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Research article

On solving a second-order scalar differential equation involving symmetry/reflection of argument: exact solution

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Abstract: This paper applies a reduction approach to solve a second-order scalar model with reflection of argument. Such types of models arise in modeling the transmission of nerve impulses in neuroscience. The proposed approach is capable of eliminating the reflection term with nonlocal classic behavior. Accordingly, an equivalent fourth-order model is deduced. The converted model is then exactly solved, basically in exponential and hyperbolic forms. Domains of valid parameters for obtaining several forms of the solution are specified. Moreover, various special cases are studied including repeated roots for the characteristic equation in which different types of solutions are determined. Furthermore, periodic solutions are derived under particular constraints. The proposed analysis reveals that the suggested approach can be extended to handle higher-order reflected models.

Keywords: reduction approach; scalar differential equation; reflection of argument; exact solution
Mathematics Subject Classification: 34B15, 47H11

1. Introduction

1.1. Current research status

First-order differential equations with reflection of arguments are usually governed by

$$y'(t) = F(t, y(t), y(-t)).$$

These forms introduce nonlocal behavior unlike classical ordinary differential equations (ODEs). Historically, the first study to analyze a boundary value problem (BVP) with argument-reflection was introduced by the authors [1]. Later, several authors investigated some versions of BVPs involving argument-reflection [2, 3] and their extensions [4, 5]. Reflection arguments often generate mixed initial-boundary value formulations and require adapted analytical tools, including operator methods, transformation techniques, and fixed-point approaches [6–8].

The solutions of some initial value problems (IVPs) under prescribed initial conditions (ICs) have been presented in Refs. [9–11]. In Refs. [12–14], the authors transformed this kind of IVPs into second-order ODEs and also to first-order systems subject to modified ICs. Additionally, the asymptotic and boundedness properties of the solutions of such first-order IVPs were addressed in Refs. [15, 16], respectively.

Moreover, some results were reported in [17–19] on second-order BVPs under different boundary conditions. Besides, existence analysis were addressed in Ref. [20] for higher-order IVPs. In Refs. [21–23], various techniques were applied to solve different IVPs/BVPs models with reflection of the argument. Solvability of two-point BVPs with reflection of the argument were also investigated in [24, 25] while some other problems with involution can be found in [26–28]. Studying such kind of problems may develop our understanding about the transmission of nerve impulses along axons, which are governed by differential equations with reflection of argument [29].

1.2. Methods of second-order reflected IVPs

Although both IVPs and BVPs arise in the theory of functional differential equations with reflection including the forms $y'(t) - ay(-t) = 0$ and $y'(t) + ay(t) + by(-t) = g(t)$, the present paper is exclusively devoted to the analytical treatment of the following second-order IVP with argument-reflection:

$$y''(t) + cy'(t) = ay(t) + by(-t), \quad t \in \mathbb{R}, \tag{1.1}$$

subject to

$$y(0) = \alpha, \quad y'(0) = \beta, \tag{1.2}$$

where a, b, c, α, β are real constants. The present model incorporates both local dynamics and a reflection term. The presence of $y(-t)$ introduces nonlocal behavior, making classical solution techniques insufficient without suitable modifications. Although a number of analytical techniques [30–32] in addition to the Laplace transform (LT) [33, 34] were found effective to deal with some scalar models, the reduction approach [35] is preferred for its simplicity in eliminating the reflection term. The objective of this work is to solve the IVPs (1.1) and (1.2) in an exact form. The proposed procedure transforms the original reflected differential equation into an equivalent higher-order ODE through differentiation and elimination of the reflected term. This transformation allows classical analytical techniques to be applied while preserving the equivalence between the two formulations. Various types of solutions are to be determined including periodic ones. It will be declared that each type of solution is subjected to specific constraints of the model's parameters.

2. Reduction approach

This section transforms the problem governed by Eqs (1.1) and (1.2) into an equivalent fourth-order ODE without reflection. The main advantage of this approach lies in its capability of solving the transformed model utilizing standard methods. Before launching, let us rewrite Eq (1.1) in the form

$$y^{(2)}(t) + c\, y^{(1)}(t) = a\, y(t) + b\, y(-t). \tag{2.1}$$

Differentiating both sides of Eq (2.1) with respect to t

$$y^{(3)}(t) + c\,y^{(2)}(t) = a\,y^{(1)}(t) - b\,y^{(1)}(-t). \tag{2.2}$$

Differentiating once more gives

$$y^{(4)}(t) + c\,y^{(3)}(t) = a\,y^{(2)}(t) + b\,y^{(2)}(-t). \tag{2.3}$$

Replacing t by $-t$ in Eq (1.1), it then follow that

$$y^{(2)}(-t) = -c\,y^{(1)}(-t) + a\,y(-t) + b\,y(t). \tag{2.4}$$

Substituting Eq (2.4) into Eq (2.3) leads to

$$y^{(4)}(t) + c\,y^{(3)}(t) = a\,y^{(2)}(t) + b[-c\,y^{(1)}(-t) + a\,y(-t) + b\,y(t)]. \tag{2.5}$$

Using Eqs (2.1) and (2.2) to eliminate $y(-t)$ and its derivatives $y^{(1)}(-t)$, we find

$$y(-t) = \frac{1}{b}[y^{(2)}(t) + c\,y^{(1)}(t) - a\,y(t)], \quad b \neq 0, \tag{2.6}$$

and

$$y^{(1)}(-t) = \frac{1}{b}[a\,y^{(1)}(t) - y^{(3)}(t) - c\,y^{(2)}(t)], \quad b \neq 0. \tag{2.7}$$

Inserting Eqs (2.6) and (2.7) into Eq (2.5) and simplifying, we obtain the fourth-order ODE:

$$y^{(4)}(t) - (2a + c^2)\,y^{(2)}(t) + (a^2 - b^2)\,y(t) = 0, \quad b \neq 0. \tag{2.8}$$

2.1. Characteristic equation

The characteristic equation corresponding to the ODE in Eq (2.8) is

$$\lambda^4 - (2a + c^2)\lambda^2 + (a^2 - b^2) = 0. \tag{2.9}$$

Let $\mu = \lambda^2$, then

$$\mu^2 - (2a + c^2)\mu + (a^2 - b^2) = 0. \tag{2.10}$$

Thus

$$\mu = \frac{1}{2}\left(2a + c^2 \pm \sqrt{(2a + c^2)^2 - 4(a^2 - b^2)}\right). \tag{2.11}$$

