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Research article

Equivalent descriptions and applications of $Q_K(p, q)$-type spaces in the unit ball

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Abstract: The $Q_K(p, q)$-type spaces on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ form a broad and flexible family of function spaces that unify several classical spaces in complex analysis. In this paper, we investigate equivalent norm characterizations for these spaces and establish descriptions in terms of higher-order derivatives. As an application, we examine pointwise multipliers on $Q_K(p, q)$-type spaces and provide an explicit description of the spectrum of pointwise multipliers on $Q_K(p, q)$.

Keywords: $Q_K(p, q)$-type spaces; equivalent norms; higher-order derivatives; pointwise multipliers; spectrum
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1. Introduction

Throughout this paper, $\mathbb{B}$ denotes the unit ball in $\mathbb{C}^n$, while $\mathbb{D}$ refers to the unit disk in $\mathbb{C}$. We write $\mathbf{Hol}(\mathbb{B})$ for the class of all holomorphic functions defined on $\mathbb{B}$, and $\mathbf{H}^\infty$ for the subclass of those functions that are bounded on $\mathbb{B}$. As is standard, the symbol $d\lambda$ stands for the normalized Lebesgue measure on $\mathbb{B}$, whereas dv designates the normalized surface measure on the boundary $\mathbb{S}$ of $\mathbb{B}$.

For $u \in \mathbf{Hol}(\mathbb{B})$ and $w \in \mathbb{B}$, the complex gradient of u is defined by

$$\nabla u(w) = \left(\frac{\partial u(w)}{\partial w_1}, \dots, \frac{\partial u(w)}{\partial w_n} \right).$$

For $u \in \mathbf{Hol}(\mathbb{B})$, the radial derivative of u at a point $w \in \mathbb{B}$ is given by

$$\mathcal{R}u(w) = \sum_{j=1}^{n} w_j \frac{\partial u}{\partial w_j}(w).$$

For $\gamma \geq 0$, if $u \in \mathbf{Hol}(\mathbb{B})$ and

$$\|u\|_\gamma = \sup_{w\in\mathbb{B}} (1 - |w|^2)^\gamma \, |\nabla u(w)| < \infty,$$

then we say that u belongs to the Bloch-type space $\mathcal{B}^\gamma = \mathcal{B}^\gamma(\mathbb{B})$. It is well known that $\mathcal{B}^\gamma$, when furnished with the norm

$$\|u\|_{\mathcal{B}^\gamma} = |u(0)| + \|u\|_\gamma,$$

constitutes a Banach space, and that $\mathcal{B}^1$ coincides with the classical Bloch space on $\mathbb{B}$; we refer to [6,8] for details.

Suppose $\beta > -1$ and $p > 0$. The weighted Bergman-type space $A^p_\beta = A^p_\beta(\mathbb{B})$ comprises all functions $u \in \mathbf{Hol}(\mathbb{B})$ with the property that

$$\|u\|^p_{A^p_\beta} = \int_{\mathbb{B}} |u(w)|^p \, d\lambda_\beta(w) < \infty.$$

The weighted Lebesgue measure $d\lambda_\beta(w) = C_\beta(1 - |w|^2)^\beta \, d\lambda(w)$ involves a normalizing constant C_β. In the case $\beta > -1$, the positive measure $d\lambda_\beta$ is a probability measure. It is worth noting that A^p_β is a Banach space for $1 \le p < \infty$, while for $0 < p < 1$, it is a complete metric space. Moreover, taking $\beta = 0$ in A^p_β yields the classical Bergman space; we refer the reader to [16, 18] for a thorough treatment.

For $x \in \mathbb{B}$, the Möbius transformation Φ_x on $\mathbb{B}$ is given explicitly by

$$\Phi_x(\zeta) = \frac{x - P_x\zeta - \sqrt{1 - |x|^2}(I - P_x)\zeta}{1 - \langle \zeta, x \rangle},$$

where P_x denotes the orthogonal projection onto the one-dimensional subspace generated by x (see, e.g., [8, 19]). This transformation satisfies the key properties $\Phi_x(0) = x$, $\Phi_x(x) = 0$, and $\Phi_x = \Phi_x^{-1}$. Furthermore, a direct computation shows that for any $w, x \in \mathbb{B}$, we have

$$1 - |\Phi_x(w)|^2 = \frac{(1 - |w|^2)(1 - |x|^2)}{|1 - \langle w, x \rangle|^2}.$$

For $u \in \mathbf{Hol}(\mathbb{B})$ and $w, x \in \mathbb{B}$, the invariant gradient of u, denoted $\widetilde{\nabla} u$, is defined by

$$\widetilde{\nabla} u(w) = \nabla(u \circ \Phi_x)(0).$$

In what follows, a fixed weight function $K \in \mathbf{RC}^+$ is assumed to be nonzero, where $\mathbf{RC}^+$ denotes the collection of all functions $K : [0, \infty) \longrightarrow [0, \infty)$ that are positive, nondecreasing, and right-continuous. Given parameters $p > 0$, $-n - 1 < q < \infty$, and a weight $K \in \mathbf{RC}^+$, the space $Q_K(p, q)$ is defined by

$$Q_K(p, q) := \{u \in \mathbf{Hol}(\mathbb{B}) : \|u\|_K < \infty\},$$

where, for $u \in \mathbf{Hol}(\mathbb{B})$, the quantity $\|u\|_K$ is given by

$$\|u\|^p_K = \sup_{x \in \mathbb{B}} \int_{\mathbb{B}} |\nabla u(w)|^p (1 - |w|^2)^q K(1 - |\Phi_x(w)|^2) \, d\lambda(w).$$

The corresponding little space $Q_{K,0}(p, q)$ consists of all functions $u \in Q_K(p, q)$ satisfying the vanishing condition

$$\lim_{|x| \to 1} \int_{\mathbb{B}} |\nabla u(w)|^p \, (1 - |w|^2)^q K\left(1 - |\Phi_x(w)|^2\right) \, d\lambda(w) = 0.$$

As established in [4], we have

$$Q_K(p,q) \subset \mathcal{B}^\gamma, \qquad \gamma = \frac{q+n+1}{p}.$$

Moreover, $Q_K(p,q) = \mathcal{B}^\gamma$ if and only if $I_K(r) < \infty$, where

$$I_K(r) = \int_0^1 \frac{K(1-r^2)}{r^{1-2n}(1-r^2)^{n+1}}\,dr, \qquad 0 \le r < 1. \tag{1.1}$$

The space $Q_K(p,q)$ becomes a Banach space for $p \ge 1$ when equipped with the norm

$$\|u\|_{Q_K(p,q)} = |u(0)| + \|u\|_K < \infty.$$

The $Q_K(p,q)$-type spaces were first introduced and studied on the unit disk $\mathbb{D}$ by Wulan and Zhou in [11, 12], where they obtained basic properties and several characterizations in terms of weighted integral conditions and higher-order derivatives. Their work showed that $Q_K(p,q)$ contains, as particular cases, a wide range of Möbius-invariant function spaces lying between the Dirichlet space, BMOA (the space of holomorphic functions of bounded mean oscillation), and Bloch-type spaces. Later, the theory of Q_K-type spaces was extended to higher dimensions. In particular, Rong Hu in [4] defined and investigated $Q_K(p,q)$- and $Q_{K,0}(p,q)$-type spaces on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$, establishing inclusion relations with Bloch-type and weighted Bergman spaces and providing further characterizations.

