Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(9): 28710–28739.
DOI: 10.3934/math.20261143
Received: 04 June 2026
Revised: 10 August 2026
Accepted: 19 August 2026
Published: 07 September 2026

Research article

Periodic mild solutions for semilinear impulsive fractional evolution equations with memory

Ahmad Al-Omari[1] **and Hanan Al-Saadi**[2,*]

[1] Department of Mathematics, Faculty of Science, Al al-Bayt University, Mafraq, Jordan; omarimutah1@yahoo.com

[2] Department of Mathematics, Faculty of Sciences, Umm Al-Qura University, Makkah 21955, Saudi Arabia

* **Correspondence:** Email: Hsssaadi@uqu.edu.sa.

Abstract: This work examines fractional evolution equations involving a piecewise Caputo derivative, periodic impulses, and a convolution memory kernel. For the associated linear impulsive periodic problem, we demonstrate under exponential stability of the semigroup the existence and uniqueness of a periodic mild solution and prove that the corresponding solution operator is linear and bounded. Turning to the semilinear setting, we establish the existence of at least one periodic mild solution under two distinct families of growth restrictions on the nonlinear terms: (i) linear growth, via Sadovskii's fixed point theorem; (ii) more general Osgood-type growth, via the Leray-Schauder alternative. The impulses are taken to be Lipschitz with a suitably small constant. Finally, we give an application to a fractional impulsive heat equation with Dirichlet boundary conditions.

Keywords: impulsive fractional evolution equations; periodic impulses; periodic mild solutions; existence; compact semigroup
Mathematics Subject Classification: 34A08, 34A37, 34C25

1. Introduction

Fractional evolution equations have become a useful tool for modeling many phenomena in science and engineering. Several studies have dealt with the existence of mild solutions for semilinear fractional evolution equations (see e.g., [1, 2] and the references therein).

Periodic problems for evolution equations arise naturally in physical models and have been extensively studied for integer-order equations (see [3–5] and the references cited there). In the stochastic setting, optimal control problems for non-autonomous second-order stochastic differential equations with delayed arguments have been investigated recently in [6], where existence of mild solutions and approximate/optimal controllability were established using evolution operators and compact semigroup theory. However, because fractional derivatives involve memory and hereditary effects, a solution of a periodic boundary value problem for a fractional differential equation cannot be extended periodically to the whole real line $\mathbb{R}$. Indeed, Ren and Wang [7] proved that nontrivial periodic solutions do not exist for fractional evolution equations.

Impulsive fractional differential equations have drawn attention for their ability to describe sudden changes in dynamical systems; various existence results have been obtained (see [8–10] and the references therein). For recent advances in this direction, we refer to the study of fractional random integro-differential inclusions with non-instantaneous impulses [11, 12], which investigates existence and approximate controllability using the Bohnenblust Karlin fixed point theorem. Nevertheless, periodic solutions for fractional differential equations with periodic impulses have been investigated only rarely.

Feckan and Wang [13] studied fractional ordinary differential equations under periodic impulse conditions and established the existence of periodic solutions for Caputo derivatives with a variable lower limit. Zhang and Xiong [14] considered a class of semilinear fractional impulsive functional differential equations with a piecewise Caputo derivative and discussed periodic solutions using properties of the Mittag-Leffler function. More recently, Ren et al. [15] examined Caputo-type fractional evolution equations with a variable lower limit and proved the existence of periodic mild solutions for fractional evolution equations with periodic impulses.

In related directions, Al-Omari and Rashid [16] investigated the local existence and regularity of mild solutions for Hadamard fractional semilinear integro-differential equations with compact semigroups. Al-Omari and Al-Saadi [17, 18] studied impulsive Hadamard fractional differential equations with (ω, ρ)-boundary value problems and impulsive fractional integro-differential equations via fractional operators. More recently, Al-Omari, Al-Saadi, and Alharbi [19] established the existence and uniqueness of (ω, c)-periodic solutions for semilinear integro-differential equations with Hadamard derivatives. These contributions, which address impulsive effects and periodic-type conditions in fractional settings, provide a complementary perspective to the framework developed in the present work.

Most of the works mentioned above require the nonlinear term to be Lipschitz. However, in complex reaction-diffusion processes, the nonlinear function often depends on time in a complicated manner and cannot be expected to satisfy only Lipschitz hypotheses. Moreover, memory effects (e.g., convolution integrals with a kernel κ) are common in applications but are rarely treated together with periodic impulses.

Motivated by these facts, we study the periodic problem for fractional impulsive evolution equations in a Banach space X that contains a memory term:

$$\begin{aligned} {}^{c}D^{q}_{x_i,x}u(x) + Au(x) &= F(x, u(x)) + \int_0^x \kappa(x-y)\, G(y, u(y))\, dy, \quad x \in (x_i, x_{i+1}), \quad x \geq 0, \\ \Delta u(x_i) &= I_i(u(x_i)), \quad i \in \mathbb{N}, \end{aligned} \tag{1.1}$$

where ${}^{c}D^{q}_{x_i,x}$ denotes the Caputo fractional derivative of order $q \in (0, 1)$ with lower limit x_i, $A : D(A) \subset X \to X$ is a closed linear operator such that $-A$ generates a C_0-semigroup $T(x)$ $(x \geq 0)$ on X, $F, G : \mathbb{R}^+ \times X \to X$ are continuous and ω-periodic in x, κ is a locally integrable kernel, $p \in \mathbb{N}$ is the number of impulse points in $[0, \omega]$, $0 < x_1 < x_2 < \cdots < x_p < \omega = x_{p+1}$ and $x_{i+p+1} = x_i + \omega$ for $i \in \mathbb{N}$,

$\Delta u(x_i) = u(x_i^+) - u(x_i^-)$ is the jump of u at x_i, and $I_i : X \to X$ are continuous with $I_{i+p+1} = I_i$. Without loss of generality, we take $x_0 = 0$ as the first impulse point.

Using operator semigroup theory together with two fixed point theorems (Sadovskii's theorem and the Leray-Schauder alternative), we investigate the existence of ω-periodic mild solutions of (1.1). The main contributions are three theorems:

- Theorem 3.1 (linear case): Under exponential stability of the semigroup, the linear impulsive problem (with a prescribed source h and fixed jumps v_i) admits a unique ω-periodic mild solution, and the associated solution operator is linear and bounded.
- Theorem 3.2 (semilinear case with linear growth): If F and G satisfy linear growth conditions and the impulses are Lipschitz with a sufficiently small constant, then at least one ω-periodic mild solution exists.
- Theorem 3.3 (semilinear case with general nonlinear growth): If F and G satisfy an Osgood-type condition (e.g., $\|F(x,t)\| \leq \phi(x)\Psi(\|t\|)$ with $\int_a^\infty \frac{1}{\Psi(y)}dy = \infty$) and the impulses remain Lipschitz with a small constant, then at least one ω-periodic mild solution still exists.

The main contributions and novelties of the present work compared with the existing literature [13–15] can be summarized as follows.

(i) Inclusion of a memory term: The models studied in [13–15] do not incorporate convolutional memory kernels. In contrast, our system (1.1) contains the term $\displaystyle\int_0^x \kappa(x-y)G(y,u(y))dy$, where κ is a general locally integrable kernel. This term is essential for modeling anomalous diffusion processes, viscoelastic materials, and population dynamics with hereditary effects, thereby making our framework significantly more realistic for physical applications.

(ii) Generalized growth conditions on nonlinearities: Previous existence results for periodic impulsive fractional equations, such as those in [13–15], rely heavily on Lipschitz continuity of the nonlinear functions F and G. In this work, we remove this restrictive assumption and establish existence under two substantially broader frameworks:

   (a) *Linear growth conditions* (Theorem 3.2), treated via Sadovskii's fixed point theorem for condensing operators;

   (b) *Osgood-type growth conditions* (Theorem 3.3), where the nonlinearities may grow slower than any power function, handled via the Leray–Schauder alternative.

   This extension covers a wide class of singular or weakly singular nonlinear models that are not accessible by the methods of [13–15].

(iii) Rigorous operator-theoretic approach: Unlike the existing works, we do not merely prove existence. We first establish, in Theorem 3.1, that the linear impulsive problem with memory admits a unique periodic mild solution and that the associated solution operator P is linear and bounded. This operator-theoretic framework, combined with the exponential stability and compactness of the semigroup, provides a unified and robust tool for handling periodic impulses together with memory and generalized nonlinear growth.

To the best of our knowledge, the combination of all three features, memory effects, generalized growth (linear and Osgood), and a bounded linear solution operator for impulsive periodic problems, has not been investigated previously in the context of fractional evolution equations.

The paper is organized as follows. Section 2 collects basic definitions and auxiliary lemmas (semigroup properties, Mittag Leffler functions, fixed point theorems). Section 3 proves the existence and uniqueness of periodic mild solutions for the linear impulsive problem with memory (Theorem 3.1). Section 4 presents the main semilinear results: Theorem 3.2 (linear growth) and Theorem 3.3 (general growth). Finally, Section 5 provides an illustrative example.

2. Preliminaries

This section recalls several fundamental notions and auxiliary results used later. We first describe properties of C_0-semigroups, in particular exponential stability and compactness, which are essential for constructing solution operators. Next we introduce the operator families $U(x)$ and $V(x)$ generated by the semigroup and the Mittag-Leffler functions; these families play a central role in the representation of mild solutions of fractional evolution equations. Finally, we state two fixed point theorems (Sadovskii and the Leray-Schauder alternative) that will be applied in the proofs of our main results.

Let X be a Banach space with norm $\|\cdot\|$, and let $A : D(A) \subset X \to X$ be a closed linear operator such that $-A$ generates a C_0-semigroup $T(x)$ $(x \geq 0)$ on X. Let θ denote the zero element of X. For such a semigroup there exist constants $M \geq 1$ and $\nu \in \mathbb{R}$ satisfying (see [20])

$$\|T(x)\| \leq Me^{\nu x}, \qquad x \geq 0. \tag{2.1}$$

If, in addition,

$$\|T(x)\| \leq M, \qquad x \geq 0, \tag{2.2}$$

the semigroup is said to be uniformly bounded.

The growth index of $T(x)$ is defined as

$$\nu_0 = \inf\{\nu \in \mathbb{R} \mid \exists M \geq 1,\ \|T(x)\| \leq Me^{\nu x}\ \forall x \geq 0\}. \tag{2.3}$$

When $\nu_0 < 0$, $T(x)$ is said to be exponentially stable. Clearly, an exponentially stable C_0-semigroup is uniformly bounded. If $T(x)$ is continuous in the uniform operator topology for every $x > 0$, then ν_0 can be expressed via the spectrum $\sigma(A)$ of A:

$$\nu_0 = -\inf\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\}. \tag{2.4}$$

If $T(x)$ is a compact semigroup, then it is continuous in the uniform operator topology for all $x > 0$.

For detailed definitions of Caputo fractional derivatives, we refer to [21, 22]; they are not repeated here. We only introduce the operator families needed in this paper.

Define $U(x), V(x)$ $(x \geq 0)$ by

$$U(x) = \int_0^\infty \xi_q(y)T(x^q y)dy, \qquad V(x) = q\int_0^\infty y\xi_q(y)T(x^q y)dy, \tag{2.5}$$

where

$$\xi_q(y) = \frac{1}{\pi q}\sum_{n=1}^{\infty}(-y)^{n-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad y \in (0,\infty), \tag{2.6}$$

is a one-sided probability density satisfying

$$\xi_q(y) \geq 0, \quad \int_0^\infty \xi_q(y)dy = 1, \quad \int_0^\infty y\xi_q(y)dy = \frac{1}{\Gamma(1+q)}. \tag{2.7}$$

Lemma 2.1. *[21–24] The families $U(x)$ and $V(x)$ defined by (2.5) possess the following properties:*

(i) Strong continuity: for any $t \in X$ and $x_1, x_2 \geq 0$,

$$\|U(x_2)t - U(x_1)t\| \to 0, \qquad \|V(x_2)t - V(x_1)t\| \to 0 \quad \text{as } x_2 - x_1 \to 0.$$

(ii) If $T(x)$ is uniformly bounded, then for each $x \in \mathbb{R}^+$, $U(x)$ and $V(x)$ are bounded linear operators with

$$\|U(x)t\| \leq M\|t\|, \qquad \|V(x)t\| \leq \frac{M}{\Gamma(q)}\|t\|, \quad t \in X. \tag{2.8}$$

(iii) If $T(x)$ is compact, then $U(x)$ and $V(x)$ are compact for every $x > 0$.
(iv) If $T(x)$ is equicontinuous, then $U(x)$ and $V(x)$ are uniformly continuous for $x > 0$.
(v) If $T(x)$ is positive, then $U(x)$ and $V(x)$ are positive operators.
(vi) If $T(x)$ is exponentially stable with growth index $\nu_0 < 0$, then for any $x \geq 0$,

$$\|U(x)\| \leq ME_q(\nu_0 x^q), \qquad \|V(x)\| \leq ME_{q,q}(\nu_0 x^q), \tag{2.9}$$

where $E_q(\cdot)$ and $E_{q,q}(\cdot)$ are Mittag-Leffler functions.

