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Research article

Generalized co-Gottlieb sets on product spaces via separable maps

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Abstract: Let X, Y, and A be based connected CW-complexes; let Z be an H-space; and let $p : X\times Y \to A$ be a based map. We study generalized co-Gottlieb sets associated with the product domain $X \times Y$ by means of separable maps. We show that the canonical restriction map $i^\sharp : [X \times Y, Z] \longrightarrow [X \vee Y, Z]$ is bijective if and only if every element of $[X \times Y, Z]$ is separable. For a grouplike H-space Z, this condition is equivalent to $[X \wedge Y, Z] = 0$. We also introduce a smash-product obstruction that measures the failure of separability and obtain an injectivity criterion for the restricted map $\widetilde{i^\sharp} : p^\top(X \times Y, Z) \longrightarrow (p \circ i)^\top(X \vee Y, Z)$ without assuming $[X \wedge Y, Z] = 0$. Under the separability condition, we obtain natural coordinate injections. Finally, if Z is a grouplike H-space and $(p \circ i)^\top(X \vee Y, Z)$ is a finite cyclic group of prime-power order, then $p^\top(X \times Y, Z)$ is a group.

Keywords: generalized co-Gottlieb set; separable map; product space; H-space; essential map
Mathematics Subject Classification: 55Q05, 55P45

1. Introduction

Classical Gottlieb theory originated in Gottlieb's study of a distinguished subgroup of the fundamental group and of evaluation subgroups of homotopy groups [1, 2]. Subsequent developments include studies of Gottlieb groups of spheres and wedges of Moore spaces, as well as cyclic element-preserving maps associated with Gottlieb groups [3–5]. Its dual theory is formulated in terms of cocyclic maps and coevaluation subgroups [6, 7]. Let X, A, and Z be based connected CW-complexes, and let $p : X \to A$ be a based map. The generalized co-Gottlieb set

$$p^\top(X, Z) = \{a \in [X, Z] \mid p\top a\}$$

was introduced by Oda [8]. Here $p\top a$ means that there exists a map $\theta : X \to A \vee Z$ such that

$$i_{A,Z} \circ \theta \simeq (p \times a) \circ \Delta_X,$$

where $i_{A,Z} : A \vee Z \to A \times Z$ is the canonical inclusion and $\Delta_X : X \to X \times X$ is the diagonal map.

This construction generalizes the dual Gottlieb set

$$DG(X, Z),$$

introduced by Varadarajan [6]. Cocyclic maps and coevaluation subgroups associated with $DG(X, Z)$ were studied by Lim [7]. Cocyclic element-preserving pair maps and related fibrations were investigated by Kim and Oda [9]. Recent extensions of co-H-space structures through intermediate DC_k^p-structures were investigated by Yoon [10]. Related homotopy comultiplications on wedges of spheres and Moore spaces were studied by Lee [11]. For an abelian group F and an Eilenberg-MacLane space $K(F, n)$, the generalized co-Gottlieb set

$$G_p^n(X; F) = p^\top(X, K(F, n))$$

was studied by Yoon [12].

Further properties of generalized co-Gottlieb sets were studied by Choi, Kim, and Oda [13]. In particular, actions and exact sequences were established. Moreover, for grouplike H-spaces Y_1 and Y_2,

$$p^\top(X, Y_1 \times Y_2) \cong p^\top(X, Y_1) \times p^\top(X, Y_2)$$

was obtained under suitable assumptions. As a consequence, several sufficient conditions under which generalized co-Gottlieb sets become groups were derived.

Following [13], a map $p : X \to A$ is called essential with respect to Z if $p^\top(X, Z)$ is a group.

Previous studies of generalized co-Gottlieb sets involving product spaces have mainly concerned products in the target. In that setting, the canonical projections provide natural coordinate maps, and generalized co-Gottlieb sets with target $Y_1 \times Y_2$ can be described in terms of the corresponding sets with targets Y_1 and Y_2 under suitable grouplike assumptions [13].

The situation is different when the domain is a product. A map from $X \times Y$ to Z is not determined in general by its restrictions to the two factors. The additional information is related to the smash-product term in the standard cofibration

$$X \vee Y \xrightarrow{i} X \times Y \xrightarrow{q} X \wedge Y.$$

Thus, the study of generalized co-Gottlieb sets with product domain requires an analysis of the restriction map

$$i^\sharp : [X \times Y, Z] \longrightarrow [X \vee Y, Z],$$

and of the contribution arising from $[X \wedge Y, Z]$.

Let Z be an H-space. An element $\alpha \in [X \times Y, Z]$ is called separable if

$$\alpha = (\alpha \circ i_X \circ \pi_X) + (\alpha \circ i_Y \circ \pi_Y).$$

We say that $[X \times Y, Z]$ satisfies the separability condition if all of its elements are separable.

