k_1])$ satisfies the symmetric requirements, thus, confirming that $[h]\mathcal{R}_2[g]$.

Finally, we address the transitivity of the relation. Let $[g],[h],[t] \in \Sigma/\mathcal{R}_1$ such that $[g]\mathcal{R}_2[h]$ and $[h]\mathcal{R}_2[t]$. By hypothesis, there exist finite sequences $([k_1],\dots,[k_n])$ and $([t_1],\dots,[t_m])$ which establish these respective connections. Taking into account that $[k_n] = [t_1] = [h]$, a straightforward verification shows that the concatenated sequence

$$([k_1],\dots,[k_n],[t_2],\dots,[t_m])$$

constitutes a valid connection from $[g]$ to $[t]$. Consequently, $[g]\mathcal{R}_2[t]$, proving that $\mathcal{R}_2$ is a transitive relation. $\square$

We introduced a new equivalence relation $\mathcal{R}_2$ on the set $\Sigma/\mathcal{R}_1$ (see Lemma 3.1), which allows us to define the following graded linear subspaces of $\mathcal{A}$ for any $[g] \in \Sigma/\mathcal{R}_1$. First, denote

$$[[g]] := \{[h] \in \Sigma/\mathcal{R}_1 : [h]\mathcal{R}_2[g]\}$$

and so

$$(\Sigma/\mathcal{R}_1)/\mathcal{R}_2 = \{[[g]] : [g] \in (\Sigma/\mathcal{R}_1)\}.$$

Finally, for any $[[g]] \in (\Sigma/\mathcal{R}_1)/\mathcal{R}_2$, we will denote

$$\mathcal{A}_{0[[g]]} := \sum_{[h]\in[[g]]} \mathcal{A}_{0,[h]} \subset \mathcal{A}_0 \quad \text{and} \quad \mathcal{A}_{[[g]]} = \bigoplus_{[h]\in[[g]]} \mathcal{A}_{[h]} \subset \bigoplus_{g\in\Sigma} \mathcal{A}_g. \tag{3.2}$$

Observe that Eqs (3.1) and (3.2) imply that

$$\mathcal{A}_{0[[g]]} \oplus \mathcal{A}_{[[g]]} \tag{3.3}$$

is a graded linear subspace of $\mathcal{A}$ for any $[[g]] \in (\Sigma/\mathcal{R}_1)/\mathcal{R}_2$.

Proposition 3.1. *Let*

$$\mathcal{A} = \mathcal{A}_0 \oplus \left(\bigoplus_{g\in\Sigma} \mathcal{A}_g \right)$$

be a graded algebra. Then, $\mathcal{A}_{0[[g]]} \oplus \mathcal{A}_{[[g]]}$ is a graded subalgebra of $\mathcal{A}$ for any $[[g]] \in (\Sigma/\mathcal{R}_1)/\mathcal{R}_2$.

Proof. We begin by recalling that Eq (3.3) ensures that $\mathcal{A}_{0[[g]]} \oplus \mathcal{A}_{[[g]]}$ is a graded linear subspace of $\mathcal{A}$.

Now, consider the following product:

$$(\mathcal{A}_{0[[g]]} \oplus \mathcal{A}_{[[g]]})(\mathcal{A}_{0[[g]]} \oplus \mathcal{A}_{[[g]]}) \subset (\mathcal{A}_{0[[g]]}\mathcal{A}_{0[[g]]}) + (\mathcal{A}_{0[[g]]}\mathcal{A}_{[[g]]}) + (\mathcal{A}_{[[g]]}\mathcal{A}_{0[[g]]}) + (\mathcal{A}_{[[g]]}\mathcal{A}_{[[g]]}). \tag{3.4}$$

First, observe that

$$\mathcal{A}_{0[[g]]}\mathcal{A}_{0[[g]]} \subset \mathcal{A}_0 \left(\sum_{[h]\in[[g]]} \mathcal{A}_{0,[h]} \right) \subset \sum_{[h]\in[[g]]} \mathcal{A}_0\mathcal{A}_{0,[h]} \subset \sum_{[h]\in[[g]]} \mathcal{A}_{0,[h]} = \mathcal{A}_{0[[g]]} \tag{3.5}$$

and that

$$(\mathcal{A}_{0[[g]]}\mathcal{A}_{[[g]]}) + (\mathcal{A}_{[[g]]}\mathcal{A}_{0[[g]]}) \subset \mathcal{A}_0\left(\bigoplus_{[h]\in[[g]]}\mathcal{A}_{[h]}\right) + \left(\bigoplus_{[h]\in[[g]]}\mathcal{A}_{[h]}\right)\mathcal{A}_0 \subset \bigoplus_{[h]\in[[g]]}\mathcal{A}_{[h]} = \mathcal{A}_{[[g]]}. \tag{3.6}$$