Define

$$\omega = 2a + c^2, \quad \Omega = (2a + c^2)^2 - 4(a^2 - b^2), \tag{2.12}$$

then

$$\mu = \frac{1}{2}\left(\omega \pm \sqrt{\Omega}\right). \tag{2.13}$$

Therefore

$$\lambda = \pm\sqrt{\frac{1}{2}\left(\omega \pm \sqrt{\Omega}\right)}. \tag{2.14}$$

The four distinct roots of Eq (2.9) are then given by

$$\lambda_{1,2} = \pm\sqrt{\frac{1}{2}\left(\omega + \sqrt{\Omega}\right)}, \quad \lambda_{3,4} = \pm\sqrt{\frac{1}{2}\left(\omega - \sqrt{\Omega}\right)}. \tag{2.15}$$

2.2. Exact solution

The general solution of Eq (2.8) is well-known as

$$y(t) = \sum_{j=1}^{4} C_j e^{\lambda_j t}, \tag{2.16}$$

where $C_j,\ j = 1, 2, 3, 4$ are undetermined constants. In order to evaluate the constants C_j, we need to specify two further ICs in addition to the given ICs (1.2). Employing the ICs (1.2) in Eqs (2.1) and (2.2) implies

$$y^{(2)}(0) = (a + b)y(0) - c\, y^{(1)}(0) = \alpha(a + b) - \beta\, c, \tag{2.17}$$

and

$$y^{(3)}(0) = (a - b)y^{(1)}(0) - c\, y^{(2)}(0) = \beta(a - b) - c[\alpha(a + b) - \beta\, c], \tag{2.18}$$

i.e.,

$$y^{(3)}(0) = \beta(a - b) - \alpha c(a + b) + \beta\, c^2. \tag{2.19}$$

From Eq (2.16), we have

$$\begin{cases} y(t) = \sum_{j=1}^{4} C_j e^{\lambda_j t}, \\ y^{(1)}(t) = \sum_{j=1}^{4} \lambda_j C_j e^{\lambda_j t}, \\ y^{(2)}(t) = \sum_{j=1}^{4} \lambda_j^2 C_j e^{\lambda_j t}, \\ y^{(3)}(t) = \sum_{j=1}^{4} \lambda_j^3 C_j e^{\lambda_j t}. \end{cases} \tag{2.20}$$

Accordingly, the unknown constants C_j are governed by the algebraic system:

$$\begin{cases} y(0) = \sum_{j=1}^{4} C_j, \\ y^{(1)}(0) = \sum_{j=1}^{4} \lambda_j C_j, \\ y^{(2)}(0) = \sum_{j=1}^{4} \lambda_j^2 C_j, \\ y^{(3)}(0) = \sum_{j=1}^{4} \lambda_j^3 C_j. \end{cases} \tag{2.21}$$

This system is to be solved such that

$$\begin{cases} y(0) = \alpha, \\ y^{(1)}(0) = \beta, \\ y^{(2)}(0) = A, \\ y^{(3)}(0) = B, \end{cases} \tag{2.22}$$

where A and B are defined by

$$A = \alpha(a+b) - \beta c, \quad B = \beta(a-b) - \alpha c(a+b) + \beta c^2. \tag{2.23}$$

The solution of the system (2.21) and (2.22) is

$$\begin{cases} C_1 = \dfrac{B - A\lambda_4 + \lambda_3(-A+\beta\lambda_4) + \lambda_2\left[-A+\beta\lambda_4+\lambda_3(\beta-\alpha\lambda_4)\right]}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}, \\ \\ C_2 = \dfrac{-B + A\lambda_4 + \lambda_3(A-\beta\lambda_4) + \lambda_1\left[A-\beta\lambda_4+\lambda_3(-\beta+\alpha\lambda_4)\right]}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}, \\ \\ C_3 = \dfrac{B - A\lambda_4 + \lambda_2(-A+\beta\lambda_4) + \lambda_1\left[-A+\beta\lambda_4+\lambda_2(\beta-\alpha\lambda_4)\right]}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)}, \\ \\ C_4 = \dfrac{-B + A\lambda_3 + \lambda_2(A-\beta\lambda_3) + \lambda_1\left[A-\beta\lambda_3+\lambda_2(-\beta+\alpha\lambda_3)\right]}{(\lambda_3-\lambda_4)(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)}. \end{cases} \tag{2.24}$$

Since $\lambda_2 = -\lambda_1$ and $\lambda_4 = -\lambda_3$, then the constants C_j, $j = 1, 2, 3, 4$ can be simplified to take the form:

$$\begin{cases} C_1 = \dfrac{B + A\lambda_1 - (\beta+\alpha\lambda_1)\lambda_3^2}{2(\lambda_1^3 - \lambda_1\lambda_3^2)}, \\ \\ C_2 = \dfrac{-B + A\lambda_1 + (\beta-\alpha\lambda_1)\lambda_3^2}{2(\lambda_1^3 - \lambda_1\lambda_3^2)}, \\ \\ C_3 = -\dfrac{B + A\lambda_3 - \lambda_1^2(\beta+\alpha\lambda_3)}{2\lambda_3(\lambda_1^2 - \lambda_3^2)}, \\ \\ C_4 = \dfrac{B - A\lambda_3 + \lambda_1^2(-\beta+\alpha\lambda_3)}{2\lambda_1^2\lambda_3 - 2\lambda_3^3}, \end{cases} \tag{2.25}$$

where

$$\lambda_1 = \sqrt{\frac{1}{2}\left(\omega + \sqrt{\Omega}\right)}, \quad \lambda_3 = \sqrt{\frac{1}{2}\left(\omega - \sqrt{\Omega}\right)}.$$

Hence, the solution of the model given by Eqs (1.1) and (1.2) can be written as

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{-\lambda_1 t} + C_3 e^{\lambda_3 t} + C_4 e^{-\lambda_3 t}. \tag{2.26}$$

