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Idempotent-prime ideals and Peirce decompositions in noncommutative rings

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Abstract: We study idempotent-prime ideals in associative rings with identity. A proper two-sided ideal P of R is idempotent-prime if $aR(1 - a) \subseteq P$ implies $a \in P$ or $1 - a \in P$, for every $a \in R$. For $A = R/P$, the main quotient criterion shows that this condition is equivalent to the absence of nontrivial idempotents $e \in A$ satisfying $eA(1 - e) = 0$. Thus, idempotent-primeness rules out nontrivial triangular Peirce decompositions in the quotient. In the commutative case, it is equivalent to connectedness of R/P, or equivalently of $V(P) \subseteq \operatorname{Spec} R$. After developing the quotient criterion and its Peirce consequences, we treat the functorial behavior, comaximal ideals, Jacobson-radical lifting, local and semisimple Artinian quotients, torsion-theoretic and avoidance interpretations, finite direct products, matrix rings, triangular matrix rings, topological descriptions, and graph-theoretic criteria.

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1. Introduction

Let R be a commutative ring with $1 \neq 0$. In [1], the authors introduced the notion of an *idempotent-prime ideal*: a proper ideal P of R is idempotent-prime if $a(1 - a) \in P$ implies $a \in P$ or $1 - a \in P$, for every $a \in R$. Since $a(1-a) \in P$ means exactly that the image of a in R/P is an idempotent, this condition is naturally connected with the absence of nontrivial idempotents in the quotient. Equivalently, in the commutative setting, idempotent-primeness is the same as connectedness of R/P, or of $V(P) \subseteq \operatorname{Spec} R$.

This paper develops a noncommutative version of that idea. Since the single product $a(1 - a)$ does not capture two-sided behavior in a noncommutative ring, we replace it by the sandwich condition $aR(1 - a) \subseteq P$. This leads to a left-right symmetric definition and to a quotient criterion in terms of

Peirce decompositions. If $A = R/P$, then P is idempotent-prime if and only if A has no nontrivial idempotent e for which the Peirce component $eA(1-e)$ vanishes. Equivalently, by symmetry, no nontrivial idempotent satisfies $(1-e)Ae = 0$.

The results are arranged as follows: Section 2 gives the definition, the quotient criterion, and the Peirce-theoretic consequences, including the semiprime case; Section 3 studies the ideal-theoretic behavior: homomorphic images, quotient rings, comaximal ideals, Jacobson-radical lifting, local and semisimple Artinian quotients, torsion-theoretic interpretations, and avoidance phenomena; Section 4 treats standard constructions such as finite direct products, matrix rings, and triangular matrix rings; and Section 5 gives topological and graph-theoretic descriptions, with the commutative case phrased in terms of connectedness and the noncommutative case treated via the two-sided prime spectrum and Peirce digraphs. Section 6 summarizes the main conclusions.

2. Foundations and quotient criteria

This section introduces the noncommutative definition and translates it into a condition on the quotient ring. The quotient form is the central tool of the paper: It shows that idempotent-primeness is the exclusion of one-sided triangular Peirce decompositions.

Throughout, rings are associative with identity $1 \neq 0$, and ring homomorphisms are unital. Unless otherwise stated, an ideal means a two-sided ideal. The word *proper* is explicitly stated when it is required.

The following definition is used throughout the paper.

Definition 2.1. Let R be a ring, and let $P \lhd R$ be a proper ideal. We say that P is an *idempotent-prime ideal* if, for every $a \in R$, $aR(1-a) \subseteq P$ implies $a \in P$ or $1-a \in P$.

Remark 2.2. We record three elementary consequences of Definition 2.1.

(1) The terminology comes from the quotient $A = R/P$. Indeed, Definition 2.1 is equivalent to saying that every idempotent $e \in A$ satisfying $eA(1-e) = 0$ is trivial, that is, $e = 0$ or $e = 1$. In the reviewer's terminology, this means that A is *anti-semiabelian*. Thus, P is idempotent-prime if and only if R/P is anti-semiabelian. In particular, if R/P is prime, then P is idempotent-prime. In the commutative case, this is equivalent to the connectedness of R/P; see Section 5.

(2) Definition 2.1 is equivalent to the form $aR(a+1) \subseteq P \Rightarrow a \in P$ or $a+1 \in P$. Indeed, replacing a by $-a$, we have $(-a)R(1+a) \subseteq P$ if and only if $aR(a+1) \subseteq P$, and the conclusion $-a \in P$ or $1+a \in P$ is equivalent to $a \in P$ or $a+1 \in P$. We shall use these equivalent formulations interchangeably.

(3) The definition is left-right symmetric. More precisely, for a proper ideal $P \lhd R$, the following two conditions are equivalent:

   (a) For every $a \in R$, $aR(1-a) \subseteq P$ implies $a \in P$ or $1-a \in P$;

   (b) For every $a \in R$, $(1-a)Ra \subseteq P$ implies $a \in P$ or $1-a \in P$.

   Indeed, applying the first condition to $b = 1-a$ gives $bR(1-b) = (1-a)Ra$. The conclusion $b \in P$ or $1-b \in P$ is exactly $1-a \in P$ or $a \in P$. The converse is identical.

The next proposition compares the new condition with ordinary primeness.

Proposition 2.3. *Every prime ideal of R is idempotent-prime.*

Proof. Let $P \lhd R$ be prime, and suppose $aR(a+1) \subseteq P$. By the elementwise characterization of prime ideals in unital rings, either $a \in P$ or $a+1 \in P$. Thus, P is idempotent-prime. $\square$

The quotient criterion is the main elementary characterization of the definition.

Proposition 2.4. *Let $P \lhd R$ be a proper ideal, and put $A = R/P$. The following are equivalent:*

(1) P is idempotent-prime in R;

(2) The zero ideal is idempotent-prime in A;

(3) For every $x \in A$, $xA(1-x) = 0$ implies either $x = 0$ or $x = 1$; and

(4) A has no nontrivial idempotent e satisfying $eA(1-e) = 0$.

Proof. Let $\pi : R \to A$ be the canonical surjection. The equivalence of (1) and (2) follows directly from

$$\pi(aR(1-a)) = \pi(a)A(1-\pi(a)).$$

Statement (2) is exactly statement (3), written in the quotient ring.

Assume (3). If $e^2 = e$ and $eA(1-e) = 0$, then applying (3) to $x = e$ gives $e = 0$ or $e = 1$. Hence (4) holds.

Conversely, assume (4), and let $xA(1-x) = 0$. Since $1 \in A$, we get $x(1-x) = 0$, and hence $x^2 = x$. Thus, x is an idempotent satisfying $xA(1-x) = 0$. By (4), either $x = 0$ or $x = 1$. Hence, (3) holds. $\square$

The following example shows that the converse of Proposition 2.3 fails even for a familiar matrix ring.

Example 2.5. Let $R = M_2(\mathbb{Z})$, and let $P = M_2(4\mathbb{Z})$. Then, P is idempotent-prime, but P is not prime.

Put $A = R/P \cong M_2(\mathbb{Z}/4\mathbb{Z})$. We show that A has no nontrivial idempotent e that satisfies $eA(1-e) = 0$. Let $e = e^2 \in A$, and suppose that $eA(1-e) = 0$. Then, $(AeA)(A(1-e)A) = 0$. Now, $2A$ is a nilpotent ideal of A, and $A/2A \cong M_2(\mathbb{F}_2)$, which is a simple Artinian ring, hence prime. Passing to the quotient $A/2A$, the equality above gives $\overline{AeA}\,\overline{A(1-e)A} = 0$. Since $A/2A$ is prime, one of these two ideals is zero. Therefore, either $AeA \subseteq 2A$ or $A(1-e)A \subseteq 2A$.

If $AeA \subseteq 2A$, then $e \in 2A$. Hence, $e = 2f$ for some $f \in A$. Since e is idempotent, $e = e^2 = (2f)^2 = 4f^2 = 0$. Thus, $e = 0$.