We first prove that $i^\sharp$ is bijective if and only if $[X \times Y, Z]$ satisfies the separability condition. If Z is a grouplike H-space, this condition is equivalent to

$$[X \wedge Y, Z] = 0.$$

We also investigate the case in which separability fails. For each homotopy class, we define an obstruction arising from $\operatorname{Im} q^\sharp$ and show that its vanishing is equivalent to separability. Furthermore, we characterize the injectivity of the restricted map

$$\widetilde{i}^\sharp : p^\top(X \times Y, Z) \longrightarrow (p \circ i)^\top(X \vee Y, Z)$$

without requiring $[X \wedge Y, Z] = 0$.

Under the separability condition, we obtain natural injections from $p^\top(X \times Y, Z)$ into $(p \circ i)^\top(X \vee Y, Z)$ and into

$$p_X^\top(X, Z) \times p_Y^\top(Y, Z).$$

Finally, if Z is a grouplike H-space and $(p \circ i)^\top(X \vee Y, Z)$ is a finite cyclic group of prime-power order, then $p^\top(X \times Y, Z)$ is a group. Examples illustrate the injectivity results and the group criterion.

Henceforth, all spaces are assumed to be based connected CW-complexes, and all maps and homotopies are basepoint-preserving. The set of based homotopy classes of based maps from X to Y is denoted by $[X, Y]$. Notationally, a map $f : X \to Y$ and its homotopy class $[f] \in [X, Y]$ will not be distinguished.

2. Preliminaries

For $K = X, Y$, let $i_K : K \to X \times Y$ and $j_K : K \to X \vee Y$ denote the canonical inclusions and let $\pi_K : X \times Y \to K$ denote the canonical projection. Let

$$i : X \vee Y \to X \times Y$$

be the canonical inclusion. The cofibration

$$X \vee Y \xrightarrow{i} X \times Y \xrightarrow{q} X \wedge Y$$

induces an exact sequence of pointed sets

$$[X \wedge Y, Z] \xrightarrow{q^\sharp} [X \times Y, Z] \xrightarrow{i^\sharp} [X \vee Y, Z],$$

for every based space Z. Here

$$i^\sharp : [X \times Y, Z] \to [X \vee Y, Z]$$

is defined by $i^\sharp(\alpha) = \alpha \circ i$.

Let Z be an H-space with multiplication $m_Z : Z \times Z \to Z$. For any based CW-complex B, the multiplication m_Z induces a binary operation on $[B, Z]$. For $f, g \in [B, Z]$, define

$$f + g = m_Z \circ (f \times g) \circ \Delta_B,$$

where $\Delta_B : B \to B \times B$ is the diagonal map. An H-space Z is called grouplike if its multiplication is homotopy associative and it admits a homotopy inverse. In this case, the induced operation makes $[B, Z]$ a group. Thus, the results involving only the addition of homotopy classes require Z to be an H-space, whereas the results involving inverses or arbitrary integer multiples require Z to be a grouplike H-space. We refer to [14] for the standard terminology and basic properties of H-spaces and grouplike

H-spaces. A based space C is called a co-H-space if it admits a comultiplication $C \to C \vee C$ whose two coordinate maps are homotopic to 1_C. Every suspension is a co-H-space. It is well known that

$$DG(C, Z) = [C, Z],$$

for every based space Z; see [7]. Consequently, for every based map $r : C \to A$,

$$r^{\top}(C, Z) = [C, Z].$$

By the universal property of wedge sums, there is a bijection

$$\tau : [X \vee Y, Z] \to [X, Z] \times [Y, Z]$$

defined by

$$\tau(\beta) = (\beta \circ j_X, \beta \circ j_Y).$$

The map τ is compatible with the binary operations induced by m_Z. More precisely, for $\beta, \gamma \in [X \vee Y, Z]$,

$$\tau(\beta + \gamma) = (\beta \circ j_X + \gamma \circ j_X,\, \beta \circ j_Y + \gamma \circ j_Y).$$

Hence, if Z is a grouplike H-space, then τ is an isomorphism of groups, where the operation on $[X, Z] \times [Y, Z]$ is defined coordinatewise.

By [14, Corollary 1.3.7], there is a bijection

$$\theta : [B, X] \times [B, Y] \longrightarrow [B, X \times Y]$$

defined by

$$\theta(f, g) = (f \times g) \circ \Delta_B,$$

where $\Delta_B : B \to B \times B$ denotes the diagonal map.

In the definition of μ below, we use the same notation θ for the bijection

$$\theta : [X \times Y, Z] \times [X \times Y, Z] \longrightarrow [X \times Y, Z \times Z].$$

For $\alpha \in [X \times Y, Z]$, we have

$$(\tau \circ i^{\sharp})(\alpha) = (\alpha \circ i \circ j_X, \alpha \circ i \circ j_Y).$$

Since $i \circ j_K = i_K$ for $K = X, Y$, it follows that

$$(\tau \circ i^{\sharp})(\alpha) = (\alpha \circ i_X, \alpha \circ i_Y).$$

For $K = X, Y$, the projection map $\pi_K : X \times Y \to K$ induces a map

$$\pi_K^{\sharp} : [K, Z] \to [X \times Y, Z]$$

defined by

$$\pi_K^{\sharp}(\beta_K) = \beta_K \circ \pi_K.$$

Hence, there is a map

$$\pi_X^\sharp \times \pi_Y^\sharp : [X, Z] \times [Y, Z] \to [X \times Y, Z] \times [X \times Y, Z].$$

