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Research article

Polynomial Gosper multipliers for hypergeometric terms

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Abstract: We study polynomial multipliers that transform hypergeometric terms into Gosper-summable terms. For a broad class of hypergeometric terms whose consecutive-term ratio has a constant numerator and a polynomial denominator, we introduce a linear operator whose image coincides with the full space of polynomial Gosper multipliers, thereby obtaining a complete characterization of all such multipliers. We construct a normalized basis for this space and derive explicit recurrence relations for the basis coefficients and the associated inverse change-of-basis matrices. These results yield closed-form evaluations of the corresponding finite sums and are illustrated by several classical factorial-type hypergeometric terms.

Keywords: hypergeometric terms; Gosper summation; telescoping sums; polynomial multipliers; closed-form evaluations
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1. Introduction

Symbolic summation and telescoping algorithms play an important role in computer algebra and special function theory. A fundamental result in this area is Gosper's algorithm [1, 2], which decides indefinite summability for hypergeometric terms and, when summability holds, constructs a hypergeometric antidifference. Subsequent developments have led to a broad theory of creative telescoping and symbolic summation. Telescoping methods have also been applied to summation formulas and inversion pairs [3], to q-series [4], and, more recently, to hypergeometric multiple sums [5, 6].

Let $\mathbb{K}$ be a field of characteristic zero, $\mathbb{K}[k]$ the polynomial ring in k over $\mathbb{K}$, and $\mathbb{K}(k)$ the field of rational functions in k. A nonzero sequence $(a_k)_{k\geq 0}$ is called a hypergeometric term (over $\mathbb{K}$) if

$$\frac{a_{k+1}}{a_k} \in \mathbb{K}(k).$$

A hypergeometric term a_k is said to be Gosper summable [1, 2] if there exists a rational function $R(k) \in \mathbb{K}(k)$, called a Gosper certificate, such that, with $g_k = R(k)a_k$, one has $a_k = g_{k+1} - g_k$. In this case, the finite sum telescopes:

$$\sum_{k=0}^{n} a_k = g_{n+1} - g_0.$$

Therefore, Gosper summability provides an explicit antidifference and reduces the evaluation of a finite hypergeometric sum to its boundary terms. Such a representation is particularly useful when one seeks structural information in n, rather than a term-by-term evaluation of the sum.

A related but distinct problem arises when the original hypergeometric term a_k is not itself Gosper summable. One may then ask whether multiplying a_k by a suitable rational or polynomial function can make the resulting term Gosper summable. This leads to the study of Gosper-summable multipliers.

Recent work has begun to investigate this multiplier problem systematically. Hou and Li [7] studied the Gosper summability of rational multiples of hypergeometric terms, obtaining structural results and degree bounds for summable multipliers. Dougherty-Bliss [8] obtained explicit polynomial multiplier constructions for several basic Pochhammer-type hypergeometric terms. These works address important questions concerning the existence, degree, and explicit form of Gosper multipliers. Comparatively less attention has been paid to the algebraic structure of the space formed by all polynomial multipliers for a fixed hypergeometric term. For important classes of hypergeometric terms, a systematic description of this vector space and a canonical construction of a basis do not appear to be available.

Motivated by this problem, we consider, for a fixed hypergeometric term a_k, the vector space

$$V(a) = \{p(k) \in \mathbb{K}[k] \mid a_k p(k) \text{ is Gosper summable}\}.$$

Elements of $V(a)$ transform the summand a_k into a telescoping hypergeometric term and hence yield finite sums in closed form. Such polynomials will be called Gosper multipliers for a_k.

The present paper develops a linear-operator framework for characterizing $V(a)$ and constructing normalized bases for this space. We focus on the class of hypergeometric terms satisfying

$$\frac{a_{k+1}}{a_k} = \frac{z}{\prod_{j=1}^{m}(k + c_j + 1)}.$$

For this class, we introduce the linear operator

$$L(q)(k) = q(k)\prod_{j=1}^{m}(k + c_j) - zq(k + 1), \qquad q(k) \in \mathbb{K}[k]$$

and prove that $V(a) = \operatorname{Im} L$. Therefore, every polynomial Gosper multiplier arises from the operator L, which provides a unified framework for constructing polynomial multipliers and deriving explicit telescoping identities.

Indeed, for every polynomial $q(k) \in \mathbb{K}[k]$,

$$\sum_{k=0}^{n} a_k L(q)(k) = q(0)\prod_{j=1}^{m} c_j - za_n q(n + 1).$$

Consequently, every polynomial Gosper multiplier in $V(a)$ is generated by the operator L and admits an explicit telescoping evaluation.

For example, when

$$a_k = \frac{1}{(2k)!},$$

our results completely characterize the vector space of polynomial Gosper multipliers. The normalized basis constructed in this paper provides a systematic description of all such multipliers. One basis element is

$$k^2 - \frac{1}{2}k - \frac{1}{4},$$

from which the telescoping identity

$$\sum_{k=0}^{n} \frac{4k^2 - 2k - 1}{(2k)!} = -\frac{1}{(2n)!}$$

is obtained. Systematic constructions for factorial, odd-factorial, triple-factorial, and squared-factorial hypergeometric terms are presented in Section 5.

Building upon this operator-theoretic characterization, we construct a normalized basis $\{q_r(k)\}_{r\geq m}$ for $V(a)$. This basis provides a canonical family of polynomial Gosper multipliers, admits explicit recursive descriptions, gives rise to explicit change-of-basis matrices, and leads to closed-form evaluations of the corresponding finite sums.

The normalization eliminates all intermediate powers and reduces each basis element to its leading term together with only m lower-degree coefficients. Consequently, the construction of each normalized multiplier is reduced to the recursive determination of a fixed finite set of coefficients.

The paper is organized as follows. In Section 2, we introduce the linear operator associated with hypergeometric terms and prove that, for the class under consideration, its image coincides with the space $V(a)$ of polynomial Gosper multipliers. Section 3 constructs a normalized basis for $V(a)$ and derives recurrence relations for its coefficients. In Section 4, we study the associated change-of-basis matrices and obtain explicit closed-form evaluations. Section 5 presents several illustrative examples and applications. Finally, Section 6 contains concluding remarks and directions for future research.