For $q, q' > 0$, the two-parameter Mittag-Leffler function is defined by

$$E_{q,q'}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(q' + qn)}, \qquad z \in \mathbb{C}, \tag{2.10}$$

and we write $E_q = E_{q,1}$. According to [25],

$$E_q(-z) = \int_0^\infty \xi_q(y)e^{-zy}dy, \qquad E_{q,q}(-z) = q\int_0^\infty y\xi_q(y)e^{-zy}dy. \tag{2.11}$$

The Mittag-Leffler function has the following additional properties.

Lemma 2.2. *[26]*

$$\frac{d}{dz}(z^q E_{q,q+1}(\mu z^q)) = z^{q-1}E_{q,q}(\mu z^q), \qquad q, \mu, z \in \mathbb{R}. \tag{2.12}$$

Lemma 2.3. *[27] Let $q \in (0,1)$ and $\mu \in \mathbb{R}$. Then:*

(i) If $\mu < 0$ and $x_2 \geq x_1 \geq 0$, then $E_q(\mu x_2^q) \leq E_q(\mu x_1^q)$.
(ii) $E_q(0) = 1$, $E_{q,q}(0) = 1/\Gamma(q)$, and for $x > 0$ we have $E_q(\mu x^q) > 0$, $E_{q,q}(\mu x^q) > 0$.

Lemma 2.4. *[14] If $\mu > 0$ and $q \in (0,1)$, then*

(i) For $x > 0$,

$$\frac{d}{dx}(x^q E_{q,q+1}(-\mu x^q)) = x^{q-1}E_{q,q}(-\mu x^q) \geq 0. \tag{2.13}$$

(ii) For $x > 0$,

$$\lim_{x\to\infty} x^q E_{q,q+1}(-\mu x^q) = \frac{1}{\mu}, \qquad x^q E_{q,q+1}(-\mu x^q) \in [0, \tfrac{1}{\mu}). \tag{2.14}$$

The following fixed point theorems will be needed.

Lemma 2.5. *[28] Let X be a Banach space and $\overline{\Omega} \subset X$ a bounded convex closed set. If $Q : \overline{\Omega} \to \overline{\Omega}$ is a condensing operator, then Q has at least one fixed point in $\overline{\Omega}$.*

Lemma 2.6. *[29] Let X be a Banach space and let $Q_1, Q_2 : X \to X$ be such that Q_1 is completely continuous and Q_2 is a contraction. Then, either*

(i) the operator equation $u = Q_1 u + Q_2 u$ has a solution, or
(ii) the set $\Upsilon = \{u \in X \mid u = \lambda Q_1 u + \lambda Q_2(u/\lambda),\ \lambda \in (0,1)\}$ is unbounded.

3. Periodic mild solutions of impulsive periodic problems

Let

$$\tilde{D} = \{x_1, x_2, \dots, x_p\} \subset [0, \omega], \qquad 0 = x_0 < x_1 < x_2 < \cdots < x_p < x_{p+1} = \omega,$$

and extend periodically: $x_{i+p+1} = x_i + \omega$ for all $i \in \mathbb{N}$. Denote

$$J_i = (x_i, x_{i+1}], \quad i = 0, 1, \dots, p.$$

The space $PC([0,\omega], X)$ consists of functions $u : [0,\omega] \to X$ such that

- u is continuous on each interval J_i,
- u is left-continuous at each x_i ($u(x_i^-) = u(x_i)$),
- the right limits $u(x_i^+)$ exist.

The space $PC_\omega(\mathbb{R}^+, X)$ is the set of functions $u : \mathbb{R}^+ \to X$ that are ω-periodic ($u(x+\omega) = u(x)$ for all $x \geq 0$), and whose restriction to $[0,\omega]$ belongs to $PC([0,\omega], X)$. Its norm is

$$\|u\|_{PC} = \max\Big\{ \sup_{x\in[0,\omega]} \|u(x^-)\|,\ \sup_{x\in[0,\omega]} \|u(x^+)\| \Big\}.$$

Define the operators $U(x), V(x)$ ($x \geq 0$) via the one-sided probability density $\xi_q(y)$:

$$U(x) = \int_0^\infty \xi_q(y) T(x^q y) dy, \qquad V(x) = q \int_0^\infty y \xi_q(y) T(x^q y) dy. \tag{3.1}$$

These families satisfy the well-known properties (see Lemma 2.1):

- Strong continuity: $\|U(x)t - U(y)t\| \to 0$ and $\|V(x)t - V(y)t\| \to 0$ as $x \to y$ for each $t \in X$.
- Boundedness: if $\|T(x)\| \leq Me^{\nu x}$, then $\|U(x)\| \leq ME_q(\nu x^q)$ and $\|V(x)\| \leq ME_{q,q}(\nu x^q)$.
- For exponentially stable semigroups, there exists an equivalent norm such that $|U(x)| \leq E_q(-\nu x^q) < 1$.

Theorem 3.1. *Assume that $-A$ generates an exponentially stable C_0-semigroup $T(x)$ $(x \geq 0)$ on the Banach space X with growth index $\nu_0 < 0$. Let $\kappa \in L^1_{\text{loc}}(\mathbb{R}^+)$ be a given kernel. Let $h, w \in PC_\omega(\mathbb{R}^+, X)$ be fixed ω-periodic piecewise continuous functions, and let $\{\nu_i\}_{i\in\mathbb{N}} \subset X$ satisfy $\nu_{i+p+1} = \nu_i$ for all i (periodic impulses). Then, the impulsive evolution equation*

$$\begin{aligned} {}^cD^q_{x_i,x}u(x) + Au(x) =& h(x) + \int_0^x \kappa(x-y)\, w(y)\, dy, \quad x \in (x_i, x_{i+1}), \\ \Delta u(x_i) =& \nu_i, \quad i \in \mathbb{N}, \end{aligned} \tag{3.2}$$

has a unique ω-periodic mild solution $u \in PC_\omega(\mathbb{R}^+, X)$.

The product space $PC_\omega(\mathbb{R}^+, X) \times PC_\omega(\mathbb{R}^+, X) \times X^p$ is equipped with the norm

$$\|(h, w, \{\nu_i\})\| := \|h\|_{PC} + \|w\|_{PC} + \sum_{k=1}^{p} \|\nu_k\|.$$

Moreover, the solution operator

$$P : PC_\omega(\mathbb{R}^+, X) \times PC_\omega(\mathbb{R}^+, X) \times X^p \longrightarrow PC_\omega(\mathbb{R}^+, X)$$

defined by $(h, w, \{\nu_i\}) \mapsto u$ is bounded with respect to this norm.

Proof. Given $\phi(x) = h(x) + \int_0^x \kappa(x-y)w(y)dy \in PC([0, \omega], X)$, we write the mild solution of the impulsive Cauchy problem

$$\begin{aligned} {}^cD^q_{x_i,x}u(x) + Au(x) =& h(x) + \int_0^x \kappa(x-y)\, w(y)\, dy, \quad x \in J_i, \\ u(x_i^+) =& u(x_i^-) + \nu_i, \end{aligned}$$

as given by the variation of constants formula:

$$u(x) = U(x - x_i)\big(u(x_i^-) + \nu_i\big) + \int_{x_i}^x (x-y)^{q-1} V(x-y) \left(h(y) + \int_0^y \kappa(y-\tau) w(\tau) d\tau \right) dy, \qquad t \in J_i. \tag{3.3}$$

The variation of constants formula (3.3) for the piecewise Caputo fractional derivative with impulses is standard in the theory of fractional evolution equations; see, for example, [21, 22, 24]. For the convenience of the reader, we recall that it is obtained by applying the Laplace transform on each subinterval $(x_i, x_{i+1}]$ and using the semigroup property of $T(x)$ together with the identity

$$\mathcal{L}\{{}^cD^q_{x_i}u(x)\}(s) = s^q \mathcal{L}\{u(x)\}(s) - s^{q-1}u(x_i^+),$$

which is the Caputo analogue of the classical first-order case.

For $x \in [0, x_1]$, we have $x_i = 0$, $u(x_i^-) = u(0)$, and we set $\nu_0 = 0$ (no impulse at 0). Formula (3.3) gives

$$u(x) = U(x)u(0) + \int_0^x (x-y)^{q-1} V(x-y) \left(h(y) + \int_0^y \kappa(y-\tau) w(\tau) d\tau \right) dy, \qquad x \in [0, x_1]. \tag{3.4}$$

For $x_i \leq y \leq x \leq x_{i+1}$, define simply $\mathcal{U}(x, y) = U(x - y)$.

For general times $x_{j-1} \leq y \leq x_j \leq \cdots \leq x_i \leq x \leq x_{i+1}$, define

$$\mathcal{U}(x, y) = U(x - x_i) \prod_{k=j}^{i-1} U(x_{k+1} - x_k)\, U(x_j - y). \tag{3.5}$$

Here, $\mathcal{U}(x, y)$ denotes the propagation operator that evolves the solution from time y to time x across the impulse points, while $U(\cdot)$ stands for the fundamental solution operator defined in (2.5).

This operator propagates the solution from time y to time x across the jumps.

On J_i for $i \geq 1$, assume that we know $u(x)$ for $x \leq x_i$, and that we have already computed the effect of all previous impulses. Using (3.3) repeatedly, we obtain for $x \in (x_i, x_{i+1}]$,

$$\begin{aligned} u(x) = {} & \mathcal{U}(x, 0)u(0) + \sum_{k=1}^{i} \mathcal{U}(x, x_k)\nu_k \\ & + \sum_{k=1}^{i} \mathcal{U}(x, x_k) \int_{x_{k-1}}^{x_k} (x_k - y)^{q-1} V(x_k - y) \left(h(y) + \int_0^y \kappa(y - \tau) w(\tau) d\tau \right) dy \\ & + \int_{x_i}^{x} (x - y)^{q-1} V(x - y) \left(h(y) + \int_0^y \kappa(y - \tau) w(\tau) d\tau \right) dy. \end{aligned} \tag{3.6}$$

For $i = 1$ (i.e., $x \in (x_1, x_2]$), we have

$$u(x) = U(x - x_1)u(x_1^+) + \int_{x_1}^{x} (x - y)^{q-1} V(x - y) \left(h(y) + \int_0^y \kappa(y - \tau) w(\tau) d\tau \right) dy.$$

But, $u(x_1^+) = u(x_1^-) + \nu_1$, and from (3.4) with $x = x_1$, we get

$$u(x_1^-) = U(x_1)u(0) + \int_0^{x_1} (x_1 - y)^{q-1} V(x_1 - y) \left(h(y) + \int_0^y \kappa(y - \tau) w(\tau) d\tau \right) dy.$$

Hence,

$$u(x) = U(x - x_1)\Big[U(x_1)u(0) + \int_0^{x_1} (x_1 - y)^{q-1} V(x_1 - y) \left(h(y) + \int_0^y \kappa(y - \tau) w(\tau) d\tau \right) dy + \nu_1 \Big] + \int_{x_1}^{x} \cdots$$

which is exactly (3.6) with $i = 1$ because $\mathcal{U}(x, 0) = U(x - x_1)U(x_1)$ and $\mathcal{U}(x, x_1) = U(x - x_1)$. The induction step is similar and omitted for brevity.

We look for a solution such that $u(0) = u(\omega) =: t_0$ (the periodic condition). Set $x = \omega = x_{p+1}$ in (3.6). Note that $\mathcal{U}(\omega, 0) = U(\omega - x_p) \prod_{k=1}^{p} U(x_{k+1} - x_k)$, and the sum over $k = 1$ to p includes all subintervals. We obtain

$$\begin{aligned} t_0 = {} & \mathcal{U}(\omega, 0)t_0 + \sum_{k=1}^{p} \mathcal{U}(\omega, x_k)\nu_k \\ & + \sum_{k=1}^{p} \mathcal{U}(\omega, x_k) \int_{x_{k-1}}^{x_k} (x_k - y)^{q-1} V(x_k - y) \left(h(y) + \int_0^y \kappa(y - \tau) w(\tau) d\tau \right) dy \\ & + \int_{x_p}^{\omega} (\omega - y)^{q-1} V(\omega - y) \left(h(y) + \int_0^y \kappa(y - \tau) w(\tau) d\tau \right) dy. \end{aligned}$$

Bring $\mathcal{U}(\omega,0)t_0$ to the left side:

$$\begin{aligned}(I-\mathcal{U}(\omega,0))t_0 &= \sum_{k=1}^{p}\mathcal{U}(\omega,x_k)\nu_k\\ &+\sum_{k=1}^{p}\mathcal{U}(\omega,x_k)\int_{x_{k-1}}^{x_k}(x_k-y)^{q-1}V(x_k-y)\left(h(y)+\int_0^y \kappa(y-\tau)w(\tau)d\tau\right)dy\\ &+\int_{x_p}^{\omega}(\omega-y)^{q-1}V(\omega-y)\left(h(y)+\int_0^y \kappa(y-\tau)w(\tau)d\tau\right)dy. \end{aligned} \tag{3.7}$$

Since $T(x)$ is exponentially stable, there exist constants $M\geq 1$ and $\nu>0$ such that

$$\|T(x)\|\leq Me^{-\nu x},\qquad x\geq 0. \tag{3.8}$$

Define an equivalent norm on X by

$$|t|=\sup_{x\geq 0}\|e^{\nu x}T(x)t\|. \tag{3.9}$$

It is easy to verify that $\|t\|\leq |t|\leq M\|t\|$. Let $|T(x)|$ denote the operator norm of $T(x)$ in this new norm. Then, from (3.9) we get

$$|T(x)|\leq e^{-\nu x},\qquad x\geq 0. \tag{3.10}$$

Now recall that $U(x)=\int_0^\infty \xi_q(y)T(x^q y)dy$. In the norm $|\cdot|$,

$$|U(x)|\leq \int_0^\infty \xi_q(y)|T(x^q y)|dy\leq \int_0^\infty \xi_q(y)e^{-\nu x^q y}dy=E_q(-\nu x^q). \tag{3.11}$$