If $A(1-e)A \subseteq 2A$, then $1-e \in 2A$. Hence, $1-e = 2g$ for some $g \in A$. Since $1-e$ is idempotent, $1-e = (1-e)^2 = (2g)^2 = 4g^2 = 0$. Thus, $e = 1$.

Therefore, every idempotent $e \in A$ satisfying $eA(1-e) = 0$ is either 0 or 1. By Proposition 2.4, P is idempotent-prime.

On the other hand, P is not prime. Indeed,

$$M_2(2\mathbb{Z}) \not\subseteq P, \qquad M_2(2\mathbb{Z})M_2(2\mathbb{Z}) \subseteq M_2(4\mathbb{Z}) = P.$$

Hence, P is idempotent-prime but not prime.

The left-right symmetry gives the corresponding criterion for the opposite Peirce component.

Corollary 2.6. *Let $P \lhd R$ be a proper ideal, and put $A = R/P$. Then, P is idempotent-prime if and only if A has no nontrivial idempotent e satisfying* $(1-e)Ae = 0$.

Proof. This follows from Remark 2.2(3) applied in the quotient ring, or equivalently from Proposition 2.4 applied to $1-e$. □

Remark 2.7. Let $A = R/P$. Proposition 2.4 gives a Peirce-theoretic interpretation of the idempotent-prime condition. If $e = e^2 \in A$, then e induces the Peirce decomposition

$$A = eAe \oplus eA(1-e) \oplus (1-e)Ae \oplus (1-e)A(1-e),$$

or equivalently the generalized matrix form

$$A = \begin{pmatrix} eAe & eA(1-e) \\ (1-e)Ae & (1-e)A(1-e) \end{pmatrix}.$$

Thus an idempotent-prime quotient has no nontrivial idempotent e for which $eA(1-e) = 0$. By Corollary 2.6, it also has no nontrivial idempotent e for which $(1-e)Ae = 0$. Hence, idempotent-primeness forbids a nontrivial triangular Peirce form.

The vanishing of one Peirce corner is stronger than Peirce triviality. Recall that e is inner Peirce trivial if $eA(1-e)Ae = 0$, outer Peirce trivial if $(1-e)AeA(1-e) = 0$, and Peirce trivial if both conditions hold. If $eA(1-e) = 0$, then $eA(1-e)Ae = 0$ and $(1-e)AeA(1-e) = (1-e)A(eA(1-e)) = 0$. Similarly, if $(1-e)Ae = 0$, then $(1-e)AeA(1-e) = 0$ and $eA(1-e)Ae = eA((1-e)Ae) = 0$. Thus, every triangular Peirce decomposition gives a Peirce-trivial idempotent, although the converse need not hold.

The first consequence is that an idempotent-prime quotient cannot split as a nontrivial direct product.

Corollary 2.8. *Let $P \lhd R$ be an idempotent-prime ideal, and put $A = R/P$. Then, A has no nontrivial central idempotents. Consequently, A is indecomposable as a direct product of two nonzero rings.*

Proof. If $e = e^2 \in Z(A)$, then $eA(1-e) = 0$. By Proposition 2.4, either $e = 0$ or $e = 1$. The final assertion follows from the standard correspondence between direct product decompositions of a unital ring and central idempotents. □

The next example shows why the absence of central idempotents is not enough in general.

Example 2.9. The converse of Corollary 2.8 is false. Let k be a field, and let $A = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$. The center of A consists of scalar diagonal matrices, so A has no nontrivial central idempotents.

Nevertheless, the zero ideal of A is not idempotent-prime. Indeed, let $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then, e is a nontrivial idempotent and, for every $X = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in A$, one has $eX(1-e) = 0$. Thus, $eA(1-e) = 0$, and Proposition 2.4 shows that (0) is not idempotent-prime.

Recall that a ring is called *Abelian* if every idempotent is central, and it is called *connected* if its only idempotents are 0 and 1.

When every idempotent is central, the quotient criterion reduces to connectedness.

Corollary 2.10. *Let $P \lhd R$ be a proper ideal, and put $A = R/P$. If A is Abelian, then P is idempotent-prime in R if and only if A is connected.*

Proof. Since A is Abelian, every idempotent $e \in A$ is central, and hence $eA(1 - e) = 0$. Now, Proposition 2.4 gives the result. □

We recall that a ring R is *semiperfect* if $R/\operatorname{Jac}(R)$ is semisimple Artinian and idempotents lift modulo $\operatorname{Jac}(R)$. For abelian semiperfect quotients, the preceding connectedness condition is equivalent to the quotient being local.

Corollary 2.11. *Let $P \lhd R$ be a proper ideal, and put $A = R/P$. Assume that A is Abelian and semiperfect. Then, the following are equivalent:*

(1) P is idempotent-prime in R;

(2) A is connected; and

(3) A is local.

Proof. The equivalence of (1) and (2) is Corollary 2.10. By [2, Proposition 2.6], an Abelian semiperfect ring is a finite direct product of connected local rings. Such a product is connected if and only if it has exactly one factor, and this is equivalent to being local. Hence, (2) and (3) are equivalent. □

Recall that a ring A is semiprime if it has no nonzero nilpotent ideals. Following [3, 4], let $\operatorname{Pt}(A)$ denote the set of Peirce trivial idempotents of A. Thus,

$$\operatorname{Pt}(A) = \{e = e^2 \in A \mid eA(1 - e)Ae = 0 = (1 - e)AeA(1 - e)\}.$$

We say that A is a 1-Peirce ring if $A \neq 0$ and $\operatorname{Pt}(A) = \{0, 1\}$. For $n > 1$, the notion of an n-Peirce ring is recursively defined as in [4].

For semiprime quotients, the quotient criterion becomes equivalent to several standard indecomposability conditions.

Theorem 2.12. *Let $P \lhd R$ be a proper ideal, and put $A = R/P$. Assume that A is semiprime. Then the following are equivalent:*

(1) P is idempotent-prime in R;

(2) A has no nontrivial central idempotents;

(3) A is indecomposable as a direct product of two nonzero rings; and

(4) A is a 1*-Peirce ring.*

Proof. The implication (1)⇒(2) is Corollary 2.8, and (2)⇔(3) is the standard central-idempotent/direct-product correspondence.

We prove (2)⇒(1). Assume that A has no nontrivial central idempotents, and let $e = e^2 \in A$ satisfy $eA(1-e) = 0$. Set $L = (1-e)Ae$. The hypothesis $eA(1-e) = 0$ implies that L is a two-sided ideal: for $a \in A$, $a(1-e) = (1-e)a(1-e)$ and $ea = eae$, so $aL \subseteq L$ and $La \subseteq L$. Additionally, $L^2 = 0$. Since A is semiprime, $L = 0$. Hence, both off-diagonal Peirce components vanish, so $ea = ae$ for all $a \in A$. Thus e is central, and by (2), it is either 0 or 1. Now, Proposition 2.4 shows that P is idempotent-prime.

Next, we prove (2)⇔(4). Assume (2), and let $e \in \mathrm{Pt}(A)$. Then, $eA(1-e)Ae = 0$ and $(1-e)AeA(1-e) = 0$. Put $I = AeA(1-e)A$. Then, $I \lhd A$, and

$$I^2 = AeA(1-e)AAeA(1-e)A \subseteq A(eA(1-e)Ae)A(1-e)A = 0.$$

Since A is semiprime, $I = 0$, and therefore, $eA(1-e) = 0$. Similarly, with $J = A(1-e)AeA$, the equality $J^2 = 0$ gives $(1-e)Ae = 0$. Hence, e is central. By (2), either $e = 0$ or $e = 1$, so $\mathrm{Pt}(A) = \{0, 1\}$.

Conversely, assume $\mathrm{Pt}(A) = \{0, 1\}$. If $e = e^2 \in Z(A)$, then $eA(1-e) = 0 = (1-e)Ae$, so e is Peirce trivial. Thus, $e = 0$ or $e = 1$, and (2) holds. □

This gives the following formulation in the language of Peirce rings.

