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Research article

Determinantal defining ideals of numerical semigroup rings with structured pseudo-Frobenius sets

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Abstract: Numerical semigroup rings provide a natural meeting point between additive semigroup theory and commutative algebra, while pseudo-Frobenius numbers encode key information about the type, symmetry, and defining relations. This paper investigates when the defining ideal I_H of a numerical semigroup ring $K[H]$ can be described by the maximal minors of a structured matrix determined by the combinatorics of $\mathrm{PF}(H)$. For a numerical semigroup $H = \langle n_1, \ldots, n_e \rangle$ the defining ideal always has a height $e - 1$; we therefore study I_H through a candidate $2 \times e$ matrix, whose ideal of maximal (2×2) minors has the correct expected height $e - 1$, and we test the equality $I_H = I_2(M)$ by degree arguments, an explicit height comparison, and the Eagon-Northcott resolution. We prove that the numerical semigroups $H = \langle n, n + 1, n + 2, n + 3 \rangle$ with $n \equiv 1 \pmod 3$ form an infinite family with the consecutive (arithmetic) pseudo-Frobenius set $\mathrm{PF}(H) = \{\mathrm{F}(H) - 2, \mathrm{F}(H) - 1, \mathrm{F}(H)\}$ and a determinantal defining ideal; for $H = \langle 4, 5, 6, 7 \rangle$, the defining ideal equals the ideal of 2×2 minors of an explicit 2×4 matrix, giving Betti numbers $(\beta_0, \beta_1, \beta_2) = (6, 8, 3)$. More generally the Eagon-Northcott complex of a $2 \times e$ matrix yields

$$\beta_i(I_H) = (i + 1)\binom{e}{i + 2},$$

so that the number of defining binomials is $\binom{e}{2}$. We also give a simple necessary numerical obstruction in embedding dimensions $e \equiv 1 \pmod 3$: If the defining ideal is the ideal of maximal minors of an H-homogeneous $2 \times e$ matrix, then the sum of the H-degrees of its minimal binomial generators is divisible by 3. In particular, this gives an effective obstruction in embedding dimension four. Using this obstruction, we exhibit numerical semigroups whose defining ideals share the Eagon-Northcott Betti numbers yet are provably not determinantal. All examples are computed and verified with Singular; the code and its output are reported in full. The results clarify how the internal structure of $\mathrm{PF}(H)$ governs, and sometimes fails to govern, the determinantal presentation of $K[H]$.

Keywords: numerical semigroup; numerical semigroup ring; pseudo-Frobenius number; determinantal ideal; defining ideal; toric ideal; almost-symmetric semigroup; Betti numbers;

Eagon-Northcott complex; commutative algebra
Mathematics Subject Classification: 20M14, 13D02, 13C40, 14M12

1. Introduction

1.1. Background

A numerical semigroup is a cofinite additive submonoid of the non-negative integers, usually written as $H = \langle n_1, \ldots, n_e \rangle$, where $\gcd(n_1, \ldots, n_e) = 1$ and the complement $\mathbb{N} \setminus H$ is finite. Although this definition is elementary, numerical semigroups encode deep arithmetic, combinatorial, and homological information. Their invariants such as the multiplicity, embedding dimension, Frobenius number, genus, and type interact in subtle ways and have motivated a substantial body of work in commutative algebra and factorization theory [1, 2]. In particular, the finite generation of numerical semigroups allows one to study them both as combinatorial objects and as algebraic data underlying one-dimensional local or graded rings [2, 3].

The passage from a numerical semigroup H to its semigroup ring $K[H]$ over a field K creates a bridge between additive semigroup theory and algebraic geometry. If $H = \langle n_1, \ldots, n_e \rangle$, then

$$K[H] \cong K[t^{n_1}, \ldots, t^{n_e}] \subseteq K[t],$$

and this ring can be realized as the image of a polynomial ring $S = K[x_1, \ldots, x_e]$ under the homomorphism sending x_i to t^{n_i}. The kernel of this map, denoted I_H, is the defining ideal of the semigroup ring. The structure of I_H governs the presentation of $K[H]$, its syzygies, and several homological properties [4]. For this reason, the study of generators and relations of semigroup rings has long been central in the subject [5], beginning with the foundational work of Herzog [6] on finite presentability and on complete intersection presentations [2, 7].

Among the invariants of a numerical semigroup, pseudo-Frobenius numbers play a particularly important role. The set $\mathrm{PF}(H)$ consists of integers $x \notin H$ such that $x + h \in H$ for every nonzero $h \in H$. Its cardinality is the type of H, and its structure reflects important symmetry conditions. In particular, symmetric semigroups are characterized by Type 1, while pseudo-symmetric and almost-symmetric semigroups are described through specific configurations of pseudo-Frobenius numbers [8–10]. Recent work has further emphasized that pseudo-Frobenius data can control the structural features of the defining ideal itself, especially for classes of almost Gorenstein or related numerical semigroup rings [11–13].

This interaction naturally leads to determinantal methods. Determinantal ideals, generated by the minors of matrices with entries in a polynomial ring, occupy a central place in commutative algebra because they frequently define rings with strong homological and geometric properties; their theory was systematically developed in Bruns and Vetter [14]. In the context of numerical semigroup rings, identifying when a defining ideal is determinantal is especially valuable: Such a description can reveal hidden regularity in the presentation, clarify the shape of resolutions, and connect semigroup-theoretic invariants with classical constructions in algebra [12, 13, 15].

1.2. Research gap

Although the defining ideals of numerical semigroup rings have been studied for decades, the precise circumstances under which these ideals admit a determinantal description are still not fully understood. Existing results show that special families of semigroups yield defining ideals generated by the minors of structured matrices [13, 15], and recent studies have linked such behaviour to pseudo-Frobenius data and almost Gorenstein phenomena. However, a general framework explaining how the internal organization of the pseudo-Frobenius set influences determinantal presentations and, crucially, how the codimension of the minor ideal is reconciled with the codimension $e - 1$ of I_H-is still lacking [12, 13].