Finally, by Lemma 2.4 and Eq (3.1), we also have

$$\mathcal{A}_{[[g]]}\mathcal{A}_{[[g]]} = \left(\bigoplus_{[h]\in[[g]]}\mathcal{A}_{[h]}\right)\left(\bigoplus_{[h]\in[[g]]}\mathcal{A}_{[h]}\right) \subset \mathcal{A}_{[[g]]}. \tag{3.7}$$

Equations (3.4)–(3.7) allow us to assert

$$(\mathcal{A}_{0[[g]]} \oplus \mathcal{A}_{[[g]]})(\mathcal{A}_{0[[g]]} \oplus \mathcal{A}_{[[g]]}) \subset \mathcal{A}_{0[[g]]} \oplus \mathcal{A}_{[[g]]}$$

as desired. □

Proposition 3.2. *Let*

$$\mathcal{A} = \mathcal{A}_0 \oplus \left(\bigoplus_{g\in\Sigma}\mathcal{A}_g\right)$$

be a graded algebra. Then,

$$\bigoplus_{g\in\Sigma}\mathcal{A}_g = \bigoplus_{[[g]]\in(\Sigma/\mathcal{R}_1)/\mathcal{R}_2}\mathcal{A}_{[[g]]}$$

with $\mathcal{A}_{[[g]]}\mathcal{A}_{[[h]]} = 0$ *for any* $[[g]], [[h]] \in (\Sigma/\mathcal{R}_1)/\mathcal{R}_2$ *such that* $[[g]] \neq [[h]]$.

Proof. The decomposition as a direct sum of linear subspaces is a consequence of Eq (2.3) and Lemma 2.5, while the fact that $\mathcal{A}_{[[g]]}\mathcal{A}_{[[h]]} = 0$ when $[[g]] \neq [[h]]$ follows from Lemma 2.5. □

Lemma 3.2. *Let* $[[g]], [[h]] \in (\Sigma/\mathcal{R}_1)/\mathcal{R}_2$ *such that* $[[g]] \neq [[h]]$. *Then, the following assertions hold:*

(i) $\mathcal{A}_{0,[[g]]}\mathcal{A}_{[[h]]} + \mathcal{A}_{[[h]]}\mathcal{A}_{0,[[g]]} = 0$;

(ii) $\mathcal{A}_{0,[[g]]}\mathcal{A}_{0,[[h]]} = 0$.

Proof. We begin by observing that

$$\mathcal{A}_{0,[g']}\mathcal{A}_{[h']} + \mathcal{A}_{[h']}\mathcal{A}_{0,[g']} = 0 \tag{3.8}$$

and

$$\mathcal{A}_{0,[g']}\mathcal{A}_{0,[h']} = 0 \tag{3.9}$$

for any $[g'] \in [[g]]$ and $[h'] \in [[h]]$.

Indeed, in the opposite case, the connection sequence $([g'], [h'])$ would imply that $[g']$ is related to $[h']$ through $\mathcal{R}_2$, which would yield $[[g]] = [[h]]$, which is a contradiction. Now, the results follow from Eqs (3.8) and (3.9). □

We recall that the **annihilator** of an algebra $\mathcal{A}$ is defined as the ideal

$$\mathrm{Ann}(\mathcal{A}) = \{x \in \mathcal{A} : x\mathcal{A} + \mathcal{A}x = 0\}.$$

Proposition 3.3. *Suppose*

$$\mathcal{A}_0 = \sum_{g\in\Sigma} \mathcal{A}_g\mathcal{A}_{-g}$$

and $\mathrm{Ann}(\mathcal{A}) = 0$. *Then,*

$$\mathcal{A}_0 = \bigoplus_{[[g]]\in(\Sigma/\mathcal{R}_1)/\mathcal{R}_2} \mathcal{A}_{0,[[g]]}.$$

Proof. Since

$$\mathcal{A}_0 = \sum_{g\in\Sigma} \mathcal{A}_g\mathcal{A}_{-g},$$

we clearly have

$$\mathcal{A}_0 = \sum_{[[g]]\in(\Sigma/\mathcal{R}_1)/\mathcal{R}_2} \mathcal{A}_{0,[[g]]}. \tag{3.10}$$

Hence, we only have to verify the direct character of the sum.