Corollary 2.13. *Let $P \lhd R$ be a proper ideal, and put $A = R/P$. If A is semiprime, then P is idempotent-prime in R if and only if A is a 1-Peirce ring. In particular, if A is a semiprime n-Peirce ring with $n > 1$, then P is not idempotent-prime.*

Proof. The equivalence is Theorem 2.12. If $n > 1$, then by [4, Theorem 3.1], A decomposes as a direct product of n semiprime 1-Peirce rings, and hence has a nontrivial central idempotent. Therefore P cannot be idempotent-prime by Theorem 2.12. □

Additionally, the quotient criterion can be expressed by excluding nontrivial semicentral idempotents.

Corollary 2.14. *Let $P \lhd R$ be a proper ideal, and put $A = R/P$. The following are equivalent:*

(1) P is idempotent-prime in R;

(2) A has no nontrivial idempotent e such that $eA = eAe$;

(3) A has no nontrivial idempotent e such that $Ae = eAe$; and

(4) A has no nontrivial idempotent e for which one of the Peirce components $eA(1-e)$ or $(1-e)Ae$ vanishes.

Proof. For an idempotent $e \in A$, the equality $eA = eAe$ is equivalent to $eA(1-e) = 0$, since $eA = eAe + eA(1-e)$. Likewise, $Ae = eAe$ is equivalent to $(1-e)Ae = 0$. The result follows from Proposition 2.4 and Corollary 2.6. □

The next result connects idempotent–prime ideals with the noncommutative primary-type ideals studied in [5]. Recall that a proper ideal $I \lhd R$ is called *right primary* if, whenever $A, B \lhd R$ satisfy $AB \subseteq I$, then $A \subseteq I$ or $B^n \subseteq I$ for some positive integer n. It is called *principally right primary* if the same condition is only required for principal ideals A and B. Similarly, an ideal $I \lhd R$ is called *nilary* if, whenever $A, B \lhd R$ satisfy $AB \subseteq I$, then $A^m \subseteq I$ or $B^n \subseteq I$ for some positive integers m, n. It is called *principally nilary* if this condition is only required for principal ideals A and B. Every right or left primary ideal is nilary [5, Definition 1.1].

Proposition 2.15. *Let R be a ring with identity, and let $P \lhd R$ be a proper ideal. If P is principally nilary, then P is idempotent–prime. In particular, every nilary ideal of R is idempotent–prime. Consequently, every right or left primary ideal of R is idempotent–prime.*

Proof. Let $a \in R$ and suppose that $aR(a+1) \subseteq P$. Set $I = RaR$ and $J = R(a+1)R$. Then, $IJ = RaRR(a+1)R \subseteq R(aR(a+1))R \subseteq P$. Since P is principally nilary, either $I^m \subseteq P$ for some positive integer m or $J^n \subseteq P$ for some positive integer n.

Put $A = R/P$. Since $a(a+1) \in P$, we have $\overline{a}(\overline{a}+1) = 0$. Hence, $e = -\overline{a}$ is an idempotent of A because $e^2 = (-\overline{a})^2 = \overline{a}^2 = -\overline{a} = e$.

First, assume that $I^m \subseteq P$. Then, $(A\overline{a}A)^m = 0$. However, $A\overline{a}A = AeA$, and AeA is an idempotent ideal since $(AeA)^2 = AeA$. Therefore, $A\overline{a}A = 0$. In particular, $\overline{a} = 0$, and hence $a \in P$.

Next, assume that $J^n \subseteq P$. Since $\overline{a+1} = 1 - e$, we have $A\overline{(a+1)}A = A(1-e)A$. Again, $A(1-e)A$ is an idempotent ideal. Since $(A\overline{(a+1)}A)^n = 0$, it follows that $A\overline{(a+1)}A = 0$. Hence, $\overline{a+1} = 0$, and so $a + 1 \in P$.

Thus, $aR(a+1) \subseteq P$ implies $a \in P$ or $a+1 \in P$. Therefore, P is idempotent–prime. □

3. Ideal-theoretic and torsion-theoretic behavior

This section gathers properties that are best understood at the level of ideals and module categories. The results show how idempotent-primeness behaves under homomorphisms and quotients, how it interacts with comaximal ideals and radical extensions, and how the Peirce obstruction gives torsion-theoretic and avoidance interpretations.

The first stability result records the behavior under surjective homomorphisms.

Proposition 3.1. *Let $\varphi : R \twoheadrightarrow S$ be a surjective unital ring homomorphism:*

(1) If $Q \lhd S$ is idempotent-prime, then $\varphi^{-1}(Q)$ is idempotent-prime in R; and

(2) If $P \lhd R$ is idempotent-prime and $\ker\varphi \subseteq P$*, then $\varphi(P)$ is idempotent-prime in S.*

Proof. For (1), suppose $aR(a+1) \subseteq \varphi^{-1}(Q)$. Applying φ and using surjectivity gives $\varphi(a)S(\varphi(a)+1) \subseteq Q$. Since Q is idempotent-prime, $\varphi(a) \in Q$ or $\varphi(a)+1 \in Q$, and hence $a \in \varphi^{-1}(Q)$ or $a+1 \in \varphi^{-1}(Q)$.

For (2), let $s \in S$ and suppose $sS(s+1) \subseteq \varphi(P)$. Choose $a \in R$ with $\varphi(a) = s$. For every $r \in R$, $\varphi(ar(a+1)) \in \varphi(P)$, so $ar(a+1) \in \varphi^{-1}(\varphi(P)) = P + \ker\varphi = P$. Hence $aR(a+1) \subseteq P$. Since P is idempotent-prime, either $a \in P$ or $a+1 \in P$. Applying φ, we get $s \in \varphi(P)$ or $s+1 \in \varphi(P)$. □

The quotient version immediately follows.

Corollary 3.2. *Let $J \subseteq P \lhd R$, where P is proper. Then, P is idempotent-prime in R if and only if P/J is idempotent-prime in R/J.*

Proof. Apply Proposition 3.1 to the quotient map $R \twoheadrightarrow R/J$. □

Recall that two ideals $I, J \lhd R$ are said to be *comaximal* if $I + J = R$. The following theorem gives an ideal-theoretic characterization in terms of comaximal ideals.

Theorem 3.3. *Let $P \lhd R$ be a proper ideal. The following are equivalent:*

(1) P is idempotent-prime; and

(2) Whenever $I, J \lhd R$ are comaximal ideals with $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$.

Proof. Assume (1), and let $I + J = R$ with $IJ \subseteq P$. Choose $i \in I$ and $j \in J$ such that $i + j = 1$. Put $a = -i$. Then, $a + 1 = j$, and $aR(a + 1) = (-i)Rj \subseteq IJ \subseteq P$. Hence, $i \in P$ or $j \in P$. If $i \in P$ and $x \in I$, then $x = x(i + j) = xi + xj \in P$, so $I \subseteq P$. If $j \in P$, then the same argument using $y = (i + j)y$ for $y \in J$ gives $J \subseteq P$.

Conversely, assume (2), and let $aR(a + 1) \subseteq P$. Define $I = RaR + P$ and $J = R(a + 1)R + P$. Then, $I + J = R$, because $1 = (a + 1) - a \in I + J$. Moreover,

$$IJ \subseteq RaR\,R(a + 1)R + P \subseteq R(aR(a + 1))R + P \subseteq P.$$

By (2), either $I \subseteq P$ or $J \subseteq P$. Thus, $a \in P$ or $a + 1 \in P$, and P is idempotent-prime. □

Local rings provide a large source of examples.

Proposition 3.4. *Let R be local. Every proper ideal of R is idempotent-prime.*

Proof. Let $aR(a + 1) \subseteq P$. In a local ring, for every $x \in R$, either x or $1 - x$ is a unit. Applying this to $x = -a$, either a or $a + 1$ is a unit. If a is a unit, then $aa^{-1}(a + 1) = a + 1 \in P$. If $a + 1$ is a unit, then $a(a + 1)^{-1}(a + 1) = a \in P$. Hence, P is idempotent-prime. □

The next theorem treats Jacobson-radical extensions.