The Mittag-Leffler function $E_q(-\nu x^q)$ satisfies $0<E_q(-\nu x^q)<1$ for all $x>0$ (see Lemma 2.3). Hence, $|U(x)|<1$ for every $x>0$. However, to conclude that $|\mathcal{U}(\omega,0)|<1$, we need a uniform estimate over all subintervals. Since $E_q(-\nu t^q)$ is continuous and strictly decreasing on $[0,\omega]$ with $E_q(0)=1$, and since $b:=\min_{0\leq k\leq p}(x_{k+1}-x_k)>0$, we have

$$\max_{0\leq k\leq p}|U(x_{k+1}-x_k)|\leq E_q(-\nu b^q)=:\alpha<1.$$

Using the representation (3.5) of $\mathcal{U}$, it follows that

$$|\mathcal{U}(x,0)|\leq \prod_{k=0}^{p}|U(x_{k+1}-x_k)|\leq \alpha^{p+1}<1,\qquad x>0. \tag{3.12}$$

Consequently, the operator $I-\mathcal{U}(\omega,0)$ is invertible and its inverse is bounded in the norm $|\cdot|$, hence also in the original norm $\|\cdot\|$. We denote

$$\mathcal{R}=(I-\mathcal{U}(\omega,0))^{-1}\in\mathcal{L}(X). \tag{3.13}$$

From (3.7) and (3.13), we obtain

$$t_0 = (I - \mathcal{U}(\omega,0))^{-1}\left[\sum_{k=1}^{p}\mathcal{U}(\omega,x_k)\nu_k \right.$$
$$+\sum_{k=1}^{p}\mathcal{U}(\omega,x_k)\int_{x_{k-1}}^{x_k}(x_k-y)^{q-1}V(x_k-y)\left(h(y)+\int_0^y\kappa(y-\tau)w(\tau)d\tau\right)dy$$
$$\left.+\int_{x_p}^{\omega}(\omega-y)^{q-1}V(\omega-y)\left(h(y)+\int_0^y\kappa(y-\tau)w(\tau)d\tau\right)dy.\right] := B(h). \tag{3.14}$$

This value is unique because the right-hand side depends only on the given data $h, w, \{\nu_i\}$, not on u.

Thus, we obtain a unique mild solution on $[0,\omega]$ satisfying $u(0) = u(\omega) = t_0$. Insert the obtained t_0 into formulas (3.4) and (3.6). This defines a function $u(x)$ for all $x \in [0,\omega]$. By construction, on each subinterval it satisfies the integral equation (3.2) with the explicit source term

$$\phi(y) = h(y) + \int_0^y \kappa(y-\tau)w(\tau)d\tau$$

and the jumps $\Delta u(x_i) = \nu_i$. Moreover, $u(0) = t_0$ and $u(\omega) = x_0$ by the way we defined x_0. Hence, u is a mild solution of (3.2) on $[0,\omega]$ with the periodic condition.

If another mild solution $\tilde{u}$ satisfied the same periodic condition, then $\tilde{u}(0)$ would also satisfy (3.7) because $I - \mathcal{U}(\omega,0)$ is injective, so $\tilde{u}(0) = t_0$. Then, the uniqueness of the mild solution on each subinterval (which follows from the contraction property of the integral representation) forces $\tilde{u} = u$.

We now prove that the solution we constructed on $[0,\omega]$ automatically satisfies $u(x+\omega) = u(x)$ for all $x \in [0,\omega]$, and therefore can be extended periodically to $\mathbb{R}^+$.

Case (1): $x \in [0, x_1]$**.** For such x, $x+\omega \in [\omega, x_1+\omega] = [x_{p+1}, x_{p+2}]$. Using formula (3.4) for $u(x+\omega)$ but starting from x_{p+1} (which is ω), we have

$$u(x+\omega) = U(x+\omega-x_{p+1})u(\omega) + \int_{x_{p+1}}^{x+\omega}(x+\omega-y)^{q-1}V(x+\omega-y)\left(h(y)+\int_0^y\kappa(y-\tau)w(\tau)d\tau\right)dy.$$

Since $u(\omega) = u(0) = x_0$ and $x+\omega-x_{p+1} = x$, this becomes

$$u(x+\omega) = U(x)x_0 + \int_0^x (x-r)^{q-1}V(x-r)\left(h(r+\omega)+\int_0^{r+\omega}\kappa(r+\omega-\tau)w(\tau)d\tau\right)dr,$$

where we made the substitution $r = y-\omega$ (so $y = r+\omega$). Because h is ω-periodic, $h(r+\omega) = h(r)$. For the memory term, we split the integral:

$$\int_0^{r+\omega}\kappa(r+\omega-\tau)w(\tau)d\tau = \int_{-\omega}^{r}\kappa(r-\sigma)w(\sigma+\omega)d\sigma$$

(by setting $\sigma = \tau-\omega$). Since w is ω-periodic, $w(\sigma+\omega) = w(\sigma)$. Moreover, $\kappa(\cdot)$ is defined to be zero for negative arguments (or we extend it by zero). Thus, the integration from $-\omega$ to 0 gives zero, and we obtain

$$\int_0^{r+\omega}\kappa(r+\omega-\tau)w(\tau)d\tau = \int_0^r \kappa(r-\sigma)w(\sigma)d\sigma.$$

Therefore,

$$u(x+\omega) = U(x)x_0 + \int_0^x (x-r)^{q-1} V(x-r)\left(h(r) + \int_0^r \kappa(r-\sigma)w(\sigma)d\sigma\right)dr = u(x).$$

Case (2): $x \in [x_i, x_{i+1}]$ **with** $i = 1, \dots, p$**.** Then, $x + \omega \in [x_i + \omega, x_{i+1} + \omega] = [x_{i+p+1}, x_{i+p+2}]$. We will show that the expression (3.6) for $u(x+\omega)$ with i replaced by $i+p+1$ reduces exactly to the expression for $u(x)$.

First, from the formula (3.5), we obtain

$$\mathcal{U}(x+\omega,\omega) = U(x+\omega-x_{i+p+1})\prod_{k=p+1}^{i+p} U(x_{k+1}-x_k) = U(x-x_i)\prod_{k=1}^{i-1} U(x_{k+1}-x_k) = \mathcal{U}(t,0). \quad (3.15)$$

Similarly, for any $k = 1, \dots, i$,

$$\mathcal{U}(x+\omega, x_{k+p+1}) = U(x+\omega-x_{i+p+1})\prod_{j=k+p+1}^{i+p} U(x_{j+1}-x_j) = U(x-x_i)\prod_{j=k}^{i-1} U(x_{j+1}-x_j) = \mathcal{U}(x, x_k). \quad (3.16)$$

Now write formula (3.6) for $u(x+\omega)$ with i replaced by $i+p+1$:

$$\begin{aligned} u(x+\omega) &= \mathcal{U}(x+\omega,\omega)u(\omega) + \sum_{k=1}^{i+p+1} \mathcal{U}(x+\omega, x_k)v_k \\ &\quad + \sum_{k=1}^{i+p+1} \mathcal{U}(x+\omega, x_k)\int_{x_{k-1}}^{x_k} (x_k-y)^{q-1}V(x_k-y)\left(h(y) + \int_0^y \kappa(y-\tau)w(\tau)d\tau\right)dy \\ &\quad + \int_{x_{i+p+1}}^{x+\omega} (x+\omega-y)^{q-1}V(x+\omega-y)\left(h(y) + \int_0^y \kappa(y-\tau)w(\tau)d\tau\right)dy. \end{aligned}$$

Because of the periodicity of the impulses, $v_{k+p+1} = v_k$. Also, the intervals $[x_{k-1}, x_k]$ for $k > p+1$ are just translates of those for $k \le p+1$: for $k' = k-p-1$ we have $x_{k-1} = x_{k'+p} = x_{k'} + \omega$ and $x_k = x_{k'+p+1} = x_{k'+1} + \omega$. Using the periodicity of h, w, κ and the change of variable $y' = y - \omega$ in the integrals, we obtain

$$\begin{aligned} &\int_{x_{k-1}}^{x_k} (x_k-y)^{q-1}V(x_k-y)\left(h(y) + \int_0^y \kappa(y-\tau)w(\tau)d\tau\right)dy \\ &= \int_{x_{k'-1}}^{x_{k'}} (x_{k'}-y')^{q-1}V(x_{k'}-y')\left(h(y') + \int_0^{y'} \kappa(y'-\tau)w(\tau)d\tau\right)dy'. \end{aligned}$$

For $x \in [x_i, x_{i+1}]$ with $i = 1, \dots, p$, we have $x+\omega \in [x_{i+p+1}, x_{i+p+2}]$. Applying formula (3.6) to $u(x+\omega)$ with i replaced by $i+p+1$, we obtain

$$\begin{aligned} u(x+\omega) &= \mathcal{U}(x+\omega,\omega)u(\omega) + \sum_{k=1}^{i+p+1} \mathcal{U}(x+\omega, x_k)v_k \\ &\quad + \sum_{k=1}^{i+p+1} \mathcal{U}(x+\omega, x_k)\int_{x_{k-1}}^{x_k} (x_k-y)^{q-1}V(x_k-y)\phi(y)\,dy \\ &\quad + \int_{x_{i+p+1}}^{x+\omega} (x+\omega-y)^{q-1}V(x+\omega-y)\phi(y)\,dy. \end{aligned}$$

Since $u(\omega) = u(0)$ and $\mathcal{U}(x+\omega, \omega) = \mathcal{U}(x, 0)$ by (3.16), the first term becomes $\mathcal{U}(x, 0)u(0)$. For the sums, we split the index set $\{1, \ldots, i+p+1\}$ into three parts:

(i) For $k = 1, \ldots, p$, using $\mathcal{U}(x+\omega, x_k) = \mathcal{U}(x, x_k)$ and the periodicity of ν_k and ϕ, these terms are exactly the first-period contributions:

$$\sum_{k=1}^{p} \mathcal{U}(x, x_k)\nu_k + \sum_{k=1}^{p} \mathcal{U}(x, x_k) \int_{x_{k-1}}^{x_k} (x_k - y)^{q-1} V(x_k - y)\phi(y)\, dy.$$

(ii) For $k = p+1$, this corresponds to $x_{p+1} = \omega$. Since there is no impulse at ω (the jumps are only at $x_1, \ldots, x_p$ within $[0, \omega]$), the term ν_{p+1} is not present. Equivalently, this contribution is absorbed into the initial term $\mathcal{U}(x, 0)u(0)$ already accounted for.

(iii) For $k = p+2, \ldots, i+p+1$, set $k' = k - (p+1)$, so that $k' \in \{1, \ldots, i\}$. Using the identities

$$\mathcal{U}(x+\omega, x_{k'+p+1}) = \mathcal{U}(x, x_{k'}), \qquad \nu_{k'+p+1} = \nu_{k'},$$

and the change of variable $y' = y - \omega$ in the integrals (noting that $x_{k-1} = x_{k'-1} + \omega$ and $x_k = x_{k'} + \omega$), these terms become

$$\sum_{k'=1}^{i} \mathcal{U}(x, x_{k'})\nu_{k'} + \sum_{k'=1}^{i} \mathcal{U}(x, x_{k'}) \int_{x_{k'-1}}^{x_{k'}} (x_{k'} - y')^{q-1} V(x_{k'} - y')\phi(y')\, dy'.$$

Finally, the last integral in the expression for $u(x+\omega)$, namely $\int_{x_{i+p+1}}^{x+\omega} (x+\omega-y)^{q-1} V(x+\omega-y)\phi(y)\, dy$, is transformed via $y' = y - \omega$ into $\int_{x_i}^{x} (x - y')^{q-1} V(x - y')\phi(y')\, dy'$.

Adding the contributions from (i), (iii), the transformed last integral, and the initial term $\mathcal{U}(x, 0)u(0)$, we recover precisely the right-hand side of (3.6). Hence, $u(x+\omega) = u(x)$.

After shifting the summation index and using (3.15)-(3.16), the whole expression collapses to the right-hand side of (3.6). Therefore, $u(x+\omega) = u(x)$.