To do so, suppose there exists some $x \in \mathcal{A}_0$ and $[[g]] \in (\Sigma/\mathcal{R}_1)/\mathcal{R}_2$ such that

$$x \in \mathcal{A}_{0,[[g]]} \cap \left(\sum_{[[h]]\in((\Sigma/\mathcal{R}_1)/\mathcal{R}_2)\setminus\{[[g]]\}} \mathcal{A}_{0,[[h]]} \right). \tag{3.11}$$

Since $\mathrm{Ann}(\mathcal{A}) = 0$, there exists $0 \neq y \in \mathcal{A}$ such that either $yx \neq 0$ or $xy \neq 0$. Suppose $yx \neq 0$ (the other case can be argued analogously).

We can distinguish two possibilities. In the first case, we can take $y \in \mathcal{A}_k$ for $k \neq 0$. Then, Eq (3.11) gives us $y\mathcal{A}_{0,[[g]]} \neq 0$ and

$$y \left(\sum_{[[h]]\in((\Sigma/\mathcal{R}_1)/\mathcal{R}_2)\setminus\{[[g]]\}} \mathcal{A}_{0,[[h]]} \right) \neq 0.$$

From this, we obtain $[[k]] = [[g]]$ and also $[[k]] \neq [[g]]$, a contradiction.

In the second case, we have $y \in \mathcal{A}_0$. Then there exists $l \in \Sigma$ such that we can assume $y \in \mathcal{A}_l\mathcal{A}_{-l}$. As above, Eq (3.11) implies that $y\mathcal{A}_{0,[[g]]} \neq 0$ and

$$y \left(\sum_{[[h]]\in((\Sigma/\mathcal{R}_1)/\mathcal{R}_2)\setminus\{[[g]]\}} \mathcal{A}_{0,[[h]]} \right) \neq 0,$$

which yields $[[l]] = [[g]]$ and also $[[l]] \neq [[g]]$, which is another contradiction. Thus, Eq (3.11) does not hold, and the sum in Eq (3.10) is indeed a direct sum. □

4. Global decomposition theorem

Based on the construction of the equivalence relations $\mathcal{R}_1$ and $\mathcal{R}_2$, we can state the primary structural results for graded algebras. Theorem 4.1 extends [2, Theorem 2.1] from commutative algebras to arbitrary algebras, while Theorem 5.1 and its framework correspond to the extensions of [2, Theorem 2.2 and Corollary 2.1] .

Theorem 4.1. *Let*

$$\mathcal{A} = \mathcal{A}_0 \oplus \left(\bigoplus_{g \in \Sigma} \mathcal{A}_g \right)$$

be a graded algebra such that

$$\mathcal{A}_0 = \sum_{g \in \Sigma} \mathcal{A}_g \mathcal{A}_{-g} \quad \textit{and} \quad \operatorname{Ann}(\mathcal{A}) = 0.$$

Then, $\mathcal{A}$ decomposes as the direct sum

$$\mathcal{A} = \bigoplus_{[[g]] \in (\Sigma/\mathcal{R}_1)/\mathcal{R}_2} (\mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]})$$

where each $\mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]}$ is a graded ideal of $\mathcal{A}$ satisfying

$$(\mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]})(\mathcal{A}_{0,[[h]]} \oplus \mathcal{A}_{[[h]]}) = 0$$

when $[[g]] \neq [[h]]$.

Proof. We begin by observing that Propositions 3.2 and 3.3 and Lemma 3.2 allow us to assert that

$$\begin{aligned}\mathcal{A} &= \mathcal{A}_0 \oplus \left(\bigoplus_{g \in \Sigma} \mathcal{A}_g \right) = \left(\bigoplus_{[[g]] \in (\Sigma/\mathcal{R}_1)/\mathcal{R}_2} \mathcal{A}_{0,[[g]]} \right) \oplus \left(\bigoplus_{[[g]] \in (\Sigma/\mathcal{R}_1)/\mathcal{R}_2} \mathcal{A}_{[[g]]} \right) \\ &= \bigoplus_{[[g]] \in (\Sigma/\mathcal{R}_1)/\mathcal{R}_2} (\mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]})\end{aligned}$$

with

$$\mathcal{A}_{[[g]]}\mathcal{A}_{[[h]]} = 0 \tag{4.1}$$

for any $[[g]], [[h]] \in (\Sigma/\mathcal{R}_1)/\mathcal{R}_2$ such that $[[g]] \neq [[h]]$.