2. Polynomial multipliers and telescoping operators

In this section, we develop the linear-operator framework underlying the characterization of polynomial Gosper multipliers. We first associate a linear operator with an arbitrary hypergeometric term and show that its image is contained in the corresponding multiplier space $V(a)$. We then specialize to the class

$$a_k = \frac{z^k}{\prod_{j=1}^{m}(c_j + 1)_k},$$

for which the operator provides a complete characterization of $V(a)$. More precisely, we prove that the operator is a linear isomorphism from $\mathbb{K}[k]$ onto $V(a)$. This identification yields a natural basis of $V(a)$, which serves as the starting point for the normalized basis and recurrence relations developed in the next section.

Let (a_k) be a fixed hypergeometric term, so that $a_{k+1}/a_k \in \mathbb{K}(k)$. Choose nonzero polynomials $b_k, w_k \in \mathbb{K}[k]$ such that

$$\frac{a_{k+1}}{a_k} = \frac{b_k}{w_k},$$

and fix this representation throughout the following construction.

For any polynomial $q(k) \in \mathbb{K}[k]$, define the operator $L_0 : \mathbb{K}[k] \to \mathbb{K}[k]$ by

$$L_0(q)(k) = q(k)w_{k-1} - q(k+1)b_k.$$

Proposition 2.1. *With the above notation, one has* $\operatorname{Im} L_0 \subseteq V(a)$.

Proof. Let $q(k) \in \mathbb{K}[k]$, and set

$$p(k) = L_0(q)(k) = q(k)w_{k-1} - q(k+1)b_k.$$

Define

$$g_k = -q(k)w_{k-1}a_k.$$

Then

$$g_{k+1} = -q(k+1)w_k a_{k+1} = -q(k+1)b_k a_k.$$

Hence

$$g_{k+1} - g_k = a_k\big(q(k)w_{k-1} - q(k+1)b_k\big) = a_k p(k).$$

Thus $p(k) \in V(a)$, proving that $\operatorname{Im} L_0 \subseteq V(a)$. □

Since $L_0 : \mathbb{K}[k] \to \mathbb{K}[k]$ is linear, its image $\operatorname{Im} L_0$ is a vector subspace of $V(a)$. For a general hypergeometric term, however, this inclusion need not be an equality.

Remark 2.2. The inclusion $\operatorname{Im} L_0 \subseteq V(a)$ may be strict, even when b_k and w_k are coprime. For example, let

$$a_k = -\frac{k+1}{(k+2)!}, \qquad k \geq 0.$$

Since

$$a_k = \frac{1}{(k+2)!} - \frac{1}{(k+1)!},$$

the term a_k is Gosper summable, and hence $1 \in V(a)$. On the other hand,

$$\frac{a_{k+1}}{a_k} = \frac{k+2}{(k+1)(k+3)},$$

where $k+2$ and $(k+1)(k+3)$ are coprime. We may therefore take

$$b_k = k+2, \qquad w_k = (k+1)(k+3).$$

Since $w_{k-1} = k(k+2)$, the associated operator is

$$L_0(q)(k) = k(k+2)q(k) - (k+2)q(k+1).$$

If $q(k) \neq 0$ has degree r, then

$$\deg L_0(q) = r + 2.$$

Hence every nonzero polynomial in $\operatorname{Im} L_0$ has degree at least 2, so in particular $1 \notin \operatorname{Im} L_0$. Therefore, $\operatorname{Im} L_0 \subsetneq V(a)$.

Remark 2.3. Injectivity need not hold for the operator associated with a general representation

$$\frac{a_{k+1}}{a_k} = \frac{b_k}{w_k}.$$

For example, consider the shifted harmonic term

$$a_k = \frac{1}{k+1}, \qquad k \geq 0,$$

whose partial sums are the harmonic numbers H_{n+1}. Since

$$\frac{a_{k+1}}{a_k} = \frac{k+1}{k+2},$$

we may take

$$b_k = k+1, \qquad w_k = k+2.$$

Then $w_{k-1} = k+1$, and hence

$$L_0(1)(k) = w_{k-1} - b_k = 0.$$

Therefore the operator associated with a general hypergeometric term need not be injective. Together with the preceding remark, this shows that neither injectivity nor surjectivity onto $V(a)$ holds in general. We shall show that both properties hold for the special class considered below.

We now turn to the class for which a complete characterization of $V(a)$ is possible.

Let $z \neq 0$, and let $c_1, \ldots, c_m$ be parameters such that none of them is a negative integer. We define a hypergeometric term $(a_k)_{k\geq 0}$ by

$$\frac{a_{k+1}}{a_k} = \frac{z}{\prod_{j=1}^{m}(k+c_j+1)}, \qquad a_0 = 1. \tag{2.1}$$

Then

$$a_k = \frac{z^k}{(c_1+1)_k \cdots (c_m+1)_k}. \tag{2.2}$$

Here $(x)_k = x(x+1)\cdots(x+k-1)$ denotes the Pochhammer symbol. The sequence a_k is closely related to the coefficients of the generalized hypergeometric series ${}_0F_m$, up to a missing factorial factor.

This family provides a broad class of hypergeometric terms whose ratio a_{k+1}/a_k has a denominator of arbitrary degree $m \geq 1$ in k and a constant numerator.

For the present class,

$$b_k = z, \qquad w_k = \prod_{j=1}^{m}(k+c_j+1).$$

Hence $w_{k-1} = \prod_{j=1}^{m}(k+c_j)$. Specializing the general operator L_0 to this choice of b_k and w_k, we obtain the associated operator

$$L(q)(k) = q(k)\prod_{j=1}^{m}(k+c_j) - z\,q(k+1).$$

Unlike the general operator L_0, which satisfies only $\operatorname{Im} L_0 \subseteq V(a)$, the operator L completely characterizes the space of polynomial Gosper multipliers for the present class. More precisely, we shall prove that

$$V(a) = \operatorname{Im} L,$$

so that every polynomial Gosper multiplier arises uniquely from the operator L.