Consequently, for every x in the interval $[0, \omega]$, we obtain $u(x+\omega) = u(x)$. By extending u to an ω-periodic function defined on $\mathbb{R}^+$ (keeping the same notation u), we find that $u := Ph$ belongs to $PC_\omega(\mathbb{R}^+, X)$. This ω-periodic extension on $\mathbb{R}^+$ provides the unique mild solution that is ω-periodic to Eq (3.2). Moreover, using the expression of t_0, for $x \in [0, \omega]$, we have the following:

$$Ph(x) = \begin{cases} U(x)B(h) + \displaystyle\int_{x_m}^{x} (x-y)^{q-1} V(x-y)\Big(h(y) + \int_0^y \kappa(y-\tau)w(\tau)d\tau\Big)\mathrm{d}y, \; x \in [0, x_1], \\ \mathcal{U}(x, 0)B(h) + \sum_{k=1}^{i} \mathcal{U}(x, x_k) \displaystyle\int_{x_{k-1}}^{x_k} (x_k - y)^{q-1} V(x_k - y)\Big(h(y) + \int_0^y \kappa(y-\tau)w(\tau)d\tau\Big)\mathrm{d}y \\ \quad + \displaystyle\int_{x_i}^{x} (x-y)^{q-1} V(x-y)\Big(h(y) + \int_0^y \kappa(y-\tau)w(\tau)d\tau\Big)\mathrm{d}y + \sum_{k=1}^{i} \mathcal{U}(x, x_k)\nu_k, \; x \in (x_i, x_{i+1}]. \end{cases}$$

Thus, the function u defined on $[0, \omega]$ satisfies $u(0) = u(\omega)$ and $u(x+\omega) = u(x)$ for all $x \in [0, \omega]$. Its periodic extension to $\mathbb{R}^+$ (still denoted by u) belongs to $PC_\omega(\mathbb{R}^+, X)$ and is a mild solution of (3.2) on the whole half-line.

Linearity and boundedness of the solution operator P. Define the operator P by $P(h, w, \{\nu_i\}) = u$, where u is the unique periodic mild solution obtained above. From (3.4), (3.6), and (3.14), it is evident that u depends linearly on h, w, and $\{v_i\}$. Indeed, x_0 is linear in those data because

$$R = \sum_{k=1}^{p} \mathcal{U}(\omega, x_k)v_k + \sum_{k=1}^{p} \mathcal{U}(\omega, x_k) \int_{x_{k-1}}^{x_k} (x_k - y)^{q-1} V(x_k - y) \left(h(y) + \int_0^y \kappa(y-\tau)w(\tau)d\tau \right) dy$$
$$+ \int_{x_p}^{\omega} (\omega - y)^{q-1} V(\omega - y) \left(h(y) + \int_0^y \kappa(y-\tau)w(\tau)d\tau \right) dy,$$

is linear and $\mathcal{R} = (I - \mathcal{U}(\omega, 0))^{-1}$ is linear, and each integral term is linear. Hence, P is linear.

We now prove boundedness. From Lemma 2.1, there exist constants $C_U, C_V > 0$ such that

$$\|U(x)\| \le C_U E_q(v_0 x^q), \qquad \|V(x)\| \le C_V E_{q,q}(v_0 x^q), \qquad x \ge 0.$$

Moreover, since $v_0 < 0$, the Mittag-Leffler functions are bounded on $[0, \omega]$ by some constants C_U, C_V. Thus,

$$\sup_{x \in [0,\omega]} \|U(x)\| \le C_U, \qquad \sup_{x \in [0,\omega]} \|V(x)\| \le C_V. \tag{3.17}$$

From (3.5), we obtain the estimates

$$\|\mathcal{U}(x, x_j)\| \le C_U^{i-j+1}, \qquad \left\| \sum_{j=1}^{i} \mathcal{U}(x, x_j) \right\| \le \sum_{j=1}^{i} C_U^j. \tag{3.18}$$

Let $b = \min_{1 \le k \le p+1}(x_k - x_{k-1}) > 0$. Sharper bounds are possible using (3.8)-(3.9), but the constants C_U, C_V suffice for boundedness.

Now, from the explicit source term of (3.2),

$$\|\phi\|_{PC} = \left\| h(y) + \int_0^y \kappa(y-\tau)w(\tau)d\tau \right\| \le \|h(y)\| + \int_0^y |\kappa(y-\tau)| \, \|w(\tau)\| d\tau \le \|h\|_{PC} + \|\kappa\|_{L^1(0,\omega)} \|w\|_{PC}. \tag{3.19}$$

Hence, $\|\phi\|_{PC} \le C_h \|h\|_{PC} + C_w \|w\|_{PC}$ with $C_h = 1$, $C_w = \|\kappa\|_{L^1}$.

From (3.14) and the bound $\|(I - \mathcal{U}(\omega, 0))^{-1}\| \le C_0$ (some constant, because the inverse exists and is bounded), we get

$$\|x_0\| \le C_0 \Big(\sum_{k=1}^{p} \|\mathcal{U}(\omega, x_k)\| \, \|v_k\|$$
$$+ \sum_{k=1}^{p} \|\mathcal{U}(\omega, x_k)\| \left\| \int_{x_{k-1}}^{x_k} (x_k - y)^{q-1} V(x_k - y)\phi(y)dy \right\| + \left\| \int_{x_p}^{\omega} (\omega - y)^{q-1} V(\omega - y)\phi(y)dy \right\| \Big).$$

Using (3.17) and the fact that

$$\int_0^{\omega} (x-y)^{q-1} dy = \frac{x^q}{q} \le \frac{\omega^q}{q},$$

we obtain the constants K_1, K_2, and K_3 such that

$$\|t_0\| \le K_1 \sum_{k=1}^{p} \|v_k\| + K_2 \|\phi\|_{PC}.$$

Then, from (3.4) and (3.6), with similar estimates, we finally get

$$\|u\|_{PC} \leq K_4\|x_0\| + K_5 \sum_{k=1}^{p} \|v_k\| + K_6\|\phi\|_{PC}.$$

Combining with the bound for ϕ in terms of h, w gives

$$\|u\|_{PC} \leq L_1\|h\|_{PC} + L_2\|w\|_{PC} + L_3 \sum_{k=1}^{p} \|v_k\|,$$

where L_1–L_3 are constants depending only on $\omega, q, \kappa, v_0, p$, and the semigroup bounds. Hence, P is a bounded linear operator. □

Theorem 3.2. *Suppose X is a Banach space, and let $-A$ be the generator of an exponentially stable and compact semigroup $T(x)$ for $x \geq 0$ on X. Furthermore, suppose the functions $F, G : \mathbb{R}^+ \times X \to X$ are continuous and ω-periodic with respect to the first variable, and the operators $I_i \in C(X, X)$ (for $i \in \mathbb{N}$) satisfy the periodicity condition $I_{i+p+1} = I_i$ with $I_0 = I$, where p denotes the number of impulse moments lying in the interval $[0, \omega]$. We also assume that F, G, and I_i meet the following requirements:*

(H1) For any $r > 0$ there exists a positive ω-periodic function $\phi_r \in C_\omega(\mathbb{R}^+)$ such that $\sup_{\|x\|\leq r} \|F(x,t)\| \leq \phi_r(x)$ for all $x \in \mathbb{R}^+$. For each $r > 0$, the function ϕ_r is bounded on $[0, \omega]$ (which follows from its continuity and periodicity), and $\liminf_{r\to\infty} \dfrac{\|\phi_r\|_{C([0,\omega])}}{r} =: \rho_F < \infty$.

(H1) For any $r > 0$ there exists a positive ω-periodic function $\psi_r \in C_\omega(\mathbb{R}^+)$ such that $\sup_{\|x\|\leq r} \|G(x,t)\| \leq \psi_r(x)$ for all $x \in \mathbb{R}^+$. For each $r > 0$, the function ψ_r is bounded on $[0, \omega]$ (which follows from its continuity and periodicity), and $\liminf_{r\to\infty} \dfrac{\|\psi_r\|_{C([0,\omega])}}{r} =: \rho_G < \infty$.*

(H2) For each I_i, $I_i(\theta) = \theta$, there exists a positive number K such that $\|I_i(t) - I_i(s)\| \leq K\|t - s\|$ for all $t, s \in X$, $i \in \mathbb{N}$.

Set

$$\kappa_0 := \int_0^\omega |\kappa(y)|dy, \quad \rho := \rho_F + \kappa_0\rho_G,$$

and define

$$C_i := \sum_{k=1}^{i} M^k E_q^k(v_0 b^q), \quad b := \min_{1\leq k\leq p+1} (x_k - x_{k-1}).$$

Note that the sequence (C_i) is strictly increasing, since each term $M^k E_q^k(v_0 b^q)$ is positive. Consequently, condition (3.20) is most restrictive for $i = p$; if it holds for $i = p$, it automatically holds for all $i = 1, \ldots, p$. Nevertheless, we state it for all i because the contraction estimate for Q_2 is required on every subinterval $(x_i, x_{i+1}]$. If

$$\left(K + \frac{M\rho}{|v_0|}\right) C_i + \frac{M\rho}{|v_0|} < 1 \qquad \text{for all } i = 1, \ldots, p, \tag{3.20}$$

then the semilinear impulsive problem with memory

$${}^cD^q_{x_i,x}u(x) + Au(x) = F(x, u(x)) + \int_0^x \kappa(x-y)G(y, u(y))dy, \quad x \in (x_i, x_{i+1}),$$

$$\Delta u(x_i) = I_i(u(x_i^-)), \quad i \in \mathbb{N}, \quad \textit{and} \qquad u(0) = u_0, \tag{3.21}$$

has at least one ω-periodic mild solution $u \in PC_\omega(\mathbb{R}^+, X)$.

Proof. According to Theorem 3.1, define the operator Q on $PC_\omega(\mathbb{R}^+, X)$, which can be expressed on $[0, \omega]$ as

$$Qu(x) = \begin{cases} U(x)B(u) + \displaystyle\int_0^x (x-y)^{q-1} V(x-y)\Big[F(y,u(y)) + \int_0^y \kappa(y-\tau)G(\tau,u(\tau))d\tau\Big]dy, x \in [0, x_1], \\ \mathcal{U}(x,0)B(u) + \displaystyle\sum_{k=1}^{i} \mathcal{U}(x,x_k)I_k(u(x_k)) \\ \quad + \displaystyle\sum_{k=1}^{i} \mathcal{U}(x,x_k)\int_{x_{k-1}}^{x_k} (x_k-y)^{q-1} V(x_k-y)\Big[F(y,u(y)) + \int_0^y \kappa(y-\tau)G(\tau,u(\tau))d\tau\Big]dy \\ \quad + \displaystyle\int_{x_i}^{x} (x-y)^{q-1} V(x-y)\Big[F(y,u(y)) + \int_0^y \kappa(y-\tau)G(\tau,u(\tau))d\tau\Big]dy, x \in (x_i, x_{i+1}]. \end{cases} \tag{3.22}$$

Define $B(u)$ as the periodic boundary value term that determines the initial value $u(0)$ via the periodicity condition $u(0) = u(\omega)$, where

$$\begin{aligned} B(u) = (I - \mathcal{U}(\omega,0))^{-1}\Big(&\sum_{k=1}^{p} \mathcal{U}(\omega,x_k)I_k(u(x_k)) \\ &+ \sum_{k=1}^{p} \mathcal{U}(\omega,x_k)\int_{x_{k-1}}^{x_k} (x_k-y)^{q-1} V(x_k-y)\Big[F(y,u(y)) + \int_0^y \kappa(y-\tau)G(\tau,u(\tau))d\tau\Big]dy \\ &+ \int_{x_p}^{\omega} (\omega-y)^{q-1} V(\omega-y)\Big[F(y,u(y)) + \int_0^y \kappa(y-\tau)G(\tau,u(\tau))d\tau\Big]dy\Big). \end{aligned} \tag{3.23}$$

One can readily show that the operator $Q : PC_\omega(\mathbb{R}^+, X) \to PC_\omega(\mathbb{R}^+, X)$ is well-defined. Moreover, thanks to the continuity of F, G, and I_i, the operator Q turns out to be continuous as well. In view of Theorem 3.1, the existence of an ω-periodic mild solution to Eq (3.21) is equivalent to the existence of a fixed point of Q. Consequently, we shall employ a fixed point theorem to establish that Q admits a fixed point.

Given any $R > 0$, we set

$$\overline{\Omega}_R = \{u \in PC_\omega(\mathbb{R}^+, X) \mid \|u\|_{PC} \le R\}. \tag{3.24}$$

Observe that $\overline{\Omega}_R$ is a closed ball in $PC_\omega(\mathbb{R}^+, X)$ centered at the zero element θ with radius R. Our next goal is to demonstrate the existence of a positive number R satisfying $Q(\overline{\Omega}_R) \subset \overline{\Omega}_R$.

First, from (H1) and (H1*), for any $u \in \overline{\Omega}_R$, we have:

$$\|F(y,u(y))\| \le \phi_R(y), \qquad \|G(y,u(y))\| \le \psi_R(y).$$

Consequently,

$$\Big\|F(y,u(y)) + \int_0^y \kappa(y-\tau)G(\tau,u(\tau))d\tau\Big\| \le \phi_R(y) + \int_0^y |\kappa(y-\tau)|\psi_R(\tau)d\tau \le \phi_R(y) + \kappa_0\|\psi_R\|_{C([0,\omega])}.$$

Define

$$\Phi_R(y) := \phi_R(y) + \kappa_0\|\psi_R\|_{C([0,\omega])}, \qquad \|\Phi_R\|_{C([0,\omega])} \le \|\phi_R\|_{C([0,\omega])} + \kappa_0\|\psi_R\|_{C([0,\omega])},$$

and set

$$\rho_R := \frac{\|\Phi_R\|_{C([0,\omega])}}{R}.$$

Then, by the definitions of ρ_F and ρ_G,

$$\liminf_{R\to\infty} \rho_R = \rho_F + \kappa_0\rho_G =: \rho.$$

Now, using the known bounds (from Lemma 2.4 and the properties of U, V)

$$\|U(x)\| \le M E_q(\nu_0 b^q) =: c_0, \qquad \|\mathcal{U}(x,0)\| \le M^{i+1} E_q^{i+1}(\nu_0 b^q) =: c_i,$$

$$\Big\|\sum_{k=1}^{i} \mathcal{U}(x,x_k)\Big\| \le \sum_{k=1}^{i} M^k E_q^k(\nu_0 b^q) =: C_i,$$

and

$$\int_0^x (x-y)^{q-1}\|V(x-y)\|\,dy \le \frac{M}{|\nu_0|}, \qquad \int_{x_{k-1}}^{x_k} (x_k-y)^{q-1}\|V(x_k-y)\|\,dy \le \frac{M}{|\nu_0|},$$

we estimate $\|Qu(x)\|$.