From this, it remains only to prove that each $\mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]}$ is a graded ideal of $\mathcal{A}$. Consider the product

$$\mathcal{A}(\mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]}) = \left(\bigoplus_{[[h]] \in (\Sigma/\mathcal{R}_1)/\mathcal{R}_2} (\mathcal{A}_{0,[[h]]} \oplus \mathcal{A}_{[[h]]}) \right) (\mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]}). \tag{4.2}$$

By Lemma 3.2, Eq (4.1), and the grading, we have

$$\left(\bigoplus_{[[h]] \in (\Sigma/\mathcal{R}_1)/\mathcal{R}_2} (\mathcal{A}_{0,[[h]]} \oplus \mathcal{A}_{[[h]]}) \right) (\mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]}) \subset (\mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]})(\mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]}). \tag{4.3}$$

Now, Proposition 3.1 yields

$$(\mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]})(\mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]}) \subset \mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]}. \tag{4.4}$$

From Eqs (4.2)–(4.4), we obtain

$$\mathcal{A}(\mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]}) \subset \mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]}.$$

In a similar way, we have $(\mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]})\mathcal{A} \subset \mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]}$, showing that $\mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]}$ is an ideal of $\mathcal{A}$. Finally, its graded character follows from Eq (3.3). $\square$

5. The maximal length case

In this section, we consider graded algebras of maximal length and characterize the simplicity of a graded algebra.

Definition 5.1. *A graded algebra*

$$\mathcal{A} = \mathcal{A}_0 \oplus \left(\bigoplus_{g\in\Sigma} \mathcal{A}_g \right)$$

is said to be of **maximal length** *if* $\dim(\mathcal{A}_g) = 1$ *for any* $g \in \Sigma$.

Definition 5.2. *A graded algebra*

$$\mathcal{A} = \mathcal{A}_0 \oplus \left(\bigoplus_{g\in\Sigma} \mathcal{A}_g \right)$$

of maximal length with

$$\mathcal{A}_0 = \sum_{g\in\Sigma} \mathcal{A}_g \mathcal{A}_{-g}$$

is said to be a **division graded algebra** *if the following conditions hold:*

- *For any* $g, h \in \Sigma$ *with* $g + h \neq 0$, *we have*

$$\mathcal{A}_g \mathcal{A}_h = \mathcal{A}_{g+h} \iff \mathcal{A}_g = \mathcal{A}_{g+h}\mathcal{A}_{-h} \iff \mathcal{A}_h = \mathcal{A}_{-g}\mathcal{A}_{g+h}.$$

- *For any* $g, h \in \Sigma$, *we have*

$$\mathcal{A}_g \mathcal{A}_{-g} = \mathcal{A}_h \mathcal{A}_{-h} \iff \mathcal{A}_g = (\mathcal{A}_h \mathcal{A}_{-h})\mathcal{A}_g \iff \mathcal{A}_{-g} = \mathcal{A}_{-g}(\mathcal{A}_h \mathcal{A}_{-h}).$$

- *For any* $g, h, k \in \Sigma$, *we have*

$$\begin{aligned}(\mathcal{A}_h \mathcal{A}_{-h})(\mathcal{A}_g \mathcal{A}_{-g}) = \mathcal{A}_k A_{-k} \iff \mathcal{A}_h \mathcal{A}_{-h} \\ = (\mathcal{A}_k A_{-k})(\mathcal{A}_g \mathcal{A}_{-g}) \iff \mathcal{A}_g \mathcal{A}_{-g} \\ = (\mathcal{A}_h \mathcal{A}_{-h})(\mathcal{A}_k A_{-k}).\end{aligned}$$