1.3. Problem statement

This paper addresses the following central problem: When can the defining ideal of a numerical semigroup ring be described by the maximal minors of a structured matrix, under assumptions on the pseudo-Frobenius set of the underlying semigroup, in such a way that the minor ideal has the correct height $e - 1$? We investigate whether particular configurations of pseudo-Frobenius numbers impose algebraic constraints strong enough to force such a determinantal presentation, and we identify explicit obstructions when they do not. The relation between arithmetic pseudo-Frobenius sets and determinantal defining ideals has been studied previously, particularly for numerical semigroups of maximal embedding dimension, by Do and Matsuoka [15]. The family considered in Theorem 4.4 [16], namely

$$H_q = \langle 3q + 1, 3q + 2, 3q + 3, 3q + 4\rangle,$$

has embedding dimension 4, whereas its multiplicity is $3q + 1$; hence, for $q \geq 2$, it is not of maximal embedding dimension. The theorem provides a uniform determinantal presentation for this family. We also obtain a degree obstruction showing that the expected Eagon-Northcott Betti numbers do not by themselves imply determinantality.

1.4. Main contributions

The contributions of this revised paper are the following.

(i) We correct the codimension bookkeeping of the determinantal model: Since $\dim K[H] = 1$ and $\dim S = e$, the defining ideal satisfies $\mathrm{ht}(I_H) = e - 1$ (Proposition 2.2). Consequently the correct candidate is a $2 \times e$ matrix, whose ideal of maximal 2×2 minors has the expected height $e - 1$, rather than a $2 \times (e - 1)$ matrix (whose maximal minors have a height of at most $e - 2$).
(ii) We prove that $H = \langle 4, 5, 6, 7\rangle$ is determinantal by exhibiting an explicit 2×4 matrix whose 2×2 minors generate I_H (Theorem 4.3), and we prove that the semigroups $\langle n, n + 1, n + 2, n + 3\rangle$ with $n \equiv 1 \pmod 3$ form an infinite family with a consecutive pseudo-Frobenius set and the determinantal defining ideal (Theorem 4.4).
(iii) We compute the correct homological data via the Eagon-Northcott complex of a $2 \times e$ matrix, obtaining

$$\beta_i(I_H) = (i + 1)\binom{e}{i + 2}$$

(Theorem 4.1); in particular, the number of defining binomials is

$$\binom{e}{2}.$$

(iv) We give a simple, checkable necessary condition for determinantality (Proposition 4.8) and use it to prove that certain semigroups are not determinantal even though their Betti numbers coincide with the Eagon-Northcott numbers (Example 4.9).
(v) Every example is computed and independently verified in Singular; the scripts and their exact output are reproduced in Section 3.5 and Appendix.

1.5. Aim and objectives

Aim

To identify the implications of structured pseudo-Frobenius sets on the algebraic description of numerical semigroup rings, and to understand when the defining ideals admit a determinantal description of the correct height.

Objectives

(1) To characterize classes of numerical semigroups with structured pseudo-Frobenius sets;
(2) To construct $2 \times e$ matrices whose maximal minors may generate the defining ideals of the corresponding semigroup rings;
(3) To establish, and to rigorously verify criteria for determinantal presentations in terms of semigroup invariants and pseudo-Frobenius data;
(4) To illustrate the results through computed examples and to delimit the framework with explicit counterexamples;
(5) To clarify the homological consequences (height, resolution, Betti numbers) of these presentations.

2. Preliminaries

This section fixes notation and records the standard facts used throughout.

2.1. Numerical semigroups

Let $H = \langle n_1, \ldots, n_e \rangle \subseteq \mathbb{N}$ be a numerical semigroup, minimally generated by

$$1 < n_1 < n_2 < \cdots < n_e, \qquad \gcd(n_1, \ldots, n_e) = 1.$$

The integer $e = e(H)$ is the embedding dimension. The largest integer not in H is the Frobenius number $\mathrm{F}(H) = \max(\mathbb{Z} \setminus H)$, and $g(H) = \#(\mathbb{N} \setminus H)$ is the genus. The pseudo-Frobenius set is

$$\mathrm{PF}(H) = \{\, x \notin H \mid x + h \in H \text{ for all } h \in H \setminus \{0\} \,\},$$

and $\mathrm{t}(H) = \#\,\mathrm{PF}(H)$ is the type. It is standard [2, Ch. 1] that $x \in \mathrm{PF}(H)$ if and only if $x \notin H$ and $x + n_i \in H$ for every minimal generator n_i; this is the criterion used in all computations below.

2.2. *The semigroup ring and its defining ideal*

Let K be a field and $S = K[x_1, \ldots, x_e]$, graded by $\deg(x_i) = n_i$ (the H-grading). The map

$$\varphi\colon S \longrightarrow K[H] = K[t^{n_1}, \ldots, t^{n_e}], \qquad \varphi(x_i) = t^{n_i},$$

is a surjective homomorphism of graded K-algebras, and its kernel

$$I_H = \ker\varphi$$

is the defining ideal. It is a prime, H-homogeneous binomial ideal; a binomial $x^a - x^b$ lies in I_H if and only if $\sum_i a_i n_i = \sum_i b_i n_i$ [2, 6]. Homogeneity with respect to the H-grading is used repeatedly, so we record it explicitly.

Lemma 2.1. *A binomial $x^a - x^b \in S$ belongs to I_H if and only if it is H-homogeneous, i.e.,*

$$\deg_H(x^a) = \deg_H(x^b),$$

where $\deg_H(x^c) = \sum_i c_i n_i$. Consequently, every minimal generating set of I_H consists of H-homogeneous binomials, and any matrix presenting I_H by its minors must be H-homogeneous.

Proof. Since

$$\varphi(x^a - x^b) = t^{\deg_H(x^a)} - t^{\deg_H(x^b)},$$

the binomial lies in $\ker\varphi$ exactly when the two exponents have equal H-degree. As I_H is H-homogeneous and prime, it has a minimal generating set of homogeneous binomials [2]. □

2.3. *Height of the defining ideal*

The following elementary fact is the pivot of the whole paper.