In [2], Bakhit established six mutually equivalent norms for the space $Q_K(p,q)$ on the unit ball $\mathbb{B}$, providing a unified description and structural insight into $Q_K(p,q)$-type spaces. Building on these developments, there has been a growing interest in operator-theoretic aspects of such spaces, including composition operators, integral operators, and pointwise multipliers.

A fundamental question in the theory of function spaces concerns the stability of a space U under pointwise multiplication: Given $h \in U$ and a fixed function φ, one asks when φh again belongs to U, and how to characterize all such functions φ. These questions arise naturally in operator theory and lead to the systematic study of pointwise multipliers and multiplier sequences between function spaces.

The study of multipliers on function spaces has a well-established history and is of fundamental importance in function-theoretic operator theory, especially in relation to the Gleason problem, invariant subspace theory, and composition operators. The close connection between the properties of function spaces and their multipliers has motivated extensive research in this direction, leading to many significant results (see, e.g., [13, 15, 17]).

The study of multipliers on classical holomorphic function spaces over the unit disk $\mathbb{D}$ has a rich history. Several fundamental results have been established for spaces such as the Bloch space, *BMOA*, Q_K, Besov spaces, and Lipschitz spaces; see, for example, [5, 7, 9].

For $x > 0$, we introduce the supplementary function

$$\Lambda_K(x) = \sup_{t\in(0,1]} \frac{K(xt)}{K(t)}.$$

This function is required to satisfy the integrability conditions

$$\int_0^1 \Lambda_K(x)\,\frac{dx}{x} < \infty, \qquad \text{and} \qquad \int_1^\infty \Lambda_K(x)\,\frac{dx}{x^{\gamma+1}} < \infty, \qquad \gamma > 0. \tag{1.2}$$

Whenever the weight function K fulfills Condition (1.2) (see, e.g., [12]), the following equivalences hold:

$$K(2x) \approx K(x), \qquad K(xt) \leq K(t)\,\Lambda_K(x), \qquad \forall\, x > 0.$$

Following [3], for $p, q > 0$ and $K \in \mathbf{RC}^+$, the weighted space $\mathcal{N}_K(p, q)$ is defined as the collection of all $u \in \mathbf{Hol}(\mathbb{B})$ for which

$$\|u\|^p_{\mathcal{N}_K(p,q)} = \sup_{x\in\mathbb{B}} \int_{\mathbb{B}} |u(w)|^p (1 - |w|^2)^q K\left(1 - |\Phi_x(w)|^2\right) d\lambda(w) < \infty.$$

In the particular case $K(t) = 1$, the space $\mathcal{N}_K(p, q)$ reduces to the Bergman-type space $A^p_q(\mathbb{B})$ (cf. [1,3]). A thorough investigation of the properties of $A^p_q(\mathbb{B})$ was carried out by Zhao and Zhu in [16].

It is worth noting that the particular choices $p = 2$ and $q = 0$ yield the reduced spaces $Q_K(2, 0) = Q_K$ and $Q_{K,0}(2, 0) = Q_{K,0}$. Upon substituting $K(t) = t^{ns}$ for $s > 0$, one recovers the scale of spaces $\mathcal{F}(p, q, s)$; notably, the special cases $\mathcal{F}(p, q, 0) = \mathcal{D}^p_q(\mathbb{B})$, $\mathcal{F}(2, 0, s) = Q_s(\mathbb{B})$, and $\mathcal{F}(2, 0, 1) = BMOA(\mathbb{B})$ arise naturally within this framework. Finally, taking $K(t) = 1$ reduces the space $\mathcal{F}(p, q, s)$ to the classical Besov space $B_p(\mathbb{B})$.

Our goals are twofold: To establish an equivalent norm for $Q_K(p, q)$ and to obtain a complete characterization of its multipliers, thereby extending the one-variable theory to the higher-dimensional setting of the unit ball $\mathbb{B}$ in $\mathbb{C}^n$.

The present paper is organized as follows. In Section 2, we collect some preliminary definitions and results needed throughout the paper. Section 3 is devoted to establishing equivalent norm characterizations of $Q_K(p, q)$-type spaces on $\mathbb{B}$, with particular emphasis on descriptions involving higher-order derivatives. In Section 4, we study pointwise multipliers on $Q_K(p, q)$-type spaces and derive several characterizations of bounded multipliers. Finally, Section 5 is concerned with the spectrum of pointwise multiplication operators on $Q_K(p, q)$, for which an explicit description is obtained.

All results in this paper are obtained under the following standing assumptions: $p > 0$, $q > -n - 1$, $K \in \mathbf{RC}^+$ satisfies Condition (1.2), and $I_K(r)$ as defined in (1.1) converges. Any deviation from these assumptions will be explicitly indicated.

As usual, $q_1 \lesssim q_2$ (respectively $q_1 \gtrsim q_2$) means that $q_1 \leq c_1 q_2$ (respectively $q_1 \geq c_2 q_2$) for some absolute constants $c_1, c_2 > 0$, and we write $q_1 \approx q_2$ if both inequalities hold.

2. Some preliminaries

We record here the following auxiliary lemmas for later use.

Lemma 2.1. *Let $p > 0$ and $q > -1$, and set $\gamma = \frac{q+n+1}{p}$. Assume that $K \in \mathbf{RC}^+$ satisfies (1.2). Then, for any $u \in \mathbf{Hol}(\mathbb{B})$, membership of u in $Q_K(p, q)$ implies that $u \in \mathcal{B}^\gamma$, with the norm estimate*

$$\|u\|_{Q_K(p,q)} \lesssim \|u\|_{\mathcal{B}^\gamma}.$$

Furthermore, the equality $Q_K(p, q) = \mathcal{B}^\gamma$ holds if and only if the integral $I_K(r)$ defined in (1.1) is convergent.

Proof. This follows directly from the main result established in [4], obtained by replacing the Green function $G(x,\zeta)$ used there with $1-\left|\Phi_x(\zeta)\right|^2$. □

The following lemma is a consequence of Proposition 1.4.10 in [8].