Theorem 2.4. *Let $m \geq 1$, let $z \neq 0$, and let $c_1, \ldots, c_m$ be such that none of them is a negative integer. For the hypergeometric term*

$$a_k = \frac{z^k}{\prod_{j=1}^{m}(c_j+1)_k},$$

the associated operator $L : \mathbb{K}[k] \longrightarrow V(a)$ is a linear isomorphism. Equivalently,

$$\operatorname{Im} L = V(a) \qquad \textit{and} \qquad \ker L = \{0\}.$$

Proof. We first prove that L is injective. Let $q(k) \in \mathbb{K}[k]$ be nonzero, with $\deg q = r \geq 0$, and let α_r denote its leading coefficient. Then

$$q(k)\prod_{j=1}^{m}(k+c_j)$$

has leading term $\alpha_r k^{r+m}$, whereas $zq(k+1)$ has degree r. Hence

$$\deg L(q) = r + m,$$

and therefore $L(q) \neq 0$. Thus

$$\ker L = \{0\}.$$

It remains to prove that L is surjective onto $V(a)$. The inclusion

$$\operatorname{Im} L \subseteq V(a)$$

was established above, so it suffices to prove the reverse inclusion. Let $p(k) \in V(a)$. Since $a_k p(k)$ is Gosper summable, there exists $R(k) \in \mathbb{K}(k)$ such that

$$a_k p(k) = R(k+1)a_{k+1} - R(k)a_k,$$

where the polynomial factor $p(k)$ has been absorbed into the rational Gosper certificate. Hence, using Eq (2.1),

$$p(k) = \frac{zR(k+1)}{\prod_{j=1}^{m}(k+c_j+1)} - R(k).$$

Define

$$q(k) = -\frac{R(k)}{\prod_{j=1}^{m}(k+c_j)}.$$

Then

$$p(k) = q(k)\prod_{j=1}^{m}(k+c_j) - zq(k+1) = L(q)(k).$$

It remains to show that $q(k) \in \mathbb{K}[k]$.

Write

$$q(k) = \frac{N(k)}{D(k)},$$

where $N, D \in \mathbb{K}[k]$, $\gcd(N, D) = 1$, and D is monic. Since $p(k) = L(q)(k)$ is a polynomial, we have

$$p(k)D(k)D(k+1) = N(k)D(k+1)\prod_{j=1}^{m}(k+c_j) - zN(k+1)D(k).$$

Reducing this identity modulo $D(k+1)$, we obtain

$$zN(k+1)D(k) \equiv 0 \pmod{D(k+1)}.$$

Since $z \neq 0$ and

$$\gcd(N(k+1), D(k+1)) = 1,$$

it follows that

$$D(k+1) \mid D(k).$$

The polynomials $D(k)$ and $D(k+1)$ are monic and have the same degree; hence

$$D(k+1) = D(k).$$

Let $d = \deg D$. If $d \geq 1$, write

$$D(k) = k^d + a_{d-1}k^{d-1} + \cdots.$$

Then

$$D(k+1) = k^d + (d + a_{d-1})k^{d-1} + \cdots.$$

Since $\mathbb{K}$ has characteristic zero, the identity $D(k+1) = D(k)$ forces $d = 0$. Thus D is constant, and hence $q(k) \in \mathbb{K}[k]$.

Therefore $p(k) = L(q)(k) \in \operatorname{Im} L$, proving that

$$V(a) \subseteq \operatorname{Im} L.$$

Consequently,

$$V(a) = \operatorname{Im} L.$$

Together with $\ker L = \{0\}$, this shows that L is a linear isomorphism from $\mathbb{K}[k]$ onto $V(a)$. □

For later use, we write

$$\prod_{\ell=1}^{m}(k + c_\ell) = \sum_{\ell=0}^{m} e_\ell k^{m-\ell}, \tag{2.3}$$

where $e_\ell = e_\ell(c_1, \ldots, c_m)$ denotes the ℓ-th elementary symmetric polynomial in $c_1, \ldots, c_m$. We use the conventions

$$e_0 = 1, \qquad e_\ell = 0 \quad \text{for } \ell > m.$$

In particular,

$$e_m = \prod_{\ell=1}^{m} c_\ell.$$

Since $L(q)(k) \in V(a)$ for every polynomial $q(k)$, the corresponding telescoping relation yields the following explicit finite-sum evaluation.

Proposition 2.5. *For every polynomial $q(k) \in \mathbb{K}[k]$, the identity*

$$\sum_{k=0}^{n} a_k L(q)(k) = q(0)e_m - z\, a_n\, q(n+1) \tag{2.4}$$

holds. Equivalently,

$$\sum_{k=0}^{n} a_k L(q)(k) = q(0)e_m - \frac{z^{n+1} q(n+1)}{\prod_{\ell=1}^{m}(c_\ell+1)_n}.$$

Proof. Define

$$G_k = q(k)\, a_k \prod_{\ell=1}^{m}(k+c_\ell).$$

Using Eq (2.1), we obtain

$$G_{k+1} = q(k+1)a_{k+1} \prod_{\ell=1}^{m}(k+c_\ell+1) = z\, a_k\, q(k+1).$$

Hence,

$$G_k - G_{k+1} = a_k \left(q(k) \prod_{\ell=1}^{m}(k+c_\ell) - z\, q(k+1) \right) = a_k L(q)(k).$$

Summing from $k = 0$ to n, we obtain

$$\sum_{k=0}^{n} a_k L(q)(k) = G_0 - G_{n+1}.$$

Now

$$G_0 = q(0)\, a_0 \prod_{\ell=1}^{m} c_\ell = q(0)e_m.$$

Moreover,

$$G_{n+1} = q(n+1)a_{n+1} \prod_{\ell=1}^{m}(n+1+c_\ell) = z\, a_n\, q(n+1).$$

This proves the result. □

We next apply the linear isomorphism L to the monomial basis of $\mathbb{K}[k]$. Since

$$L : \mathbb{K}[k] \to V(a)$$

is a linear isomorphism, the image

$$\{L(k^r)\}_{r\geq 0}$$

of the monomial basis forms a natural basis of $V(a)$. We refer to it as the natural basis of $V(a)$. This basis serves as the starting point for constructing the normalized basis in the next section.

In particular, taking $q(k) = k^r$ with $r \geq 0$ in Proposition 2.5, we obtain

$$\sum_{k=0}^{n} a_k L(k^r) = e_m \delta_{0,r} - z\, a_n\, (n+1)^r = e_m \delta_{0,r} - \frac{z^{n+1}(n+1)^r}{\prod_{\ell=1}^{m}(c_\ell+1)_n}, \tag{2.5}$$

where $\delta_{i,j}$ denotes the Kronecker delta, with the convention that $0^0 = 1$.