Proposition 2.2. *For every numerical semigroup H of embedding dimension e*

$$\mathrm{ht}(I_H) = e - 1.$$

Proof. The ring $K[H] = K[t^{n_1}, \ldots, t^{n_e}]$ is a one-dimensional domain, being a subring of $K[t]$ that contains t^{n_1} and is module-finite over $K[t^{n_1}]$; hence $\dim K[H] = 1$. Because $I_H = \ker\varphi$ and $S/I_H \cong K[H]$, we have $\dim S/I_H = 1$. As I_H is a prime ideal of the Cohen-Macaulay ring S with $\dim S = e$

$$\mathrm{ht}(I_H) = \dim S - \dim(S/I_H) = e - 1.$$ □

Remark 2.3. Proposition 2.2 forces the choice of matrix size. For a $2 \times m$ matrix M with entries in S, the ideal $I_2(M)$ of maximal (2×2) minors has, when it is nonzero and of the expected (generic) codimension, height $m - 1$ [14]. To match $\mathrm{ht}(I_H) = e - 1$, we therefore need $m = e$, i.e., a $2 \times e$ matrix. A $2 \times (e - 1)$ matrix can only produce a height of at most $e - 2$, so its maximal minors can never generate I_H.

2.4. Determinantal ideals and the Eagon-Northcott complex

Let M be a $2\times e$ matrix over S whose ideal $I_2(M)$ of maximal minors has the expected height $e-1$. Then $S/I_2(M)$ is a perfect (Cohen-Macaulay) ideal of grade $e-1$, and its minimal free resolution is the Eagon-Northcott complex of M [14]. For the maximal minors of a $2\times e$ matrix, this complex has length $e-1$ and, in the generic case, ranks as follows:

$$\beta_i(S/I_2(M)) = i\binom{e}{i+1}, \qquad i = 1,\dots,e-1,$$

with $\beta_0 = 1$. Re-indexing for the ideal itself, we have

$$\beta_i(I_2(M)) = \beta_{i+1}(S/I_2(M)) = (i+1)\binom{e}{i+2}.$$

When $e = 4$, the complex has length 3 and reduces to a Hilbert-Burch-type resolution; for $e \geq 5$, it is genuinely longer.

3. Methodology

Methodology is understood here, as in pure mathematics, as the algebraic setup, the structural hypotheses on H, the construction of the associated matrices and ideals, and the plan of proof. In addition, every assertion about a specific semigroup below is checked by an explicit computation whose code and output are given in Section 3.5; nothing in the examples is hypothetical.

3.1. Algebraic setup

With the same notation as in Section 2, for each semigroup, we record

$$\begin{aligned} e(H) &= \text{embedding dimension},\\ \mathrm{F}(H) &= \max(\mathbb{Z}\setminus H),\\ \mathrm{PF}(H) &= \{\, f \notin H \mid f+h \in H \text{ for all } h \in H\setminus\{0\}\,\},\\ \mathrm{t}(H) &= \#\,\mathrm{PF}(H). \end{aligned}$$

These invariants are the input to the structural criteria developed below [3, 10].

3.2. Structured pseudo-Frobenius sets

Since $\mathrm{PF}(H)$ controls the type and reflects symmetry properties, it is a natural place to look for hidden regularity in the defining relations [8, 12, 17].

Definition 3.1. A numerical semigroup H has a structured pseudo-Frobenius set if

$$\mathrm{PF}(H) = \{f_1,\dots,f_t\},$$

listed in increasing order, satisfies one of the following patterns.
(A) **Arithmetic type:**

$$\mathrm{PF}(H) = \{a,\ a+d,\ \dots,\ a+(t-1)d\}, \qquad d>0.$$

(B) **Centrally symmetric type:**

$$f_i + f_{t+1-i} = c$$

for all i, where $c \in \mathbb{Z}$ is constant.

(C) **Generator-linked type:** $\alpha > 0$ exists such that

$$\mathrm{PF}(H) \subseteq \{\, \alpha n_1 - n_i \mid 2 \le i \le e \,\}.$$

(D) **Almost-symmetric type:**

$$\mathrm{PF}(H) \setminus \{F(H)\}$$

pairs compatibly with the almost-symmetric behaviour of H.

These conditions are not assumed to be equivalent; they define admissible families within which the determinantal behaviour of I_H is investigated. Table 1 records, for each pattern, the matrix construction to which it is matched. The rule "one pattern, one construction" is fixed here once and for all, so that no ambiguity remains about which matrix applies to which family.

Table 1. The four structured patterns and the matrix construction assigned to each. All constructions use a $2 \times e$ matrix so that $I_2(M)$ has the correct height $e-1$ (Remark 2.3).

Pattern	Form of PF(H)	Assigned $2 \times e$ construction
(A) Arithmetic	$\{a, a+d, \dots, a+(t-1)d\}$	Hankel/scroll matrix with a single "wrap" entry (Theorem 4.3)
(B) Central symmetry	$\{u, c-u, v, c-v, \mathrm{F}(H)\}$	Symmetric $2 \times e$ matrix
(C) Generator-linked	$\{\alpha n_1 - n_2, \dots, \alpha n_1 - n_e\}$	Generator-linked $2 \times e$ matrix (Proposition 4.6)
(D) Almost-symmetric	Almost-symmetric configuration	Near-Hilbert-Burch $2 \times e$ matrix

3.3. *Construction of the candidate matrix*

Let $H = \langle n_1, \dots, n_e \rangle$ have a structured pseudo-Frobenius set. We seek a $2 \times e$ matrix M over S whose maximal minors are candidates for the generators of I_H. By Lemma 2.1, M must be H-homogeneous: Writing $p_c = \deg_H(M_{1c})$ and $q_c = \deg_H(M_{2c})$ for the two entries of column c, the minor

$$\Delta_{ij} = M_{1i}M_{2j} - M_{1j}M_{2i}$$

is H-homogeneous if and only if $p_i + q_j = p_j + q_i$, if and only if

$$p_c - q_c = \delta \quad \text{is independent of the column } c. \tag{3.1}$$

Condition (3.1) is the exact homogeneity requirement that any determinantal presentation of I_H must satisfy; it is what makes the obstruction of Proposition 4.8 possible. The candidate ideal is $J_M = I_2(M)$, and the central question is whether $I_H = J_M$.

3.4. *Proof strategy*

The verification of $I_H = I_2(M)$ proceeds in four steps.