Theorem 3.5. *Let A be a ring, and let $J \lhd A$ satisfy $J \subseteq \mathrm{Jac}(A)$. If the zero ideal of A/J is idempotent-prime, then the zero ideal of A is idempotent-prime.*

Proof. Let $e = e^2 \in A$ satisfy $eA(1 - e) = 0$. Then, the image $\overline{e} \in A/J$ satisfies $\overline{e}(A/J)(1 - \overline{e}) = 0$. Since the zero ideal of A/J is idempotent-prime, we have $\overline{e} = 0$ or $\overline{e} = 1$.

If $\overline{e} = 0$, then $e \in J \subseteq \mathrm{Jac}(A)$. Since an idempotent in the Jacobson radical is zero, $e = 0$. If $\overline{e} = 1$, then $1 - e \in J \subseteq \mathrm{Jac}(A)$. Since $1 - e$ is an idempotent in the Jacobson radical, $1 - e = 0$, and hence $e = 1$.

Therefore, A has no nontrivial idempotent e satisfying $eA(1 - e) = 0$. Hence, the zero ideal of A is idempotent-prime. □

A simple Artinian semisimple quotient gives an immediate criterion.

Corollary 3.6. *Let A be a ring. If $A/\mathrm{Jac}(A)$ is simple Artinian, then the zero ideal of A is idempotent-prime.*

Proof. Since $A/\mathrm{Jac}(A)$ is simple Artinian, its zero ideal is prime, hence idempotent-prime. Applying Theorem 3.5 with $J = \mathrm{Jac}(A)$, we conclude that the zero ideal of A is idempotent-prime. □

In the semisimple Artinian case, idempotent-primeness is exactly simplicity.

Corollary 3.7. *Let A be a semisimple Artinian ring. Then, the zero ideal of A is idempotent-prime if and only if A is simple Artinian.*

Proof. If A is simple Artinian, then the zero ideal of A is prime, hence idempotent-prime.

Conversely, suppose that A is semisimple Artinian but not simple Artinian. Then, A is a direct product of at least two nonzero simple Artinian rings. Hence, A has a nontrivial central idempotent. This central idempotent e satisfies $eA(1 - e) = 0$, so the zero ideal of A is not idempotent-prime. □

Corollary 3.8. *Let $P \lhd R$ be a proper ideal. If $(R/P)/\operatorname{Jac}(R/P)$ is simple Artinian, then P is idempotent-prime in R. If R/P is semisimple Artinian, then P is idempotent-prime in R if and only if R/P is simple Artinian.*

Proof. Apply Corollaries 3.6 and 3.7 to $A = R/P$, together with the quotient criterion. □

Now, we explain the torsion-theoretic content of the quotient criterion. We use the standard terminology of torsion theories in module categories; for example, see [6]. A torsion theory on Mod-A is a pair $(\mathcal{T}, \mathcal{F})$ of classes of right A-modules such that $\operatorname{Hom}_A(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$, and every right A-module M has a largest submodule $t(M) \in \mathcal{T}$ with $M/t(M) \in \mathcal{F}$. The torsion theory is hereditary if $\mathcal{T}$ is closed under submodules.

The result below does not identify idempotent-prime ideals with prime torsion theories in the usual sense. Instead, it shows that idempotent-prime quotients have no nontrivial hereditary torsion theory forced by an idempotent $e \in A$ with $eA(1 - e) = 0$. Such an idempotent is precisely the obstruction appearing in the quotient criterion, and it naturally gives rise to a hereditary torsion theory on Mod-A.

Proposition 3.9. *Let A be a ring, and let $e = e^2 \in A$ satisfy $eA(1 - e) = 0$. Define two classes of right A-modules by $\mathcal{T}_e = \{M \in \text{Mod-}A \mid M(1 - e) = 0\}$ and $\mathcal{F}_e = \{M \in \text{Mod-}A \mid Me = 0\}$. Then, $(\mathcal{T}_e, \mathcal{F}_e)$ is a hereditary torsion theory on Mod-A. Its torsion radical is given by $t_e(M) = Me$.*

Proof. Since $eA(1 - e) = 0$, we have $eA = eAe$. Hence, Me is a submodule of M because $(me)a = m(ea) \in Me$ for all $m \in M$ and $a \in A$. Additionally, $(Me)(1 - e) = 0$, so $Me \in \mathcal{T}_e$.

The quotient M/Me belongs to $\mathcal{F}_e$ because $(m + Me)e = 0$ for every $m \in M$. Thus, every right A-module M fits into a short exact sequence $0 \to Me \to M \to M/Me \to 0$ with $Me \in \mathcal{T}_e$ and $M/Me \in \mathcal{F}_e$.

It remains to check orthogonality. Let $T \in \mathcal{T}_e$, $F \in \mathcal{F}_e$, and let $f : T \to F$ be an A-homomorphism. For $x \in T$, we have $x = xe$ because $x(1 - e) = 0$. Hence, $f(x) = f(xe) = f(x)e = 0$, since $Fe = 0$. Therefore, $\operatorname{Hom}_A(T, F) = 0$. Thus $(\mathcal{T}_e, \mathcal{F}_e)$ is a torsion theory. Since $\mathcal{T}_e$ is closed under submodules, this torsion theory is hereditary. □

The corresponding Gabriel filter can be written explicitly.

Proposition 3.10. *With the notation of Proposition 3.9, the right Gabriel filter corresponding to $(\mathcal{T}_e, \mathcal{F}_e)$ is $\mathfrak{G}_e = \{I \leq A_A \mid A(1 - e)A \subseteq I\}$. In particular, this torsion theory is generated by the idempotent two-sided ideal $A(1 - e)A$.*

Proof. For a right ideal $I \leq A_A$, the cyclic module A/I belongs to $\mathcal{T}_e$ if and only if $(A/I)(1 - e) = 0$. This is equivalent to $A(1 - e) \subseteq I$, and since I is a right ideal, it is equivalent to $A(1 - e)A \subseteq I$. Hence, the associated Gabriel filter is the stated family.

The ideal $A(1-e)A$ is idempotent because it is generated as a two-sided ideal by the idempotent $1-e$. Thus, the corresponding torsion theory is one of the standard idempotent-ideal torsion theories. □

The torsion-theoretic form of the quotient criterion is as follows.

Theorem 3.11. *Let $P \lhd R$ be a proper ideal, and put $A = R/P$. Then P is idempotent-prime in R if and only if every Peirce torsion theory $(\mathcal{T}_e, \mathcal{F}_e)$ on Mod-A arising from an idempotent $e \in A$ with $eA(1 - e) = 0$ is trivial.*

Proof. By the quotient criterion, P is idempotent-prime if and only if A has no nontrivial idempotent e satisfying $eA(1-e) = 0$.

If such a nontrivial idempotent e exists, then Proposition 3.9 gives a hereditary torsion theory $(\mathcal{T}_e, \mathcal{F}_e)$. This torsion theory is nontrivial: The module Ae belongs to $\mathcal{T}_e$ and is nonzero, while A/Ae is nonzero whenever $e \neq 1$.

Conversely, every nontrivial Peirce torsion theory of the displayed form comes from a nontrivial idempotent $e \in A$ satisfying $eA(1-e) = 0$. By the quotient criterion, the existence of such an idempotent is exactly the failure of idempotent-primeness. □

Remark 3.12. Theorem 3.11 should not be read as saying that idempotent-prime ideals are the same as prime torsion theories. Prime torsion theories are a different notion in the literature. The present condition only detects torsion theories arising from triangular Peirce idempotents in the quotient ring.

The usual prime avoidance lemma does not directly carry over to idempotent-prime ideals. The reason is that idempotent-primeness only governs Peirce idempotents, rather than arbitrary products of elements or ideals. Even so, one still obtains an avoidance theorem for ideals generated by central idempotents.

First, we record the elementary two-subgroup avoidance lemma in additive form.