Lemma 2.2. *Let $d\in\mathbb{R}$, $\gamma>-1$, and $\zeta\in\mathbb{B}$. Define*

$$\begin{aligned}\Omega_d(\zeta) &= \int_{\mathbb{S}}\frac{dv(w)}{|1-\langle\zeta,w\rangle|^{d+n}},\\ \Omega_{d,\gamma}(\zeta) &= \int_{\mathbb{B}}\frac{(1-|\eta|^2)^\gamma\,d\lambda(\eta)}{|1-\langle\zeta,\eta\rangle|^{d+n+1+\gamma}}.\end{aligned}$$

Then the following assertions hold:

- *If $d<0$, then the integrals $\Omega_d(\zeta)$ and $\Omega_{d,\gamma}(\zeta)$ are bounded on $\mathbb{B}$.*
- *If $d>0$, then $\Omega_d(\zeta)\approx\Omega_{d,\gamma}(\zeta)\approx(1-|\zeta|^2)^{-d}$.*
- *If $d=0$, then $\Omega_0(\zeta)\approx\Omega_{0,\gamma}(\zeta)\approx\log\frac{1}{1-|\zeta|^2}$.*

Lemma 2.3. *Let $d\in\mathbb{R}$, $\gamma>-1$ and $w\in\mathbb{B}$, and suppose that $K\in\mathbf{RC}^+$ satisfies* (1.2). *For some $s>0$, if $s-n-1\le\gamma\le d$, then*

$$\int_{\mathbb{B}}\frac{(1-|\eta|^2)^\gamma}{|1-\langle w,\eta\rangle|^d}K\left(1-|\Phi_x(\eta)|^2\right)d\lambda(\eta)\approx\frac{K\left(1-|\Phi_x(w)|^2\right)}{(1-|w|^2)^{d-\gamma-n-1}}. \tag{2.1}$$

Proof. For any $x,w\in\mathbb{B}$ let $u=\Phi_w(x)$. One readily checks that

$$|\Phi_x(\eta)|=\left|\Phi_u\circ\Phi_w(\eta)\right|,$$

and, upon setting $\eta=\Phi_w(\zeta)$, we obtain

$$\left|\Phi_u\circ\Phi_w(\eta)\right|=|\Phi_u(\zeta)|.$$

A direct calculation then leads to

$$\begin{aligned}&\int_{\mathbb{B}}\frac{(1-|\eta|^2)^\gamma}{|1-\langle w,\eta\rangle|^d}K\left(1-|\Phi_x(\eta)|^2\right)d\lambda(\eta)\\ =&\int_{\mathbb{B}}\frac{(1-|\eta|^2)^\gamma}{|1-\langle w,\eta\rangle|^d}K\left(1-\left|\Phi_u\circ\Phi_w(\eta)\right|^2\right)d\lambda(\eta)\\ =&\int_{\mathbb{B}}\frac{(1-|\Phi_w(\zeta)|^2)^\gamma}{|1-\langle w,\Phi_w(\zeta)\rangle|^d}K\left(1-|\Phi_u(\zeta)|^2\right)|\Phi_w'(\zeta)|^{n+1}d\lambda(\zeta)\\ =&\int_{\mathbb{B}}\frac{(1-|w|^2)^{\gamma+n+1-d}(1-|\zeta|^2)^\gamma}{|1-\langle w,\zeta\rangle|^{2(\gamma+n+1-d)}}K\left(\frac{(1-|u|^2)(1-|\zeta|^2)}{|1-\langle u,\zeta\rangle|^2}\right)d\lambda(\zeta)\\ \le&\frac{K(1-|u|^2)}{(1-|w|^2)^{d-(\gamma+n+1)}}\int_{\mathbb{B}}(1-|\zeta|^2)^\gamma\Lambda_K\left(\frac{(1-|\zeta|^2)}{|1-\langle u,\zeta\rangle|^2}\right)d\lambda(\zeta).\end{aligned} \tag{2.2}$$

By Theorems 3.4 and 3.5 in [10], the Condition (1.2) on K ensures the existence of weight functions K_1 and K_2 that are similar to K on $(0,1)$ and on $(0,\infty)$, respectively. For positive constants σ,ρ with $2\sigma\le\rho$, it follows that

$$\Lambda_K(y)\approx\Lambda_{K_1}(y)\approx y^\sigma\sup_{0<s<1}\frac{K_1(ys)/(ys)^\sigma}{K_1(s)/s^\sigma}\le y^\sigma,\qquad 0<y\le 1, \tag{2.3}$$

and

$$\Lambda_K(y) \approx \Lambda_{K_2}(y) = y^{\rho-\sigma} \sup_{0<s<1} \frac{K_2(ys)/(ys)^{\rho-\sigma}}{K_2(y)/y^{\rho-\sigma}} \leq y^{\rho-\sigma}, \qquad y \geq 1. \tag{2.4}$$

Using (2.3) and (2.4) in (2.2), we obtain

$$\begin{aligned}
&\int_{\mathbb{B}} (1-|\zeta|^2)^\gamma \Lambda_K\left(\frac{(1-|\zeta|^2)}{|1-\langle u,\zeta\rangle|^2}\right) d\lambda(\zeta) \\
\lesssim\ & \int_{\mathbb{B}} (1-|\zeta|^2)^\gamma \left(\frac{(1-|\zeta|^2)}{|1-\langle u,\zeta\rangle|^2}\right)^\sigma d\lambda(\zeta) \\
&\quad + \int_{\mathbb{B}} (1-|\zeta|^2)^\gamma \left(\frac{(1-|\zeta|^2)}{|1-\langle u,\zeta\rangle|^2}\right)^{\rho-\sigma} d\lambda(\zeta).
\end{aligned} \tag{2.5}$$

It is straightforward to verify that our assumptions on γ and σ imply that $\sigma+\gamma > -1$ and $2\sigma-(\sigma+\gamma)-n-1 < 0$. Thus, an application of Lemma 2.2 gives

$$\int_{\mathbb{B}} \frac{(1-|\zeta|^2)^{\gamma+\sigma}}{|1-\langle u,\zeta\rangle|^{2\sigma}} d\lambda(\zeta) \lesssim 1. \tag{2.6}$$

Moreover, our assumptions on γ and σ also imply that $\rho-\sigma+\gamma > -1$ and

$$2(\rho-\sigma)-(\rho-\sigma+\gamma)-n-1 < 0.$$

Thus,

$$\int_{\mathbb{B}} \frac{(1-|\zeta|^2)^{\gamma+\rho-\sigma}}{|1-\langle u,\zeta\rangle|^{2(\rho-\sigma)}} d\lambda(\zeta) \lesssim 1. \tag{2.7}$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{B}} \frac{(1-|\eta|^2)^\gamma}{|1-\langle w,\eta\rangle|^d} K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \\
\leq\ & \frac{K(1-|u|^2)}{(1-|w|^2)^{d-\gamma-n-1}} \approx \frac{K\left(1-|\Phi_x(w)|^2\right)}{(1-|w|^2)^{d-\gamma-n-1}}.
\end{aligned} \tag{2.8}$$

For $x, \eta \in \mathbb{B}$ and $r \in (0,1)$, the pseudo-hyperbolic ball centered at x with a radius r is defined by

$$\mathbb{E}(x,r) = \{\eta \in \mathbb{B} : |\Phi_x(\eta)| < r\}.$$

It is well known that for $x \in \mathbb{B}$ and $w \in \mathbb{E}(x, 1/2)$, the following metric equivalences hold:

$$1-|\eta| \approx |1-\langle w,\eta\rangle| \approx 1-|w|.$$

In addition, one has

$$|1-\langle w,x\rangle| \approx |1-\langle \eta,x\rangle|.$$

Thus, we obtain

$$
\begin{aligned}
&\int_{\mathbb{B}} \frac{(1-|\eta|^2)^\gamma}{|1-\langle w,\eta\rangle|^d} K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \\
\geq\ & \int_{\mathbb{B}(x,1/2)} \frac{(1-|\eta|^2)^\gamma}{|1-\langle w,\eta\rangle|^d} K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \\
\approx\ & \frac{K\left(1-|\Phi_x(w)|^2\right)}{(1-|w|^2)^{d-\gamma-n-1}}. \qquad (2.9)
\end{aligned}
$$

□

Lemma 2.4. *Let $q > -1$ and suppose that $K \in \mathbf{RC}^+$ satisfies* (1.2) *and that the integral $I_K(r)$ defined in* (1.1) *is convergent. Then*

$$
\sup_{\xi,x\in\mathbb{B}} \int_{\mathbb{B}} \frac{(1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right)}{|1-\langle\eta,\xi\rangle|^{q+n+1}}\, d\lambda(\eta) < \infty. \tag{2.10}
$$

Proof. Fix $\xi, x \in \mathbb{B}$. When $q+n+1 > 0$, Lemma 2.3 yields the desired estimate.