Step 1 (minors lie in I_H).

Each maximal minor Δ_{ij} is an H-homogeneous binomial by (3.1), hence, $\Delta_{ij} \in I_H$ by Lemma 2.1. Thus $I_2(M) \subseteq I_H$.

Step 2 (reverse containment).

The reverse inclusion

$$I_H \subseteq I_2(M)$$

must be established separately for the family under consideration. It may be proved by a Gröbner-basis or elimination argument when an explicit generating set is available. For the infinite family of Theorem 4.4, we instead use a quotient dimension argument: After setting $x_1 = 0$, we compute the dimension of the resulting finite-dimensional quotient and compare it with the corresponding quotient of the numerical semigroup ring. This gives the reverse containment uniformly for all members of the family.

Step 3 (height).

By Step 1–2, if the reverse containment holds then $I_H = I_2(M)$; by Proposition 2.2, both ideals have height $e-1$, which is exactly the expected height of the maximal minors of a $2\times e$, matrix (Remark 2.3). No auxiliary "dependent column relation" is needed once the matrix has e columns.

Step 4 (resolution).

When $I_H = I_2(M)$ for a $2\times e$, matrix of the expected height, S/I_H is resolved by the Eagon-Northcott complex of M [14], which yields the Betti numbers of Theorem 4.1.

3.5. Computational verification

All computations below were carried out in Singular 4.3.2 [18, 19] using elimination to obtain I_H, minbase for a minimal binomial generating set, and mres/betti for the minimal free resolution; pseudo-Frobenius sets were checked independently with the criterion $x + n_i \in H$ for all i (and cross-checked against NumericalSgps [20]). The script used for each semigroup is

```
ring R = 0,(t,x(1..e)),dp;
ideal J = x(1)-t^(n1), ..., x(e)-t^(ne);
ideal I0 = eliminate(J,t);              // defining ideal I_H
ring S = 0,(x(1..e)),wp(n1,...,ne);     // H-graded polynomial ring
ideal I = minbase(imap(R,I0));          // minimal binomial generators
resolution re = mres(I,0);
print(betti(re),"betti");               // Betti numbers
print(nvars(S)-dim(std(I)));            // height of I_H
```

The concrete outputs are collected in Table 2 and reproduced in full in Appendix. Every entry of Table 2 is a completed computation, not an illustration.

Table 2. Verified data for the worked semigroups. Here $\beta = (\beta_0, \beta_1, \ldots)$ are the Betti numbers of I_H; "E-N?" records whether β equals the Eagon-Northcott numbers of a $2 \times e$ matrix; "Det.?" records whether I_H is actually the ideal of maximal minors of a graded $2 \times e$ matrix (Proposition 4.8).

H	$\mathrm{PF}(H)$	e	$\mathrm{ht}\, I_H$	$\beta(I_H)$	E–N?	Det.?
$\langle 4, 5, 6, 7\rangle$	$\{1, 2, 3\}$	4	3	$(6, 8, 3)$	Yes	Yes
$\langle 7, 8, 9, 10\rangle$	$\{11, 12, 13\}$	4	3	$(6, 8, 3)$	Yes	Yes
$\langle 10, 11, 12, 13\rangle$	$\{27, 28, 29\}$	4	3	$(6, 8, 3)$	Yes	Yes
$\langle 6, 8, 9, 11\rangle$	$\{3, 10, 13\}$	4	3	$(6, 8, 3)$	Yes	No
$\langle 7, 10, 12, 13\rangle$	$\{15, 16, 18\}$	4	3	$(4, 6, 3)$	No	No
$\langle 9, 13, 14, 16\rangle$	$\{21, 33\}$	4	3	$(5, 6, 2)$	No	No
$\langle 8, 11, 14, 15, 17\rangle$	$\{18, 20, 21\}$	5	4	$(7, 12, 9, 3)$	No	No

4. Results

Throughout, $H = \langle n_1, \ldots, n_e\rangle$ with $\deg(x_i) = n_i$, and φ, I_H are as in Section 2.

4.1. The homological model

Theorem 4.1. *Suppose $I_H = I_2(M)$ for a $2 \times e$ matrix M over S whose ideal of maximal minors has the expected height $e - 1$. Then S/I_H is resolved by the Eagon-Northcott complex of M, and*

$$\beta_i(I_H) = (i + 1)\binom{e}{i + 2}, \qquad i = 0, 1, \ldots, e - 2.$$

In particular, I_H is minimally generated by

$$\beta_0(I_H) = \binom{e}{2}$$

binomials, and

$$\beta_1(I_H) = 2\binom{e}{3}.$$

Proof. Because $\mathrm{ht}\, I_2(M) = e - 1$ equals the codimension of the generic determinantal variety, $I_2(M)$ is a perfect ideal of grade $e - 1$ and the Eagon-Northcott complex of M is its minimal free resolution [14]. For a $2 \times e$ matrix the terms of that complex have the ranks

$$\beta_i(S/I_H) = i\binom{e}{i + 1}, \qquad i = 1, \ldots, e - 1,$$

re-indexing to I_H gives

$$\beta_i(I_H) = (i + 1)\binom{e}{i + 2}.$$

The stated special values follow by setting $i = 0, 1$. □

Remark 4.2. The values in Table 3 replace the numbers $(3, 6, 2), (6, 8, 3), (10, 10, 4), (15, 12, 5)$ quoted in the previous version. Those numbers came from a $2 \times (e - 1)$ matrix and, as one checks, do not even satisfy the rank alternation $\sum_i (-1)^i \beta_i = 1$ required of a resolution of a rank-one ideal; e.g., for $e = 4$ they give $3 - 6 + 2 = -1 \neq 1$. The corrected values satisfy $\sum_i (-1)^i \beta_i = 1$ in every case, e.g., $6 - 8 + 3 = 1$ and $10 - 20 + 15 - 4 = 1$.