The multiplication m_Z induces the map

$$m_{Z_\sharp} : [X \times Y, Z \times Z] \to [X \times Y, Z]$$

defined by

$$m_{Z_\sharp}(\gamma) = m_Z \circ \gamma.$$

Define

$$\mu = m_{Z_\sharp} \circ \theta \circ (\pi_X^\sharp \times \pi_Y^\sharp) : [X, Z] \times [Y, Z] \to [X \times Y, Z].$$

Explicitly, for $(\beta_X, \beta_Y) \in [X, Z] \times [Y, Z]$,

$$\mu(\beta_X, \beta_Y) = (\beta_X \circ \pi_X) + (\beta_Y \circ \pi_Y).$$

We summarize the above maps in the following diagram.

$$\begin{tikzcd} {[X \times Y, Z]} \arrow[r, "i^\sharp"] & {[X \vee Y, Z]} \arrow[d, "\tau"] \\ & {[X, Z] \times [Y, Z]} \arrow[ul, "\mu"] \end{tikzcd} .$$

3. Separable maps on product spaces

We begin with the following computations.

For $\alpha \in [X \times Y, Z]$, we have

$$\begin{aligned}
(\mu \circ \tau \circ i^\sharp)(\alpha) &= \mu(\tau(\alpha \circ i)) \\
&= \mu(\alpha \circ i \circ j_X, \alpha \circ i \circ j_Y) \\
&= \mu(\alpha \circ i_X, \alpha \circ i_Y) \\
&= (m_{Z_\sharp} \circ \theta \circ (\pi_X^\sharp \times \pi_Y^\sharp))(\alpha \circ i_X, \alpha \circ i_Y) \\
&= m_{Z_\sharp} \circ \theta((\alpha \circ i_X \circ \pi_X), (\alpha \circ i_Y \circ \pi_Y)) \\
&= m_{Z_\sharp}\Big(((\alpha \circ i_X \circ \pi_X) \times (\alpha \circ i_Y \circ \pi_Y)) \circ \Delta_{X \times Y}\Big) \\
&= m_Z((\alpha \circ i_X \circ \pi_X) \times (\alpha \circ i_Y \circ \pi_Y)) \circ \Delta_{X \times Y} \\
&= (\alpha \circ i_X \circ \pi_X) + (\alpha \circ i_Y \circ \pi_Y).
\end{aligned}$$

Next, let $\beta \in [X \vee Y, Z]$. Then,

$$\begin{aligned}
(i^\sharp \circ \mu \circ \tau)(\beta) &= i^\sharp(\mu(\beta \circ j_X, \beta \circ j_Y)) \\
&= i^\sharp\Big((m_{Z_\sharp} \circ \theta \circ (\pi_X^\sharp \times \pi_Y^\sharp))(\beta \circ j_X, \beta \circ j_Y)\Big) \\
&= i^\sharp\Big(m_{Z_\sharp} \circ \theta(\beta \circ j_X \circ \pi_X, \beta \circ j_Y \circ \pi_Y)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= i^\sharp\Big(m_{Z_\sharp}\Big(((\beta \circ j_X \circ \pi_X) \times (\beta \circ j_Y \circ \pi_Y)) \circ \Delta_{X\times Y}\Big)\Big)\\
&= i^\sharp\Big(m_Z \circ ((\beta \circ j_X \circ \pi_X) \times (\beta \circ j_Y \circ \pi_Y)) \circ \Delta_{X\times Y}\Big)\\
&= i^\sharp\Big((\beta \circ j_X \circ \pi_X) + (\beta \circ j_Y \circ \pi_Y)\Big)\\
&= \Big((\beta \circ j_X \circ \pi_X) + (\beta \circ j_Y \circ \pi_Y)\Big) \circ i\\
&= (\beta \circ j_X \circ \pi_X \circ i) + (\beta \circ j_Y \circ \pi_Y \circ i).
\end{aligned}
$$

Proposition 1. *Let Z be an H-space. Then,*

$$i^\sharp \circ \mu \circ \tau = 1_{[X\vee Y,Z]}.$$

Consequently,

$$i^\sharp : [X \times Y, Z] \to [X \vee Y, Z]$$

is surjective.

Proof. Let $\beta \in [X \vee Y, Z]$. Then,

$$(i^\sharp \circ \mu \circ \tau)(\beta) = (\beta \circ j_X \circ \pi_X \circ i) + (\beta \circ j_Y \circ \pi_Y \circ i).$$

Since

$$\pi_X \circ i \circ j_X = 1_X, \quad \pi_Y \circ i \circ j_Y = 1_Y,$$

and

$$\pi_Y \circ i \circ j_X = *, \quad \pi_X \circ i \circ j_Y = *,$$

applying τ, we obtain

$$
\begin{aligned}
&\tau((i^\sharp \circ \mu \circ \tau)(\beta))\\
=&\Big((\beta \circ j_X \circ \pi_X \circ i \circ j_X) + (\beta \circ j_Y \circ \pi_Y \circ i \circ j_X), (\beta \circ j_X \circ \pi_X \circ i \circ j_Y) + (\beta \circ j_Y \circ \pi_Y \circ i \circ j_Y)\Big)\\
=&(\beta \circ j_X + *, * + \beta \circ j_Y)\\
=&(\beta \circ j_X, \beta \circ j_Y)\\
=&\tau(\beta).
\end{aligned}
$$