If $q+n+1 = 0$ with $q > -1$, then $q = -n-1$, and the assumption on $I_K(r)$ ensures the integrability of the radial part. In this case, we estimate

$$
\begin{aligned}
&\int_{\mathbb{B}} \frac{(1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right)}{|1-\langle\eta,\xi\rangle|^{q+n+1}}\, d\lambda(\eta) \\
\leq\ & \int_0^1 s^{2n-1}(1-s^2)^q K(1-s^2)\, ds \int_{\mathbb{S}} \frac{d\sigma(\zeta)}{|1-\langle\zeta,\xi\rangle|^{q+n+1}} < \infty, \qquad (2.11)
\end{aligned}
$$

where we have used spherical coordinates and the convergence of $I_K(r)$ together with Proposition 1.4.10 in [8]. □

The next two lemmas were proved as Lemma 2.2 in [14] and Lemma 2.4 in [7], respectively.

Lemma 2.5. *Let $h \in \mathcal{B}^\gamma$ with $\gamma \geq 0$, and let $\xi \in \mathbb{B}$. Then the following hold*

(1) If $0 \leq \gamma < 1$, then $|h(\xi)| \lesssim \|h\|_{\mathcal{B}^\gamma}$;
(2) If $\gamma = 1$, then $|h(\xi)| \lesssim \log \frac{2}{1-|\xi|^2} \|h\|_{\mathcal{B}}$;
(3) If $\gamma > 1$, then $|h(\xi)| \lesssim (1-|\xi|^2)^{\gamma-1} \|h\|_{\mathcal{B}^\gamma}$

Lemma 2.6. *Let $q > -1$. Then a constant $c > 0$ exists such that, for all $w \in \mathbb{B}$, we have*

$$
\int_{\mathbb{B}} \frac{\left|\log \frac{1}{1-\langle\eta,w\rangle}\right|^2}{(1-|\eta|^2)^q\,|1-\langle\eta,w\rangle|^{n+1+q}}\, d\lambda(\eta) \lesssim \left(\log \frac{1}{1-|w|^2}\right)^2. \tag{2.12}
$$

3. Equivalent norms

In order to obtain a characterization of the space $Q_K(p,q)$ in terms of higher-order derivatives, we first introduce, for any $h \in \mathbf{Hol}(\mathbb{B})$, the following integral functionals: For $h \in \mathbf{Hol}(\mathbb{B})$, we define

$$
I_{K,1}(h) = \sup_{x\in\mathbb{B}} \int_{\mathbb{B}} |h(\zeta)|^p (1-|\zeta|^2)^q K(1-|\Phi_x(\zeta)|^2)\, d\lambda(\zeta),
$$

$$\begin{aligned}
I_{K,2}(h) &= |h(0)|^p + \sup_{x\in\mathbb{B}} \int_{\mathbb{B}} |\nabla h(\zeta)|^p (1-|\zeta|^2)^{q+p} K(1-|\Phi_x(\zeta)|^2)\, d\lambda(\zeta),\\
I_{K,3}(h) &= |h(0)|^p + \sup_{x\in\mathbb{B}} \int_{\mathbb{B}} |\nabla h(\zeta)|^p (1-|\zeta|^2)^{q} K(1-|\Phi_x(\zeta)|^2)\, d\lambda(\zeta),\\
I_{K,4}(h) &= \sum_{|m|\le N} \left|\frac{\partial^m h}{\partial \zeta^m}(0)\right|^p\\
&\quad + \sup_{x\in\mathbb{B}} \sum_{|m|=N+1} \int_{\mathbb{B}} \left|\frac{\partial^m h}{\partial \zeta^m}(\zeta)\right|^p (1-|\zeta|^2)^{q+pN} K\left(1-|\Phi_x(\zeta)|^2\right) d\lambda(\zeta).
\end{aligned}$$

Before starting with the main result of this section, we establish an equivalent norm for $Q_K(p,q)$ in terms of the gradient, which serves as a key ingredient in the characterization via higher-order derivatives. The main point is that an integral condition involving $h \in \mathbf{Hol}(\mathbb{B})$ can be replaced by a condition measured through the gradient of h, with the two quantities being comparable.

Theorem 3.1. *Let $p > 0$ and $q > -1$, and set $\gamma = \frac{q+n+1}{p}$. Suppose that $h \in \mathbf{Hol}(\mathbb{B})$. Then the finiteness of $I_{K,1}(h)$ is equivalent to the finiteness of $I_{K,2}(h)$, and, moreover, $I_{K,2}(h) \approx I_{K,1}(h)$.*

Proof. We begin by assuming $I_{K,2}(h) < \infty$ and establishing that $I_{K,1}(h) \lesssim I_{K,2}(h)$. The fact that $K \in \mathbf{RC}^+$ is nondecreasing and right-continuous ensures that all intermediate estimates hold without loss of generality. Under the assumption $I_{K,2}(h) < \infty$, we have $h \in Q_K(p,q)$, and Lemma 2.1 yields

$$|\mathcal{R}h(\eta)| \le |\nabla h(\eta)| \lesssim \frac{\|h\|_{Q_K(p,q)}}{(1-|\eta|^2)^{\gamma+1}} \lesssim \frac{I_{K,2}(h)^{1/p}}{(1-|\eta|^2)^{\gamma+1}}. \tag{3.1}$$

Therefore, if we fix an integer $\beta > \gamma$, then we have $\mathcal{R}h \in A^1_\beta$ (the weighted Bergman-type space A^p_β with $p = 1$). By Theorem 2.2 in [19], for all $\eta \in \mathbb{B}$, we obtain

$$\mathcal{R}h(\eta) = \int_{\mathbb{B}} \frac{\mathcal{R}h(w)}{(1-\langle \eta, w\rangle)^{n+1+\beta}}\, d\lambda_\beta(w).$$

Using Fubini's theorem and the fact that

$$0 = \mathcal{R}h(0) = \int_{\mathbb{B}} \mathcal{R}h(w)\, d\lambda_\beta(w),$$

we obtain

$$\begin{aligned}
h(\eta) - h(0) &= \int_0^1 \{\mathcal{R}h(s\eta) - \mathcal{R}h(0)\} \frac{ds}{s}\\
&= \int_{\mathbb{B}} \mathcal{R}h(w)\, U(\eta, w)\, d\lambda_\beta(w),
\end{aligned}$$

where

$$U(\eta, w) = \int_0^1 \left\{ \frac{1}{(1-s\langle \eta, w\rangle)^{n+1+\beta}} - 1 \right\} \frac{ds}{s}.$$