3. Normalized bases and recurrence relations

In the previous section, we proved that $L : \mathbb{K}[k] \longrightarrow V(a)$ is a linear isomorphism. Consequently, the image of the monomial basis $\{k^r\}_{r\geq 0}$ under L, namely $\{L(k^r)\}_{r\geq 0}$, forms a natural basis of $V(a)$.

For each $r \geq 0$, the polynomial

$$L(k^r) = k^r \prod_{j=1}^{m}(k + c_j) - z(k + 1)^r$$

is monic of degree $r + m$. This triangular degree structure allows us to replace the natural basis by a uniquely determined normalized basis.

Proposition 3.1. *For every integer $r \geq m$, there exists a unique polynomial $q_r(k) \in V(a)$ of the form*

$$q_r(k) = k^r + p_r(k), \qquad \deg p_r \leq m - 1.$$

Moreover, the family $\{q_r(k)\}_{r\geq m}$ forms a basis of $V(a)$.

Proof. For each $j \geq 0$, the polynomial $L(k^j)$ is monic of degree $m+j$. Hence the natural basis $\{L(k^j)\}_{j\geq 0}$ is unit triangular with respect to the monomials $k^m, k^{m+1}, \ldots$. Successively eliminating the terms of degrees $r-1, r-2, \ldots, m$ from $L(k^{r-m})$ yields a unique polynomial $q_r(k) = k^r + p_r(k)$ with $\deg p_r \leq m-1$. The resulting change-of-basis transformation is unit triangular, and therefore $\{q_r(k)\}_{r\geq m}$ is again a basis of $V(a)$. □

Writing

$$p_r(k) = \sum_{i=0}^{m-1} s_i(r)k^i,$$

we obtain the normalized basis elements in the form

$$q_r(k) = k^r + \sum_{i=0}^{m-1} s_i(r)k^i.$$

Therefore, the construction of $q_r(k)$ reduces to determining the m lower-degree coefficients $s_i(r)$.

We next determine the expansion of the natural basis $\{L(k^r)\}_{r\geq 0}$ with respect to the normalized basis $\{q_j(k)\}_{j\geq m}$. The resulting coefficients will subsequently yield recurrence relations for the lower-degree coefficients $s_i(r)$.

Proposition 3.2. *Let $m \geq 1$ and $r \geq 0$. In the expansion of $L(k^r)$ with respect to the basis $\{q_j(k)\}_{j\geq m}$,*

$$L(k^r) = \sum_{j=m}^{m+r} \alpha_{r,j}q_j(k),$$

and the coefficients are given by

$$\alpha_{r,j} = e_{m+r-j} - z\binom{r}{j}, \qquad m \leq j \leq m + r,$$

where we adopt the convention $\binom{r}{j} = 0$ *for* $j > r$. *Equivalently,*

$$L(k^r) = \sum_{j=m}^{m+r} \left(e_{m+r-j} - z\binom{r}{j} \right) q_j(k).$$

Proof. By definition,

$$L(k^r) = k^r \prod_{\ell=1}^{m} (k + c_\ell) - z(k+1)^r.$$

Using Eq (2.3) and $(k+1)^r = \sum_{\ell=0}^{r} \binom{r}{\ell} k^\ell$, we obtain

$$L(k^r) = \sum_{\ell=0}^{m} e_\ell k^{m+r-\ell} - z \sum_{\ell=0}^{r} \binom{r}{\ell} k^\ell.$$

Since

$$q_j(k) = k^j + \sum_{i=0}^{m-1} s_i(j) k^i,$$

the coefficient of k^j in the expansion with respect to the normalized basis is determined entirely by the coefficient of k^j in $L(k^r)$ for $m \le j \le m + r$. Hence

$$\alpha_{r,j} = e_{m+r-j} - z\binom{r}{j}, \qquad m \le j \le m + r.$$

This proves the assertion. □

The coefficients arising from the expansion of the natural basis $\{L(k^r)\}$ in the normalized basis $\{q_j(k)\}$ define a change-of-basis matrix A for $V(a)$. Its inverse $B = A^{-1}$ expresses the normalized basis in terms of the natural basis.

We now determine the coefficients $s_i(r)$ by comparing the lower-degree terms in the expansion of $L(k^r)$. This yields the following recurrence relation.

Proposition 3.3. *For integers* $r \ge 0$ *and* $0 \le i \le m - 1$, *the coefficients* $s_i(r)$ *satisfy the recurrence*

$$s_i(r + m) = -\sum_{\ell=1}^{m} e_\ell \, s_i(r + m - \ell) + z \sum_{j=0}^{r} \binom{r}{j} s_i(j),$$

with initial values

$$s_i(j) = -\delta_{i,j}, \qquad 0 \le j \le m - 1.$$

Proof. Using Proposition 3.2, we substitute the expression

$$\alpha_{r,j} = e_{m+r-j} - z\binom{r}{j}$$

into

$$\sum_{\ell=0}^{m} e_\ell k^{m+r-\ell} - z \sum_{j=0}^{r} \binom{r}{j} k^j = \sum_{j=m}^{m+r} \alpha_{r,j} \left(k^j + \sum_{i=0}^{m-1} s_i(j) k^i \right),$$

and comparing the coefficients of k^i for $0 \le i \le m-1$, we obtain

$$-z\binom{r}{i} = \sum_{j=m}^{m+r}\left(e_{m+r-j} - z\binom{r}{j}\right)s_i(j), \qquad 0 \le i \le m-1.$$

Separating the term $j = m + r$ and using $e_0 = 1$ and $\binom{r}{m+r} = 0$, this yields

$$s_i(m+r) = -z\binom{r}{i} - \sum_{j=m}^{m+r-1}\left(e_{m+r-j} - z\binom{r}{j}\right)s_i(j).$$

Although the normalized basis $q_r(k)$ is defined only for $r \ge m$, it is convenient to extend the coefficients $s_i(r)$ to $0 \le r \le m-1$ by setting

$$s_i(r) = -\delta_{i,r}, \qquad 0 \le r \le m-1.$$

With this convention, the recurrence relation can be written in the uniform form

$$s_i(r+m) = -\sum_{\ell=1}^{m} e_\ell\, s_i(r+m-\ell) + z\sum_{j=0}^{r}\binom{r}{j}s_i(j), \qquad r \ge 0,\ 0 \le i \le m-1.$$

□

Define

$$S_i(t) = \sum_{r\ge 0} s_i(r)\frac{t^r}{r!}.$$

Then $S_i(t)$ satisfies the differential equation

$$\left(\prod_{\ell=1}^{m}\left(\frac{d}{dt} + c_\ell\right)\right)S_i(t) = \sum_{j=0}^{m} e_j S_i^{(m-j)}(t) = ze^t S_i(t), \tag{3.1}$$

with initial conditions

$$S_i^{(j)}(0) = -\delta_{i,j}, \qquad 0 \le j \le m-1.$$

Although these differential equations admit explicit closed-form solutions in certain special cases, such expressions are often less suitable for coefficient extraction than the recurrence relations. Accordingly, the recurrence for $s_i(r)$ will serve as the primary computational tool for constructing the normalized basis in the examples of Section 5.