Since τ is injective,

$$(i^\sharp \circ \mu \circ \tau)(\beta) = \beta.$$

Hence,

$$i^\sharp \circ \mu \circ \tau = 1_{[X\vee Y,Z]}.$$

Therefore, $i^\sharp$ is surjective. □

Definition 1. *Let Z be an H-space. An element $\alpha \in [X \times Y, Z]$ is said to be separable if*

$$\alpha = (\alpha \circ i_X \circ \pi_X) + (\alpha \circ i_Y \circ \pi_Y).$$

We say that $[X \times Y, Z]$ satisfies the separability condition if all of its elements are separable.

Theorem 1. *Let Z be an H-space. Then, the map*

$$i^\sharp : [X \times Y, Z] \to [X \vee Y, Z]$$

is a bijection if and only if $[X \times Y, Z]$ satisfies the separability condition.

Proof. Suppose that $[X \times Y, Z]$ satisfies the separability condition. By Proposition 1, $i^\sharp$ is surjective. To prove injectivity, let $\alpha, \alpha' \in [X \times Y, Z]$ and suppose that

$$i^\sharp(\alpha) = i^\sharp(\alpha').$$

Then,

$$\alpha \circ i = \alpha' \circ i.$$

Hence,

$$\alpha \circ i_K = \alpha' \circ i_K,$$

for $K = X, Y$.

Since $[X \times Y, Z]$ satisfies the separability condition,

$$\alpha = (\alpha \circ i_X \circ \pi_X) + (\alpha \circ i_Y \circ \pi_Y) = (\alpha' \circ i_X \circ \pi_X) + (\alpha' \circ i_Y \circ \pi_Y) = \alpha'.$$

Thus, $i^\sharp$ is injective. Therefore, $i^\sharp : [X \times Y, Z] \to [X \vee Y, Z]$ is a bijection.

Conversely, suppose that $i^\sharp : [X \times Y, Z] \to [X \vee Y, Z]$ is a bijection.

Let $\alpha \in [X \times Y, Z]$. By the computation at the beginning of this section,

$$(\mu \circ \tau \circ i^\sharp)(\alpha) = (\alpha \circ i_X \circ \pi_X) + (\alpha \circ i_Y \circ \pi_Y).$$

Applying $i^\sharp$ to the equality above, we compute the left-hand side as follows:

$$i^\sharp((\mu \circ \tau \circ i^\sharp)(\alpha)) = (i^\sharp \circ \mu \circ \tau \circ i^\sharp)(\alpha) = (i^\sharp \circ \mu \circ \tau)(i^\sharp(\alpha)) = i^\sharp(\alpha),$$

where the last equality follows from Proposition 1.

On the other hand, the right-hand side gives

$$i^\sharp\Big((\alpha \circ i_X \circ \pi_X) + (\alpha \circ i_Y \circ \pi_Y)\Big).$$

Hence,

$$i^\sharp(\alpha) = i^\sharp\Big((\alpha \circ i_X \circ \pi_X) + (\alpha \circ i_Y \circ \pi_Y)\Big).$$

Since $i^\sharp$ is injective, it follows that

$$\alpha = (\alpha \circ i_X \circ \pi_X) + (\alpha \circ i_Y \circ \pi_Y).$$

Thus α is separable. Since α was arbitrary, $[X \times Y, Z]$ satisfies the separability condition. □

Corollary 1. *Let Z be a grouplike H-space. Then, the following are equivalent.*

(1) The map $i^\sharp : [X \times Y, Z] \to [X \vee Y, Z]$ is an isomorphism.
(2) $[X \times Y, Z]$ satisfies the separability condition.

(3) $[X \wedge Y, Z] = 0$.

Proof. By Theorem 1, (1) and (2) are equivalent.

Since Z is a grouplike H-space, the standard mapping-cone argument yields a short exact sequence of groups

$$0 \longrightarrow [X \wedge Y, Z] \xrightarrow{q^\sharp} [X \times Y, Z] \xrightarrow{i^\sharp} [X \vee Y, Z] \longrightarrow 0.$$

Hence, $i^\sharp$ is an isomorphism if and only if $[X \wedge Y, Z] = 0$.

Thus, (1) and (3) are equivalent. □

We next consider the failure of the separability condition. Assume that Z is a grouplike H-space. Define

$$\sigma = \mu \circ \tau \circ i^\sharp : [X \times Y, Z] \longrightarrow [X \times Y, Z].$$

Thus, for every $\alpha \in [X \times Y, Z]$, $\sigma(\alpha) = (\alpha \circ i_X \circ \pi_X) + (\alpha \circ i_Y \circ \pi_Y)$.