Proposition 5.1. *Let*

$$\mathcal{A} = \mathcal{A}_0 \oplus \left(\bigoplus_{g\in\Sigma} \mathcal{A}_g \right)$$

be a graded algebra of maximal length such that

$$\mathcal{A}_0 = \sum_{g\in\Sigma} \mathcal{A}_g \mathcal{A}_{-g}$$

and $\mathrm{Ann}(\mathcal{A}) = 0$. *If the grading of* $\mathcal{A}$ *is of division, then any ideal* $\mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]}$ *for* $[[g]] \in (\Sigma/\mathcal{R}_1)/\mathcal{R}_2$ *in the decomposition of Theorem 4.1 is graded simple.*

Proof. We begin by showing that

$$\text{if } \mathcal{A}_g \subset I \text{ for some } g \in \Sigma$$

and some non-zero graded ideal I of $\mathcal{A}$, then

$$0 \neq \mathcal{A}_{-g} \subset I. \tag{5.1}$$

Indeed, since $\mathrm{Ann}(\mathcal{A}) = 0$, we have three cases. In the first case, there exists $h \in \Sigma$ with $g + h \neq 0$ such that

$$0 \neq \mathcal{A}_g\mathcal{A}_h = \mathcal{A}_{g+h} \subset I.$$

Thus, the division character of the grading allows us to assert

$$0 \neq \mathcal{A}_h = \mathcal{A}_{-g}\mathcal{A}_{g+h} \subset I,$$

from which it follows that

$$0 \neq \mathcal{A}_{-g} = \mathcal{A}_h\mathcal{A}_{-g-h} \subset I.$$

In the second case,

$$0 \neq \mathcal{A}_g\mathcal{A}_{-g} \subset I.$$

By the division character of the grading we get that the equality

$$\mathcal{A}_g\mathcal{A}_{-g} = \mathcal{A}_g\mathcal{A}_{-g} \subset I$$

implies

$$\mathcal{A}_{-g} = \mathcal{A}_{-g}(\mathcal{A}_g\mathcal{A}_{-g}) \subset I.$$

In the third case, there exists $h \in \Sigma$ such that

$$0 \neq \mathcal{A}_g(\mathcal{A}_h\mathcal{A}_{-h}) = \mathcal{A}_g \subset I.$$

From this, we have

$$0 \neq \mathcal{A}_h\mathcal{A}_{-h} = \mathcal{A}_{-g}\mathcal{A}_g \subset I$$

and then

$$0 \neq \mathcal{A}_{-g} = (\mathcal{A}_h\mathcal{A}_{-h})\mathcal{A}_{-g} \subset I.$$

Thus, we have shown that $0 \neq \mathcal{A}_{-g} \subset I$ in either case.

Second, let us consider the transition map in Eq (2.1) and suppose there exists $g \in \Sigma$ such that $\mathcal{A}_g \subset I$ for a non-zero graded ideal of $\mathcal{A}$. Then, we have: if $\nu(g, h) = g + h \neq 0$ for $h \in \Sigma$, then

$$\mathcal{A}_g\mathcal{A}_h \neq 0 \quad \text{or} \quad \mathcal{A}_h\mathcal{A}_g \neq 0.$$

By the maximal length of the grading,

$$\mathcal{A}_g\mathcal{A}_h = \mathcal{A}_{g+h} \quad \text{or} \quad \mathcal{A}_h\mathcal{A}_g = \mathcal{A}_{g+h},$$

which implies $\mathcal{A}_{g+h} \subset I$.

If

$$\nu(g, \bar{h}) = g - h \neq 0$$

for $h \in \Sigma$, then,

$$\mathcal{A}_{g-h}\mathcal{A}_h \neq 0 \quad \text{or} \quad \mathcal{A}_h\mathcal{A}_{g-h} \neq 0.$$

By the maximal length of the grading,

$$\mathcal{A}_{g-h}\mathcal{A}_h = \mathcal{A}_g \quad \text{or} \quad \mathcal{A}_h\mathcal{A}_{g-h} = \mathcal{A}_g.$$

Now, the division character of the grading implies

$$\mathcal{A}_{g-h} = \mathcal{A}_g\mathcal{A}_{-h} \subset I \quad \text{or} \quad \mathcal{A}_{g-h} = \mathcal{A}_{-h}\mathcal{A}_g \subset I.$$