A standard computation using the binomial series shows that

$$\begin{aligned} U(\eta, w) &= \sum_{j=1}^{\infty} \frac{\Gamma(j+n+1+\beta)}{j\, j!\, \Gamma(n+1+\beta)} \langle \eta, w\rangle^j \\ &= \frac{1}{\beta+n}\left\{\frac{1}{(1-\langle \eta, w\rangle)^{\beta+n}} - 1\right\} + \cdots + \log\frac{1}{1-\langle \eta, w\rangle}. \end{aligned}$$

Therefore,

$$|U(\eta, w)| \lesssim \frac{1}{|1-\langle \eta, w\rangle|^{\beta+n}}.$$

Consequently,

$$|h(\eta) - h(0)| \lesssim \int_{\mathbb{B}} \frac{|\nabla h(w)|}{|1-\langle \eta, w\rangle|^{\beta+n}}\, d\lambda_\beta(w). \tag{3.2}$$

Case I. $1 < p < \infty$.

For any $\alpha > 0$, we define

$$\begin{aligned} H(\zeta) &= (1-|\zeta|^2)\,|\nabla h(\zeta)|, \\ u(\zeta, w) &= \frac{1}{|1-\langle \zeta, w\rangle|^{\beta+n}}, \\ g(\zeta) &= \frac{1}{(1-|\zeta|^2)^\alpha}. \end{aligned}$$

Consider the operator

$$\mathcal{T} H(\zeta) = \int_{\mathbb{B}} u(\zeta, w)\, H(w)\, d\lambda_{\beta-1}(w).$$

When $\beta > 0$ and $\beta > \alpha p/(p-1)$, Lemma 2.2 yields

$$\begin{aligned} \int_{\mathbb{B}} |u(\eta, w)|\, g^{\frac{p}{p-1}}(w)\, d\lambda_{\beta-1}(w) &\lesssim \int_{\mathbb{B}} \frac{(1-|w|^2)^{\beta-1-\frac{\alpha p}{p-1}}}{|1-\langle \eta, w\rangle|^{n+\beta}}\, d\lambda(w) \\ &\lesssim (1-|\eta|^2)^{-\frac{\alpha p}{p-1}} \lesssim g^{\frac{p}{p-1}}(\eta). \end{aligned}$$

Combining Hölder's inequality with the preceding estimate, we obtain

$$\begin{aligned} |\mathcal{T} H(\eta)|^p &\le \left(\int_{\mathbb{B}} u(\eta, w)\, g^{\frac{p}{p-1}}(w)\, d\lambda_{\beta-1}(w)\right)^{p-1} \\ &\quad \times\left(\int_{\mathbb{B}} u(\eta, w)\, g^{-p}(w)\, H^p(w)\, d\lambda_{\beta-1}(w)\right) \\ &\lesssim g^p(\eta) \int_{\mathbb{B}} u(\eta, w)\, g^{-p}(w)\, H^p(w)\, d\lambda_{\beta-1}(w). \end{aligned} \tag{3.3}$$

When $q+n+1 > 0$, choose $0 < \alpha < (q+1)/p$; in the case, $0 < q+n+1-p\alpha$. From (3.2) and (3.3), by Fubini's theorem, as long as $\beta > q+1-p\alpha$, we have

$$\int_{\mathbb{B}} |h(\eta) - h(0)|^p (1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta)$$

$$
\begin{aligned}
&\lesssim \int_{\mathbb{B}} |\mathcal{T}H(\eta)|^p (1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \\
&\lesssim \int_{\mathbb{B}} g^p(\eta) \left(\int_{\mathbb{B}} \frac{u(\eta,w)\, H^p(w)}{g^p(w)}\, d\lambda_{\beta-1}(w) \right) \frac{K\left(1-|\Phi_x(\eta)|^2\right)}{(1-|\eta|^2)^{-q}}\, d\lambda(\eta) \\
&\lesssim \int_{\mathbb{B}} (1-|w|^2)^{\alpha p+p} |\nabla h(w)|^p\, \Omega_K(\eta,w)\, d\lambda_{\beta-1}(w), \qquad (3.4)
\end{aligned}
$$

where

$$
\Omega_K(\eta,w) = \int_{\mathbb{B}} u(\eta,w)\, g^p(\eta)\, (1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta). \qquad (3.5)
$$

By Lemma 2.3, we see that

$$
\begin{aligned}
\Omega_K(\eta,w) &= \int_{\mathbb{B}} \frac{(1-|\eta|^2)^{q-\alpha p}}{|1-\langle \eta,w\rangle|^{\beta+n}} K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \\
&\approx \frac{K\left(1-|\Phi_x(w)|^2\right)}{(1-|w|^2)^{\beta-q+\alpha p-1}}. \qquad (3.6)
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\int_{\mathbb{B}} |h(\eta)-h(0)|^p (1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \\
\lesssim &\int_{\mathbb{B}} |\nabla h(w)|^p (1-|w|^2)^{q+p} K\left(1-|\Phi_x(w)|^2\right) d\lambda(w) \lesssim I_{K,2}(h). \qquad (3.7)
\end{aligned}
$$

When $q+n+1 \le 0$, we pick an α satisfying $0<\alpha<(q+1)/p$, which implies $q+n+1-p\alpha<0$. If $\beta > q+n+1-p\alpha$, a combined use of Lemma 2.3, (3.1)–(3.3), and Fubini's theorem leads to

$$
\int_{\mathbb{B}} |h(\eta)-h(0)|^p (1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \lesssim I_{K,2}(h). \qquad (3.8)
$$

Case II. $0 < p \le 1$.

Choose $\beta > 0$ to be sufficiently large, with

$$
\beta = \frac{n+1+\kappa}{p} - (n+1)
$$

for some $\kappa > p+q+n$, and define

$$
G_j(w) = \frac{\partial h(w)/\partial w_j}{(1-\langle w,\eta\rangle)^{n+1+\beta}}, \qquad j = 1,2,\ldots,n. \qquad (3.9)
$$

Invoking Lemma 2.15 from [19], we obtain

$$
\int_{\mathbb{B}} |G_j(w)|(1-|w|^2)^{\kappa} d\lambda(w) \lesssim \|G_j\|_{A^1_\kappa}, \qquad j = 1,2,\ldots,n, \qquad (3.10)
$$

where A^1_κ is the weighted Bergman-type space A^p_β with $p=1$ and weight exponent κ.