Define the smash-product obstruction of α by

$$\omega(\alpha) = -\sigma(\alpha) + \alpha.$$

Proposition 2. *For every $\alpha \in [X \times Y, Z]$, the following statements hold.*

(1) $\sigma(\alpha)$ is separable.
(2) $\omega(\alpha) \in \ker i^\sharp = \operatorname{Im} q^\sharp$.
(3) $\alpha = \sigma(\alpha) + \omega(\alpha)$.
(4) The element α is separable if and only if $\omega(\alpha) = 0$.

Proof. By Proposition 1,

$$i^\sharp \circ \mu \circ \tau = 1_{[X \vee Y, Z]}.$$

Therefore,

$$\sigma(\sigma(\alpha)) = (\mu \circ \tau \circ i^\sharp)((\mu \circ \tau \circ i^\sharp)(\alpha)) = \mu \circ \tau\Big((i^\sharp \circ \mu \circ \tau)(i^\sharp(\alpha))\Big) = \mu \circ \tau(i^\sharp(\alpha)) = \sigma(\alpha).$$

Since an element $\gamma \in [X \times Y, Z]$ is separable precisely when $\sigma(\gamma) = \gamma$, it follows that $\sigma(\alpha)$ is separable.

Moreover,

$$i^\sharp(\omega(\alpha)) = i^\sharp(-\sigma(\alpha) + \alpha) = -i^\sharp(\sigma(\alpha)) + i^\sharp(\alpha) = -i^\sharp(\alpha) + i^\sharp(\alpha) = 0.$$

Hence, $\omega(\alpha) \in \ker i^\sharp$.

By exactness of $[X \wedge Y, Z] \xrightarrow{q^\sharp} [X \times Y, Z] \xrightarrow{i^\sharp} [X \vee Y, Z]$, we have $\ker i^\sharp = \operatorname{Im} q^\sharp$. Thus $\omega(\alpha) \in \operatorname{Im} q^\sharp$.

Since $[X \times Y, Z]$ is a group, $\sigma(\alpha) + \omega(\alpha) = \sigma(\alpha) + (-\sigma(\alpha) + \alpha) = \alpha$.

Finally, α is separable if and only if $\sigma(\alpha) = \alpha$, which is equivalent to $-\sigma(\alpha) + \alpha = 0$.

Therefore, α is separable if and only if $\omega(\alpha) = 0$. □

Remark 1. *The obstruction $\omega(\alpha)$ measures the part of α that cannot be recovered from its restrictions to the two factors. In particular, separability may fail only through a class arising from the smash-product term*

$$[X \wedge Y, Z].$$

4. Splitting of generalized co-Gottlieb sets

Let $p : X \times Y \to A$ be a map. Put $p_X = p \circ i_X$ and $p_Y = p \circ i_Y$. Recall that $i_{A,Z} : A \vee Z \to A \times Z$ denotes the canonical inclusion. Throughout this section, let Z be an H-space.

Lemma 1. *If $\alpha \in p^\top(X \times Y, Z)$, then $\alpha \circ i_X \in p_X^\top(X, Z)$ and $\alpha \circ i_Y \in p_Y^\top(Y, Z)$.*

Proof. Suppose that $\alpha \in p^\top(X \times Y, Z)$. Then, there exists a map $L : X \times Y \to A \vee Z$ such that

$$i_{A,Z} \circ L \simeq (p \times \alpha) \circ \Delta_{X \times Y}.$$

By the naturality of diagonal maps, $\Delta_{X\times Y} \circ i_X = (i_X \times i_X) \circ \Delta_X$.
Hence,

$$(p \times \alpha) \circ \Delta_{X\times Y} \circ i_X = (p \times \alpha) \circ (i_X \times i_X) \circ \Delta_X = ((p \circ i_X) \times (\alpha \circ i_X)) \circ \Delta_X = (p_X \times (\alpha \circ i_X)) \circ \Delta_X.$$

Therefore,

$$i_{A,Z} \circ L \circ i_X \simeq (p_X \times (\alpha \circ i_X)) \circ \Delta_X.$$

Thus $\alpha \circ i_X \in p_X^\top(X, Z)$. Similarly, $\alpha \circ i_Y \in p_Y^\top(Y, Z)$. □

Therefore, there is a map

$$\Phi : p^\top(X \times Y, Z) \longrightarrow p_X^\top(X, Z) \times p_Y^\top(Y, Z)$$

defined by

$$\Phi(\alpha) = (\alpha \circ i_X, \alpha \circ i_Y).$$

Proposition 3. *Suppose that $[X \times Y, Z]$ satisfies the separability condition. Then, the map*

$$\Phi : p^\top(X \times Y, Z) \to p_X^\top(X, Z) \times p_Y^\top(Y, Z)$$

is injective.