From the above discussion and Eq (5.1), we can assert that

$$\text{if } \mathcal{A}_g \subset I$$

for a non-zero graded ideal I of $\mathcal{A}$, and

$$\nu(\epsilon g, \alpha) = g_1$$

for some $\epsilon \in \{1, -1\}$ and $\alpha \in \Sigma \cup \bar{\Sigma}$, then, box

$$\mathcal{A}_{g_1} \subset I. \tag{5.2}$$

Third, consider now $[g] \in \Sigma/\mathcal{R}_1$ for some $g \in \Sigma$ satisfying $\mathcal{A}_g \subset I$ for a non-zero graded ideal I of $\mathcal{A}$. Since any $h \in [g]$ is connected to g (see Definition 2.1), Eqs (5.1) and (5.2) show that $\mathcal{A}_h \subset I$, therefore

$$\mathcal{A}_{0,[g]} \oplus \mathcal{A}_{[g]} \subset I \tag{5.3}$$

(see Eq (3.1)).

Fourth, consider $[g] \in \Sigma/\mathcal{R}_1$ for some $g \in \Sigma$ satisfying $\mathcal{A}_{0,[g]} \oplus \mathcal{A}_{[g]} \subset I$ for a non-zero graded ideal I of $\mathcal{A}$, and take any $[h] \in \Sigma/\mathcal{R}_1$ such that

$$\mathcal{A}_{0,[g]}\mathcal{A}_{[h]} + \mathcal{A}_{[h]}\mathcal{A}_{0,[g]} + \mathcal{A}_{0,[h]}\mathcal{A}_{[g]} + \mathcal{A}_{[g]}\mathcal{A}_{0,[h]} + \mathcal{A}_{0,[g]}\mathcal{A}_{0,[h]} + \mathcal{A}_{0,[h]}\mathcal{A}_{0,[g]} \neq 0. \tag{5.4}$$

Let us distinguish three cases:

Case 1: If $\mathcal{A}_{0,[g]}\mathcal{A}_{[h]} + \mathcal{A}_{[h]}\mathcal{A}_{0,[g]} \neq 0$, then

$$0 \neq \mathcal{A}_{0,[g]}\mathcal{A}_{[h]} + \mathcal{A}_{[h]}\mathcal{A}_{0,[g]} \subset \mathcal{A}_{[h]} \quad \text{and} \quad 0 \neq \mathcal{A}_{0,[g]}\mathcal{A}_{[h]} + \mathcal{A}_{[h]}\mathcal{A}_{0,[g]} \subset I.$$

Hence, Eq (5.3) allows us to assert that

$$\mathcal{A}_{0,[h]} \oplus \mathcal{A}_{[h]} \subset I.$$

Case 2: If $\mathcal{A}_{0,[h]}\mathcal{A}_{[g]} + \mathcal{A}_{[g]}\mathcal{A}_{0,[h]} \neq 0$, then there exist $g' \in [g]$ and $h' \in [h]$ such that

$$(\mathcal{A}_{h'}\mathcal{A}_{-h'})\mathcal{A}_{g'} + \mathcal{A}_{g'}(\mathcal{A}_{h'}\mathcal{A}_{-h'}) \neq 0.$$

Thus,

$$(\mathcal{A}_{h'}\mathcal{A}_{-h'})\mathcal{A}_{g'} + \mathcal{A}_{g'}(\mathcal{A}_{h'}\mathcal{A}_{-h'}) = \mathcal{A}_{g'}.$$

The division character of the grading implies that either

$$\mathcal{A}_{h'}\mathcal{A}_{-h'} = \mathcal{A}_{g'}\mathcal{A}_{-g'} \subset I \quad \text{or} \quad \mathcal{A}_{h'}\mathcal{A}_{-h'} = \mathcal{A}_{-g'}\mathcal{A}_{g'} \subset I,$$

which in turn yields

$$\mathcal{A}_{h'} = (\mathcal{A}_{g'}\mathcal{A}_{-g'})\mathcal{A}_{h'} \subset I \quad \text{or} \quad \mathcal{A}_{h'} = (\mathcal{A}_{-g'}\mathcal{A}_{g'})\mathcal{A}_{h'} \subset I.$$