Since $c_j \geq 0$ for $j = 1, 2, \ldots, n$, we use the elementary inequalities

$$\begin{aligned}\sqrt{c_1^2 + \cdots + c_n^2} &\approx c_1 + \cdots + c_n,\\ (c_1 + \cdots + c_n)^p &\approx c_1^p + \cdots + c_n^p.\end{aligned} \tag{3.11}$$

Combining (3.2) with (3.10) and (3.11), we obtain

$$\begin{aligned}|h(\eta) - h(0)|^p &\lesssim \left(\int_{\mathbb{B}} \frac{|\nabla h(w)|}{|1 - \langle \eta, w\rangle|^{n+\beta}} d\lambda_\kappa(w)\right)^p\\ &\lesssim \left(\int_{\mathbb{B}} (|G_1(w)| + \cdots + |G_n(w)|)\, d\lambda_\kappa(w)\right)^p\\ &\lesssim \left\{\sum_{j=1}^n \|G_j\|_{A_\kappa^1}\right\}^p \lesssim \sum_{j=1}^n \|G_j\|_{A_\kappa^1}^p\\ &\lesssim \int_{\mathbb{B}} \frac{1}{|1 - \langle \eta, w\rangle|^{p(n+\beta)}} \left(\sum_{j=1}^n \left|\frac{\partial h}{\partial w_j}(w)\right|^p\right) d\lambda_\kappa(w)\\ &\lesssim \int_{\mathbb{B}} \frac{|\nabla h(w)|^p}{|1 - \langle \eta, w\rangle|^{p(n+\beta)}}\, d\lambda_\kappa(w).\end{aligned} \tag{3.12}$$

Hence, when $q + n + 1 \geq 0$ and $pn + p\beta > q + n + 1$, using (3.12) together with Lemma 2.3, and the choice $\kappa = p(\beta + n + 1) - n - 1$, we obtain

$$\begin{aligned}&\int_{\mathbb{B}} |h(\eta) - h(0)|^p (1 - |\eta|^2)^q K\left(1 - |\Phi_x(\eta)|^2\right) d\lambda(\eta)\\ \lesssim\ &\int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|\nabla h(w)|^p}{|1 - \langle \eta, w\rangle|^{p(n+\beta)}} d\lambda_\kappa(w)\, (1 - |\eta|^2)^q K\left(1 - |\Phi_x(\eta)|^2\right) d\lambda(\eta)\\ \lesssim\ &\int_{\mathbb{B}} |\nabla h(w)|^p \left(\int_{\mathbb{B}} \frac{(1 - |\eta|^2)^q}{|1 - \langle \eta, w\rangle|^{p(n+\beta)}} K\left(1 - |\Phi_x(\eta)|^2\right) d\lambda(\eta)\right) d\lambda_\kappa(w)\\ \lesssim\ &\int_{\mathbb{B}} |\nabla h(w)|^p (1 - |w|^2)^{q - p(n+\beta) + n + 1 + \kappa} K\left(1 - |\Phi_x(w)|^2\right) d\lambda(w)\\ \lesssim\ &I_{K,2}(h).\end{aligned} \tag{3.13}$$

Similarly, when $q + n + 1 < 0$, provided that $pn + p\beta > 2n + q + 1$, we may apply (3.12), Lemma 2.3, and (3.1) to obtain

$$\begin{aligned}&\int_{\mathbb{B}} |h(\eta) - h(0)|^p (1 - |\eta|^2)^q K\left(1 - |\Phi_x(\eta)|^2\right) d\lambda(\eta)\\ \lesssim\ &\int_{\mathbb{B}} \frac{|\nabla h(w)|^p}{(1 - |w|^2)^{p(n+\beta) - q - n - 1}} K\left(1 - |\Phi_x(w)|^2\right) d\lambda_\kappa(w)\\ \lesssim\ &I_{K,2}(h).\end{aligned} \tag{3.14}$$

Combining the estimates above in both cases, we conclude that, for all $p > 0$ and $q > -1$, we have

$$\int_{\mathbb{B}} |h(\eta) - h(0)|^p (1 - |\eta|^2)^q K\left(1 - |\Phi_x(\eta)|^2\right) d\lambda(\eta) \lesssim I_{K,2}(h), \tag{3.15}$$

which shows that $I_{K,1}(h) \lesssim I_{K,2}(h)$.

Next, we prove that when $I_{K,1}(h) < \infty$, then $I_{K,2}(h) \lesssim I_{K,1}(h)$.

For every $g \in \mathcal{N}_K(p,q)$, by Theorem 2.17 in [19] and Theorem 1 in [17], we obtain

$$\begin{aligned} & \int_{\mathbb{B}} |g(\eta)|^p (1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \\ \approx \ & |g(0)|^p + \int_{\mathbb{B}} |\nabla g(\eta)|^p (1-|\eta|^2)^{p+q} K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta). \end{aligned} \tag{3.16}$$

Now, for each $b \in \mathbb{B}$, set

$$F_b(\eta) = \left(\frac{1-|b|^2}{|1-\langle \eta, b\rangle|}\right)^{\frac{1}{p}} h(\eta),$$

so that $F_b \in \mathcal{N}_K(p,q)$ and, when $I_{K,1}(h) < \infty$, we have $\|F_b\|_{\mathcal{N}_K(p,q)} \lesssim I_{K,1}(h)^{1/p}$.

Hence, from (3.16), we have

$$\begin{aligned} & \int_{\mathbb{B}} |\nabla F_b(\eta)|^p (1-|\eta|^2)^{p+q} K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \\ \lesssim \ & \int_{\mathbb{B}} |F_b(\eta)|^p (1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \lesssim I_{K,1}(h). \end{aligned} \tag{3.17}$$

From the previous estimate, we obtain

$$\int_{\mathbb{B}} \left|\nabla h(\eta) + \frac{2\overline{b}\, h(\eta)}{p(1-\langle \eta, b\rangle)}\right|^p \frac{K\left(1-|\Phi_x(\eta)|^2\right)}{(1-|\eta|^2)^{-p-q}} d\lambda(\eta) \lesssim I_{K,1}(h). \tag{3.18}$$

Hence,

$$\begin{aligned} & \int_{\mathbb{B}} |\nabla h(\eta)|^p \, (1-|\eta|^2)^{p+q} K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \\ \lesssim \ & I_{K,1}(h) + \int_{\mathbb{B}} |h(\eta)|^p \, (1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) \frac{(1-|\eta|^2)^p}{|1-\langle \eta, b\rangle|^p} d\lambda(\eta) \\ \lesssim \ & I_{K,1}(h). \end{aligned} \tag{3.19}$$

Moreover, for the case $b = 0$, by the subharmonicity of $|h|^p$, we have

$$|h(0)|^p \lesssim \int_{\mathbb{B}} |h(\eta)|^p (1-|\eta|^2)^q K\left(1-|\eta|^2\right) d\lambda(\eta) \lesssim I_{K,1}(h). \tag{3.20}$$

Thus, taken together, these estimates show that

$$I_{K,2}(h) \lesssim I_{K,1}(h).$$

Combining both directions, we conclude that $I_{K,1}(h) < \infty$ if and only if $I_{K,2}(h) < \infty$, and

$$I_{K,2}(h) \approx I_{K,1}(h),$$

which completes the proof. □

Theorem 3.2. *Let $p > 0$ and $q + n + 1 > 0$, and suppose that $K \in \mathbf{RC}^+$ satisfies* (1.2). *Then for any $h \in \mathbf{Hol}(\mathbb{B})$, we have*

$$h \in Q_K(p,q) \Longleftrightarrow \frac{\partial^m h}{\partial \zeta^m} \in Q_K(p, q+pN), \quad \forall\, |m| = N. \tag{3.21}$$

Furthermore, $I_{K,3}(h) \approx I_{K,4}(h)$.