Table 3. Corrected Betti numbers $\beta(I_H) = (\beta_0, \ldots, \beta_{e-2})$ predicted by Theorem 4.1 for a $2 \times e$ determinantal presentation. The entry for $e = 4$ was verified directly (Table 2).

e	$\beta(I_H) = \left((i+1)\binom{e}{i+2}\right)_{i \geq 0}$
4	(6, 8, 3)
5	(10, 20, 15, 4)
6	(15, 40, 45, 24, 5)
7	(21, 70, 105, 84, 35, 6)

4.2. A rigorous determinantal example and an infinite family

Theorem 4.3. *Let $H = \langle 4, 5, 6, 7 \rangle$, so $S = K[x_1, x_2, x_3, x_4]$ with $\deg x_i = i + 3$. Let*

$$M = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_1^2 \end{pmatrix}.$$

Then $I_H = I_2(M)$, so I_H is determinantal of height 3, minimally generated by the six 2×2 minors

$$\begin{array}{lll} x_2^2 - x_1x_3, & x_2x_3 - x_1x_4, & x_3^2 - x_2x_4, \\ x_1^2x_2 - x_3x_4, & x_1^2x_3 - x_4^2, & x_1^3 - x_2x_4, \end{array}$$

and S/I_H is resolved by the Eagon-Northcott complex of M, giving $\beta(I_H) = (6, 8, 3)$. Moreover $\mathrm{PF}(H) = \{1, 2, 3\}$ is an arithmetic progression.

Proof. Each of the six displayed binomials is a maximal minor of M and is H-homogeneous: For the columns of M, one has $(p_c) = (4, 5, 6, 7)$ and $(q_c) = (5, 6, 7, 8)$, so $p_c - q_c = -1$ is constant, and (3.1) holds; hence, every minor lies in I_H and $I_2(M) \subseteq I_H$. Conversely, a direct elimination shows that these six binomials already minimally generate I_H (Step 2 of Section 3.4); equivalently, a Gröbner basis of $I_2(M)$ equals one of I_H. Both inclusions were verified in Singular by reducing each generating set modulo by a standard basis of the other and obtaining zero (Appendix). Since M is 2×4 and $\operatorname{ht} I_H = 3$ by Proposition 2.2, the ideal has the expected determinantal height, so the Eagon-Northcott complex resolves S/I_H and $\beta(I_H) = (6, 8, 3)$ by Theorem 4.1. Finally $\mathrm{F}(H) = 3$ and a check of the criterion $x + n_i \in H$ gives $\mathrm{PF}(H) = \{1, 2, 3\}$. □

Theorem 4.4. *Let $q \geq 1$, put*

$$n = 3q + 1,$$

and let

$$H_q = \langle n, n+1, n+2, n+3 \rangle.$$

Then H_q has embedding dimension 4*, and*

$$\mathrm{PF}(H_q) = \{qn-3,\ qn-2,\ qn-1\} = \left\{F(H_q)-2,\ F(H_q)-1,\ F(H_q)\right\}.$$

In particular, $\mathrm{PF}(H_q)$ *is an arithmetic progression of length* 3 *and common difference* 1.

Moreover, let

$$S = K[x_1,\ x_2,\ x_3,\ x_4], \qquad \deg_H(x_i) = n+i-1,$$

and consider the matrix

$$M_q = \begin{pmatrix} x_1 & x_2 & x_3 & x_4^q \\ x_2 & x_3 & x_4 & x_1^{q+1} \end{pmatrix}.$$

Then

$$I_{H_q} = I_2(M_q).$$

Consequently, I_{H_q} is a determinantal ideal of height 3*, minimally generated by six binomials, and*

$$\beta(I_{H_q}) = (6, 8, 3).$$

Proof. Since $n = 3q+1$, the four integers

$$n, \quad n+1, \quad n+2, \quad n+3$$

form a minimal generating set of H_q. Hence

$$e(H_q) = 4.$$

We first determine the Apéry set of H_q with respect to n. For $1 \leq r \leq n-1 = 3q$, set

$$w_r = \left\lceil \frac{r}{3} \right\rceil n + r.$$

We claim that w_r is the least element of H_q congruent to r modulo n.

Indeed, let $h \in H_q$ satisfy

$$h \equiv r \pmod{n}.$$

Write

$$h = a_0 n + a_1(n+1) + a_2(n+2) + a_3(n+3), \qquad a_i \in \mathbb{N}.$$

Put

$$k = a_0 + a_1 + a_2 + a_3$$

and

$$s = a_1 + 2a_2 + 3a_3.$$

Then

$$h = kn + s.$$

Since $h \equiv r \pmod{n}$, an integer $t \geq 0$ exists such that

$$s = r + tn.$$

Moreover

$$s = a_1 + 2a_2 + 3a_3 \le 3(a_0 + a_1 + a_2 + a_3) = 3k.$$

Therefore

$$k \ge \left\lceil \frac{r + tn}{3} \right\rceil \ge \left\lceil \frac{r}{3} \right\rceil.$$

Consequently

$$h = kn + r + tn \ge \left\lceil \frac{r}{3} \right\rceil n + r = w_r.$$

Conversely, r can be written as a sum of $\lceil r/3 \rceil$ elements from $\{1, 2, 3\}$. Therefore, w_r is a non-negative integral combination of $n + 1, n + 2, n + 3$, and hence

$$w_r \in H_q.$$

Thus w_r is the least element of H_q in its residue class modulo n. Hence

$$\mathrm{Ap}(H_q, n) = \{0\} \cup \left\{ \left\lceil \frac{r}{3} \right\rceil n + r \;:\; 1 \le r \le 3q \right\}.$$

We now determine the maximal elements of the Apéry set with respect to the semigroup order. First observe that

$$w_1 < w_2 < \cdots < w_{3q}.$$

This follows directly from

$$w_r = \left\lceil \frac{r}{3} \right\rceil n + r.$$

For

$$1 \le r \le 3q - 3,$$

we have

$$w_{r+3} = \left\lceil \frac{r + 3}{3} \right\rceil n + r + 3 = w_r + (n + 3).$$

Since $n + 3 \in H_q \setminus \{0\}$, it follows that

$$w_r <_{H_q} w_{r+3}.$$

Hence, none of

$$w_1, \ldots, w_{3q-3}$$

is maximal in $\mathrm{Ap}(H_q, n)$.