Proof. Let $\alpha, \beta \in p^\top(X \times Y, Z)$ and suppose that

$$\Phi(\alpha) = \Phi(\beta).$$

Then,

$$\alpha \circ i_X \simeq \beta \circ i_X \quad \text{and} \quad \alpha \circ i_Y \simeq \beta \circ i_Y.$$

Since $[X \times Y, Z]$ satisfies the separability condition,

$$\alpha = (\alpha \circ i_X \circ \pi_X) + (\alpha \circ i_Y \circ \pi_Y) \simeq (\beta \circ i_X \circ \pi_X) + (\beta \circ i_Y \circ \pi_Y) = \beta.$$

Hence, $\alpha \simeq \beta$. Therefore, Φ is injective. □

Lemma 2. *The restriction of $i^\sharp$ defines a map*

$$\widetilde{i^\sharp} : p^\top(X \times Y, Z) \longrightarrow (p \circ i)^\top(X \vee Y, Z)$$

given by $\widetilde{i^\sharp}(\alpha) = \alpha \circ i$.

Proof. Let $\alpha \in p^{\top}(X \times Y, Z)$. Then, there exists a map $L : X \times Y \longrightarrow A \vee Z$ such that

$$i_{A,Z} \circ L \simeq (p \times \alpha) \circ \Delta_{X \times Y}.$$

By the naturality of diagonal maps, $\Delta_{X\times Y} \circ i = (i \times i) \circ \Delta_{X\vee Y}$.
Therefore,

$$i_{A,Z} \circ L \circ i \simeq (p \times \alpha) \circ \Delta_{X\times Y} \circ i = (p \times \alpha) \circ (i \times i) \circ \Delta_{X\vee Y} = ((p \circ i) \times (\alpha \circ i)) \circ \Delta_{X\vee Y}.$$

Hence, $\alpha \circ i \in (p \circ i)^{\top}(X \vee Y, Z)$. Thus, $\widetilde{i^{\sharp}}$ is well-defined. □

By [12, Lemma 2.17], the map τ restricts to a bijection

$$\tau : (p \circ i)^{\top}(X \vee Y, Z) \longrightarrow p_X^{\top}(X, Z) \times p_Y^{\top}(Y, Z).$$

Moreover, $\Phi = \tau \circ \widetilde{i^{\sharp}}$.

Indeed, for every $\alpha \in p^{\top}(X \times Y, Z)$, $\tau(\widetilde{i^{\sharp}}(\alpha)) = (\alpha \circ i_X, \alpha \circ i_Y) = \Phi(\alpha)$.

Thus, $\widetilde{i^{\sharp}}$ records the restriction to the wedge, whereas Φ records the two coordinate restrictions separately.

For a grouplike H-space Z, define the difference set

$$\mathcal{D}_p(X \times Y, Z) = \left\{-\alpha + \beta \;\middle|\; \alpha, \beta \in p^{\top}(X \times Y, Z)\right\}.$$

Proposition 4. *Let Z be a grouplike H-space. Then,*

$$\widetilde{i^{\sharp}} : p^{\top}(X \times Y, Z) \longrightarrow (p \circ i)^{\top}(X \vee Y, Z)$$

is injective if and only if

$$\mathcal{D}_p(X \times Y, Z) \cap \operatorname{Im} q^{\sharp} = \{0\}.$$

Proof. Suppose first that $\widetilde{i^{\sharp}}$ is injective. Let $\delta \in \mathcal{D}_p(X \times Y, Z) \cap \operatorname{Im} q^{\sharp}$.

Then, $\delta = -\alpha + \beta$ for some $\alpha, \beta \in p^{\top}(X \times Y, Z)$.

Since $\delta \in \operatorname{Im} q^{\sharp} = \ker i^{\sharp}$, we have $0 = i^{\sharp}(\delta) = -i^{\sharp}(\alpha) + i^{\sharp}(\beta)$.

Hence,

$$\widetilde{i^{\sharp}}(\alpha) = \widetilde{i^{\sharp}}(\beta).$$

By injectivity, $\alpha = \beta$, and therefore $\delta = 0$.

Since the constant homotopy class belongs to $p^{\top}(X \times Y, Z)$, the zero element belongs to the intersection. Thus,

$$\mathcal{D}_p(X \times Y, Z) \cap \operatorname{Im} q^{\sharp} = \{0\}.$$

Conversely, suppose that $\mathcal{D}_p(X \times Y, Z) \cap \operatorname{Im} q^{\sharp} = \{0\}$.

Let $\alpha, \beta \in p^{\top}(X \times Y, Z)$ satisfy $\widetilde{i^{\sharp}}(\alpha) = \widetilde{i^{\sharp}}(\beta)$. Then, $i^{\sharp}(-\alpha + \beta) = 0$.

By exactness,

$$-\alpha + \beta \in \ker i^{\sharp} = \operatorname{Im} q^{\sharp}.$$

By definition,

$$-\alpha + \beta \in \mathcal{D}_p(X \times Y, Z).$$

Hence, $-\alpha + \beta = 0$. Therefore, $\alpha = \beta$, and so $\widetilde{i^{\sharp}}$ is injective. □

Remark 2. *Proposition 4 does not require*

$$[X \wedge Y, Z] = 0.$$

Indeed, the image of $q^\sharp$ *may be nontrivial. The proposition requires only that no nonzero element of* $\operatorname{Im} q^\sharp$ *occur as the difference of two elements of* $p^\top(X \times Y, Z)$*. Thus it gives an injectivity criterion even when the separability condition fails.*