Lemma 3.13. *Let G be an Abelian group, and let $H, K, L \leq G$. If $H \subseteq K \cup L$, then either $H \subseteq K$ or $H \subseteq L$.*

Proof. Assume that $H \nsubseteq K$ and $H \nsubseteq L$. Choose $a \in H \setminus K$ and $b \in H \setminus L$. Since $H \subseteq K \cup L$, we have $a \in L$ and $b \in K$. Now $a+b \in H$, so either $a+b \in K$ or $a+b \in L$. If $a+b \in K$, then $a = (a+b)-b \in K$, a contradiction. If $a+b \in L$, then $b = (a+b)-a \in L$, which is again a contradiction. Hence, $H \subseteq K$ or $H \subseteq L$. □

The following example shows that ordinary finite prime avoidance does not extend to idempotent-prime ideals.

Example 3.14. Let $R = \mathbb{F}_2[x, y]/(x^2, xy, y^2)$, and let $\mathfrak{m} = (x, y)$. Then, R is a commutative local ring with maximal ideal $\mathfrak{m}$. Hence, every proper ideal of R is idempotent-prime because every quotient of R by a proper ideal is local and therefore has no nontrivial idempotents.

Set $P_1 = (x)$, $P_2 = (y)$, and $P_3 = (x+y)$. Then P_1, P_2, P_3 are idempotent-prime ideals. However, $\mathfrak{m} = P_1 \cup P_2 \cup P_3$ because $\mathfrak{m} = \{0, x, y, x+y\}$ as an $\mathbb{F}_2$-vector space. Clearly $\mathfrak{m} \nsubseteq P_i$ for $i = 1, 2, 3$. Thus, finite prime avoidance fails for idempotent-prime ideals.

The correct replacement is avoidance for ideals generated by central idempotents.

Theorem 3.15. *Let $P_1, \ldots, P_n \lhd R$ be idempotent-prime ideals, and let $J \lhd R$ be an ideal generated by central idempotents. If $J \subseteq P_1 \cup \cdots \cup P_n$, then $J \subseteq P_i$ for some i.*

Proof. Assume that $J \nsubseteq P_i$ for every i. Since J is generated by central idempotents, for each i there exists a central idempotent $c_i \in J \setminus P_i$.

Because c_i is central, we have $c_iR(1-c_i) = 0$. Since P_i is idempotent-prime and $c_i \notin P_i$, it follows that $1 - c_i \in P_i$.

Now, put $c = 1 - (1-c_1)(1-c_2)\cdots(1-c_n)$. Since the c_i's are central idempotents, c is a central idempotent. Moreover, $c \in J$ because c is the Boolean join of $c_1, \ldots, c_n$; hence, it is a finite sum of products of the c_i's.

For each i, the element $(1-c_1)\cdots(1-c_n)$ belongs to P_i because its i-th factor $1-c_i$ belongs to P_i. If $c \in P_i$, then $1 = c + (1-c_1)\cdots(1-c_n) \in P_i$, which contradicts the properness of P_i. Hence, $c \notin P_i$ for every i.

Thus, $c \in J \setminus (P_1 \cup \cdots \cup P_n)$, which contradicts $J \subseteq P_1 \cup \cdots \cup P_n$. Therefore, $J \subseteq P_i$ for some i. □

Remark 3.16. Theorem 3.15 is the natural idempotent-prime analogue of prime avoidance. Ordinary prime avoidance controls arbitrary elements by using products, while idempotent-primeness only controls idempotents e satisfying $eR(1-e) \subseteq P$. Therefore, the correct replacement is an avoidance theorem for ideals generated by central idempotents.

4. Standard constructions

The quotient criterion makes many standard constructions transparent. Direct products are controlled by central idempotents, matrix rings are controlled by the semiprime quotient criterion or by radical lifting, and triangular matrix rings are controlled by their diagonal quotient components.

Finite products behave as expected: An idempotent-prime quotient can only have one nonzero component.

Theorem 4.1. *Let $R_1, \ldots, R_n$ be rings, where $n \geq 2$, and let $P \lhd R_1 \times \cdots \times R_n$ be a proper ideal. Write $P = I_1 \times \cdots \times I_n$, with $I_i \lhd R_i$. Then, P is idempotent-prime if and only if there is a unique index k such that I_k is idempotent-prime in R_k and $I_i = R_i$ for every $i \neq k$.*

Proof. Put $A_i = R_i/I_i$. Then, $(R_1 \times \cdots \times R_n)/P \cong A_1 \times \cdots \times A_n$. If P is idempotent-prime, then Proposition 2.4 implies that this quotient has no nontrivial idempotent e with $eA(1-e) = 0$. If two of the A_i are nonzero, then the idempotent that is 1 in one of those components and 0 in the others is nontrivial and central; hence, it satisfies $eA(1-e) = 0$, which is a contradiction. Since P is proper, at least one A_i is nonzero. Therefore exactly one A_k is nonzero, equivalently $I_i = R_i$ for every $i \neq k$. Then, the quotient is isomorphic to R_k/I_k, so Proposition 2.4 shows that I_k is idempotent-prime in R_k.

Conversely, if such an index k exists, then the quotient by P is isomorphic to R_k/I_k. Since I_k is idempotent-prime, Proposition 2.4 shows that P is idempotent-prime. □

Let A and B be rings, let ${}_AM_B$ be a unital (A, B)-bimodule, and consider the upper triangular matrix ring

$$T = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}.$$

For ideals $I \lhd A$, $J \lhd B$, and an (A, B)-subbimodule $N \subseteq M$ satisfying $IM + MJ \subseteq N$, write the following:

$$\mathrm{Tri}(I, N, J) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in I,\ m \in N,\ b \in J \right\}.$$

Every ideal of T is of this form.

For a two-by-two triangular matrix ring, the quotient can only retain one diagonal side.

Theorem 4.2. *With the above notation, a proper ideal $P \lhd T$ is idempotent-prime if and only if exactly one of the following holds:*

(1) $P = \mathrm{Tri}(I, M, B)$, where $I \lhd A$ is idempotent-prime in A; and

(2) $P = \mathrm{Tri}(A, M, J)$, where $J \lhd B$ is idempotent-prime in B.

Proof. Write $P = \mathrm{Tri}(I, N, J)$. Then,

$$T/P \cong \begin{pmatrix} A/I & M/N \\ 0 & B/J \end{pmatrix}.$$

If P is idempotent-prime and both A/I and B/J are nonzero, then the image of $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is a nontrivial idempotent e in T/P satisfying $e(T/P)(1 - e) = 0$, which contradicts Proposition 2.4. Hence, $A/I = 0$ or $B/J = 0$, that is, $I = A$ or $J = B$. Since M is unital and $IM + MJ \subseteq N$, either equality forces $N = M$. Therefore P has one of the displayed forms.

If $P = \mathrm{Tri}(I, M, B)$, then $T/P \cong A/I$. Hence, by Proposition 2.4, P is idempotent-prime in T if and only if I is idempotent-prime in A. Similarly, if $P = \mathrm{Tri}(A, M, J)$, then $T/P \cong B/J$, and P is idempotent-prime in T if and only if J is idempotent-prime in B. □

Matrix rings over local rings give another basic family.

Proposition 4.3. *Let S be a local ring, and let $n \geq 1$. Then, the zero ideal of $M_n(S)$ is idempotent-prime.*

Proof. Put $A = M_n(S)$, and let $\mathrm{Jac}(S)$ denote the Jacobson radical of S. Then, $\mathrm{Jac}(A) = M_n(\mathrm{Jac}(S))$, and $A/\mathrm{Jac}(A) \cong M_n(S/\mathrm{Jac}(S))$, where $S/\mathrm{Jac}(S)$ is a division ring, see [7]. The ring $M_n(S/\mathrm{Jac}(S))$ is simple Artinian; hence, it is semiprime with no nontrivial central idempotents. By Theorem 2.12, the zero ideal of $M_n(S/\mathrm{Jac}(S))$ is idempotent-prime.