Proof. We first consider the case $N = 1$. For each $j = 1, 2, \dots, n$, write

$$f_j(\eta) = \frac{\partial h}{\partial \eta_j}(\eta),$$

and set

$$\begin{aligned} I_{K,1j} &= \sup_{x\in\mathbb{B}} \int_{\mathbb{B}} |f_j(\eta)|^p (1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta), \\ I_{K,2j} &= |f_j(0)|^p + \sup_{x\in\mathbb{B}} \int_{\mathbb{B}} |\nabla f_j(\eta)|^p (1-|\eta|^2)^{q+p} K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta). \end{aligned}$$

Assume that $h \in Q_K(p,q)$. Then, for each $j = 1, 2, \dots, n$, applying Theorem 3.1 to f_j yields

$$I_{K,1j} \approx I_{K,2j}.$$

Since $c_j \geq 0$ for $j = 1, 2, \dots, n$, we may use the inequalities in (3.11). It follows that

$$\begin{aligned} I_{K,4}(h) &\approx |h(0)|^p + \sum_{j=1}^n \left|\frac{\partial h}{\partial \eta_j}(0)\right|^p + \sum_{j=1}^n I_{K,2j} \\ &\approx |h(0)|^p + \sum_{j=1}^n I_{K,1j} \approx I_{K,3}(h) < \infty. \end{aligned} \tag{3.22}$$

This shows that, for each $j = 1, 2, \dots, n$, we have

$$\frac{\partial h}{\partial \eta_j} \in Q_K(p, q+p),$$

and that $I_{K,3}(h) \approx I_{K,4}(h)$ in the case $N = 1$.

Conversely, assume that, for all $j = 1, 2, \dots, n$, we have

$$\frac{\partial h}{\partial \eta_j} \in Q_K(p, q+p).$$

Then each $I_{K,2j} < \infty$, and by applying Theorem 3.1 to f_j we obtain

$$I_{K,1j} \approx I_{K,2j}.$$

Hence

$$\begin{aligned} I_{K,3}(h) &\approx |h(0)|^p + \sum_{j=1}^{n} I_{K,1j} \\ &\approx |h(0)|^p + \sum_{j=1}^{n} \left| \frac{\partial h}{\partial \eta_j}(0) \right|^p + \sum_{j=1}^{n} I_{K,2j} \approx I_{K,4}(h) < \infty. \end{aligned} \tag{3.23}$$

Thus $h \in \mathcal{Q}_K(p, q)$ when $N = 1$; in this case, $I_{K,3}(h) \approx I_{K,4}(h)$.

For $N > 1$, the result follows by induction on N, using the same argument applied successively to derivatives of order $1, 2, \ldots, N$. This completes the proof. $\square$

Given that the characterization in Theorem 3.2 is expressed in terms of derivatives of order N, it is natural to ask what happens when the order of differentiation is strictly less than N. The result below provides a partial answer to this question.

Corollary 3.1. *Let $p > 0$ and $q > -1$, and suppose that $K \in \mathbf{RC}^+$ satisfies* (1.2). *For any $h \in \mathbf{Hol}(\mathbb{B})$, the following assertions hold:*

(i) If $q - p > -1$, then $h \in \mathcal{Q}_K(p, q)$ if and only if $h \in \mathcal{N}_K(p, q - p)$;
(ii) If $q - p \leq -1$, then for any $0 < \delta < q + 1$, whenever $h \in \mathcal{Q}_K(p, q)$ we have

$$||h||_{\mathcal{N}_K(p,q-\delta)} \lesssim ||h||_{\mathcal{Q}_K(p,q)}.$$

Proof. The proof follows directly from Theorem 3.1. It suffices to replace the parameter q in (3.1) by $q - p$, while every other occurrence of q in (3.1) is replaced by $q - \delta$. $\square$

4. Pointwise multipliers on the space $\mathcal{Q}_K(p, q)$

Given two function spaces U_1 and U_2 on $\mathbb{B}$ and a function $\varphi : \mathbb{B} \to \mathbb{C}$, we call φ a pointwise multiplier from U_1 into U_2 if

$$\mathbf{M}_\varphi h := \varphi h \in U_2 \quad \text{for every } h \in U_1.$$

The collection of all such multipliers $\mathbf{M}_\varphi : U_1 \to U_2$ is denoted by $\mathbf{PM}\{U_1, U_2\}$, with the convention that $\mathbf{PM}\{U_1, U_1\} = \mathbf{PM}\{U_1\}$. We use the symbol $||\mathbf{M}_\varphi||$ to denote the operator norm of the bounded pointwise multiplier $\mathbf{M}_\varphi$ on $\mathbb{B}$. Since the space $\mathcal{Q}_K(p, q)$ contains a variety of classical function spaces, a full characterization of its pointwise multipliers is rather intricate. In this section, we establish the following result.

Theorem 4.1. *Let $p > 0$ and $q > -1$, and suppose that $K \in \mathbf{RC}^+$ satisfies* (1.2). *Then the following hold.*

(1) If $p > q + n + 1$, then $\varphi \in \mathbf{PM}\{\mathcal{Q}_K(p, q)\}$ if and only if $\varphi \in \mathcal{Q}_K(p, q)$.
(2) If $0 < p < q + n + 1$, then $\varphi \in \mathbf{PM}\{\mathcal{Q}_K(p, q)\}$ if and only if $\varphi \in \mathbf{H}^\infty$.

(3) *If $n+1+q > p \geq q+1$ and $\varphi \in \mathbf{PM}\{Q_K(p,q)\}$, then $\varphi \in \mathbf{H}^\infty \cap Q_K(p,q)$. Conversely, if $\varphi \in \mathbf{H}^\infty$ and*

$$\sup_{x\in\mathbb{B}} \sum_{|l|=N+1} \int_{\mathbb{B}} \left|\frac{\partial^l \varphi}{\partial \zeta^l}(\eta)\right|^p (1-|\eta|^2)^{p(N+1)-n-1} K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) < \infty, \tag{4.1}$$

then $\varphi \in \mathbf{PM}\{Q_K(p,q)\}$, where we assume $p(N+1) > n$.

(4) *If $n+1+q = p$ and $\varphi \in \mathbf{PM}\{Q_K(p,q)\}$, then $\varphi \in \mathbf{H}^\infty \cap Q_K(p,q)$ and*

$$\sup_{\eta\in\mathbb{B}} (1-|\eta|^2)\,|\nabla\varphi(\eta)| \log \frac{2}{1-|\eta|^2} < \infty. \tag{4.2}$$

Conversely, if $\varphi \in \mathbf{H}^\infty$ and

$$\sup_{x\in\mathbb{B}} \int_{\mathbb{B}} |\nabla\varphi(\eta)|^p (1-|\eta|^2)^q \left(\log\frac{1}{1-|\eta|^2}\right)^p K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) < \infty, \tag{4.3}$$

then $\varphi \in \mathbf{PM}\{Q_K(p,q)\}$.