Example 1. *Let*

$$X = Y = S^1, \qquad Z = K(\mathbb{Z}, 2),$$

and let

$$p : S^1 \times S^1 \longrightarrow *$$

be the constant map. Then,

$$p^\top(S^1 \times S^1, Z) = [S^1 \times S^1, Z] \cong H^2(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z},$$

whereas

$$(p \circ i)^\top(S^1 \vee S^1, Z) = [S^1 \vee S^1, Z] \cong H^2(S^1 \vee S^1; \mathbb{Z}) = 0.$$

Hence,

$$\widetilde{i^\sharp} : p^\top(S^1 \times S^1, Z) \longrightarrow (p \circ i)^\top(S^1 \vee S^1, Z)$$

is the zero map $\mathbb{Z} \to 0$ *and is not injective.*

Moreover, since $S^1 \wedge S^1 \simeq S^2$*, the quotient map* $q : S^1 \times S^1 \to S^2$ *induces an isomorphism*

$$q^\sharp : [S^2, K(\mathbb{Z}, 2)] \longrightarrow [S^1 \times S^1, K(\mathbb{Z}, 2)].$$

Thus,

$$\operatorname{Im} q^\sharp = \mathcal{D}_p(S^1 \times S^1, Z) = [S^1 \times S^1, Z] \cong \mathbb{Z}.$$

Consequently,

$$\mathcal{D}_p(S^1 \times S^1, Z) \cap \operatorname{Im} q^\sharp = [S^1 \times S^1, Z] \cong \mathbb{Z},$$

which is nontrivial.

Therefore, the nontrivial smash-product term is precisely the obstruction to the injectivity of $\widetilde{i^\sharp}$ *in this example.*

Proposition 5. *Suppose that* $[X \times Y, Z]$ *satisfies the separability condition. Then, the map*

$$\widetilde{i^\sharp} : p^\top(X \times Y, Z) \to (p \circ i)^\top(X \vee Y, Z)$$

is injective.

Proof. By Theorem 1, the map

$$i^\sharp : [X \times Y, Z] \longrightarrow [X \vee Y, Z]$$

is injective. Since $\widetilde{i^\sharp}$ is the restriction of $i^\sharp$ to $p^\top(X \times Y, Z)$, it is also injective. □

Example 2. *Let $X = S^1$, $Y = S^2$, and $Z = S^1$.*

Let $q : S^1 \times S^2 \longrightarrow S^1 \wedge S^2 \simeq S^3$ be the quotient map.

Since $S^1 \wedge S^2 \simeq S^3$ and $[S^3, S^1] = \pi_3(S^1) = 0$, $[S^1 \times S^2, S^1]$ satisfies the separability condition.

The quotient map q collapses the subspace $S^1 \vee S^2$ to the basepoint. Hence,

$$q \circ i \simeq *.$$

Since every map is cocyclic with respect to a null map,

$$(q \circ i)^{\top}(S^1 \vee S^2, S^1) = *^{\top}(S^1 \vee S^2, S^1) = [S^1 \vee S^2, S^1].$$

Since $[S^1 \vee S^2, S^1] \cong [S^1, S^1] \times [S^2, S^1] \cong \mathbb{Z} \times 0 \cong \mathbb{Z}$, we obtain

$$(q \circ i)^{\top}(S^1 \vee S^2, S^1) \cong \mathbb{Z}.$$

By Proposition 5, the restricted map

$$\widetilde{i^{\sharp}} : q^{\top}(S^1 \times S^2, S^1) \longrightarrow (q \circ i)^{\top}(S^1 \vee S^2, S^1) \cong \mathbb{Z}$$

is injective.

Example 3. *Let $Z = S^7$, and let $p : S^4 \times S^7 \to A$ be any map.*

Since $S^4 \wedge S^7 \simeq S^{11}$ and $[S^{11}, S^7] = \pi_{11}(S^7) = 0$ by [15], $[S^4 \times S^7, S^7]$ satisfies the separability condition.

By [12, Lemma 2.17],

$$(p \circ i)^{\top}(S^4 \vee S^7, S^7) \cong p_X^{\top}(S^4, S^7) \times p_Y^{\top}(S^7, S^7).$$

Since every sphere of positive dimension is a suspension, both S^4 and S^7 are co-H-spaces. Hence, the standard co-H-space property recalled above gives

$$p_X^{\top}(S^4, S^7) = [S^4, S^7] \quad \text{and} \quad p_Y^{\top}(S^7, S^7) = [S^7, S^7].$$

Since

$$[S^4, S^7] = 0 \quad \text{and} \quad [S^7, S^7] \cong \mathbb{Z},$$

it follows that

$$(p \circ i)^{\top}(S^4 \vee S^7, S^7) \cong \mathbb{Z}.$$

By Proposition 5, the restricted map

$$\widetilde{i^{\sharp}} : p^{\top}(S^4 \times S^7, S^7) \longrightarrow (p \circ i)^{\top}(S^4 \vee S^7, S^7) \cong \mathbb{Z}$$

is injective.