Let $e = e^2 \in A$ satisfy $eA(1 - e) = 0$. Passing modulo $\mathrm{Jac}(A)$ gives the following:

$$\overline{e}\,(A/\mathrm{Jac}(A))(1 - \overline{e}) = 0.$$

Since the zero ideal of $A/\mathrm{Jac}(A)$ is idempotent-prime, either $\overline{e} = 0$ or $\overline{e} = 1$. If $\overline{e} = 0$, then $e \in \mathrm{Jac}(A)$, and an idempotent in the Jacobson radical must be zero. If $\overline{e} = 1$, then $1 - e \in \mathrm{Jac}(A)$, so $1 - e = 0$. Thus, $e = 0$ or $e = 1$. Now, Proposition 2.4 proves the claim. □

For semiprime quotients, idempotent-primeness is stable under full matrix rings.

Theorem 4.4. *Let $P \lhd R$ be a proper ideal, let $n \geq 1$, and assume that R/P is semiprime. Then, P is idempotent-prime in R if and only if $M_n(P)$ is idempotent-prime in $M_n(R)$.*

Proof. Put $A = R/P$. Then, $M_n(R)/M_n(P) \cong M_n(A)$, and $M_n(A)$ is semiprime because A is semiprime. By Theorem 2.12, P is idempotent-prime in R if and only if A has no nontrivial central idempotents. Again by Theorem 2.12, $M_n(P)$ is idempotent-prime in $M_n(R)$ if and only if $M_n(A)$ has no nontrivial central idempotents.

The central idempotents of $M_n(A)$ are precisely the matrices cI_n, where c is a central idempotent of A. Therefore, A has no nontrivial central idempotents if and only if $M_n(A)$ has no nontrivial central idempotents. The equivalence follows. □

The zero-ideal case is the following immediate consequence.

Corollary 4.5. *Let A be a semiprime ring, and let $n \geq 1$. Then, the zero ideal of A is idempotent-prime if and only if the zero ideal of $M_n(A)$ is idempotent-prime.*

Proof. Apply Theorem 4.4 with $R = A$ and $P = 0$. □

The two-by-two triangular classification extends to finite generalized upper triangular matrix rings.

Theorem 4.6. *Let $n \geq 2$, and let*

$$T = \begin{pmatrix} R_1 & M_{12} & M_{13} & \cdots & M_{1n} \\ 0 & R_2 & M_{23} & \cdots & M_{2n} \\ 0 & 0 & R_3 & \cdots & M_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & R_n \end{pmatrix}$$

be a generalized upper triangular matrix ring with standard diagonal idempotents $e_1, \ldots, e_n$. Assume that all bimodules are unital and that the structure maps make T associative. For $1 \leq k \leq n$ and an ideal $I \lhd R_k$, define $P_k(I)$ to be the set of all elements $x = (x_{ij}) \in T$ such that $x_{kk} \in I$. Then a proper ideal $P \lhd T$ is idempotent-prime if and only if there are a unique index k and an idempotent-prime ideal $I \lhd R_k$ such that $P = P_k(I)$.

Proof. First, observe that $P_k(I)$ is an ideal of T and $T/P_k(I) \cong R_k/I$. Hence, if I is idempotent-prime in R_k, then $P_k(I)$ is idempotent-prime in T by Proposition 2.4.

Conversely, assume that $P \lhd T$ is idempotent-prime. Put $\overline{T} = T/P$, and let $\overline{e}_i$ denote the image of e_i in $\overline{T}$. For $m = 1, \ldots, n-1$, set $s_m = \overline{e}_{m+1} + \cdots + \overline{e}_n$. Since T is upper triangular, $s_m\overline{T}(1 - s_m) = 0$. By Proposition 2.4, each s_m is either 0 or 1. Additionally, set $s_0 = 1$ and $s_n = 0$. Since the sequence $s_0, s_1, \ldots, s_n$ changes from 1 to 0, there is a unique index k such that $s_{k-1} = 1$ and $s_k = 0$. It follows that $\overline{e}_k = 1$ and $\overline{e}_i = 0$ for every $i \neq k$.

Thus, $e_i \in P$ for every $i \neq k$. Hence, every block e_iTe_j with $(i, j) \neq (k, k)$ is contained in P because at least one of e_i and e_j lies in P. Define $I \subseteq R_k$ by $I = P \cap e_kTe_k$ under the identification $e_kTe_k \cong R_k$. Then, $I \lhd R_k$, and the preceding paragraph gives $P = P_k(I)$. Since P is proper, I is proper. Finally, $T/P \cong R_k/I$, so Proposition 2.4 implies that I is idempotent-prime in R_k. □

Remark 4.7. Theorem 4.6 recovers the two-by-two triangular matrix classification when $n = 2$. Moreover, it shows that in any finite generalized upper triangular matrix ring, an idempotent-prime quotient can only retain one diagonal quotient component.

5. Topological and graph-theoretic descriptions

This section gives topological and graph-theoretic interpretations of idempotent-primeness. In the commutative case, the decisive notion is connectedness: a proper ideal $P \lhd R$ is idempotent-prime if and only if $\operatorname{Spec}(R/P)$, equivalently $V(P) \subseteq \operatorname{Spec} R$, is connected. Therefore, we begin with the commutative case in these terms. Then, we turn to the noncommutative setting, where the two-sided prime spectrum gives a partial analogue, and finally record graph-theoretic reformulations in both the commutative and noncommutative situations.

5.1. Connectedness in the commutative case

For commutative rings, idempotent-primeness is exactly a connectedness condition on the corresponding closed subset of the prime spectrum.

Theorem 5.1. *Let R be a commutative ring, and let $P \lhd R$ be a proper ideal. Then, the following conditions are equivalent:*

(1) P is idempotent-prime;

(2) $\operatorname{Spec}(R/P)$ is connected; and

(3) $V(P) \subseteq \operatorname{Spec} R$ is connected.

Proof. Put $A = R/P$. Since A is commutative, every idempotent $e \in A$ satisfies $eA(1 - e) = 0$. Hence, P is idempotent-prime if and only if A has no nontrivial idempotents.

By the standard correspondence between idempotents and clopen subsets of the spectrum, A has no nontrivial idempotents if and only if $\operatorname{Spec} A$ is connected. Finally, $\operatorname{Spec}(R/P)$ is naturally homeomorphic to $V(P) \subseteq \operatorname{Spec} R$. Therefore, the three conditions are equivalent. □

Remark 5.2. In the commutative case, prime ideals correspond to irreducible closed subsets of $\operatorname{Spec} R$, whereas idempotent-prime ideals correspond to connected closed subsets. Thus, idempotent-primeness is weaker than primeness: Irreducible closed subsets are connected, but connected closed subsets need not be irreducible.

The next two examples illustrate the connected and disconnected cases.

Example 5.3. Let k be a field, and let $R = k[x, y]$. Put $P = (xy)$. Then, P is not prime since $xy \in P$ but $x \notin P$ and $y \notin P$. However, $V(P) = V(x) \cup V(y)$, and the two affine lines $V(x)$ and $V(y)$ meet at the origin. Hence, $V(P)$ is connected. By Theorem 5.1, P is idempotent-prime.

Example 5.4. Let k be a field, and let $R = k[x]$. Put $P = (x(x - 1))$. Then, $R/P \cong k \times k$, so $\operatorname{Spec}(R/P)$ is the disjoint union of two points. Equivalently, $V(P) = \{(x), (x - 1)\}$ is disconnected. Hence, P is not idempotent-prime by Theorem 5.1.

5.2. The noncommutative spectrum

Now, we pass to the noncommutative setting. Throughout this subsection, $\operatorname{Spec} A$ denotes the two-sided prime spectrum of a ring A, endowed with the Zariski topology, whose closed sets are

$$V(I) = \{\mathfrak{q} \in \operatorname{Spec} A \mid I \subseteq \mathfrak{q}\}$$

for two-sided ideals $I \lhd A$.

The two-sided prime spectrum gives a partial topological description in general, and a complete one for semiprime quotients.