This solution can be expressed in terms of hyperbolic functions as

$$y(t) = (C_1 + C_2)\cosh(\lambda_1 t) + (C_1 - C_2)\sinh(\lambda_1 t) + (C_3 + C_4)\cosh(\lambda_3 t) + (C_3 - C_4)\sinh(\lambda_3 t), \quad (2.27)$$

or

$$y(t) = D_1 \cosh(\lambda_1 t) + D_2 \sinh(\lambda_1 t) + D_3 \cosh(\lambda_3 t) + D_4 \sinh(\lambda_3 t), \quad (2.28)$$

where

$$D_1 = C_1 + C_2, \quad D_2 = C_1 - C_2, \quad D_3 = C_3 + C_4, \quad D_4 = C_3 - C_4. \quad (2.29)$$

Substituting from Eq (2.25) into Eq (2.29) yields

$$D_1 = \frac{A - \alpha\lambda_3^2}{\lambda_1^2 - \lambda_3^2}, \quad D_2 = \frac{B - \beta\lambda_3^2}{\lambda_1^3 - \lambda_1\lambda_3^2}, \quad D_3 = \frac{-A + \alpha\lambda_1^2}{\lambda_1^2 - \lambda_3^2}, \quad D_4 = -\frac{B - \beta\lambda_1^2}{\lambda_1^2\lambda_3 - \lambda_3^3}. \quad (2.30)$$

Remark 1. *In this section, the exact solution of the transformed fourth-order Eq (2.8) with the ICs (2.22) is obtained by Eq (2.28) and the accompanied coefficients (2.30). To confirm the equivalence between the original second-order IVPs (1.1) and (1.2) and transformed fourth-order Eq (2.8) with the ICs (2.22), one can easily verify that the exact solution (2.28) with the accompanied coefficients (2.30) satisfies the original IVP (1.1) and (1.2) by direct substitution.*

3. Special cases

3.1. $c = 0$

If $c = 0$, Eqs (1.1) and (1.2) reduce to the IVP:

$$y''(t) = ay(t) + by(-t), \quad y(0) = \alpha, \quad y'(0) = \beta, \quad a \neq b. \quad (3.1)$$

In this case, we have from Eq (2.12) that

$$\omega = 2a, \quad \Omega = 4b^2, \quad (3.2)$$

which implies

$$\lambda_1 = \sqrt{\frac{1}{2}\left(\omega + \sqrt{\Omega}\right)} = \sqrt{a+b}, \quad \lambda_3 = \sqrt{\frac{1}{2}\left(\omega - \sqrt{\Omega}\right)} = \sqrt{a-b}, \quad (3.3)$$

and

$$A = \alpha(a + b), \quad B = \beta(a - b). \quad (3.4)$$

The constants D_j, $j = 1, 2, 3, 4$, become

$$D_1 = \alpha, \quad D_2 = 0, \quad D_3 = 0, \quad D_4 = \frac{\beta}{\sqrt{a-b}}. \quad (3.5)$$

Inserting these values into Eq (2.28) leads to the exact solution:

$$y(t) = \alpha\cosh(\lambda_1 t) + \frac{\beta}{\sqrt{a-b}}\sinh(\lambda_3 t), \quad a > b, \quad (3.6)$$

where λ_1 and λ_3 are defined by Eq (3.3). The solution can be easily verified by directly substituting into the IVP (3.1).

3.2. $b = a$

This section contains repeated roots, hence, a separate analysis is to be introduced. Let us consider $b = a$, then the IVP (1.1) and (1.2) becomes

$$y''(t) + c\,y'(t) = a(y(t) + y(-t)), \quad y(0) = \alpha, \quad y'(0) = \beta. \tag{3.7}$$

Equation (2.12) yields

$$\omega = 2a + c^2, \quad \Omega = (2a + c^2)^2, \tag{3.8}$$

which implies $\omega = \sqrt{\Omega}$. Hence, we have the roots

$$\lambda_{1,2} = \pm\sqrt{\frac{1}{2}\left(\omega + \sqrt{\Omega}\right)} = \pm\sqrt{2a + c^2}, \quad \lambda_{3,4} = 0, \tag{3.9}$$

with

$$A = 2\alpha\,a - \beta\,c, \quad B = -2\alpha\,c\,a + \beta\,c^2. \tag{3.10}$$

In this case, the general solution is given by

$$y(t) = h_1 e^{\lambda_1 t} + h_2 e^{-\lambda_1 t} + h_3 + h_4 t, \tag{3.11}$$

where h_j, $j = 1, 2, 3, 4$, are constants to be determined by solving the system:

$$\begin{cases} h_1 + h_2 + h_3 = \alpha, \\ h_1\lambda_1 - h_2\lambda_1 + h_4 = \beta, \\ h_1\lambda_1^2 + h_2\lambda_1^2 = A, \\ h_1\lambda_1^3 - h_2\lambda_1^3 = B. \end{cases} \tag{3.12}$$

Thus

$$h_1 = \frac{1}{2}\left(\frac{A}{\lambda_1^2} + \frac{B}{\lambda_1^3}\right), \quad h_2 = \frac{1}{2}\left(\frac{A}{\lambda_1^2} - \frac{B}{\lambda_1^3}\right), \quad h_3 = \alpha - (h_1 + h_2), \quad h_4 = \beta - \lambda_1(h_1 - h_2). \tag{3.13}$$

The solution can also be written as

$$y(t) = (h_1 + h_2)\cosh(\lambda_1 t) + (h_1 - h_2)\sinh(\lambda_1 t) + h_3 + h_4 t, \tag{3.14}$$

or simply by

$$y(t) = \frac{A}{\lambda_1^2}\cosh(\lambda_1 t) + \frac{B}{\lambda_1^3}\sinh(\lambda_1 t) + h_3 + h_4 t, \tag{3.15}$$

where A, B, and λ_1 are defined above. This solution satisfies the IVP (3.7).