Lemma 3. *Let G be a finite cyclic group of order ℓ^r, where ℓ is a prime and $r \geq 1$. Suppose that S is a nonempty subset of G such that $kx \in S$ for every $x \in S$ and every $k \in \mathbb{Z}$. Then, S is a subgroup of G.*

Proof. Since G is finite, there exists an element $x_0 \in S$ having maximal order among the elements of S. The subgroups of a cyclic group of prime-power order are linearly ordered by inclusion.

Therefore, $\langle x \rangle \subseteq \langle x_0 \rangle$ for every $x \in S$. Hence, $S \subseteq \langle x_0 \rangle$.

On the other hand, since S is closed under multiplication by integers, $\langle x_0 \rangle \subseteq S$.

Thus, $S = \langle x_0 \rangle$, and hence S is a subgroup of G. □

Theorem 2. *Let Z be a grouplike H-space. Suppose that $[X \times Y, Z]$ satisfies the separability condition. If $(p \circ i)^\top(X \vee Y, Z)$ is a finite cyclic group of prime-power order, then $p^\top(X \times Y, Z)$ is a subgroup of $[X \times Y, Z]$, and hence is a group.*

Proof. By Theorem 1, the map $i^\sharp : [X \times Y, Z] \longrightarrow [X \vee Y, Z]$ is a bijection.

Since Z is a grouplike H-space and $i^\sharp$ preserves the induced group operations, $i^\sharp$ is an isomorphism of groups.

Put

$$S = \widetilde{i^\sharp}(p^\top(X \times Y, Z)).$$

By Lemma 2,

$$S \subseteq (p \circ i)^\top(X \vee Y, Z).$$

By [13, Proposition 2.1 and Theorem 2.3], the constant homotopy class belongs to $p^\top(X \times Y, Z)$ and

$$k\alpha \in p^\top(X \times Y, Z),$$

for every $\alpha \in p^\top(X \times Y, Z)$ and every $k \in \mathbb{Z}$. Since $i^\sharp$ is a homomorphism, S is nonempty and closed under multiplication by integers.

By Lemma 3, S is a subgroup of $(p \circ i)^\top(X \vee Y, Z)$.

We now show that

$$p^\top(X \times Y, Z) = (i^\sharp)^{-1}(S).$$

One inclusion is immediate. Conversely, suppose that $\beta \in [X \times Y, Z]$ and $i^\sharp(\beta) \in S$. Then, there exists $\alpha \in p^\top(X \times Y, Z)$ such that $i^\sharp(\beta) = i^\sharp(\alpha)$.

Since $i^\sharp$ is injective, $\beta = \alpha$, and hence $\beta \in p^\top(X \times Y, Z)$.

Therefore, $p^\top(X \times Y, Z)$ is the inverse image of the subgroup S under the group isomorphism $i^\sharp$. Hence, $p^\top(X \times Y, Z)$ is a subgroup of $[X \times Y, Z]$. □

Following Choi, Kim, and Oda [13], we make the following definition.

Definition 2. *A map $p : X \to A$ is called essential with respect to Z if $p^\top(X, Z)$ is a group.*

Corollary 2. *Let Z be a grouplike H-space. Suppose that $[X \times Y, Z]$ satisfies the separability condition. If $(p \circ i)^\top(X \vee Y, Z)$ is a finite cyclic group of prime-power order, then p is essential with respect to Z.*

Proof. This follows immediately from Theorem 2. □

Example 4. *Let ℓ be a prime, and let $n, m \geq 1$ with $n \neq m$. Put $Z = K(\mathbb{Z}/\ell, n)$, and let*

$$p : S^n \times S^m \longrightarrow A$$

be any based map. The space Z is a grouplike H-space.

Moreover, $S^n \wedge S^m \simeq S^{n+m}$*, and hence* $[S^n \wedge S^m, Z] \cong [S^{n+m}, K(\mathbb{Z}/\ell, n)] = 0$.

Therefore, by Corollary 1, $[S^n \times S^m, Z]$ *satisfies the separability condition.*

Let $i : S^n \vee S^m \longrightarrow S^n \times S^m$ *be the canonical inclusion. Since* $S^n \vee S^m$ *is a co-H-space, the standard co-H-space property recalled above gives*

$$(p \circ i)^{\top}(S^n \vee S^m, Z) = [S^n \vee S^m, Z].$$

By the universal property of the wedge,

$$[S^n \vee S^m, K(\mathbb{Z}/\ell, n)] \cong [S^n, K(\mathbb{Z}/\ell, n)] \times [S^m, K(\mathbb{Z}/\ell, n)] \cong \mathbb{Z}/\ell \times 0 \cong \mathbb{Z}/\ell,$$

where the second isomorphism follows from $n \neq m$.

Thus, $(p \circ i)^{\top}(S^n \vee S^m, Z) \cong \mathbb{Z}/\ell$*, which is a finite cyclic group of prime-power order (in fact, of prime order). Therefore, by Theorem 2,*

$$p^{\top}(S^n \times S^m, K(\mathbb{Z}/\ell, n))$$

is a group. Hence, every based map $p : S^n \times S^m \longrightarrow A$ *is essential with respect to* $K(\mathbb{Z}/\ell, n)$.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest.

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