Theorem 5.5. *Let $P \lhd R$ be a proper ideal, and put $A = R/P$. If $\operatorname{Spec} A$ is connected, then P is idempotent-prime. If A is semiprime, then the converse also holds. Thus, when R/P is semiprime, P is idempotent-prime if and only if $\operatorname{Spec}(R/P)$ is connected.*

Proof. First, we prove the contrapositive of the first assertion. Assume that P is not idempotent-prime. Then the zero ideal of A is not idempotent-prime, so there exists a nontrivial idempotent $e \in A$ such that $eA(1-e) = 0$.

Let $\mathfrak{q} \in \operatorname{Spec} A$. Since

$$(AeA)(A(1-e)A) = 0 \subseteq \mathfrak{q},$$

the primeness of $\mathfrak{q}$ implies that $AeA \subseteq \mathfrak{q}$ or $A(1-e)A \subseteq \mathfrak{q}$. Equivalently, either $e \in \mathfrak{q}$ or $1-e \in \mathfrak{q}$. These two alternatives are mutually exclusive because $\mathfrak{q}$ is proper and $e + (1-e) = 1$.

Therefore,

$$\operatorname{Spec} A = V(AeA) \amalg V(A(1-e)A).$$

Both subsets are closed and complementary, hence clopen. Additionally, they are nonempty. Indeed, if $AeA = A$, then $A(1-e) = AeA(1-e) = 0$, so $1-e = 0$, which contradicts $e \neq 1$. Similarly, if $A(1-e)A = A$, then $eA = eA(1-e)A = 0$, so $e = 0$, which contradicts $e \neq 0$. Hence, $\operatorname{Spec} A$ is disconnected. This proves that the connectedness of $\operatorname{Spec} A$ implies idempotent-primeness of P.

Now, assume that A is semiprime and that $\operatorname{Spec} A$ is disconnected. Then, there exist ideals $I, J \lhd A$ such that $\operatorname{Spec} A = V(I) \amalg V(J)$, where both $V(I)$ and $V(J)$ are nonempty.

Since $V(I) \cup V(J) = \operatorname{Spec} A$, we have $V(IJ) = \operatorname{Spec} A$. Hence, IJ is contained in every prime ideal of A, so IJ is contained in the prime radical of A. Since A is semiprime, its prime radical is zero. Therefore, $IJ = 0$. Similarly, $JI = 0$.

Additionally, since $V(I) \cap V(J) = \emptyset$, we have $V(I+J) = \emptyset$. Hence, $I + J = A$. Moreover, $(I \cap J)^2 \subseteq IJ = 0$. Since A is semiprime, this implies $I \cap J = 0$. Thus, $A = I \oplus J$ as a direct sum of ideals.

Choose $c \in I$ and $d \in J$ such that $c + d = 1$. Since $IJ = JI = 0$, we have $cd = dc = 0$. Therefore,

$$c = c(c+d) = c^2 + cd = c^2, \qquad d = d(c+d) = dc + d^2 = d^2.$$

Thus, c and d are orthogonal idempotents.

We claim that c is central. Let $a \in A$. Since $c + d = 1$, we have $ca = ca(c+d) = cac + cad = cac$, because $cad \in IJ = 0$. Similarly, $ac = (c+d)ac = cac + dac = cac$ because $dac \in JI = 0$. Hence $ca = ac$ for every $a \in A$, so $c \in Z(A)$.

The idempotent c is nontrivial. If $c = 1$, then $1 \in I$, so $I = A$, which contradicts the nonemptiness of $V(I)$. If $c = 0$, then $d = 1$, so $J = A$, which contradicts the nonemptiness of $V(J)$. Thus, $c \neq 0, 1$.

Hence, A has a nontrivial central idempotent c. In particular, $cA(1-c) = 0$. Therefore, the zero ideal of A is not idempotent-prime. Equivalently, P is not idempotent-prime in R.

This proves that when A is semiprime, disconnectedness of $\operatorname{Spec} A$ implies failure of idempotent-primeness. Together with the first part, the theorem follows. □

The following example shows that the semiprime hypothesis in Theorem 5.5 cannot be omitted: A ring may have disconnected two-sided prime spectrum and still have idempotent-prime zero ideal.

Example 5.6. Let k be a field, and let $A = kQ/J^2$, where Q is the quiver with vertices $1, 2$ and arrows $a : 1 \to 2$, $b : 2 \to 1$, and where $J = (a, b)$ denotes the arrow ideal. In A, the image of J satisfies $J^2 = 0$. Thus J is nilpotent, so J is contained in every two-sided prime ideal of A. Therefore, the canonical map $A \to A/J$ induces a homeomorphism $\operatorname{Spec}(A/J) \cong \operatorname{Spec} A$.

Since quotienting by J kills the arrows and leaves only the two vertex idempotents, we have the following:

$$A/J \cong ke_1 \oplus ke_2 \cong k \times k.$$

Hence $\operatorname{Spec} A \cong \operatorname{Spec}(k \times k)$, which is the disjoint union of the two points $k \times 0$ and $0 \times k$. Therefore, $\operatorname{Spec} A$ is disconnected.

However, the zero ideal of A is idempotent-prime. Every element of A has a unique form

$$x = \alpha e_1 + \beta e_2 + \lambda a + \mu b$$

with $\alpha, \beta, \lambda, \mu \in k$. Since $J^2 = 0$, every product of two arrows is zero in A. Using

$$e_1 a = a = a e_2, \qquad e_2 b = b = b e_1,$$

and all other incompatible vertex-arrow products being zero, we obtain

$$x^2 = \alpha^2 e_1 + \beta^2 e_2 + \lambda(\alpha + \beta)a + \mu(\alpha + \beta)b.$$

Thus $x^2 = x$ if and only if

$$\alpha^2 = \alpha, \qquad \beta^2 = \beta, \qquad \lambda(\alpha + \beta) = \lambda, \qquad \mu(\alpha + \beta) = \mu.$$

Since k is a field, $\alpha, \beta \in \{0, 1\}$. If $(\alpha, \beta) = (0, 0)$, then $\lambda = \mu = 0$, so $x = 0$. If $(\alpha, \beta) = (1, 1)$, then again $\lambda = \mu = 0$, so $x = 1$. If $(\alpha, \beta) = (1, 0)$, then $x = e_1 + \lambda a + \mu b$. If $(\alpha, \beta) = (0, 1)$, then $x = e_2 + \lambda a + \mu b$.

Therefore, every idempotent of A is either 0, 1, or has the form $e_1 + \lambda a + \mu b$ or $e_2 + \lambda a + \mu b$, with $\lambda, \mu \in k$. If $e = e_1 + \lambda a + \mu b$, then $ea(1-e) = a \neq 0$. If $e = e_2 + \lambda a + \mu b$, then $eb(1-e) = b \neq 0$. Hence no nontrivial idempotent $e \in A$ satisfies $eA(1-e) = 0$. Therefore, the zero ideal of A is idempotent-prime.

5.3. A graph criterion in the commutative Noetherian case

For commutative Noetherian rings, connectedness may be encoded in a finite graph.

Definition 5.7. Let R be a commutative Noetherian ring, and let $P \lhd R$ be a proper ideal. Let $\operatorname{Min}(P) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ be the set of minimal prime ideals over P. The *minimal-prime intersection graph* of P, denoted $\Gamma(P)$, is the undirected graph with vertex set $\operatorname{Min}(P)$, where two distinct vertices $\mathfrak{p}_i$ and $\mathfrak{p}_j$ are joined by an edge if $V(\mathfrak{p}_i) \cap V(\mathfrak{p}_j) \neq \emptyset$. Equivalently, $\mathfrak{p}_i$ and $\mathfrak{p}_j$ are joined by an edge if $\mathfrak{p}_i + \mathfrak{p}_j \neq R$.

The next result is the finite combinatorial form of Theorem 5.1.

Theorem 5.8. *Let R be a commutative Noetherian ring, and let $P \lhd R$ be a proper ideal. Then, the following conditions are equivalent:*

(1) $V(P)$ is connected;

(2) $\Gamma(P)$ is connected; and

(3) P is idempotent-prime.