For $x \in [0, x_1]$,

$$\begin{aligned}
\|Qu(x)\| &\le \|U(x)\|\,\|B(u)\| + \int_0^x (x-y)^{q-1}\|V(x-y)\|\,\Phi_R(y)\,dy \\
&\le c_0\|B(u)\| + \frac{M}{|\nu_0|}\|\Phi_R\|_{C([0,\omega])}.
\end{aligned}$$

For $x \in (x_i, x_{i+1}]$,

$$\begin{aligned}
\|Qu(x)\| \le{}& \|\mathcal{U}(x,0)\|\,\|B(u)\| + \sum_{k=1}^{i} \|\mathcal{U}(x,x_k)\|\,\|I_k(u(x_k))\| \\
&+ \sum_{k=1}^{i} \|\mathcal{U}(x,x_k)\| \int_{x_{k-1}}^{x_k} (x_k-y)^{q-1}\|V(x_k-y)\|\,\Phi_R(y)\,dy \\
&+ \int_{x_i}^{x} (x-y)^{q-1}\|V(x-y)\|\,\Phi_R(y)\,dy \\
\le{}& c_i\|B(u)\| + K\sum_{k=1}^{i} \|\mathcal{U}(x,x_k)\|\,\|u(x_k)\| \\
&+ \frac{MC_i}{|\nu_0|}\|\Phi_R\|_{C([0,\omega])} + \frac{M}{|\nu_0|}\|\Phi_R\|_{C([0,\omega])}.
\end{aligned}$$

Since $\|u(x_k)\| \le R$ and $\sum_{k=1}^{i} \|\mathcal{U}(x,x_k)\| \le C_i$, we obtain

$$\|Qu(x)\| \leq c_i\|B(u)\| + KC_iR + \frac{M(C_i+1)}{|\nu_0|}\|\Phi_R\|_{C([0,\omega])}.$$

Now we need an estimate for $\|B(u)\|$. From (3.23) and the same bounds,

$$\begin{aligned}
\|B(u)\| \leq{}& \|(I-\mathcal{U}(\omega,0))^{-1}\|\bigg(\sum_{k=1}^{p}\|\mathcal{U}(\omega,x_k)\|\,\|I_k(u(x_k))\| \\
&+\sum_{k=1}^{p}\|\mathcal{U}(\omega,x_k)\|\int_{x_{k-1}}^{x_k}(x_k-y)^{q-1}\|V(x_k-y)\|\,\Phi_R(y)\,dy \\
&+\int_{x_p}^{\omega}(\omega-y)^{q-1}\|V(\omega-y)\|\,\Phi_R(y)\,dy\bigg) \\
\leq{}& L\bigg(K\sum_{k=1}^{p}\|u(x_k)\| + \frac{M}{|\nu_0|}\|\Phi_R\|_{C([0,\omega])}\sum_{k=1}^{p}\|\mathcal{U}(\omega,x_k)\| + \frac{M}{|\nu_0|}\|\Phi_R\|_{C([0,\omega])}\bigg) \\
\leq{}& L\Big(K_pR + \frac{M}{|\nu_0|}\|\Phi_R\|_{C([0,\omega])}(C_p+1)\Big),
\end{aligned}$$

where $L := \|(I-\mathcal{U}(\omega,0))^{-1}\|$ is a constant.

Because $\|\Phi_R\|_{C([0,\omega])} = \rho_R R$, the inequalities become

$$\|Qu(x)\| \leq c_0L\Big(K_pR + \frac{M}{|\nu_0|}\rho_R R(C_p+1)\Big) + \frac{M}{|\nu_0|}\rho_R R$$

for $x \in [0, x_1]$, and for $x \in (x_i, x_{i+1}]$

$$\|Qu(x)\| \leq c_iL\Big(K_pR + \frac{M}{|\nu_0|}\rho_R R(C_p+1)\Big) + KC_iR + \frac{M(C_i+1)}{|\nu_0|}\rho_R R.$$

Condition (3.20) guarantees that for sufficiently large R (so that ρ_R is close enough to ρ), the right-hand sides are $\leq R$. Indeed, one can choose R so large that

$$c_0L\Big(K_p + \frac{M}{|\nu_0|}\rho_R(C_p+1)\Big) + \frac{M}{|\nu_0|}\rho_R \leq 1,$$

$$c_iL\Big(K_p + \frac{M}{|\nu_0|}\rho_R(C_p+1)\Big) + KC_i + \frac{M(C_i+1)}{|\nu_0|}\rho_R \leq 1,$$

which is possible because $\rho_R \to \rho$ and (3.20) implies the corresponding strict inequalities. Therefore, $Q(\overline{\Omega}_R) \subset \overline{\Omega}_R$.

To prove that the operator Q has a fixed point in $\overline{\Omega}_R$, we decompose Q as $Q = Q_1 + Q_2$, which can be expressed on $[0,\omega]$ as

$$Q_1u(x)=\begin{cases}U(x)B(u)+\displaystyle\int_0^x(x-y)^{q-1}V(x-y)\Big[F(y,u(y))+\int_0^y\kappa(y-\tau)G(\tau,u(\tau))d\tau\Big]dy,\ x\in[0,x_1],\\ \mathcal{U}(x,0)B(u)+\displaystyle\int_{x_i}^x(x-y)^{q-1}V(x-y)\Big[F(y,u(y))+\int_0^y\kappa(y-\tau)G(\tau,u(\tau))d\tau\Big]dy\\ \quad+\displaystyle\sum_{k=1}^{i}\mathcal{U}(x,x_k)\int_{x_{k-1}}^{x_k}(x_k-y)^{q-1}V(x_k-y)\Big[F(y,u(y))\\ \quad+\int_0^y\kappa(y-\tau)G(\tau,u(\tau))d\tau\Big]dy,\ x\in(x_i,x_{i+1}],\end{cases}\tag{3.25}$$

$$Q_2u(x)=\begin{cases}0, & x\in[0,x_1],\\ \displaystyle\sum_{k=1}^{i}\mathcal{U}(x,x_k)I_k(u(x_k)), & x\in(x_i,x_{i+1}].\end{cases}\tag{3.26}$$

Next, we will prove that Q_1 is a completely continuous operator and Q_2 is a contraction operator. For each $u\in\overline{\Omega}_R$, by the definition of Q and the periodicity of u, we only consider the domain $[0,\omega]$. Although $u\in PC_\omega(\mathbb{R}^+,X)$ may have jumps at the impulse points x_k, the integral terms defining Q_1 are continuous in x. Indeed, for a bounded integrand (since $F(y,u(y))$ is bounded on $\overline{\Omega}_R$ by Φ_R) and an integrable kernel $(x-y)^{q-1}$, the map

$$x\mapsto\int_a^x(x-y)^{q-1}V(x-y)\Phi_R(y)\,dy$$

is continuous on $[a,b]$ by the dominated convergence theorem. Moreover, equicontinuity of $Q_1(\overline{\Omega}_R)$ is verified separately on each subinterval $[x_i,x_{i+1}]$, on which u is continuous. The isolated jumps of u do not affect the uniform continuity of the integral terms, as the integrals are absolutely continuous functions of the upper limit. It is clear that Q_1 is continuous and sends $\overline{\Omega}_R$ into a bounded subset of $PC_\omega(\mathbb{R}^+,X)$. We now turn to showing that the family $Q_1(\overline{\Omega}_R)$ is equicontinuous on the interval $[0,\omega]$.

- If $\tau_1,\tau_2\in[0,x_1]$ and $\tau_1<\tau_2$, we can get

$$\begin{aligned}\|Q_1u(\tau_2)-Q_1u(\tau_1)\|\le{}&\|U(\tau_2)-U(\tau_1)\|\cdot\|B(u)\|\\ &+\int_0^{\tau_1}((\tau_2-y)^{q-1}-(\tau_1-y)^{q-1})\|V(\tau_2-y)\|\cdot\Phi_R(y)\,dy\\ &+\int_0^{\tau_1}(\tau_1-y)^{q-1}\|V(\tau_2-y)-V(\tau_1-y)\|\cdot\Phi_R(y)\,dy\\ &+\int_{\tau_1}^{\tau_2}(\tau_2-y)^{q-1}\|V(\tau_2-y)\|\cdot\Phi_R(y)\,dy\\ &:=I_1+I_2+I_3+I_4.\end{aligned}$$

We verify that when $\tau_2-\tau_1\to0$, I_n $(n=1,2,3,4)$ tends to 0 independently of $u\in\overline{\Omega}_R$. According to Lemma 2.1(i), when $\tau_2-\tau_1\to0$, $I_1\to0$. Using condition (H1) and Lemma 2.1(ii),

$$
\begin{aligned}
I_2 &\le \frac{M\|\Phi_R\|_{C([0,\omega])}}{\Gamma(q)} \int_0^{\tau_1} ((\tau_2 - y)^{q-1} - (\tau_1 - y)^{q-1}) dy \\
&\le \frac{M\|\Phi_R\|_{C([0,\omega])}}{\Gamma(q)} (\tau_1^q - \tau_2^q + (\tau_2 - \tau_1)^q) \\
&\le \frac{2M\|\Phi_R\|_{C([0,\omega])}}{\Gamma(q)} (\tau_2 - \tau_1)^q \to 0, \quad \tau_2 - \tau_1 \to 0.
\end{aligned}
$$

$$
I_4 \le \frac{M\|\Phi_R\|_{C([0,\omega])}}{\Gamma(q)} \int_{\tau_1}^{\tau_2} (\tau_2 - y)^{q-1} dy = \frac{M\|\Phi_R\|_{C([0,\omega])}}{\Gamma(q+1)} (\tau_2 - \tau_1)^q \to 0, \quad \tau_2 - \tau_1 \to 0.
$$

If $\tau_1 = 0$ and $\tau_2 > 0$, then $I_3 = 0$. If $\tau_1 > 0$, for sufficiently small $\varepsilon > 0$, by (H1) and Lemma 2.1(iv),

$$
\begin{aligned}
I_3 &\le \|\Phi_R\|_{C([0,\omega])} \int_0^{\tau_1 - \varepsilon} (\tau_1 - y)^{q-1} \|V(\tau_2 - y) - V(\tau_1 - y)\| dy \\
&\quad + \|\Phi_R\|_{C([0,\omega])} \int_{\tau_1 - \varepsilon}^{\tau_1} (\tau_1 - y)^{q-1} \|V(\tau_2 - y) - V(\tau_1 - y)\| dy \\
&\le \sup_{y \in [0, \tau_1 - \varepsilon]} \|V(\tau_2 - y) - V(\tau_1 - y)\| \cdot \frac{\|\Phi_R\|_{C([0,\omega])} (\tau_1^q - \varepsilon^q)}{q} \\
&\quad + \frac{2M\|\Phi_R\|_{C([0,\omega])}}{\Gamma(q+1)} \varepsilon^q \to 0, \quad \varepsilon \to 0,\ \tau_2 - \tau_1 \to 0.
\end{aligned}
$$

Thus, $\|Q_1 u(\tau_2) - Q_1 u(\tau_1)\| \to 0$ uniformly for all $u \in \overline{\Omega}_R$ as $\tau_2 - \tau_1 \to 0$. Hence, $Q_1(\overline{\Omega}_R)$ is equicontinuous on $[0, x_1]$. A similar argument holds for $x \in (x_i, x_{i+1}]$.

- If $\tau_1, \tau_2 \in (x_i, x_{i+1}]$, by Lemma 2.1(ii) we have

$$
\begin{aligned}
\|Q_1 u(\tau_2) - Q_1 u(\tau_1)\| &\le \|\mathcal{U}(\tau_2, 0) - \mathcal{U}(\tau_1, 0)\| \cdot \|B(u)\| \\
&\quad + \Big\| \sum_{k=1}^{i} \mathcal{U}(\tau_2, x_k) \int_{x_{k-1}}^{x_k} (x_k - y)^{q-1} V(x_k - y) \big[F + \kappa G\big] dy \\
&\qquad - \sum_{k=1}^{i} \mathcal{U}(\tau_1, x_k) \int_{x_{k-1}}^{x_k} (x_k - y)^{q-1} V(x_k - y) \big[F + \kappa G\big] dy \Big\| \\
&\quad + \Big\| \int_{x_i}^{\tau_2} (\tau_2 - y)^{q-1} V(\tau_2 - y) \big[F + \kappa G\big] dy \\
&\qquad - \int_{x_i}^{\tau_1} (\tau_1 - y)^{q-1} V(\tau_1 - y) \big[F + \kappa G\big] dy \Big\| \\
&=: I_1 + I_2 + I_3 + I_4 + I_5,
\end{aligned}
$$

with the same definitions of $I_1, \ldots, I_5$ as in the original proof but with ϕ_R replaced by Φ_R. Since $U(x)$ and $V(x)$ are uniformly continuous for $x \ge 0$, when $\tau_2 - \tau_1 \to 0$ and ε is sufficiently small, each I_n tends to 0. This proves the equicontinuity for $x \in (x_i, x_{i+1}]$ ($i = 1, \ldots, p$). Therefore, $Q_1(\overline{\Omega}_R)$ is equicontinuous on $[0, \omega]$.