In the next section, we study the associated change-of-basis matrices, which provide explicit representations of the normalized basis and lead to closed-form evaluations of the corresponding finite sums.

4. Matrix representations and explicit evaluations

To obtain explicit closed-form evaluations, we study the transition matrices between the natural basis $\{L(k^r)\}_{r\geq 0}$ and the normalized basis $\{q_{m+r}(k)\}_{r\geq 0}$.

For convenience, we reindex the basis elements by setting

$$P_r(k) := L(k^r), \qquad Q_r(k) := q_{m+r}(k), \qquad r \geq 0.$$

From the change-of-basis formula obtained above, we have

$$P_r(k) = \sum_{i=0}^{r} A_{r,i} Q_i(k),$$

where

$$A_{r,i} = e_{r-i} - z\binom{r}{m+i}, \qquad 0 \leq i \leq r.$$

Here we use the conventions $e_0 = 1$, $e_\ell = 0$ for $\ell > m$ and $\binom{r}{s} = 0$ for $s > r$.

Since

$$A_{r,r} = 1,$$

the matrix $A = (A_{r,i})_{r,i\geq 0}$ is unit lower triangular and therefore invertible, with a unit lower triangular inverse.

Let

$$B = A^{-1} = (B_{i,j})_{i,j\geq 0}.$$

Then

$$Q_i(k) = \sum_{j=0}^{i} B_{i,j} P_j(k).$$

Therefore, the inverse matrix B expresses the normalized basis $\{Q_i(k)\}$ in terms of the natural basis $\{P_i(k)\}$, making it possible to transfer the telescoping evaluations of $P_i(k)$ to the normalized basis.

We now derive a recurrence relation for the entries $B_{i,j}$.

Proposition 4.1. *The entries of $B = A^{-1}$ satisfy*

$$B_{i,i} = 1, \qquad B_{i,j} = 0 \quad (i < j),$$

where the latter convention is extended by setting $B_{p,j} = 0$ whenever $p < j$ for $p \in \mathbb{Z}$ and $j \geq 0$. For $i > j$,

$$B_{i,j} = -\sum_{\ell=1}^{m} e_\ell B_{i-\ell,j} + z \sum_{k=0}^{i-j-m} \binom{i}{k} B_{i-m-k,j},$$

where an empty sum is understood to be zero.

Proof. Since $AB = I$, we have

$$\sum_{\ell=j}^{i} A_{i,\ell} B_{\ell,j} = \delta_{i,j}.$$

For $i = j$, this gives $B_{i,i} = 1$. For $i > j$, since $A_{i,i} = 1$, we obtain

$$B_{i,j} = -\sum_{\ell=j}^{i-1} A_{i,\ell} B_{\ell,j}.$$

Substituting the expression for $A_{i,\ell}$, we get

$$B_{i,j} = -\sum_{\ell=j}^{i-1} \left(e_{i-\ell} - z\binom{i}{m+\ell} \right) B_{\ell,j}.$$

Replacing ℓ by $i - \ell$, this becomes

$$B_{i,j} = -\sum_{\ell=1}^{i-j} \left(e_{\ell} - z\binom{i}{m+i-\ell} \right) B_{i-\ell,j}.$$

We now separate the two terms in the summation:

$$B_{i,j} = -\sum_{\ell=1}^{i-j} e_{\ell} B_{i-\ell,j} + z\sum_{\ell=1}^{i-j} \binom{i}{m+i-\ell} B_{i-\ell,j}.$$

Since $e_\ell = 0$ for $\ell > m$, and since $B_{p,j} = 0$ whenever $p < j$, the first sum may be written as

$$-\sum_{\ell=1}^{m} e_{\ell} B_{i-\ell,j}.$$

For the second sum, we use the symmetry of binomial coefficients $\binom{i}{r} = \binom{i}{i-r}$ to obtain

$$\binom{i}{m+i-\ell} = \binom{i}{\ell-m}.$$

Set $k = \ell - m$. Then $\ell = m + k$, and

$$B_{i-\ell,j} = B_{i-m-k,j}.$$

The range $1 \le \ell \le i - j$ becomes $1 - m \le k \le i - j - m$. Hence

$$z\sum_{\ell=1}^{i-j} \binom{i}{m+i-\ell} B_{i-\ell,j} = z\sum_{k=1-m}^{i-j-m} \binom{i}{k} B_{i-m-k,j}.$$

Since $\binom{i}{k} = 0$ for $k < 0$, this reduces to

$$z\sum_{k=0}^{i-j-m} \binom{i}{k} B_{i-m-k,j}.$$

This completes the proof. □

The recurrence relation for $B_{i,j}$ exhibits a convolutional structure analogous to that satisfied by the coefficients $s_i(r)$ in Proposition 3.3.

Combining the inverse change-of-basis formula with the general telescoping identity in Proposition 2.5, we obtain the following explicit summation formula for the normalized basis $\{q_r(k)\}_{r\geq m}$.