From this, Eq (5.3) gives us

$$\mathcal{A}_{0,[h]} \oplus \mathcal{A}_{[h]} \subset I.$$

Case 3: If $\mathcal{A}_{0,[g]}\mathcal{A}_{0,[h]} + \mathcal{A}_{0,[h]}\mathcal{A}_{0,[g]} \neq 0$, we can choose $g' \in [g]$ and $h' \in [h]$ such that

$$(\mathcal{A}_{g'}\mathcal{A}_{-g'})(\mathcal{A}_{h'}\mathcal{A}_{-h'}) + (\mathcal{A}_{h'}\mathcal{A}_{-h'})(\mathcal{A}_{g'}\mathcal{A}_{-g'}) \neq 0.$$

Then, by the maximal length of the grading, there exist $g' \in [g]$, $h' \in [h]$, and $k \in \Sigma$ such that

$$(\mathcal{A}_{h'}\mathcal{A}_{-h'})(\mathcal{A}_{g'}\mathcal{A}_{-g'}) + (\mathcal{A}_{g'}\mathcal{A}_{-g'})(\mathcal{A}_{h'}\mathcal{A}_{-h'}) = \mathcal{A}_k\mathcal{A}_{-k} \subset I.$$

As argued above, the division character of the grading implies that $\mathcal{A}_{h'} + \mathcal{A}_{-h'} \subset I$, and consequently $\mathcal{A}_{0,[h]} \oplus \mathcal{A}_{[h]} \subset I$.

Consequently, we have shown that for any $[g] \in \Sigma/\mathcal{R}_1$ such that $\mathcal{A}_{0,[g]} \oplus \mathcal{A}_{[g]} \subset I$ for a non-zero graded ideal I of $\mathcal{A}$, if $[h] \in \Sigma/\mathcal{R}_1$ satisfies Eq (5.4), then

$$\mathcal{A}_{0,[h]} \oplus \mathcal{A}_{[h]} \subset I.$$

From this, we conclude that if $[k] \in \Sigma/\mathcal{R}_1$ is such that $[g]\mathcal{R}_2[k]$ (see Definition 3.1), then,

$$\mathcal{A}_{0,[k]} \oplus \mathcal{A}_{[k]} \subset I.$$

Therefore (see Eq (3.3)),

$$\mathcal{A}_{0[[g]]} \oplus \mathcal{A}_{[[g]]} = I,$$

which proves that $\mathcal{A}_{0[[g]]} \oplus \mathcal{A}_{[[g]]}$ is graded simple. □

Theorem 5.1. *Let*

$$\mathcal{A} = \mathcal{A}_0 \oplus \left(\bigoplus_{g \in \Sigma} \mathcal{A}_g \right)$$

be a graded algebra of maximal length and with a division grading, such that

$$\mathcal{A}_0 = \sum_{g \in \Sigma} \mathcal{A}_g\mathcal{A}_{-g} \quad \textit{and} \quad \mathrm{Ann}(\mathcal{A}) = 0.$$

Then, $\mathcal{A}$ decomposes as the direct sum

$$\mathcal{A} = \bigoplus_{[[g]] \in (\Sigma/\mathcal{R}_1)/\mathcal{R}_2} (\mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]})$$

where each $\mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]}$ is a graded simple ideal of $\mathcal{A}$ such that

$$(\mathcal{A}_{0,[[g]]} \oplus \mathcal{A}_{[[g]]})(\mathcal{A}_{0,[[h]]} \oplus \mathcal{A}_{[[h]]}) = 0$$

when $[[g]] \neq [[h]]$.

Proof. This is a direct consequence of Theorem 4.1 and Proposition 5.1. □

6. Conclusions

In this paper, we provided a systematic framework for the structural analysis of G-graded algebras without restrictions on dimensionality or the underlying base field. By shifting the focus from individual homogeneous components to connectivity classes, we have demonstrated the following:

(1) The transition map ν serves as an effective tool to track the algebraic "flow" across the support Σ, allowing for the identification of disjoint subspaces $\mathcal{A}_{[g]}$.
(2) The role of the zero-degree component $\mathcal{A}_0$ is twofold: It serves as a local core for each connectivity class and as a global mediator that couples distinct classes through the relation $\mathcal{R}_2$.
(3) The resulting ideal decomposition simplifies the classification problem of graded algebras, as it allows for the separate study of mutually orthogonal components.
(4) In the case of a maximal length grading, we can establish a framework to decompose a graded algebra as the direct sum of its graded simple ideals.

These results lay the groundwork for future investigations into simple and semisimple structures within the category of graded algebras, particularly in the infinite-dimensional case where standard Lie-theoretic methods may not directly apply.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

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