It remains to consider

$$w_{3q-2}, \qquad w_{3q-1}, \qquad w_{3q}.$$

Since the sequence (w_r) is strictly increasing, an element of the Apéry set that could dominate one of these elements must be one of the later elements among these three. Their positive pairwise differences are 1 or 2. Since the multiplicity of H_q is $n \ge 4$, neither 1 nor 2 belongs to H_q. Thus the three elements are pairwise incomparable with respect to $\le_{H_q}$. Consequently, we have

$$\mathrm{Max}_{\le_{H_q}} \mathrm{Ap}(H_q, n) = \left\{ w_{3q-2},\ w_{3q-1},\ w_{3q} \right\}.$$

By the standard description of pseudo-Frobenius numbers in terms of the maximal elements of an Apéry set, we obtain

$$\mathrm{PF}(H_q) = \left\{w_{3q-2} - n,\ w_{3q-1} - n,\ w_{3q} - n\right\}.$$

Since

$$\left\lceil \frac{3q - j}{3} \right\rceil = q, \qquad j = 0,\ 1,\ 2,$$

we have

$$w_{3q-j} = qn + (3q - j).$$

Using $3q = n - 1$, we obtain

$$w_{3q-2} - n = qn - 3,$$
$$w_{3q-1} - n = qn - 2,$$

and

$$w_{3q} - n = qn - 1.$$

Therefore

$$\mathrm{PF}(H_q) = \{qn - 3,\ qn - 2,\ qn - 1\}.$$

In particular,

$$F(H_q) = qn - 1,$$

and hence

$$\mathrm{PF}(H_q) = \{F(H_q) - 2,\ F(H_q) - 1,\ F(H_q)\}.$$

We next prove the determinantal statement. Consider

$$M_q = \begin{pmatrix} x_1 & x_2 & x_3 & x_4^q \\ x_2 & x_3 & x_4 & x_1^{q+1} \end{pmatrix}.$$

For the first three columns, the difference between the H-degree of the lower entry and that of the upper entry is 1. For the fourth column, we have

$$\begin{aligned} \deg_H(x_1^{q+1}) - \deg_H(x_4^q) &= (q + 1)n - q(n + 3) \\ &= n - 3q \\ &= 1. \end{aligned}$$

Thus all column degree differences are equal. Hence, every 2×2 minor of M_q is H-homogeneous and belongs to I_{H_q}. Therefore, if

$$J_q := I_2(M_q),$$

then

$$J_q \subseteq I_{H_q}.$$

The six maximal minors of M_q, up to sign, are

$$x_2^2 - x_1x_3,$$
$$x_2x_3 - x_1x_4,$$

$$
\begin{gathered}
x_3^2 - x_2x_4,\\
x_1^{q+2} - x_2x_4^q,\\
x_1^{q+1}x_2 - x_3x_4^q,\\
x_1^{q+1}x_3 - x_4^{q+1}.
\end{gathered}
$$

It remains to prove the reverse inclusion. For modulo x_1, we obtain

$$J_q + (x_1) = \big(x_1,\ x_2^2,\ x_2x_3,\ x_3^2 - x_2x_4,\ x_2x_4^q,\ x_3x_4^q,\ x_4^{q+1}\big).$$

A K-basis of

$$S/(J_q + (x_1))$$

is given by 1, together with

$$
\begin{aligned}
x_4^j, &\quad 1 \le j \le q;\\
x_3x_4^j, &\quad 0 \le j \le q-1;\\
x_2x_4^j, &\quad 0 \le j \le q-1.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\dim_K S/(J_q + (x_1)) &= 1 + q + q + q\\
&= 3q + 1\\
&= n.
\end{aligned}
$$

Since

$$J_q \subseteq I_{H_q}$$

and

$$\dim_K S/(J_q + (x_1)) = n,$$

while

$$\dim_K S/(I_{H_q} + (x_1)) = \dim_K K[H_q]/(t^n) = \#\,\mathrm{Ap}(H_q, n) = n,$$

the natural surjection

$$S/(J_q + (x_1)) \twoheadrightarrow S/(I_{H_q} + (x_1))$$

is an isomorphism. Hence

$$J_q + (x_1) = I_{H_q} + (x_1).$$

Thus

$$I_{H_q} = J_q + x_1 I_{H_q}.$$

Since I_{H_q}/J_q is a finitely generated positively graded S-module and x_1 has positive H_q-degree, the graded Nakayama lemma implies

$$I_{H_q}/J_q = 0.$$

Hence

$$J_q = I_{H_q}.$$

Finally, since $e(H_q) = 4$,

$$\mathrm{ht}(I_{H_q}) = 4 - 1 = 3.$$

Thus I_{H_q} has the expected height for the ideal of maximal minors of a 2×4, matrix. The Eagon-Northcott complex gives the minimal free resolution, and consequently

$$\beta(I_{H_q}) = (6, 8, 3).$$

This completes the proof. $\square$

Remark 4.5. Theorem 4.4 gives a uniform proof for every $q \geq 1$; equivalently, for every $n \equiv 1 \pmod 3$. The computations for $n = 4, 7, 10$ are therefore used only as explicit computational checks and examples; they are not required for the proof of the infinite-family statement.

4.3. Generator-linked presentations

For Pattern (C), we record the explicit generators.

Proposition 4.6. *Let*

$$H = \langle n_1, \ldots, n_e \rangle$$

satisfy

$$\mathrm{PF}(H) = \{\alpha n_1 - n_2, \ldots, \alpha n_1 - n_e\}$$

for some integer $\alpha > 0$. Assume that there is a bijection

$$\rho \colon \{2, \ldots, e\} \longrightarrow \{2, \ldots, e\}$$

such that the column-degree differences

$$n_i - ((\alpha - 1)n_1 + n_{\rho(i)})$$

are independent of i.

Then the binomials

$$\Delta_{ij} = x_i x_1^{\alpha-1} x_{\rho(j)} - x_j x_1^{\alpha-1} x_{\rho(i)}, \qquad 2 \leq i < j \leq e,$$

are the 2×2 minors of

$$M_H = \begin{pmatrix} x_2 & x_3 & \cdots & x_e \\ x_1^{\alpha-1} x_{\rho(2)} & x_1^{\alpha-1} x_{\rho(3)} & \cdots & x_1^{\alpha-1} x_{\rho(e)} \end{pmatrix},$$

and each Δ_{ij} lies in I_H.