3.3. $b = -a$

A second case of repeated roots is offered in this section, which follows the preceding analysis in the previous section. The assumption $b = -a$ transforms the IVP (1.1) and (1.2) to

$$y''(t) + c\,y'(t) = a(y(t) - y(-t)), \quad t \in \mathbb{R}, \quad y(0) = \alpha, \quad y'(0) = \beta. \tag{3.16}$$

From Eq (2.12), we obtain

$$\omega = 2a + c^2, \quad \Omega = (2a + c^2)^2. \tag{3.17}$$

Similar to the previous section, we notice that $\omega = \sqrt{\Omega}$, hence

$$\lambda_{1,2} = \pm\sqrt{2a + c^2}, \quad \lambda_{3,4} = 0, \tag{3.18}$$

and

$$A = -\beta\,c, \quad B = 2\beta\,a + \beta\,c^2. \tag{3.19}$$

Accordingly, the general solution is given by

$$y(t) = k_1 e^{\lambda_1 t} + k_2 e^{-\lambda_1 t} + k_3 + k_4 t, \tag{3.20}$$

where k_j, $j = 1, 2, 3, 4$ are unknown constants. The ICs (2.22) lead to the system:

$$\begin{cases} k_1 + k_2 + k_3 = \alpha, \\ k_1\lambda_1 - k_2\lambda_1 + k_4 = \beta, \\ k_1\lambda_1^2 + k_2\lambda_1^2 = A, \\ k_1\lambda_1^3 - k_2\lambda_1^3 = B. \end{cases} \tag{3.21}$$

Thus

$$k_1 = \frac{1}{2}\left(\frac{A}{\lambda_1^2} + \frac{B}{\lambda_1^3}\right), \quad k_2 = \frac{1}{2}\left(\frac{A}{\lambda_1^2} - \frac{B}{\lambda_1^3}\right), \quad k_3 = \alpha - (k_1 + k_2), \quad k_4 = \beta - \lambda_1(k_1 - k_2). \tag{3.22}$$

In terms of hyperbolic functions, the solution reads

$$y(t) = (k_1 + k_2)\cosh(\lambda_1 t) + (k_1 - k_2)\sinh(\lambda_1 t) + k_3 + k_4 t, \tag{3.23}$$

i.e.,

$$y(t) = \frac{A}{\lambda_1^2}\cosh(\lambda_1 t) + \frac{B}{\lambda_1^3}\sinh(\lambda_1 t) + k_3 + k_4 t. \tag{3.24}$$

As seen from Eq (3.24), the solution has the same structure of the previous special case but with different A and B defined by Eq (3.19). The solution given by Eq (2.24) satisfies the IVP (3.16).

Remark 2. *In the absence of the reflection term $y(-t)$, i.e., at $b = 0$, the IVP (1.1) and (1.2) reduces to the regular form:*

$$y''(t) + c\,y'(t) - ay(t) = 0, \quad y(0) = \alpha, \quad y'(0) = \beta. \tag{3.25}$$

The solution of this IVP can not be directly obtained from our results by just inserting $b = 0$ into the general solution determined in Section 2. This is because the obtained results are basically based *on assuming that $b \neq 0$. However, this IVP can be solved utilizing standard methods in differential calculus. The characteristic equation corresponding to the problem (3.25) is*

$$\rho^2 + c\rho - a = 0, \tag{3.26}$$

with the roots

$$\rho_{1,2} = \frac{1}{2}\left(-c \pm \sqrt{c^2 + 4a}\right). \tag{3.27}$$

The solution is

$$y(t) = \gamma_1 e^{\rho_1 t} + \gamma_2 e^{\rho_2 t}, \tag{3.28}$$

where γ_1 and γ_2 are governed by

$$\gamma_1 + \gamma_2 = \alpha, \quad \gamma_1\rho_1 + \gamma_2\rho_1 = \beta. \tag{3.29}$$

Solving this system for γ_1 and γ_2 and substituting the results in Eq (3.28) finalizes the solution.

4. Characteristics of solution

In Section 2, the solution was exactly determined in terms of hyperbolic functions in the form:

$$y(t) = D_1\cosh(\lambda_1 t) + D_2\sinh(\lambda_1 t) + D_3\cosh(\lambda_3 t) + D_4\sinh(\lambda_3 t), \tag{4.1}$$

where D_1, D_2, D_3, and D_4 were already given by Eq (2.30). Using the relation $\lambda_1^2 - \lambda_3^2 = \sqrt{\Omega}$, these coefficients can be further simplified to take the forms

$$D_1 = \frac{A - \alpha\lambda_3^2}{\sqrt{\Omega}}, \quad D_2 = \frac{B - \beta\lambda_3^2}{\lambda_1\sqrt{\Omega}}, \quad D_3 = \frac{-A + \alpha\lambda_1^2}{\sqrt{\Omega}}, \quad D_4 = -\frac{B - \beta\lambda_1^2}{\lambda_3\sqrt{\Omega}}. \tag{4.2}$$

It can be seen from Eqs (4.1) and (4.2) that the solution depends mainly on the nature of the roots λ_1 and λ_3 in addition to the coefficients D_j, $j = 1, 2, 3, 4$. In this section, several types of exact solutions are analyzed including the combined hyperbolic/trigonometric-form solution, the pure hyperbolic-form solution, the combined hyperbolic/polynomial-form solution, the combined trigonometric/polynomial-form solution, and the pure trigonometric-form solution. These types of solutions require specifying further conditions on the model's parameters a, b, c while the initial values α and β do not contribute in this context. This section discusses the required conditions on the involved parameters for which the above types of solutions can be derived.