Theorem 4.2. *For every integer $r \geq m$,*

$$\sum_{k=0}^{n} a_k q_r(k) = e_m B_{r-m,0} - z\, a_n \sum_{j=0}^{r-m} B_{r-m,j}(n+1)^j.$$

Equivalently,

$$\sum_{k=0}^{n} a_k q_r(k) = e_m B_{r-m,0} - \frac{z^{n+1}}{\prod_{\ell=1}^{m}(c_\ell+1)_n} \sum_{j=0}^{r-m} B_{r-m,j}(n+1)^j.$$

Proof. By the inverse change-of-basis formula and the linearity of L,

$$q_r(k) = \sum_{j=0}^{r-m} B_{r-m,j} L(k^j) = L\left(\sum_{j=0}^{r-m} B_{r-m,j} k^j\right).$$

Set

$$h_r(k) = \sum_{j=0}^{r-m} B_{r-m,j} k^j.$$

Then $q_r = L(h_r)$. Applying Proposition 2.5, we obtain

$$\sum_{k=0}^{n} a_k q_r(k) = e_m h_r(0) - z\, a_n h_r(n+1).$$

Since $h_r(0) = B_{r-m,0}$, the desired formula follows. □

Since $\{q_r(k)\}_{r\geq m}$ is a basis of $V(a)$, the theorem, together with linearity, yields an explicit finite-sum evaluation for every polynomial Gosper multiplier in $V(a)$.

5. Examples and applications

In this section, we illustrate the main results through five classical hypergeometric terms. For each example, we specialize the recurrence relations for the coefficients $s_i(r)$, describe the corresponding exponential generating functions, and construct explicit normalized polynomial multipliers $q_r(k)$. These constructions then yield closed-form telescoping evaluations of the associated finite sums. The examples include factorial, even- and odd-factorial, triple-factorial, and squared-factorial denominators, thereby illustrating the scope of the general framework.

The case $a_k = \frac{1}{k!}$ is closely related to Bell numbers and has also appeared in the context of Gosper summation; see, for example, [8]. We begin with this classical example to illustrate the general construction.

Example 1. *Let $m = 1$, $z = 1$, and $c_1 = 0$. Then*

$$a_k = \frac{1}{k!}.$$

In this case,

$$q_r(k) = k^r + s_0(r), \qquad r \geq 1.$$

By Proposition 3.3, the coefficients $s_0(r)$ satisfy

$$s_0(r+1) = \sum_{j=0}^{r} \binom{r}{j} s_0(j),$$

with initial value

$$s_0(0) = -1.$$

The corresponding exponential generating function

$$S_0(t) = \sum_{r \geq 0} s_0(r) \frac{t^r}{r!}$$

satisfies

$$S_0'(t) = e^t S_0(t), \qquad S_0(0) = -1.$$

Hence

$$S_0(t) = -\exp(e^t - 1).$$

Therefore

$$s_0(r) = -\mathcal{B}_r,$$

where $\mathcal{B}_r$ denotes the r-th Bell number. Thus

$$q_r(k) = k^r - \mathcal{B}_r.$$

The first few normalized multipliers are

$$q_1(k) = k - 1, \quad q_2(k) = k^2 - 2, \quad q_3(k) = k^3 - 5, \quad \textit{and} \quad q_4(k) = k^4 - 15.$$

The coefficients $B_{i,j}$ satisfy

$$B_{i,j} = \sum_{k=0}^{i-j-1} \binom{i}{k} B_{i-1-k,j}.$$

Interestingly, the inverse matrix also produces several classical integer sequences. The first column of the inverse matrix $B = (B_{i,j})$ yields the Gould numbers

$$1, 1, 3, 9, 31, 121, 523, \ldots,$$

which coincide with sequence A040027 in the OEIS [9].

By Theorem 4.2, we obtain

$$\sum_{k=0}^{n} \frac{k-1}{k!} = -\frac{1}{n!},$$

$$\sum_{k=0}^{n} \frac{k^2-2}{k!} = -\frac{n+2}{n!},$$

$$\sum_{k=0}^{n} \frac{k^3-5}{k!} = -\frac{n^2+3n+5}{n!},$$

and

$$\sum_{k=0}^{n} \frac{k^4-15}{k!} = -\frac{n^3+4n^2+9n+15}{n!}.$$

The next example concerns even factorials and leads to a second-order recurrence structure.

Example 2. *Let $m = 2$, $z = \frac{1}{4}$, and*

$$(c_1, c_2) = \left(-\frac{1}{2}, 0\right).$$

Then

$$a_k = \frac{1}{(2k)!}.$$

In this case,

$$q_r(k) = k^r + s_0(r) + s_1(r)k, \qquad r \geq 2.$$

By Proposition 3.3, the coefficients $s_0(r)$ and $s_1(r)$ satisfy

$$s_i(r+2) = \frac{1}{2}s_i(r+1) + \frac{1}{4}\sum_{j=0}^{r}\binom{r}{j}s_i(j), \qquad i = 0, 1,$$

with initial values

$$s_i(j) = -\delta_{i,j}, \qquad 0 \leq j \leq 1.$$

The exponential generating functions $S_0(t)$ and $S_1(t)$ satisfy

$$\left(\frac{d}{dt} - \tfrac{1}{2}\right)\frac{d}{dt}S_i(t) = \frac{1}{4}e^t S_i(t),$$

with

$$S_0(t) = -\cosh(1 - e^{t/2})$$

and

$$S_1(t) = 2\sinh(1 - e^{t/2}).$$

Using the generating functions above, the coefficients $s_0(r)$ and $s_1(r)$ can be expressed in terms of Bell numbers and Touchard polynomials [10]. Recall that $T_r(1) = \mathcal{B}_r$, where $T_r(x)$ denotes the r-th Touchard polynomial defined by

$$\sum_{n=0}^{\infty} \frac{T_n(x)}{n!}t^n = \exp(x(e^t - 1)).$$

Consequently,

$$2^r q_r(k) = 2^r k^r + (T_r(-1) - \mathcal{B}_r)k - \frac{T_r(-1) + \mathcal{B}_r}{2}.$$

The first few normalized multipliers are

$$4q_2(k) = 4k^2 - 2k - 1, \quad 8q_3(k) = 8k^3 - 4k - 3, \quad 8q_4(k) = 8k^4 - 7k - 4,$$

and

$$32q_5(k) = 32k^5 - 54k - 25.$$

The coefficients $B_{i,j}$ satisfy

$$B_{i,j} = \frac{1}{2}B_{i-1,j} + \frac{1}{4}\sum_{k=0}^{i-j-2}\binom{i}{k}B_{i-2-k,j}.$$

Applying Theorem 4.2, we obtain

$$\sum_{k=0}^{n}\frac{4k^2 - 2k - 1}{(2k)!} = -\frac{1}{(2n)!},$$

$$\sum_{k=0}^{n}\frac{8k^3 - 4k - 3}{(2k)!} = -\frac{2n + 3}{(2n)!},$$

$$\sum_{k=0}^{n}\frac{8k^4 - 7k - 4}{(2k)!} = -\frac{2n^2 + 5n + 4}{(2n)!},$$

and

$$\sum_{k=0}^{n}\frac{32k^5 - 54k - 25}{(2k)!} = -\frac{8n^3 + 28n^2 + 36n + 25}{(2n)!}.$$

The odd-factorial case produces a different family of normalized multipliers, reflecting the asymmetry introduced by the shifted factorial denominator.