Proof. By Theorem 5.1, it suffices to prove that $V(P)$ is connected if and only if $\Gamma(P)$ is connected.

Since R is Noetherian, the closed set $V(P)$ has only finitely many irreducible components. These are precisely $V(\mathfrak{p}_1), \dots, V(\mathfrak{p}_n)$, where $\mathrm{Min}(P) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Each $V(\mathfrak{p}_i)$ is irreducible, hence connected.

A finite union of connected subspaces is connected if and only if its intersection graph is connected. Therefore, $V(P)$ is connected if and only if $\Gamma(P)$ is connected. $\square$

The next two examples recover the connected and disconnected pictures from Examples 5.3 and 5.4 in finite graph form.

Example 5.9. Let k be a field, and let $R = k[x, y]$. Put $P = (xy)$. The minimal primes over P are (x) and (y). Since $(x) + (y) = (x, y) \neq R$, the graph $\Gamma(P)$ has two vertices joined by an edge. Hence, $\Gamma(P)$ is connected, so $V(P)$ is connected and P is idempotent-prime by Theorem 5.8.

Example 5.10. Let k be a field, and let $R = k[x]$. Put $P = (x(x - 1))$. The minimal primes over P are (x) and $(x - 1)$. Since $(x) + (x - 1) = R$, the graph $\Gamma(P)$ has two isolated vertices. Hence, $\Gamma(P)$ is disconnected, so $V(P)$ is disconnected and P is not idempotent-prime by Theorem 5.8.

Remark 5.11. Theorem 5.8 shows that, in the commutative Noetherian case, idempotent-primeness is a connectedness condition on the graph of irreducible components. Thus, prime ideals correspond to irreducible closed sets, while idempotent-prime ideals correspond to connected closed sets.

5.4. *Peirce digraphs*

We conclude with a graph obstruction in the noncommutative setting, derived from Peirce decompositions.

Definition 5.12. Let A be a ring, and let $E = \{e_1, \dots, e_n\}$ be a complete set of nonzero pairwise orthogonal idempotents so that $1 = e_1 + \cdots + e_n$. The *Peirce digraph* $\Gamma_E(A)$ is the directed graph with vertices $1, \dots, n$, where there is an arrow $i \to j$ if $e_i A e_j \neq 0$.

Idempotent-primeness forces this Peirce digraph to be strongly connected, that is, for any two vertices i and j, there exists a directed path from i to j and also a directed path from j to i.

Proposition 5.13. *Let $P \lhd R$ be a proper ideal, put $A = R/P$, and let $E = \{e_1, \dots, e_n\}$ be a complete set of nonzero pairwise orthogonal idempotents in A. If P is idempotent-prime in R, then the Peirce digraph $\Gamma_E(A)$ is strongly connected.*

Proof. Assume that $\Gamma_E(A)$ is not strongly connected. Then, there exists a nonempty proper subset $S \subsetneq \{1, \dots, n\}$ such that there is no arrow from a vertex in S to a vertex outside S. Put $e_S = \sum_{i \in S} e_i$. The absence of arrows from S to its complement means exactly that $e_S A(1 - e_S) = 0$.

Since S is nonempty and proper, e_S is a nontrivial idempotent of A. Therefore, the zero ideal of A is not idempotent-prime. By the quotient criterion, P is not idempotent-prime in R, which is a contradiction. Hence, $\Gamma_E(A)$ must be strongly connected. $\square$

We recall that an idempotent $e^2 = e \in R$ is called *right semicentral* if $eR(1 - e) = 0$ and *left semicentral* if $(1 - e)Re = 0$. Equivalently, these are precisely the idempotents that yield triangular Peirce decompositions.

Idempotents $f \in A$ satisfying $fA(1 - f) = 0$ are exactly the right semicentral idempotents. Under a visibility hypothesis for such idempotents, the converse also holds.

Theorem 5.14. *Let $P \lhd R$ be a proper ideal, put $A = R/P$, and let $E = \{e_1, \ldots, e_n\}$ be a complete set of nonzero pairwise orthogonal idempotents in A. Assume that every right semicentral idempotent $f \in A$ is conjugate to $\sum_{i \in S} e_i$ for some subset $S \subseteq \{1, \ldots, n\}$. Then, P is idempotent-prime in R if and only if the Peirce digraph $\Gamma_E(A)$ is strongly connected.*

Proof. The forward implication follows from Proposition 5.13.

Conversely, assume that $\Gamma_E(A)$ is strongly connected. Suppose that P is not idempotent-prime. Then, there exists a nontrivial idempotent $f \in A$ such that $fA(1 - f) = 0$. By the hypothesis, f is conjugate to $e_S = \sum_{i \in S} e_i$ for some subset $S \subseteq \{1, \ldots, n\}$. Since f is nontrivial, S is nonempty and proper.

Because conjugation preserves the condition $fA(1 - f) = 0$, we have $e_S A(1 - e_S) = 0$. Hence, there is no arrow in $\Gamma_E(A)$ from S to its complement. This contradicts strong connectedness. Therefore, P is idempotent-prime. ☐

Remark 5.15. The converse of Proposition 5.13 need not hold without an additional hypothesis. Strong connectedness of one chosen Peirce digraph rules out idempotents that are visible as sums of the chosen Peirce idempotents, but an arbitrary ring may have other idempotents that are not detected by that graph.

6. Conclusions

We have studied idempotent-prime ideals in associative rings with identity through the structure of their quotient rings. The central criterion shows that a proper two-sided ideal P is idempotent-prime precisely when $A = R/P$ has no nontrivial idempotent e satisfying $eA(1-e) = 0$ (Proposition 2.4). Thus, the condition excludes nontrivial triangular Peirce decompositions. The characterization by products of comaximal ideals in Theorem 3.3 gives an equivalent ideal-theoretic formulation and clarifies how idempotent-primeness weakens ordinary primeness.

In the commutative case, idempotent-primeness is equivalent to connectedness of $V(P)$. For semiprime quotients, it is equivalent to the absence of nontrivial central idempotents, direct-product indecomposability, the 1-Peirce property, and connectedness of the two-sided prime spectrum (Theorems 2.12 and 5.5). The examples show that these distinctions matter outside the semiprime setting: direct-product indecomposability alone does not imply idempotent-primeness, while an idempotent-prime quotient may have disconnected two-sided prime spectrum.

The quotient viewpoint also explains the behavior under surjective homomorphisms and intermediate quotients, as well as lifting through ideals contained in the Jacobson radical. The classifications for finite direct products and finite generalized upper triangular matrix rings show that an idempotent-prime quotient retains exactly one nonzero component or diagonal quotient component, respectively (Theorems 4.1 and 4.6). For full matrix rings, idempotent-primeness is preserved and reflected when the underlying quotient is semiprime (Theorem 4.4), while matrix rings over local rings provide further examples through radical lifting.

The torsion-theoretic and avoidance results delineate the scope of the condition. Idempotent-prime quotients exclude nontrivial hereditary torsion theories arising from triangular Peirce idempotents; this is not an identification with prime torsion theories. Likewise, ordinary finite prime avoidance fails, but an avoidance theorem remains valid for ideals generated by central idempotents (Theorem 3.15).

Finally, the graph descriptions provide finite combinatorial tests in the settings considered. For commutative Noetherian rings, the minimal-prime intersection graph is connected exactly when the ideal is idempotent-prime. In the noncommutative case, idempotent-primeness forces strong connectedness of every Peirce digraph associated with a finite complete set of nonzero orthogonal idempotents. The converse for a fixed digraph holds under the visibility hypothesis of Theorem 5.14. Together, these results relate ideal-theoretic conditions to quotient structure, spectral connectedness, and the interaction of Peirce components.

Author contributions

Conceptualization, A.A. and H.K.; methodology, A.A. and H.K.; formal analysis, A.A. and H.K.; investigation, A.A. and H.K.; validation, A.A. and H.K.; writing—original draft preparation, A.A. and H.K.; writing—review and editing, A.A. and H.K. Both authors contributed equally to this work and have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors state no conflicts of interest.

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