It remains to prove that for any fixed $t \in [0, \omega]$, the set $(Q_1 \overline{\Omega}_R)(x)$ is relatively compact in X.

- For $x \in [0, x_1]$, define the set $(Q^{(0)}_{\varepsilon,\delta}\overline{\Omega}_R)(x)$ as

$$(Q^{(0)}_{\varepsilon,\delta}\overline{\Omega}_R)(x) := \{(Q^{(0)}_{\varepsilon,\delta}u)(x) \mid u \in \overline{\Omega}_R,\ 0 < \varepsilon < x < x_1\},$$

where

$$\begin{aligned}
&(Q^{(0)}_{\varepsilon,\delta}u)(x) = U(x)B(u)\\
&+q\int_0^{x-\varepsilon}\int_\delta^x \sigma(x-y)^{q-1}\xi_q(\sigma)T((x-y)^q\sigma)\Big[F(y,u(y)) + \int_0^y \kappa(y-\tau)G(\tau,u(\tau))d\tau\Big]d\sigma dy\\
&= U(x)B(u)+qT(\varepsilon^q\delta)\int_0^{x-\varepsilon}\int_\delta^x \sigma(x-y)^{q-1}\xi_q(\sigma)T((x-y)^q\sigma-\varepsilon^q\delta)\Big[F+\kappa G\Big]d\sigma dy.
\end{aligned}$$

By the compactness of $U(x)$ and $T(\varepsilon^q\delta)$, for $x \in [0, x_1]$, $(Q^{(0)}_{\varepsilon,\delta}\overline{\Omega}_R)(x)$ is relatively compact in X. Moreover, for each $u \in \overline{\Omega}_R$ and $x \in [0, x_1]$,

$$\begin{aligned}
\|Qu(x) - Q^{(0)}_{\varepsilon,\delta}u(x)\| &\le \Big\|q\int_0^x\int_0^\delta \sigma(x-y)^{q-1}\xi_q(\sigma)T((x-y)^q\sigma)\Big[F+\kappa G\Big]d\sigma dy\Big\|\\
&\quad + \Big\|q\int_{x-\varepsilon}^x\int_\delta^\infty \sigma(x-y)^{q-1}\xi_q(\sigma)T((x-y)^q\sigma)\Big[F+\kappa G\Big]d\sigma dy\Big\|\\
&\le qM\|\Phi_R\|_{C([0,\omega])}\Big(\int_0^x (x-y)^{q-1}dy\int_0^\delta \sigma\xi_q(\sigma)d\sigma\\
&\quad + \int_{x-\varepsilon}^x (x-y)^{q-1}dy\int_\delta^\infty \tau\xi_q(\tau)d\tau\Big)\\
&\le M\|\Phi_R\|_{C([0,\omega])}x^q\int_0^\delta \sigma\xi_q(\sigma)d\sigma + \frac{M\|\Phi_R\|_{C([0,\omega])}}{\Gamma(1+q)}\varepsilon^q\\
&\to 0, \quad \varepsilon \to 0,\ \delta \to 0.
\end{aligned}$$

Hence, $(Q_1\overline{\Omega}_R)(x)$ is relatively compact in X.

- For $x \in (x_i, x_{i+1}]$ $(i = 1, 2, \ldots, p)$, define the set $(Q^{(i)}_{\epsilon,\delta}\overline{\Omega}_R)(x)$ as

$$(Q^{(i)}_{\epsilon,\delta}\overline{\Omega}_R)(x) := \{(Q^{(i)}_{\epsilon,\delta}u)(x) \mid u \in \overline{\Omega}_R,\ x_i < x-\epsilon < x < x_{i+1}\},$$

where

$$
\begin{aligned}
(Q^{(i)}_{\epsilon,\delta}u)(x) =&\mathcal{U}(x,0)B(u) + \sum_{k=1}^{i}\mathcal{U}(x,x_k)\int_{x_{k-1}}^{x_k}(x_k-y)^{q-1}V(x_k-y)\Big[F(y,u(y))\\
&+\int_0^y\kappa(y-\tau)G(\tau,u(\tau))d\tau\Big]dy\\
&+q\int_{x_i}^{x-\epsilon}\int_\delta^x\sigma(x-y)^{q-1}\xi_q(\sigma)T((x-y)^q\sigma)\Big[F(y,u(y))\\
&+\int_0^y\kappa(y-\tau)G(\tau,u(\tau))d\tau\Big]d\sigma dy\\
=&U(x-x_i)\prod_{k=1}^{i}U(x_k-x_{k-1})B(u)\\
&+\sum_{k=1}^{i}U(x-x_i)\prod_{j=k}^{i-1}U(x_{j+1}-x_j)\int_{x_{k-1}}^{x_k}(x_k-y)^{q-1}V(x_k-y)\Big[F(y,u(y))\\
&+\int_0^y\kappa(y-\tau)G(\tau,u(\tau))d\tau\Big]dy\\
&+qT(\epsilon^q\delta)\int_{x_i}^{x-\epsilon}\int_\delta^x\sigma(x-y)^{q-1}\xi_q(\sigma)T((x-y)^q\sigma-\epsilon^q\delta)\Big[F(y,u(y))\\
&+\int_0^y\kappa(y-\tau)G(\tau,u(\tau))d\tau\Big]d\sigma dy.
\end{aligned}
$$

By the compactness of $U(x)$ and $T(\epsilon^q\delta)$, for $x\in(x_i,x_{i+1}]$, $(Q^{(i)}_{\epsilon,\delta}\overline{\Omega}_R)(x)$ is relatively compact in X. Moreover, for each $u\in\overline{\Omega}_R$ and $x\in(x_i,x_{i+1}]$, by the inequality

$$
\begin{aligned}
&\|Qu(x)-Q^{(i)}_{\epsilon,\delta}u(x)\|\\
&\le\left\|q\int_{x_i}^{x}\int_0^\delta\sigma(x-y)^{q-1}\xi_q(\sigma)T((x-y)^q\sigma)\Big[F(y,u(y))+\int_0^y\kappa(y-\tau)G(\tau,u(\tau))d\tau\Big]d\sigma dy\right\|\\
&+\left\|q\int_{x-\epsilon}^{x}\int_\delta^\infty\sigma(x-y)^{q-1}\xi_q(\sigma)T((x-y)^q\sigma)\Big[F(y,u(y))+\int_0^y\kappa(y-\tau)G(\tau,u(\tau))d\tau\Big]d\sigma dy\right\|\\
&\le qM\|\Phi_R\|_{C([0,\omega])}\int_{x_i}^x(x-y)^{q-1}dy\int_0^\delta\sigma\xi_q(\sigma)d\sigma\\
&\quad+qM\|\Phi_R\|_{C([0,\omega])}\int_{x-\epsilon}^x(x-y)^{q-1}dy\int_\delta^\infty\tau\xi_q(\tau)d\tau\\
&\le M\|\Phi_R\|_{C([0,\omega])}(x-x_i)^q\int_0^\delta\sigma\xi_q(\sigma)d\sigma+\frac{M\|\Phi_R\|_{C([0,\omega])}}{\Gamma(1+q)}\epsilon^q\\
&\to0,\quad \epsilon\to0,\ \delta\to0.
\end{aligned}
$$

Therefore, for all $x\in[0,\omega]$, $(Q_1\overline{\Omega}_R)(x)$ is relatively compact in X. Hence, by the Arzelà–Ascoli theorem, Q_1 is completely continuous.

Contraction property of Q_2. Let $u,v\in\overline{\Omega}_R$. If $x\in[0,x_1]$, then $\|Q_2u(x)-Q_2v(x)\|=0$. For $x\in(x_i,x_{i+1}]$, from (3.26) and condition (H2),

$$\begin{aligned}\|Q_2u(x)-Q_2v(x)\| &= \Big\|\sum_{k=1}^{i}\mathcal{U}(x,x_k)(I_k(u(x_i))-I_k(v(x_i)))\Big\| \\ &\leq \Big\|\sum_{k=1}^{i}\mathcal{U}(x,x_k)\Big\|\cdot\|I_k(u(x_i))-I_k(v(x_i))\| \\ &\leq K\sum_{k=1}^{i}M^kE_q^k(\nu_0b^q)\|u-v\|_{PC}=KC_i\|u-v\|_{PC}.\end{aligned}$$

Thus, $\|Q_2u-Q_2v\|_{PC}\leq KC_i\|u-v\|_{PC}$. Condition (3.22) implies $KC_i<1$, so Q_2 is a contraction.

Based on the above argument, we know that the operator $Q=Q_1+Q_2$ is a condensing operator (sum of a completely continuous map and a contraction). Hence, by Lemma 2.5 (Sadovskii's fixed point theorem), Q has a fixed point $u\in\overline{\Omega}_R$, i.e., Eq (3.21) has an ω-periodic mild solution. □

Theorem 3.3. *Let X be a Banach space, and let $-A$ generate an exponentially stable compact semigroup $T(x)$ $(x\geq 0)$ in X. Assume that $F,G:\mathbb{R}^+\times X\to X$ are continuous, ω-periodic in x, and $I_i\in C(X,X)$ $(i\in\mathbb{N})$ satisfy $I_{i+p+1}=I_i$, $I_0=I$, where p is the number of impulse points in $[0,\omega]$. Suppose that I_i satisfy condition (H2) (Lipschitz with constant K, $I_i(\theta)=\theta$), and that F,G satisfy:*

(H3) There exist continuous functions $\phi_F,\phi_G:\mathbb{R}^+\to\mathbb{R}^+$ and continuous nondecreasing functions $\Psi_F,\Psi_G:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$\|F(x,t)\|\leq\phi_F(x)\Psi_F(\|t\|),\qquad \|G(x,t)\|\leq\phi_G(x)\Psi_G(\|t\|),\quad x\geq 0,\ t\in X.$$

Moreover, for all $x\geq 0$, the functions $s\mapsto(x-y)^{q-1}\phi_F(y)$ and $y\mapsto(x-y)^{q-1}\phi_G(y)$ belong to $L^1([0,x])$. Also, there exist constants $a_F,a_G\geq 0$ such that

$$\int_{a_F}^{\infty}\frac{1}{\Psi_F(y)}dy=\infty,\qquad \int_{a_G}^{\infty}\frac{1}{\Psi_G(y)}dy=\infty.$$

(H3') In addition to (H3), assume that the function $\widetilde{\Psi}(r):=\Psi_F(r)+\Psi_G(r)$ satisfies the Osgood condition

$$\int_a^{\infty}\frac{dr}{\widetilde{\Psi}(r)}=\infty$$

for some $a\geq 0$. This assumption holds, for instance, if Ψ_F and Ψ_G have the same asymptotic growth rate, or if one dominates the other in the sense that $\Psi_F(r)\leq C\Psi_G(r)$ for all sufficiently large r.

(H4) The kernel κ satisfies $\kappa_0:=\int_0^{\omega}|\kappa(y)|dy<\infty$.

Define $C_i=\sum_{k=1}^{i}M^kE_q^k(\nu_0b^q)$ $(i=1,\dots,p)$, $b=\min_{1\leq k\leq p+1}(x_k-x_{k-1})$. If

$$KC_i<1\qquad \textit{for all } i=1,\dots,p,$$

then the semilinear impulsive problem with memory

$$\begin{cases}{}^cD^q_{x_i,x}u(x)+Au(x)=F(x,u(x))+\displaystyle\int_0^x\kappa(x-y)G(y,u(y))dy, & x\in(x_i,x_{i+1}),\\ \Delta u(x_i)=I_i(u(x_i^-)), & i\in\mathbb{N},\\ u(0)=u_0,\end{cases}\tag{3.27}$$

has at least one ω-periodic mild solution $u \in PC_\omega(\mathbb{R}^+, X)$.

Proof. Let Q, Q_1, and Q_2 be defined by (3.22), (3.25), and (3.26), respectively, with the full source term containing F and the memory integral. Using the integral formula derived from the linear case (without directly applying the theorem), we define the operator Q in (3.22), and we prove that its fixed points correspond to the solutions of the semilinear problem.

From Theorem 3.1 (extended to the memory term as shown in the linear case), the ω-periodic mild solution of (3.21) is equivalent to a fixed point of Q.

From the proof of Theorem 3.2, we have that Q_1 is completely continuous and Q_2 is a contraction (the arguments for complete continuity use only boundedness on bounded sets, which holds because Ψ_F, Ψ_G are continuous and ϕ_F, ϕ_G are bounded on $[0, \omega]$; the contraction property of Q_2 is unchanged). Therefore, we only need to prove that the set

$$\Upsilon = \left\{ u \in PC_\omega(\mathbb{R}^+, X) \;\middle|\; u = \lambda Q_1 u + \lambda Q_2\left(\frac{u}{\lambda}\right),\ \lambda \in (0,1) \right\} \tag{3.28}$$

is bounded. Then, by Lemma 2.6 (Leray-Schauder alternative), $Q = Q_1 + Q_2$ has a fixed point, which is the desired solution.