4.1. Combined hyperbolic/trigonometric-form solution

Let us begin with the special case $a = 0$. In this case, we have

$$\omega = c^2 > 0 \quad \text{and} \quad \Omega = \sqrt{c^4 + 4b^2} > \omega > 0 \quad \text{for every } c, b.$$

This implies that $\omega + \sqrt{\Omega} > 0$ and $\omega - \sqrt{\Omega} < 0$, hence λ_1 is real while λ_3 becomes complex, where

$$\lambda_1 = \sqrt{\frac{1}{2}\left(\omega + \sqrt{\Omega}\right)} > 0,$$

$$
\begin{aligned}
\lambda_3 &= \sqrt{\frac{1}{2}\left(\omega - \sqrt{\Omega}\right)} = i\sqrt{\frac{1}{2}\left(\sqrt{\Omega} - \omega\right)} = i\lambda_5,\\
\lambda_5 &= \sqrt{\frac{1}{2}\left(\sqrt{\Omega} - \omega\right)} > 0,\\
i &= \sqrt{-1}.
\end{aligned}
$$

Accordingly, Eq (2.28) reveals that the first two terms of the solution are of hyperbolic forms while the last two terms are of trigonometric forms whatever the values of b and c. Substituting $\lambda_3 = i\lambda_5$ into Eqs (4.1) and (4.2) and simplifying gives

$$
y(t) = D_1 \cosh(\lambda_1 t) + D_2 \sinh(\lambda_1 t) + D_3 \cos(\lambda_5 t) + D_5 \sin(\lambda_5 t), \tag{4.3}
$$

where D_1, D_2, D_3, and D_5 are

$$
\begin{aligned}
D_1 &= \frac{A + \alpha\lambda_5^2}{\sqrt{\Omega}},\\
D_2 &= \frac{B + \beta\lambda_5^2}{\lambda_1\sqrt{\Omega}},\\
D_3 &= \frac{-A + \alpha\lambda_1^2}{\sqrt{\Omega}},\\
D_5 &= -\frac{B - \beta\lambda_1^2}{\lambda_5\sqrt{\Omega}}.
\end{aligned} \tag{4.4}
$$

4.2. Pure hyperbolic-form solution

A pure hyperbolic-form solution requires real values for the roots λ_1 and λ_3 in addition to the coefficients D_1, D_2, D_3, and D_4. Hence, the pure hyperbolic-form solution requires that $\Omega > 0$ and $\omega > 0$ such that $\omega - \sqrt{\Omega} > 0$. These requirements lead to the restrictions $0 < \Omega < \omega^2$ and $a > -c^2/2$. Since ω and Ω depend on a, b, and c, then at a given value for c, there will be specific domains for a and b to obtain a pure hyperbolic-form solution. For declaration, Figures 1 and 2 show the valid domains of a and b to obtain a pure hyperbolic-form solution at $c = 2$ and $c = 4$, respectively. For example, one can select the values $a = -1$ and $b = 1/2$ from Figure 1 at $c = 2$. If the initial values are considered fixed as $\alpha = 1$ and $\beta = 1$, then we obtain the pure hyperbolic-form solution:

$$
y(t) = -3\cosh\left(\sqrt{\frac{3}{2}}\,t\right) + \sqrt{6}\sinh\left(\sqrt{\frac{3}{2}}\,t\right) + 4\cosh\left(\sqrt{\frac{1}{2}}\,t\right) - 2\sqrt{2}\sinh\left(\sqrt{\frac{1}{2}}\,t\right). \tag{4.5}
$$

Similarly, selecting the values $a = -4$ and $b = -3$ from Figure 2 at $c = 4$ leads to the pure hyperbolic-form solution:

$$
y(t) = 3\cosh(t) - 6\sinh(t) - 2\cosh(\sqrt{7}\,t) + \sqrt{7}\sinh(\sqrt{7}\,t). \tag{4.6}
$$

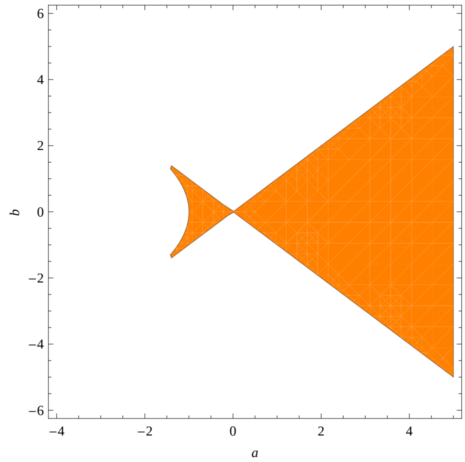

Figure 1. Domains of valid a and b in the region $a \times b = [-4, 5] \times [-6, 6]$ for pure hyperbolic-form solutions at $c = 2$.

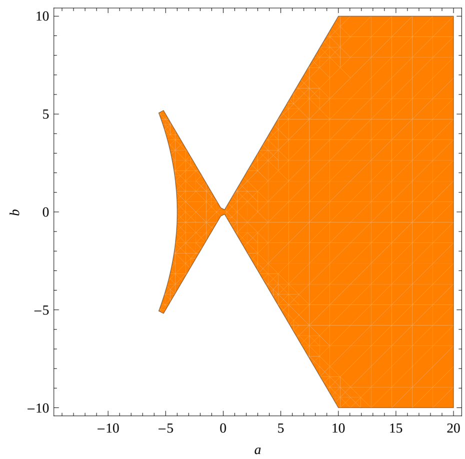

Figure 2. Domains of valid a and b in the region $a \times b = [-14, 20] \times [-10, 10]$ for pure hyperbolic-form solutions at $c = 4$.

4.3. Combined hyperbolic/polynomial-form solution

From Section 3.2, we have obtained the solution for the problems (1.1) and (1.2) under the condition $b = a$. From Eq (3.15), the solution is given by

$$y(t) = \frac{A}{\lambda_1^2} \cosh(\lambda_1 t) + \frac{B}{\lambda_1^3} \sinh(\lambda_1 t) + h_3 + h_4 t, \tag{4.7}$$

where the coefficients h_3 and h_4 can be explicitly obtained from Eq (3.13) as

$$h_3 = \alpha - \frac{A}{\lambda_1^2}, \quad h_4 = \beta - \frac{B}{\lambda_1^2}, \quad \lambda_1 = \sqrt{2a + c^2}, \quad A = 2\alpha\, a - \beta\, c, \quad B = -2\alpha\, c\, a + \beta\, c^2. \tag{4.8}$$

The combined hyperbolic/polynomial-form solution requires that $\lambda_1 \in \mathbb{R}$, which is satisfied if $a > -c^2/2$. An example can be given once c is assigned. For $c = 2$, the parameter a must be chosen such that $a > -2$.