Proof. By hypothesis, the differences

$$\deg_H(x_i) - \deg_H(x_1^{\alpha-1} x_{\rho(i)})$$

are constant as i varies. Hence, the two entries in each column of M_H satisfy the constant row-gap condition (1). Therefore every 2×2 minor of M_H is H-homogeneous. By Lemma 2.1, each such minor belongs to I_H.

The displayed binomials are precisely the 2×2 minors of M_H. $\square$

Remark 4.7. Whether the minors of Proposition 4.6 generate all of I_H (rather than merely lying in it) depends on the semigroup and is not automatic. This is the point at which the previous version over-reached: The reverse inclusion was asserted for all generator-linked semigroups but holds only under the height match $\operatorname{ht} I_2(M_H) = e - 1$, which must be checked. We therefore state generation as a hypothesis to be verified case by case (as in Table 2) rather than as a theorem, and record the genuinely proven cases in Theorems 4.3 and 4.4.

4.4. A necessary obstruction to determinantality

The following elementary but useful test detects non-determinantal semigroups directly from the degrees of the defining binomials.

Proposition 4.8. *Let*

$$H = \langle n_1, \ldots, n_e \rangle$$

be a numerical semigroup with

$$e \equiv 1 \pmod 3.$$

Suppose that I_H is minimally generated by

$$\binom{e}{2}$$

binomials and that

$$I_H = I_2(M)$$

for some H-homogeneous $2 \times e$ matrix M. If

$$d_1, \ldots, d_{\binom{e}{2}}$$

are the H-degrees of the minimal generators, then

$$\sum_k d_k \equiv 0 \pmod 3.$$

Consequently, if $e \equiv 1 \pmod 3$, I_H has $\binom{e}{2}$ minimal generators, and

$$\sum_k d_k \not\equiv 0 \pmod 3,$$

then I_H cannot be the ideal of maximal minors of an H-homogeneous $2 \times e$ matrix.

Proof. Let p_c and q_c be the H-degrees of the upper and lower entries, respectively, in Column c of M. Since M is H-homogeneous, a constant δ exists such that

$$q_c = p_c - \delta, \qquad c = 1, \ldots, e.$$

The minor determined by columns $i < j$ has the H-degree

$$p_i + q_j = p_i + p_j - \delta.$$

Summing over all $\binom{e}{2}$ pairs gives

$$\sum_k d_k = \sum_{i<j}(p_i + p_j - \delta) = (e-1)\sum_{c=1}^{e} p_c - \binom{e}{2}\delta.$$

Since

$$e \equiv 1 \pmod 3,$$

we have

$$3 \mid (e-1)$$

and also

$$3 \mid \binom{e}{2}.$$

Therefore

$$\sum_k d_k \equiv 0 \pmod 3.$$

□

4.5. Boundary and failure cases

Example 4.9 (A Betti-matching but non-determinantal semigroup)**.** Let $H = \langle 6, 8, 9, 11\rangle$. The pseudo-Frobenius set is

$$\mathrm{PF}(H) = \{3, 10, 13\}$$

(and not $\{13, 15, 16\}$, as erroneously stated before). Computation gives $\operatorname{ht} I_H = 3$ and $\beta(I_H) = (6, 8, 3)$: The ideal has exactly $\binom{4}{2} = 6$ minimal generators and its Betti numbers coincide with the Eagon-Northcott numbers of a 2×4 matrix. Nevertheless, I_H is not determinantal. The six minimal generators have H-degrees $\{17, 18, 20, 22, 24, 27\}$, whose sum is $128 \equiv 2 \pmod 3$; by 4.8 no H-homogeneous 2×4 matrix has these binomials as its maximal minors. Thus matching the Eagon-Northcott Betti numbers is necessary but not sufficient for determinantality a subtlety that the previous version missed and that this example makes precise.

Example 4.10 (A structured PF-set is not sufficient)**.** Let $H = \langle 7, 10, 12, 13\rangle$, with $\mathrm{PF}(H) = \{15, 16, 18\}$. Here, I_H has only 4 minimal generators (not $\binom{4}{2} = 6$), with $\beta(I_H) = (4, 6, 3)$; it is an almost complete intersection of height 3 and is not determinantal. Since I_H has only four minimal generators, whereas a 2×4 determinantal presentation of the expected height requires six maximal minors, such a determinantal presentation is impossible. Thus, a small and apparently regular pseudo-Frobenius set alone does not force a determinantal presentation; compatibility with the number and degrees of the minimal generators is also required.

Example 4.11 (A type-two semigroup)**.** Let $H = \langle 9, 13, 14, 16\rangle$, with $\mathrm{PF}(H) = \{21, 33\}$ and $\mathrm{t}(H) = 2$. Then $\beta(I_H) = (5, 6, 2)$ and $\operatorname{ht} I_H = 3$; the ideal is not determinantal (it has five generators and $\beta_2 = 2$). This illustrates the failure predicted when $\mathrm{PF}(H)$ matches none of the patterns of Definition 3.1.

Example 4.12 (Embedding dimension five)**.** Let $H = \langle 8, 11, 14, 15, 17\rangle$, with $\mathrm{PF}(H) = \{18, 20, 21\}$. Then $\operatorname{ht} I_H = 4 = e - 1$, but I_H has 7 minimal generators, whereas a determinantal 2×5 presentation would require $\binom{5}{2} = 10$; the Betti numbers $(7, 12, 9, 3)$ differ from the Eagon Northcott numbers $(10, 20, 15, 4)$ of Table 3. Hence, I_H is not determinantal. This is the genuine five-generator boundary case replacing the earlier limitation example.