By the periodicity of u, we only consider it on $[0, \omega]$.

Let $u \in \Upsilon$ and $t \in [0, \omega]$. Then, for some $\lambda \in (0, 1)$,

$$u(x) = \lambda Q_1 u(x) + \lambda Q_2\left(\frac{u(x)}{\lambda}\right).$$

For $x \in [0, x_1]$, $Q_2 = 0$. Using the expression of Q_1 and the bounds $\|U(x)\| \le c_0$, $\|V(x)\| \le \frac{M}{\Gamma(q)}$ (Lemma 2.1(ii)), we obtain

$$\begin{aligned}
\|u(x)\| &\le \lambda \|U(x)\| \cdot \|B(u)\| \\
&\quad + \lambda \int_0^x (x-y)^{q-1} \|V(x-y)\| \cdot \left\| F(y, u(y)) + \int_0^y \kappa(y-\tau) G(\tau, u(\tau)) d\tau \right\| dy \\
&\le c_0 \|B(u)\| + \frac{M}{\Gamma(q)} \int_0^x (x-y)^{q-1} \left\| F(y, u(y)) + \int_0^y \kappa(y-\tau) G(\tau, u(\tau)) d\tau \right\| dy.
\end{aligned}$$

By (H3) and (H4),

$$\begin{aligned}
\left\| F(y, u(y)) + \int_0^y \kappa(y-\tau) G(\tau, u(\tau)) d\tau \right\| &\le \phi_F(y) \Psi_F(\|u(y)\|) + \int_0^y |\kappa(y-\tau)|\, \phi_G(\tau) \Psi_G(\|u(\tau)\|) d\tau \\
&\le \phi_F(y) \Psi_F(\|u(y)\|) + \kappa_0 \|\phi_G\|_{C([0,\omega])} \Psi_G(\|u(\tau)\|).
\end{aligned}$$

Define

$$\Phi(y) := \phi_F(y) + \kappa_0 \|\phi_G\|_{C([0,\omega])}, \qquad \widetilde{\Psi}(r) := \Psi_F(r) + \Psi_G(r).$$

Since Ψ_F, Ψ_G are nondecreasing, $\widetilde{\Psi}$ is nondecreasing. Moreover, using (H3) and (H4), we have

$$\begin{aligned}
\left\| F(y, u(y)) + \int_0^y \kappa(y-\tau) G(\tau, u(\tau)) d\tau \right\| &\le \phi_F(y) \Psi_F(\|u(y)\|) + \kappa_0 \|\phi_G\|_{C([0,\omega])} \Psi_G(\|u(y)\|) \\
&\le \Phi(y) \widetilde{\Psi}(\|u(y)\|).
\end{aligned}$$

Let $\eta(x) := \sup_{0\le y\le x}\|u(y)\|$. Because $\widetilde{\Psi}$ is nondecreasing, $\widetilde{\Psi}(\|u(y)\|) \le \widetilde{\Psi}(\eta(x))$ for all $y \le x$. Hence,

$$\|u(x)\| \le c_0\|B(u)\| + \frac{M}{\Gamma(q)}\widetilde{\Psi}(\eta(x))\int_0^x (x-y)^{q-1}\Phi(y)dy. \tag{3.29}$$

For $x \in (x_i, x_{i+1}]$ ($i = 1, \dots, p$), using the expressions of Q_1 and Q_2, the Lipschitz condition (H2), and the fact that $I_k(\theta) = \theta$, we get

$$\begin{aligned}
\|u(x)\| \le{}& \lambda\|\mathcal{U}(x,0)\|\|B(u)\| + \lambda\sum_{k=1}^{i}\|\mathcal{U}(x,x_k)\|\left\|I_k\left(\frac{u(x_k)}{\lambda}\right)\right\| \\
&+ \lambda\sum_{k=1}^{i}\|\mathcal{U}(x,x_k)\|\int_{x_{k-1}}^{x_k}(x_k-y)^{q-1}\|V(x_k-y)\| \\
&\times\left\|F(y,u(y)) + \int_0^y \kappa(y-\tau)G(\tau,u(\tau))d\tau\right\|dy \\
&+ \lambda\int_{x_i}^{x}(x-y)^{q-1}\|V(x-y)\|\left\|F(y,u(y)) + \int_0^y \kappa(y-\tau)G(\tau,u(\tau))d\tau\right\|dy.
\end{aligned}$$

Using $\|I_k(u(x_k)/\lambda)\| \le K\|u(x_k)\|/\lambda$ and the bounds $\|\mathcal{U}(x,0)\| \le c_i$, $\|\mathcal{U}(x,x_k)\| \le M^{i-k+1}E_q^{i-k+1}(\nu_0 b^q)$, we obtain

$$\begin{aligned}
\|u(x)\| \le{}& c_i\|B(u)\| + K\sum_{k=1}^{i}M^{i-k+1}E_q^{i-k+1}(\nu_0 b^q)\|u(x_k)\| \\
&+ \frac{M}{\Gamma(q)}\sum_{k=1}^{i}\|\mathcal{U}(x,x_k)\|\int_{x_{k-1}}^{x_k}(x_k-y)^{q-1}\Phi(y)\widetilde{\Psi}(\|u(y)\|)dy \\
&+ \frac{M}{\Gamma(q)}\int_{x_i}^{x}(x-y)^{q-1}\Phi(y)\widetilde{\Psi}(\|u(y)\|)dy.
\end{aligned}$$

Since $\sum_{k=1}^{i}M^{i-k+1}E_q^{i-k+1}(\nu_0 b^q) \le C_i$ and $\|u(x_k)\| \le \eta(\omega)$, the second term is $\le KC_i\eta(\omega)$. Also, $\widetilde{\Psi}(\|u(y)\|) \le \widetilde{\Psi}(\eta(x))$ for $y \le x$. Hence,

$$\|u(x)\| \le c_i\|B(u)\| + KC_i\eta(\omega) + \frac{M}{\Gamma(q)}\widetilde{\Psi}(\eta(x))\left(C_i\int_0^{\omega}(x-y)^{q-1}\Phi(y)dy + \int_{x_i}^{x}(x-y)^{q-1}\Phi(y)dy\right). \tag{3.30}$$

Define $\eta(x) = \sup_{0\le y\le x}\|u(y)\|$ for $x \in [0,\omega]$. From (3.29), and (3.30) we obtain, as in the original proof,

$$\eta(x) \le \begin{cases} c_0\|B(u)\| + \dfrac{M}{\Gamma(q)}\widetilde{\Psi}(\eta(x))\displaystyle\int_0^x (x-y)^{q-1}\Phi(y)dy, & x \in [0,x_1], \\ d_i + e_i\widetilde{\Psi}(\eta(x))\displaystyle\int_{x_i}^{x}(x-y)^{q-1}\Phi(y)dy, & x \in (x_i, x_{i+1}], \end{cases} \tag{3.31}$$

where

$$d_i = \frac{c_i\|B(u)\| + \dfrac{M}{\Gamma(q)}\displaystyle\sum_{k=1}^{i}M^{i-k+1}E_q^{i-k+1}(\nu_0 b^q)\int_{x_{k-1}}^{x_k}(x_k-y)^{q-1}\Phi(y)\widetilde{\Psi}(\|u(y)\|)dy}{1-KC_i},$$

$$e_i = \frac{M}{\Gamma(q)(1 - KC_i)}.$$

Now, set

$$\zeta(x) = \begin{cases} c_0\|B(u)\| + \dfrac{M}{\Gamma(q)}\widetilde{\Psi}(\eta(x)) \displaystyle\int_0^x (x-y)^{q-1}\Phi(y)dy, & x \in [0, x_1], \\ d_i + e_i\widetilde{\Psi}(\eta(x)) \displaystyle\int_{x_i}^x (x-y)^{q-1}\Phi(y)dy, & x \in (x_i, x_{i+1}]. \end{cases} \tag{3.32}$$

Then, $\eta(x) \le \zeta(x)$ for all $x \in [0, \omega]$. Also, $\zeta(0) = c_0\|B(u)\|$, and $\zeta(x_i) = d_i$ for $i = 1, \dots, p$.

Define

$$\Phi_i(x) := \int_{x_i}^x (x-y)^{q-1}\Phi(y)dy, \qquad i = 0, 1, \dots, p,$$

so that formally $\Phi_i'(x) = (x - x_i)^{q-1}\Phi(x)$. However, since Φ is only locally integrable, ζ is not necessarily differentiable everywhere in the classical sense. Nevertheless, on each subinterval $(x_i, x_{i+1}]$, ζ is absolutely continuous, because it is the sum of a constant and an integral of an integrable function. Therefore, by the fundamental theorem of calculus for absolutely continuous functions, ζ is differentiable almost everywhere on $(x_i, x_{i+1}]$, and the following inequality holds in the almost-everywhere sense,

$$\frac{\zeta'(x)}{\widetilde{\Psi}(\zeta(x))} \le \begin{cases} \frac{M}{\Gamma(q)}\Phi_0'(x), & x \in [0, x_1], \\ e_i\Phi_i'(x), & x \in (x_i, x_{i+1}], \end{cases}$$

where $\Phi_i'(x) = (x - x_i)^{q-1}\Phi(x)$ is understood in the L^1-sense, and $e_i = \dfrac{M}{\Gamma(q)(1 - KC_i)}$. The integration step below is justified by the absolute continuity of ζ and the monotonicity of $\widetilde{\Psi}$. Integrating the above inequality from 0 to x for $x \in [0, x_1]$, and from x_i to x for $x \in (x_i, x_{i+1}]$, gives

$$\int_{\zeta(0)}^{\zeta(x)} \frac{dr}{\widetilde{\Psi}(r)} \le \frac{M}{\Gamma(q)}\Phi_0(x), \qquad x \in [0, x_1],$$

and

$$\int_{\zeta(x_i)}^{\zeta(x)} \frac{dr}{\widetilde{\Psi}(r)} \le e_i\Phi_i(x), \qquad x \in (x_i, x_{i+1}].$$

Since the right-hand sides are finite for every fixed $x \in [0, \omega]$ (because $\Phi \in L^1[0, \omega]$ and the kernels are integrable), and since $\widetilde{\Psi}$ satisfies the Osgood condition by assumption (H3'), i.e.,

$$\int_a^\infty \frac{dr}{\widetilde{\Psi}(r)} = \infty,$$

it follows that $\zeta(x)$ cannot blow up at any finite x. Hence, there exists a constant $R > 0$, independent of u and λ, such that

$$\zeta(x) \le R \quad \text{for all } x \in [0, \omega].$$

Consequently, $\eta(x) \le \zeta(x) \le R$, which implies $\|u\|_{PC} \le R$. Therefore, the set Υ defined in (3.29) is bounded.

Since Υ is bounded, Q_1 is completely continuous, Q_2 is a contraction (because $KC_i < 1$), and Lemma 2.6 (the Leray-Schauder alternative) implies that $Q = Q_1 + Q_2$ has a fixed point $u \in \overline{\Omega}_R$. This fixed point is an ω-periodic mild solution of the original problem.

□

4. Application

Suppose $\overline{\Omega} \subset \mathbb{R}^n$ is a bounded domain with C^2 boundary $\partial\Omega$ ($n \in \mathbb{N}$). Let Δ stand for the Laplace operator, and let λ_1 denote the principal eigenvalue of $-\Delta$ under the Dirichlet boundary condition $u|_{\partial\Omega} = 0$. This eigenvalue satisfies $\lambda_1 > 0$ (see [22]).