Example 3. *Let $m = 2$, $z = \frac{1}{4}$, and $(c_1, c_2) = (1/2, 0)$. Then*

$$a_k = \frac{1}{(2k + 1)!}.$$

For this hypergeometric term, the normalized basis has the form

$$q_r(k) = k^r + s_0(r) + s_1(r)k, \qquad r \geq 2.$$

By Proposition 3.3, the coefficients $s_0(r)$ and $s_1(r)$ satisfy the recurrence

$$s_i(r + 2) = -\frac{1}{2}s_i(r + 1) + \frac{1}{4}\sum_{j=0}^{r}\binom{r}{j}s_i(j), \qquad i = 0, 1,$$

with initial values $s_i(j) = -\delta_{i,j}$ for $j = 0, 1$. The corresponding exponential generating functions $S_0(t)$ and $S_1(t)$ satisfy the differential equation

$$S_i''(t) + \frac{1}{2}S_i'(t) = \frac{1}{4}e^t S_i(t), \qquad S_i^{(j)}(0) = -\delta_{i,j}, \qquad j = 0, 1.$$

We obtain

$$S_0(t) = -e^{-t/2}e^{e^{t/2}-1}, \qquad S_1(t) = 2e^{-t/2}\sinh(1 - e^{t/2}).$$

This implies that

$$s_0(r) = \frac{1}{2^r}\sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k+1}\mathcal{B}_k,$$
$$s_1(r) = \frac{1}{2^r}\sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}\left(T_k(-1) - \mathcal{B}_k\right).$$

Therefore, the first few normalized multipliers are

$$q_2(k) = k^2 + \frac{k}{2} - \frac{1}{4}, \qquad q_3(k) = k^3 - \frac{k}{2} - \frac{1}{8},$$
$$q_4(k) = k^4 - \frac{k}{8} - \frac{1}{4}, \qquad \text{and} \quad q_5(k) = k^5 - \frac{7k}{16} - \frac{11}{32}.$$

The coefficients $B_{i,j}$ satisfy

$$B_{i,j} = -\frac{1}{2}B_{i-1,j} + \frac{1}{4}\sum_{k=0}^{i-j-2}\binom{i}{k}B_{i-2-k,j}, \qquad i > j,$$

with $B_{i,i} = 1$ and $B_{i,j} = 0$ for $i < j$.

Theorem 4.2 then yields

$$\sum_{k=0}^{n}\frac{4k^2 + 2k - 1}{(2k+1)!} = -\frac{1}{(2n+1)!}, \qquad \sum_{k=0}^{n}\frac{8k^3 - 4k - 1}{(2k+1)!} = -\frac{1}{(2n)!},$$
$$\sum_{k=0}^{n}\frac{8k^4 - k - 2}{(2k+1)!} = -\frac{2n^2 + 3n + 2}{(2n+1)!}, \qquad \text{and} \quad \sum_{k=0}^{n}\frac{32k^5 - 14k - 11}{(2k+1)!} = -\frac{8n^3 + 20n^2 + 20n + 11}{(2n+1)!}.$$

The next example illustrates how higher-order factorial growth leads to polynomial multipliers of increasing algebraic complexity.

Example 4. *Let $m = 3$, $z = \frac{1}{27}$, and $(c_1, c_2, c_3) = (-2/3, -1/3, 0)$. Then*

$$a_k = \frac{1}{(3k)!}.$$

The normalized multipliers are

$$q_r(k) = k^r + s_0(r) + s_1(r)k + s_2(r)k^2, \qquad r \geq 3.$$

By Proposition 3.3, the coefficients $s_0(r)$, $s_1(r)$, and $s_2(r)$ satisfy the recurrence

$$s_i(r+3) = s_i(r+2) - \frac{2}{9}s_i(r+1) + \frac{1}{27}\sum_{j=0}^{r}\binom{r}{j}s_i(j), \qquad i = 0, 1, 2,$$

with initial values $s_i(j) = -\delta_{i,j}$ for $j = 0, 1, 2$. The corresponding exponential generating functions $S_0(t)$, $S_1(t)$, and $S_2(t)$ satisfy the differential equation

$$S_i'''(t) - S_i''(t) + \frac{2}{9}S_i'(t) = \frac{1}{27}e^t S_i(t), \qquad S_i^{(j)}(0) = -\delta_{i,j}, \qquad j = 0, 1, 2.$$

Unlike the previous examples, the differential equation does not appear to admit a simple closed-form solution. Nevertheless, the recurrence relations remain effective for computing the normalized multipliers. The first few normalized multipliers are

$$q_3(k) = k^3 - k^2 + \frac{2}{9}k - \frac{1}{27}, \qquad q_4(k) = k^4 - \frac{7}{9}k^2 + \frac{5}{27}k - \frac{2}{27},$$
$$q_5(k) = k^5 - \frac{16}{27}k^2 + \frac{5}{81}k - \frac{25}{243}, \qquad \text{and} \quad q_6(k) = k^6 - \frac{46}{81}k^2 - \frac{20}{243}k - \frac{91}{729}.$$

The coefficients $B_{i,j}$ satisfy

$$B_{i,j} = B_{i-1,j} - \frac{2}{9}B_{i-2,j} + \frac{1}{27}\sum_{k=0}^{i-j-3}\binom{i}{k}B_{i-3-k,j}, \qquad i > j,$$

with $B_{i,i} = 1$ and $B_{i,j} = 0$ for $i < j$.