Proposition 4.13. *The following provide necessary obstructions to a determinantal presentation. Suppose that $H = \langle n_1, \ldots, n_e \rangle$. If I_H has a number of minimal generators different from $\binom{e}{2}$, then I_H cannot be the ideal of maximal minors of a $2 \times e$ matrix. Moreover, when $e \equiv 1 \pmod 3$, if I_H has $\binom{e}{2}$ minimal generators whose H-degrees have a sum not divisible by 3, then I_H cannot be the ideal of the maximal minors of an H-homogeneous $2 \times e$ matrix.*

Proof. The first assertion follows from the fact that the maximal minors of a $2 \times e$ matrix number exactly $\binom{e}{2}$. The second assertion follows directly from Proposition 4.8. Examples 4.10–4.12 illustrate the first obstruction, while Example 4.9 illustrates the second. □

Remark 4.14. Proposition 4.13 replaces the earlier Proposition 4.15, which merely restated its hypothesis. The present statement is a genuine dichotomy with a proof and with realized instances of each case.

5. Discussion

5.1. Interpretation of the results

The results confirm, in a corrected and verified form, that the determinantal behaviour of I_H is governed by the combinatorics of $\mathrm{PF}(H)$—but only within sharp numerical constraints. The decisive constraint is the height: Since $\operatorname{ht} I_H = e - 1$ always (Proposition 2.2), a maximal-minor presentation must use a $2 \times e$ matrix, and the ideal it generates has the correct height automatically, with no need for an ad hoc "dependent column relation". Within this corrected framework, consecutive (arithmetic) pseudo-Frobenius sets force a determinantal presentation, as Theorems 4.3 and 4.4 prove for an infinite family. At the same time, the obstruction of Proposition 4.8 shows that not every semigroup with the "right" Betti numbers is determinantal, so the combinatorial hypothesis genuinely works.

5.2. Comparison with previous literature

The present work sits within the line of research on defining the ideals of numerical semigroup rings initiated by Herzog [6] and systematized by Rosales and García-Sánchez [2]. It is closest in spirit to Do and Matsuoka [15] and Kumashiro, Matsuoka and Nakashima [13], who study numerical semigroup rings defined by the maximal minors of a $2 \times n$ matrix, and to Goto, et al [12], who connect the pseudo-Frobenius numbers to the defining ideals. Relative to those works, the contribution here is to organize the discussion around the shape of $\mathrm{PF}(H)$, to insist on the height match $e - 1$, and to supply a computable obstruction. The comparison also clarifies why the four patterns are natural: They are exactly the regularities of $\mathrm{PF}(H)$ that produce a constant row-gap (3.1), which is the homogeneity condition any determinantal presentation must satisfy.

5.3. Strengths and limitations

The strengths of the corrected framework are the exact height bookkeeping, an infinite family with a fully proven determinantal presentation, correct Eagon-Northcott Betti numbers, and a checkable obstruction with realized counterexamples, all independently verified by computation. The limitations are equally clear: The general characterization for Patterns (B)–(D) is not proved here and is left as Conjecture 5.1; higher embedding dimension makes both the combinatorics and the resolution harder to control; and the determinantal property can fail even when the numerical shadow (Betti numbers) is correct.

Conjecture 5.1. *For each Pattern (B)–(D) of Definition 3.1, there is an explicit* $2 \times e$ *construction and an effectively checkable condition on the generator degrees under which* $I_H = I_2(M)$*. The obstruction of Proposition 4.8 is one necessary ingredient of such a condition.*

6. Conclusions

This paper develops, in corrected and computationally verified form, a framework for studying determinantal defining ideals of numerical semigroup rings through the structure of their pseudo-Frobenius sets. The defining ideal always has a height $e - 1$, so the correct candidate is a $2 \times e$ matrix; when its maximal minors generate I_H, the Eagon-Northcott complex gives the Betti numbers

$$\beta_i(I_H) = (i+1)\binom{e}{i+2}.$$

The consecutive family $\langle n, n+1, n+2, n+3\rangle$ with $n \equiv 1 \pmod 3$ is proved to be an infinite family with an arithmetic pseudo-Frobenius set and a determinantal defining ideal. The computations for selected values serve only as independent verification of the general theoretical result. A simple degree-sum obstruction distinguishes genuinely determinantal ideals from those that merely share the Eagon-Northcott Betti numbers. Together, these results give both positive characterizations and sharp boundaries, and they indicate that pseudo-Frobenius sets provide an effective though not unconditional bridge between the combinatorics of numerical semigroups and the geometry of their semigroup rings [2, 12, 14].

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Appendix

Singular sessions and output

For reproducibility, we record the exact computation for the determinantal base case and for one nondeterminantal case; the remaining rows of Table 2 are obtained by changing the generators and weights in the same script.

Determinantal case $H = \langle 4, 5, 6, 7 \rangle$ **(Theorem 4.3).**

```
ring R = 0,(t,x(1..4)),dp;
ideal J = x(1)-t^4, x(2)-t^5, x(3)-t^6, x(4)-t^7;
ideal I0 = eliminate(J,t);
ring S = 0,(x(1..4)),wp(4,5,6,7);
ideal I = minbase(imap(R,I0));
// I = x(2)^2-x(1)*x(3), x(2)*x(3)-x(1)*x(4), x(3)^2-x(2)*x(4),
//     x(1)^3-x(2)*x(4), x(1)^2*x(2)-x(3)*x(4), x(1)^2*x(3)-x(4)^2
matrix M[2][4] = x(1),x(2),x(3),x(4), x(2),x(3),x(4),x(1)^2;
ideal Im = minor(M,2);
reduce(I, std(Im));   // -> all zero  : I subset I_2(M)
reduce(Im, std(I));   // -> all zero  : I_2(M) subset I
4 - dim(std(I));      // -> 3         : height e-1
print(betti(mres(I,0)),"betti"); // total: 1  6  8  3
```

Non-determinantal case $H = \langle 6, 8, 9, 11 \rangle$ **(Example 4.9).**

```
ring R = 0,(t,x(1..4)),dp;
ideal J = x(1)-t^6, x(2)-t^8, x(3)-t^9, x(4)-t^11;
ideal I0 = eliminate(J,t);
ring S = 0,(x(1..4)),wp(6,8,9,11);
ideal I = minbase(imap(R,I0));   // 6 generators, degrees 17,18,20,22,24,27
4 - dim(std(I));                 // -> 3
print(betti(mres(I,0)),"betti"); // total: 1  6  8  3  (Eagon Northcott shape)
// but 17+18+20+22+24+27 = 128 = 2 (mod 3): no graded 2x4 matrix (Prop. 4.8)
```