The selection $a = -1$, for example, leads to the combined hyperbolic/polynomial-form solution:

$$y(t) = -2\cosh(\sqrt{2}\,t) + 2\sqrt{2}\sinh(\sqrt{2}\,t) + 3(1-t), \tag{4.9}$$

for the problem

$$y''(t) + 2\,y'(t) = -(y(t) + y(-t)), \quad y(0) = 1, \quad y'(0) = 1. \tag{4.10}$$

Also, under the conditions $b = -a$ and $a > -c^2/2$, there is another combined hyperbolic/polynomial-form solution given from Eq (3.24) by

$$y(t) = \frac{A}{\lambda_1^2}\cosh(\lambda_1 t) + \frac{B}{\lambda_1^3}\sinh(\lambda_1 t) + k_3 + k_4 t. \tag{4.11}$$

From Eqs (3.19) and (3.22), we have

$$k_3 = \alpha - \frac{A}{\lambda_1^2}, \quad k_4 = \beta - \frac{B}{\lambda_1^2}, \quad \lambda_1 = \sqrt{2a + c^2}, \quad A = -\beta\, c, \quad B = 2\beta\, a + \beta\, c^2 = \beta\lambda_1^2. \tag{4.12}$$

The second equality in (4.12) and the fifth one reveal that k_4 vanishes at any choices of the model's parameters. Considering $c = 2$ and $a = 1$, where $a > -c^2/2$ is satisfied, we obtain

$$y(t) = -\frac{1}{3}\cosh(\sqrt{6}\,t) + \frac{1}{\sqrt{6}}\sinh(\sqrt{6}\,t) + \frac{4}{3} \tag{4.13}$$

for the problem

$$y''(t) + 2\,y'(t) = y(t) - y(-t), \quad y(0) = 1, \quad y'(0) = 1. \tag{4.14}$$

4.4. Combined trigonometric/polynomial-form solution

If $b = a$ and $a < -c^2/2$, then $2a + c^2 < 0$, which implies a complex root $\lambda_1 = i\lambda_6$ with

$$\lambda_6 = \sqrt{-(2a + c^2)} > 0.$$

Inserting $\lambda_1 = i\lambda_6$ in Eqs (4.7) and (4.8), we obtain

$$y(t) = -\frac{A}{\lambda_6^2}\cos(\lambda_6 t) - \frac{B}{\lambda_6^3}\sinh(\lambda_6 t) + h_3 + h_4 t, \tag{4.15}$$

with

$$h_3 = \alpha + \frac{A}{\lambda_6^2}, \quad h_4 = \beta + \frac{B}{\lambda_6^2}, \quad \lambda_6 = \sqrt{-(2a + c^2)} > 0, \quad A = 2\alpha\, a - \beta\, c, \quad B = -2\alpha\, c\, a + \beta\, c^2. \tag{4.16}$$

If $c = 2$, then $a < -2$. Assume that $a = -3$, then $\lambda_6 = \sqrt{2}$. Accordingly, we obtain the combined trigonometric/polynomial-form solution:

$$y(t) = 4\cos(\sqrt{2}\,t) - 4\sqrt{2}\sin(\sqrt{2}\,t) - 3 + 9t, \tag{4.17}$$

for the problem

$$y''(t) + 2\,y'(t) = -3(y(t) + y(-t)), \quad y(0) = 1, \quad y'(0) = 1. \tag{4.18}$$

Additionally, Eqs (4.11) and (4.12) lead to the combined trigonometric/polynomial-form solution:

$$y(t) = -\frac{A}{\lambda_6^2}\cos(\lambda_6 t) - \frac{B}{\lambda_6^3}\sin(\lambda_6 t) + k_3 + k_4 t, \tag{4.19}$$

under the conditions $b = -a$ and $a < -c^2/2$ with the following coefficients:

$$k_3 = \alpha + \frac{A}{\lambda_6^2}, \quad k_4 = \beta + \frac{B}{\lambda_6^2}, \quad \lambda_6 = \sqrt{-(2a + c^2)} > 0, \quad A = -\beta c, \quad B = 2\beta a + \beta c^2 = -\beta\lambda_6^2. \tag{4.20}$$

Employing the last equality in (4.20) gives $k_4 = 0$. Choosing $c = 2$ requires $a < -c^2/2 = -2$ to obtain the following combined trigonometric/polynomial-form solution at $a = -4$:

$$y(t) = \frac{1}{2}(\cos(2t) + \sin(2t) + 1) \tag{4.21}$$

for the problem

$$y''(t) + 2y'(t) = -4(y(t) - y(-t)), \quad y(0) = 1, \quad y'(0) = 1. \tag{4.22}$$

4.5. *Pure trigonometric-form solution*

By a pure trigonometric-form solution, we mean that the four components of $y(t)$ in Eq (2.28) become trigonometric functions allowing the appearance of a periodic solution. One way to achieve this point is to obtain the roots in pure complex forms. Also, the coefficients have to be real after performing the usual simplification. This requires certain relationships between the model's parameters as well. An easy way to derive the two roots in pure complex forms is to put the restrictions $0 < \Omega < \omega^2$ and $a < -c^2/2$. For a given value of c, the parameters a and b should be assigned specific values such that the above conditions are satisfied.

In this regard, the valid domains of the parameters a and b to meet the above requirements are depicted in Figures 3 and 4 at $c = \pm 1$ and $c = \pm 2$, respectively. It may be important to refer to that the domains of a and b represented in Figures 3 and 4 are valid and equivalent at $c = \pm 1$ and $c = \pm 2$, respectively. This is simply because the parameters ω and Ω involve only even powers of c. From Figure 3, four cases are selected for valid values of a and b to derive the following four periodic solutions at $c = 1$:

$$\begin{cases} a = -4,\ b = -2, & y_1(t) = 4\cos(2t) - 4\sin(2t) - 3\cos(\sqrt{3}\,t) + 3\sqrt{3}\sin(\sqrt{3}\,t), \\ a = -4,\ b = +2, & y_2(t) = \cos(\sqrt{3}\,t) + \dfrac{1}{\sqrt{3}}\sin(\sqrt{3}\,t), \\ a = -10,\ b = -4, & y_3(t) = \dfrac{1}{15}\Big(24\cos(2\sqrt{3}\,t) - 8\sqrt{3}\sin(2\sqrt{3}\,t) \\ & \qquad -9\cos(\sqrt{7}\,t) + 9\sqrt{7}\sin(\sqrt{7}\,t)\Big), \\ a = -10,\ b = +4, & y_4(t) = \cos(\sqrt{7}\,t) + \dfrac{1}{\sqrt{7}}\sin(\sqrt{7}\,t). \end{cases} \tag{4.23}$$

These solutions are displayed in Figure 5 at the indicated values in Eq (4.23).