Theorem 4.2 then yields

$$\sum_{k=0}^{n}\frac{27k^3 - 27k^2 + 6k - 1}{(3k)!} = -\frac{1}{(3n)!},$$
$$\sum_{k=0}^{n}\frac{27k^4 - 21k^2 + 5k - 2}{(3k)!} = -\frac{n+2}{(3n)!},$$
$$\sum_{k=0}^{n}\frac{243k^5 - 144k^2 + 15k - 25}{(3k)!} = -\frac{9n^2 + 27n + 25}{(3n)!}, \quad \text{and}$$
$$\sum_{k=0}^{n}\frac{729k^6 - 414k^2 - 60k - 91}{(3k)!} = -\frac{27n^3 + 108n^2 + 156n + 91}{(3n)!}.$$

We now turn to hypergeometric terms involving squared factorial denominators, which are closely related to Bessel-type generating functions.

Example 5. *Let $m = 2$, $z = 1$, and $(c_1, c_2) = (0, 0)$. Then*

$$a_k = \frac{1}{(k!)^2}.$$

The normalized multipliers are

$$q_r(k) = k^r + s_0(r) + s_1(r)k, \qquad r \geq 2.$$

By Proposition 3.3, the coefficients $s_0(r)$ and $s_1(r)$ satisfy the recurrence

$$s_i(r+2) = \sum_{j=0}^{r} \binom{r}{j} s_i(j), \qquad r \geq 0,$$

with initial values $s_i(j) = -\delta_{i,j}$ for $j = 0, 1$. The exponential generating functions $S_0(t)$ and $S_1(t)$ satisfy

$$S_i''(t) = e^t S_i(t), \qquad i = 0, 1,$$

with initial conditions

$$S_0(0) = -1, \qquad S_0'(0) = 0$$

and

$$S_1(0) = 0, \qquad S_1'(0) = -1.$$

The corresponding generating functions admit the following representations in terms of modified Bessel functions:

$$S_0(t) = -2\left[I_1(2)K_0\left(2e^{t/2}\right) + K_1(2)I_0\left(2e^{t/2}\right)\right]$$

and

$$S_1(t) = 2\left[I_0(2)K_0\left(2e^{t/2}\right) - K_0(2)I_0\left(2e^{t/2}\right)\right],$$

where I_ν and K_ν denote the modified Bessel functions of the first and second kinds, respectively.

The Bessel-function representations reveal that the generating functions belong naturally to the class of modified Bessel functions. For the computation of the coefficients $s_i(r)$, however, the recurrence relations remain considerably more effective. The first three normalized multipliers are

$$q_2(k) = k^2 - 1, \quad q_3(k) = k^3 - k - 1, \quad \textit{and} \quad q_4(k) = k^4 - 2k - 2.$$

The coefficients $B_{i,j}$ satisfy

$$B_{i,j} = \sum_{k=0}^{i-j-2} \binom{i}{k} B_{i-2-k,j}, \qquad i > j,$$

with $B_{i,i} = 1$ and $B_{i,j} = 0$ for $i < j$.

Theorem 4.2 then yields

$$\sum_{k=0}^{n} \frac{k^2-1}{(k!)^2} = -\frac{1}{(n!)^2}, \quad \sum_{k=0}^{n} \frac{k^3-k-1}{(k!)^2} = -\frac{n+1}{(n!)^2}, \quad \textit{and} \quad \sum_{k=0}^{n} \frac{k^4-2k-2}{(k!)^2} = -\frac{n^2+2n+2}{(n!)^2}.$$

In each case, the recurrence relations provide a systematic and computationally effective method for constructing the normalized basis of $V(a)$, thereby producing explicit polynomial Gosper multipliers and closed-form telescoping evaluations of the corresponding finite sums.

Taken together, these examples demonstrate that the linear-operator framework developed in this paper applies uniformly to a broad class of factorial-type hypergeometric terms. For each family, it provides a systematic construction of the normalized basis of $V(a)$, and hence of the entire space of polynomial Gosper multipliers, together with explicit recurrence relations and closed-form telescoping evaluations within a unified algebraic framework.

6. Conclusions

In this paper, we studied the vector space $V(a)$ of polynomial Gosper multipliers associated with a hypergeometric term a_k. For the class of hypergeometric terms satisfying

$$\frac{a_{k+1}}{a_k} = \frac{z}{\prod_{j=1}^{m}(k + c_j + 1)},$$

we introduced a linear operator L and proved that $V(a) = \operatorname{Im} L$. This equality provides a complete characterization of the space of polynomial Gosper multipliers associated with the given hypergeometric term, namely, the polynomial multipliers that transform the summand into a telescoping one.

We further constructed a normalized basis

$$q_r(k) = k^r + \sum_{i=0}^{m-1} s_i(r) k^i$$

for $V(a)$. The coefficients defining this basis satisfy explicit recurrence relations together with differential equations for their exponential generating functions. We also derived recurrence relations for the inverse change-of-basis matrix, leading systematically to explicit closed-form evaluations of the finite sums $\sum_{k=0}^{n} a_k q_r(k)$.

The examples illustrate that the proposed framework applies uniformly to several classical families of factorial-type hypergeometric terms. In each case, it systematically constructs the normalized basis of $V(a)$, leading to explicit polynomial Gosper multipliers and closed-form telescoping evaluations.

Several problems remain open. One natural direction is to investigate whether analogous linear-operator frameworks and normalized bases can be developed for more general hypergeometric terms whose consecutive- term ratios have nonconstant numerators. It would also be interesting to study possible connections with symbolic summation algorithms, special functions, and combinatorial generating functions.

The operator-theoretic approach developed in this paper shows that the space of polynomial Gosper multipliers possesses a remarkably simple linear structure. This viewpoint not only provides a systematic construction of normalized bases and explicit telescoping formulas, but also suggests that similar operator-theoretic descriptions may exist for broader classes of hypergeometric terms.

This operator-theoretic framework may provide a useful foundation for the systematic study of polynomial Gosper multipliers and for future developments in symbolic summation for more general classes of hypergeometric terms.

Use of Generative-AI tools declaration

The author declares that he has used Artificial Intelligence (AI) tools in the creation of this article. ChatGPT was used to assist with language editing, improving clarity and presentation, and organizing revisions to the manuscript. All mathematical results, proofs, and final revisions were reviewed and verified by the author. The AI-assisted revisions concern the language and presentation throughout the manuscript.

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Conflict of interest

The author declares no conflict of interest.

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