We consider the following time-fractional impulsive parabolic boundary value problem with a memory term:

$$\begin{cases} {}^cD^q_{x_i}u(t,x) - \Delta u(t,x) = f(x,u(t,x)) + \displaystyle\int_0^x \kappa(x-y)\, g(y,u(t,y))\, dy, & t \in \Omega,\ x \geq 0,\ x \neq x_i, \\ \Delta u(t,x_i) = I_i(u(t,x_i)), & t \in \Omega,\ x_i = i\pi,\ i \in \mathbb{N}, \\ u|_{\partial\Omega} = 0, & \end{cases} \tag{4.1}$$

where ${}^cD^q_{x_i}$ denotes the Caputo fractional derivative of order $q \in (0,1)$ with respect to the time variable x, with lower limit x_i, $f, g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $I_i : \mathbb{R} \to \mathbb{R}$ are continuous functions, f and g are 2π-periodic in x, $I_{i+2} = I_i$, and κ is a locally integrable kernel. In this application, we take $\omega = 2\pi$ and $p = 2$, with impulse points $x_1 = \pi$ and $x_2 = 2\pi = \omega$. Consequently, the periodic extension satisfies $x_{i+2} = x_i + \omega$. Hence, the impulse functions satisfy $I_{i+2} = I_i$. This is consistent with the abstract framework (where $I_{i+p+1} = I_i$ with $p = 2$) up to a harmless shift in the index, since $x_0 = 0$ is not an impulse point in the present setting; only the indices $i \geq 1$ are used. We assume the following conditions:

(A1) There exist nonnegative constants ρ_0, ρ_1 such that

$$|f(x,\xi)| \leq \rho_0 + \rho_1|\xi|, \qquad x \geq 0,\ \xi \in \mathbb{R}.$$

(A1*) There exist nonnegative constants ρ_0', ρ_1' such that

$$|g(x,\xi)| \leq \rho_0' + \rho_1'|\xi|, \qquad x \geq 0,\ \xi \in \mathbb{R}.$$

(A2) For each I_i, $I_i(0) = 0$, and there exists $K > 0$ such that

$$|I_i(\xi_1) - I_i(\xi_2)| \leq K|\xi_1 - \xi_2|, \qquad \xi_1, \xi_2 \in \mathbb{R},\ i \in \mathbb{N}.$$

(A3) There exist continuous functions $\phi_f, \phi_g : \mathbb{R}^+ \to \mathbb{R}^+$ and continuous nondecreasing functions $\Psi_f, \Psi_g : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|f(x,\xi)| \leq \phi_f(x)\Psi_f(|\xi|), \qquad |g(x,\xi)| \leq \phi_g(x)\Psi_g(|\xi|), \quad x \geq 0,\ \xi \in \mathbb{R},$$

and for all $x \geq 0$ the functions $s \mapsto (x-y)^{q-1}\phi_f(y)$ and $y \mapsto (x-y)^{q-1}\phi_g(y)$ belong to $L^1([0,x])$. Moreover, there exist $a_f, a_g \geq 0$ such that

$$\int_{a_f}^{\infty} \frac{1}{\Psi_f(y)}dy = \infty, \qquad \int_{a_g}^{\infty} \frac{1}{\Psi_g(y)}dy = \infty.$$

Set $X = L^2(\Omega)$ with norm $\|\cdot\|_2$. Define the operator $A : D(A) \subset X \to X$ by

$$D(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \qquad Au = -\Delta u. \tag{4.2}$$

It is known (see [1]) that $-A$ generates an exponentially stable analytic semigroup $T(x)$ ($x \geq 0$) on X. Moreover, $T(x)$ is contractive, i.e., $\|T(x)\| \leq M := 1$, and because A has compact resolvent, $T(x)$ is compact for $x > 0$ [20]. The growth index of $T(x)$ is $\nu_0 = -\lambda_1 < 0$.

For $u : \Omega \times \mathbb{R}^+ \to \mathbb{R}$, define $u(x)(t) = u(t,x)$, so that $u : \mathbb{R}^+ \to X$. Set

$$F(x,\xi)(t) = f(x,\xi(t)), \qquad G(x,\xi)(t) = g(x,\xi(t)), \qquad \xi \in X,\ t \in \Omega.$$

Then, $F, G : \mathbb{R}^+ \times X \to X$ are continuous and 2π-periodic in x. The problem (4.1) is equivalent to the abstract fractional impulsive evolution equation

$$\begin{cases} {}^{c}D^{q}_{x_i,x}u(x) + Au(x) = F(x,u(x)) + \displaystyle\int_0^x \kappa(x-y)\, G(y,u(y))\, dy, \quad x \in (x_i, x_{i+1}), \\ \Delta u(x_i) = I_i(u(x_i^-)), \quad i \in \mathbb{N}. \end{cases} \tag{4.3}$$

For the Nemytskii operators $F(x,\xi)(t) = f(x,\xi(t))$ and $G(x,\xi)(t) = g(x,\xi(t))$, the Osgood condition on the scalar functions f and g ensures the corresponding condition on the abstract operators F and G. Indeed, suppose f and g satisfy (A3) with functions $\phi_f, \phi_g, \Psi_f, \Psi_g$. Then, for $\xi \in X = L^2(\Omega)$ and a.e. $x \geq 0$,

$$\begin{aligned} \|F(x,\xi)\|_2 &= \left(\int_\Omega |f(x,\xi(t))|^2 dt \right)^{1/2} \\ &\leq \left(\int_\Omega \phi_f(x)^2 \Psi_f(|\xi(t)|)^2 dt \right)^{1/2} \\ &\leq \phi_f(x) \Psi_f\left(\left(\int_\Omega |\xi(t)|^2 dt \right)^{1/2} \right) \\ &= \phi_f(x)\Psi_f(\|\xi\|_2), \end{aligned}$$

where the last inequality follows from the monotonicity of Ψ_f and the fact that $\|\xi\|_2$ dominates $\|\xi(t)\|$ in the L^2-sense. The same argument applies to G with ϕ_g, Ψ_g. Consequently, (H3) in the abstract setting holds with the same functions Ψ_f, Ψ_g, and the Osgood condition from (A3) transfers directly to (H3'). Thus, the abstract framework of Theorem 3.3 applies to the fractional heat equation (4.1) under the stated assumptions (A2) and (A3).

Let $\kappa_0 = \int_0^{2\pi} |\kappa(y)| dy$ and define the combined growth constants

$$\rho_1^{\text{tot}} = \rho_1 + \kappa_0 \rho_1', \qquad \rho^{\text{tot}} = \liminf_{r\to\infty} \frac{\|\phi_f\|_C + \kappa_0 \|\phi_g\|_C}{r}$$

(when the linear growth condition (A1)-(A1*) holds). For the smallness condition we use the quantities

$$C_i = \sum_{k=1}^{i} M^k E_q^k(\nu_0 b^q), \quad b = \min\{x_1, x_2 - x_1, \ldots, 2\pi - x_p\},$$

with $M = 1$, $\nu_0 = -\lambda_1$, $b = \pi$ (since $x_1 = \pi$, $x_2 = 2\pi$ in our specific choice of impulse times $x_i = i\pi$).

Application of Theorem 3.2 (linear growth). Assume (A1), (A1*), and (A2) hold. If

$$\Big(K + \frac{M\rho_1^{\mathrm{tot}}}{|\nu_0|}\Big)C_1 + \frac{M\rho_1^{\mathrm{tot}}}{|\nu_0|} < 1,$$

then by Theorem 3.2 problem (4.1) has at least one 2π-periodic mild solution $u \in PC_{2\pi}(\mathbb{R}^+, X)$.

Application of Theorem 3.3 (general growth). Assume (A2) and (A3) hold. If $KC_1 < 1$, then by Theorem 3.3 problem (4.1) has at least one 2π-periodic mild solution.

These results extend and improve the corresponding ones in the literature (e.g., [13–15]) by including a memory term and allowing more general growth conditions on the nonlinearities.

5. Conclusions

This paper has established the existence and uniqueness (linear case) or existence (semilinear cases) of ω-periodic mild solutions for impulsive fractional evolution equations containing a memory term, assuming exponential stability and compactness of the underlying C_0-semigroup. The linear result relies on the invertibility of $I - \mathcal{U}(\omega, 0)$; the semilinear results use Sadovskii's theorem (linear growth) and the Leray-Schauder alternative (Osgood growth). Future investigations may include nonlocal initial conditions, noncompact semigroups, stability analysis, and controllability.

Author contributions

Ahmad Al-Omari: conceptualization, formal analysis, investigation, visualization, writing–original draft, writing–review and editing; Hanan Al-Saadi: conceptualization, formal analysis, methodology, validation, writing–original draft, writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors wish to thank the referees for useful comments and suggestions.

Conflict of interest

The authors declare that they have no conflicts of interest

References

1. P. Chen, X. Zhang, Y. Li, Existence and approximate controllability of fractional evolution equations with nonlocal conditions via resolvent operators, *Fract. Calc. Appl. Anal.*, **23** (2020), 268–291. https://doi.org/10.1515/fca-2020-0011

2. A. Al-Omari, M. H. M. Rashid, Semilinear Hadamard fractional integro-differential equations with nonlocal conditions and state-dependent delay under compact semigroup, *Mathematics*, **14** (2026), 3097. https://doi.org/10.3390/math14173097
3. Y. Li, Existence and asymptotic stability of periodic solution for evolution equations with delays, *J. Funct. Anal.*, **261** (2011), 1309–1324. https://doi.org/10.1016/j.jfa.2011.05.001
4. Q. Li, M. Wei, Existence and asymptotic stability of periodic solutions for neutral evolution equations with delay, *Evol. Equ. Control Theory*, **9** (2020), 753-772. https://doi.org/10.3934/eect.2020032
5. Z. Liu, N. S. Papageorgiou. Periodic systems with time dependent maximal monotone operators, *Acta Math. Sci.*, **44** (2024), 1280–1300. https://doi.org/10.1007/s10473-024-0406-6
6. H. A. Hammad, D. A. Kattan, An optimal control for non-autonomous second-order stochastic differential equations with delayed arguments, *Phys. Scr.*, **99** (2024), 125205. https://doi.org/10.1088/1402-4896/ad8604
7. L. Ren, J. Wang, M. Fečkan, Asymptotically periodic solutions for Caputo type fractional evolution equations, *Fract. Calc. Appl. Anal.*, **21** (2018), 1294–1312. https://doi.org/10.1515/fca-2018-0068
8. H. Gou, Y. Li, A study on impulsive fractional hybrid evolution equations using sequence method, *Comput. Appl. Math.*, **39** (2020), 225. https://doi.org/10.1007/s40314-020-01239-y
9. X. B. Shu, Y. Shi, A study on the mild solution of impulsive fractional evolution equations, *Appl. Math. Comput.*, **273** (2016), 465–476. https://doi.org/10.1016/j.amc.2015.10.020
10. M. Fečkan, J. R. Wang, Y. Zhou, On the new concept of solutions and existence results for impulsive fractional evolution equations, *Dyn. Partial Differ. Equ.*, **8** (2011), 345–361. https://doi.org/10.4310/DPDE.2011.v8.n4.a3
11. H. A. Hammad, M. De la Sen, Existence of a mild solution and approximate controllability for fractional random integro-differential inclusions with non-instantaneous impulses, *Alexandria Eng. J.*, **111** (2025), 306–328. https://doi.org/10.1016/j.aej.2024.10.017
12. X. Pang, X. Li, Z. Liu, Decay mild solutions of Hilfer fractional differential variational–hemivariational inequalities, *Nonlinear Anal. Real World Appl.*, **71** (2023), 103834. https://doi.org/10.1016/j.nonrwa.2023.103834
13. M. Fečkan, J. R. Wang, Periodic impulsive fractional differential equations, *Adv. Nonlinear Anal.*, **8** (2019), 482–496. https://doi.org/10.1515/anona-2017-0015
14. T. Zhang, L. Xiong, Periodic motion for impulsive fractional functional differential equations with piecewise Caputo derivative, *Appl. Math. Lett.*, **101** (2020), 106072. https://doi.org/10.1016/j.aml.2019.106072
15. L. Ren, J. R. Wang, M. Fečkan, Periodic mild solutions of impulsive fractional evolution equations, *AIMS Math.*, **5** (2020), 497–506. https://doi.org/10.3934/math.2020033
16. A. Al-Omari, M. H. M. Rashid, Local existence and regularity of mild solutions for Hadamard fractional semilinear integro-differential equations with compact semigroups, *Axioms*, **15** (2026), 350. https://doi.org/10.3390/axioms15050350
17. A. Al-Omari, H. Al-Saadi, (ω,ρ)-BVP solution of impulsive Hadamard fractional differential equations, *Mathematics*, **11** (2023), 4370. https://doi.org/10.3390/math11204370

18. A. Al-Omari, H. Al-Saadi, Impulsive fractional order integrodifferential equation via fractional operators, *PLoS ONE*, **18** (2023), e0282665. https://doi.org/10.1371/journal.pone.0282665

19. A. Al-Omari, H. Al-Saadi, F. Alharbi, (ω, c)-periodic solution to semilinear integro differential equations with Hadamard derivatives, *Fractal Fract.*, **8** (2024), 86. https://doi.org/10.3390/fractalfract8020086

20. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Berlin: Springer-Verlag, 1983. https://doi.org/10.1007/978-1-4612-5561-1

21. J. Wang, Y. Zhou, A class of fractional evolution equations and optimal controls, *Nonlinear Anal. Real World Appl.*, **12** (2011), 262–272. https://doi.org/10.1016/j.nonrwa.2010.06.013

22. P. Chen, Y. Li, Existence of mild solutions for fractional evolution equations with mixed monotone nonlocal conditions, *Z. Angew. Math. Phys.*, **65** (2014), 711–728. https://doi.org/10.1007/s00033-013-0351-z

23. C. T. Anh, K. T. Dinh, On nonlocal problems for retarded fractional differential equations in Banach spaces, *Fixed Point Theory*, **15** (2014), 373–392.

24. Y. Zhou, *Fractional evolution equations and inclusions: analysis and control*, London: Academic Press, 2016.

25. R. N. Wang, D. H. Chen, T. J. Xiao, Abstract fractional Cauchy problems with almost sectorial operators, *J. Differ. Equations*, **252** (2012), 202–235. https://doi.org/10.1016/j.jde.2011.08.048

26. A. Kilbas, H. Srivastava, J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier, 2006.

27. J. R. Wang, M. Fečkan, Y. Zhou, Nonexistence of periodic solutions and asymptotically periodic solutions for fractional differential equations, *Commun. Nonlinear Sci. Numer. Simul.*, **18** (2013), 246–256. https://doi.org/10.1016/j.cnsns.2012.07.004

28. B. N. Sadovskii, A fixed-point principle, *Funct. Anal. Appl.*, **1** (1967), 151–153. https://doi.org/10.1007/bf01076087

29. T. A. Burton, C. Kirk, A fixed point theorem of Krasnoselskii Schaefer type, *Math. Nachr.*, **189** (1998), 23–31. https://doi.org/10.1002/mana.19981890103