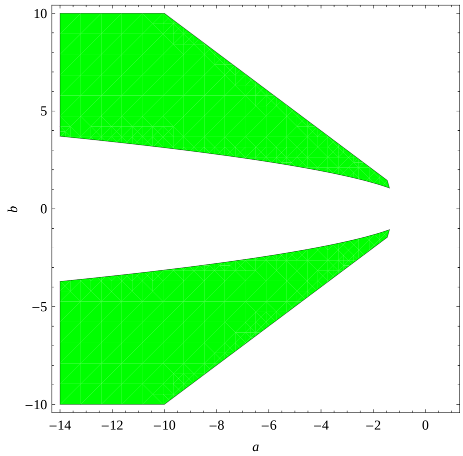

Figure 3. Domains of valid a and b in the region $a \times b = [-14, 1] \times [-10, 10]$ for pure trigonometric-form solutions at $c = \pm 1$.

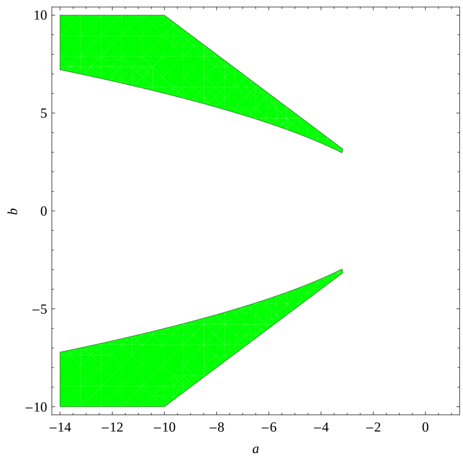

Figure 4. Domains of valid a and b in the region $a \times b = [-14, 1] \times [-10, 10]$ for pure trigonometric-form solutions at $c = \pm 2$.

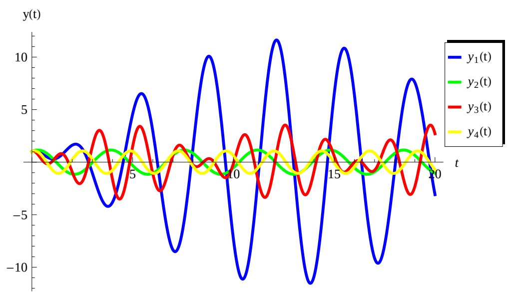

Figure 5. Representation of the solution $y(t)$ at $a = -4$, $b = -2$ ($y_1(t)$), $a = -4$, $b = 2$ ($y_2(t)$), $a = -10$, $b = -4$ ($y_3(t)$), $a = -10$ and $b = 4$ ($y_4(t)$) when $c = 1$.

At the same four cases for a and b but with $c = -1$, we obtain the following four periodic solutions:

$$\begin{cases} a=-4,\ b=-2, & y_1(t) = -\cos(2t) - \sin(2t) + \cos(\sqrt{3}\,t) + \sqrt{3}\sin(\sqrt{3}\,t), \\ a=-4,\ b=+2, & y_2(t) = -\cos(2t) + \sin(2t) + \cos(\sqrt{3}\,t) - \dfrac{1}{\sqrt{3}}\sin(\sqrt{3}\,t), \\ a=-10,\ b=-4, & y_3(t) = \dfrac{1}{15}\Big(-3\cos(2\sqrt{3}\,t) - \sqrt{3}\sin(2\sqrt{3}\,t) \\ & \qquad +3\cos(\sqrt{7}\,t) + 3\sqrt{7}\sin(\sqrt{7}\,t)\Big), \\ a=-10,\ b=+4, & y_4(t) = \dfrac{1}{35}\Big(-7\cos(2\sqrt{3}\,t) + 7\sqrt{3}\sin(2\sqrt{3}\,t) \\ & \qquad +7\cos(\sqrt{7}\,t) - \sqrt{7}\sin(\sqrt{7}\,t)\Big). \end{cases} \tag{4.24}$$

It can be easily observed from Eq (4.24) that the solutions, when $c = -1$, are completely different than those introduced in Eq (4.23) at $c = 1$, although a, b and the other parameters are the same. Additionally, the solutions (4.24) are plotted in Figure 6.

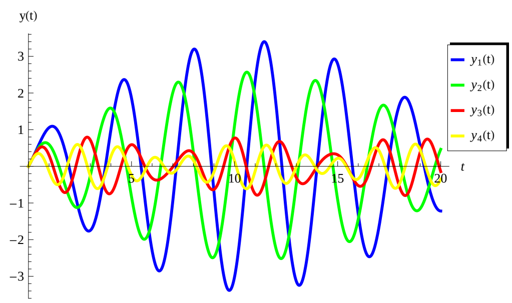

Figure 6. Representation of the solution $y(t)$ at $a = -4$, $b = -2$ ($y_1(t)$), $a = -4$, $b = 2$ ($y_2(t)$), $a = -10$, $b = -4$ ($y_3(t)$), $a = -10$ and $b = 4$ ($y_4(t)$) when $c = -1$.

Repeating these calculations at some selected values for a and b from Figure 4 at $c = 2$ and denoting

$$\begin{aligned} y_1(t) &= y(t)\Big|_{a=-6,\ b=-\frac{11}{2}}, \\ y_2(t) &= y(t)\Big|_{a=-6,\ b=\frac{11}{2}}, \\ y_3(t) &= y(t)\Big|_{a=-12,\ b=-7}, \\ y_4(t) &= y(t)\Big|_{a=-12,\ b=7}, \end{aligned}$$

we find

$$\begin{aligned} y_1(t) = \frac{1}{2\sqrt{41}}\Bigg(&(-19+\sqrt{41})\cos\left(\sqrt{4-\frac{\sqrt{41}}{2}}\,t\right) + (19+\sqrt{41})\cos\left(\sqrt{\frac{1}{2}(8+\sqrt{41})}\,t\right) \\ &+ 2\sqrt{763+103\sqrt{41}}\,\sin\left(\sqrt{4-\frac{\sqrt{41}}{2}}\,t\right) - 2\sqrt{763-103\sqrt{41}}\,\sin\left(\sqrt{\frac{1}{2}(8+\sqrt{41})}\,t\right)\Bigg), \end{aligned} \tag{4.25}$$

and

$$y_2(t) = \frac{41+3\sqrt{41}}{82}\cos\left(\sqrt{4-\frac{\sqrt{41}}{2}}\,t\right) + \left(\frac{1}{2}-\frac{3}{2\sqrt{41}}\right)\cos\left(\sqrt{\tfrac{1}{2}(8+\sqrt{41})}\,t\right)$$
$$+\frac{\sqrt{59-7\sqrt{41}}\,\sin\left(\sqrt{4-\frac{\sqrt{41}}{2}}\,t\right)+\sqrt{59+7\sqrt{41}}\,\sin\left(\sqrt{\tfrac{1}{2}(8+\sqrt{41})}\,t\right)}{\sqrt{943}}, \tag{4.26}$$

while $y_3(t)$ and $y_4(t)$ are obtained as

$$y_3(t) = \frac{1}{10}\Bigg((5-11\sqrt{5})\cos\left(\sqrt{10-\sqrt{5}}\,t\right) + (5+11\sqrt{5})\cos\left(\sqrt{10+\sqrt{5}}\,t\right)$$
$$+\sqrt{2(595+83\sqrt{5})}\,\sin\left(\sqrt{10-\sqrt{5}}\,t\right) - \sqrt{2(595-83\sqrt{5})}\,\sin\left(\sqrt{10+\sqrt{5}}\,t\right)\Bigg), \tag{4.27}$$

and

$$y_4(t) = \frac{1}{38\sqrt{10}}\Bigg(19\sqrt{2}\,(3+\sqrt{5})\cos\left(\sqrt{10-\sqrt{5}}\,t\right) + 19\sqrt{2}\,(-3+\sqrt{5})\cos\left(\sqrt{10+\sqrt{5}}\,t\right)$$
$$+2\sqrt{665+247\sqrt{5}}\,\sin\left(\sqrt{10-\sqrt{5}}\,t\right) - 2\sqrt{665-247\sqrt{5}}\,\sin\left(\sqrt{10+\sqrt{5}}\,t\right)\Bigg). \tag{4.28}$$

In Figure 7, the solutions (4.25)–(4.28) are plotted, where the periodicity of the curves is obvious. Also, the solutions

$$y_1(t) = y(t)\Big|_{a=-6,\ b=-\frac{11}{2}},$$
$$y_2(t) = y(t)\Big|_{a=-6,\ b=\frac{11}{2}},$$
$$y_3(t) = y(t)\Big|_{a=-12,\ b=-7},$$
$$y_4(t) = y(t)\Big|_{a=-12,\ b=7}$$

are reevaluated at $c = -2$ but omitted here for lengthy results. These solutions are represented in Figure 8, which confirm the periodicity of the obtained solutions under the aforementioned restrictions $0 < \Omega < \omega^2$ and $a < -c^2/2$.

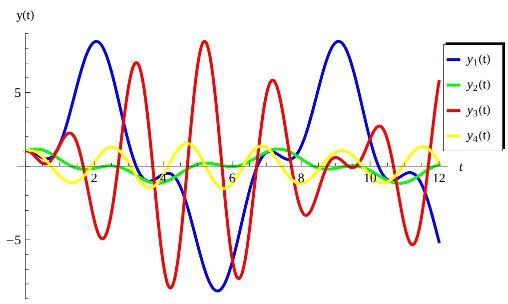

Figure 7. Representation of the solution $y(t)$ at $a = -6$, $b = -11/2$ ($y_1(t)$), $a = -6$, $b = 11/2$ ($y_2(t)$), $a = -12$, $b = -7$ ($y_3(t)$), $a = -12$ and $b = 7$ ($y_4(t)$) when $c = 2$.

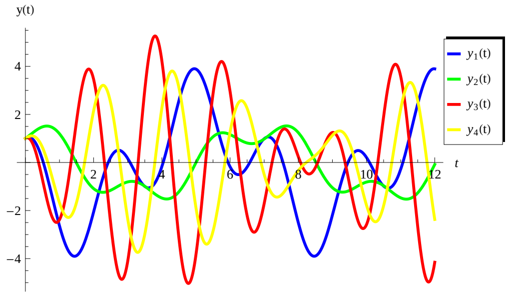

Figure 8. Representation of the solution $y(t)$ at $a = -6$, $b = -11/2$ ($y_1(t)$), $a = -6$, $b = 11/2$ ($y_2(t)$), $a = -12$, $b = -7$ ($y_3(t)$), $a = -12$ and $b = 7$ ($y_4(t)$) when $c = 2$.

5. Conclusions

In this paper, a reduction approach was successfully developed and applied to solve a second-order differential equation with a reflection argument. By transforming the original problem into an equivalent fourth-order ODE, the difficulties associated with the nonlocal reflection term were effectively eliminated. This approach enabled the derivation of exact solutions in explicit exponential and hyperbolic forms. The solution of the characteristic equation showed both types of distinct and repeated roots under some restrictions. This lead to a clear understanding of the solution structure. Several special cases were analyzed in detail, leading to simplified expressions of the solution. Moreover, the regions of valid parameters to obtain pure hyperbolic-form solutions, pure trigonometric-form solutions, and other combined-form solutions were provided through several plots. Although the solution was basically expressed in terms of four distinct exponential and hyperbolic functions, it was successfully converted under some assumptions to periodic solutions. The obtained results reflect the capability of the implemented reduction approach in handling other types of reflected differential equations [36–38] in addition to models with non-self-adjoint involution [39] and reflective delay [40]. Future works will investigate these approaches.

Author contributions

Laila F. Seddek: Methodology, validation, formal analysis, funding acquisition, investigation, writing—original draft preparation, writing—review and editing; Essam R. El-Zahar: Conceptualization, methodology, validation, formal analysis, investigation, writing—review and editing, visualization; Abdelhalim Ebaid: Conceptualization, methodology, validation, formal analysis, investigation, writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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