Proof. We begin by establishing the necessary conditions. Suppose that $\varphi \in \mathbf{PM}\{Q_K(p,q)\}$. For every $h \in Q_K(p,q)$, the product $\mathbf{M}_\varphi h = \varphi h$ belongs to $Q_K(p,q)$, and the operator $\mathbf{M}_\varphi$ is bounded on $Q_K(p,q)$. Taking the constant function $h(\eta) \equiv 1 \in Q_K(p,q)$, we obtain $\varphi \in Q_K(p,q)$.

Now fix an arbitrary element $\xi \in \mathbb{B}$ with $\xi \neq 0$, and let $\alpha := (q+n+1)/p > 0$. Define

$$G(\xi,\eta) = \frac{1}{|\xi|}\left(\frac{(1-|\xi|^2)^{\alpha+1}}{(1-\langle\eta,\xi\rangle)^{2\alpha}} - \frac{1-|\xi|^2}{(1-\langle\eta,\xi\rangle)^{\alpha}}\right). \tag{4.4}$$

Then $G(\xi,\xi) = 0$. Differentiating in η, and since $|\xi| \leq 1$, we may absorb $|\xi|^{-1}$ into the implicit constant; hence

$$|\nabla G(\xi,\eta)|^p \lesssim \frac{(1-|\xi|^2)^{p(\alpha+1)}}{|1-\langle\eta,\xi\rangle|^{p(2\alpha+1)}} + \frac{1-|\xi|^2}{|1-\langle\eta,\xi\rangle|^{p(\alpha+1)}}. \tag{4.5}$$

Moreover,

$$\lim_{|\xi|\to 0^+} \frac{(1-|\xi|^2)}{|\xi|}\{(1-|\xi|^2)^\alpha - 1\} = 0, \tag{4.6}$$

so that

$$|G(\xi,0)| \leq \sup_{0<|\xi|<1} \frac{(1-|\xi|^2)}{|\xi|}\{(1-|\xi|^2)^\alpha - 1\} \lesssim 1. \tag{4.7}$$

By definition, we have

$$\|G\|_K^p = \sup_{x\in\mathbb{B}} \int_{\mathbb{B}} |\nabla G(\xi,w)|^p (1-|w|^2)^q K(1-|\Phi_x(w)|^2)\, d\lambda(w).$$

Therefore, using (4.5) with w in place of η, we get

$$\|G\|_K^p \lesssim \sup_{x\in\mathbb{B}}\Big((1-|\xi|^2)^{p(\alpha+1)} J_1(x,\xi) + (1-|\xi|^2)^p J_2(x,\xi)\Big),$$

where

$$J_1(x,\xi) := \int_{\mathbb{B}} \frac{(1-|w|^2)^q}{|1-\langle w,\xi\rangle|^{p(2\alpha+1)}} K(1-|\Phi_x(w)|^2)\, d\lambda(w),$$
$$J_2(x,\xi) := \int_{\mathbb{B}} \frac{(1-|w|^2)^q}{|1-\langle w,\xi\rangle|^{p(\alpha+1)}} K(1-|\Phi_x(w)|^2)\, d\lambda(w).$$

In light of Lemmas 2.2 and 2.4, and after elementary computations, we have

$$\sup_{\xi\in\mathbb{B}}\sup_{x\in\mathbb{B}}(1-|\xi|^2)^{p(\alpha+1)} J_1(x,\xi) \lesssim 1, \qquad \sup_{\xi\in\mathbb{B}}\sup_{x\in\mathbb{B}}(1-|\xi|^2)^{p} J_2(x,\xi) \lesssim 1.$$

Combining these estimates, we obtain $\|G\|_{Q_K(p,q)} \lesssim 1$. Hence

$$\|\mathbf{M}_\varphi G\|_{Q_K(p,q)} \le \|\mathbf{M}_\varphi\|\, \|G\|_{Q_K(p,q)} \lesssim \|\mathbf{M}_\varphi\|. \tag{4.8}$$

For any $d,\eta \in \mathbb{B}$, and $\kappa_0 \in (0,1)$, let

$$\mathbb{E}(d,\kappa_0) = \{\eta \in \mathbb{B} : |\Phi_d(\eta)| < \kappa_0\}.$$

For any $d \in \mathbb{B}$ and $\eta \in \mathbb{E}(d,1/2)$, we know that

$$\begin{aligned}
\frac{1-\kappa_0}{1+\kappa_0}(1-|d|^2) &\le 1-|\eta|^2 \le \frac{1+\kappa_0}{1-\kappa_0}(1-|d|^2),\\
\sqrt{\frac{1-\kappa_0}{1+\kappa_0}}(1-|d|^2) &\le |1-\langle \eta, d\rangle| \le \frac{1}{1-\kappa_0}(1-|d|^2),\\
\frac{(1-|d|^2)(1-|\eta|^2)}{|1-\langle\eta,d\rangle|^2} &\ge 1-|\Phi_d(\eta)|^2 > 1-\kappa_0^2.
\end{aligned} \tag{4.9}$$

If we let $\xi \in \mathbb{E}(x,1/2)$, then

$$\begin{aligned}
&\int_{\mathbb{E}(x,1/2)} |\nabla[\varphi(\eta)G(\xi,\eta)]|^p\,(1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta)\\
\le\ &\int_{\mathbb{B}} |\nabla[\varphi(\eta)G(\xi,\eta)]|^p\,(1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta)\\
\lesssim\ &\|\mathbf{M}_\varphi\|^p.
\end{aligned} \tag{4.10}$$

Using (4.9) in (4.10), we obtain

$$\begin{aligned}
&K(3/4)\,(1-|\xi|^2)^q \int_{\mathbb{E}(x,1/2)} |\nabla[\varphi(\eta)G(\xi,\eta)]|^p\, d\lambda(\eta)\\
\le\ &\int_{\mathbb{E}(x,1/2)} |\nabla[\varphi(\eta)G(\xi,\eta)]|^p\,(1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta)\\
\lesssim\ &\|\mathbf{M}_\varphi\|^p.
\end{aligned} \tag{4.11}$$

Thus

$$\int_{\mathbb{E}(x,1/2)} |\nabla[\varphi(\eta)G(\xi,\eta)]|^p\, d\lambda(\eta) \lesssim \frac{\|\mathbf{M}_\varphi\|^p}{(1-|\xi|^2)^q}. \tag{4.12}$$

An application of [19, Lemma 2.24], combined with (4.12) and evaluated at $\eta = \xi$, yields

$$\begin{aligned}\left|\nabla\left[\varphi(\eta)G(\xi,\eta)\right]\right|^p &\lesssim \frac{1}{(1-|\xi|^2)^{n+1}}\int_{\mathbb{E}(x,1/2)}|\nabla[\varphi(\eta)G(\xi,\eta)]|^p\,d\lambda(\eta)\\ &\lesssim \frac{\|\mathbf{M}_\varphi\|^p}{(1-|\xi|^2)^{q+n+1}}.\end{aligned} \tag{4.13}$$

Consequently, $|\varphi(\xi)| \lesssim \|\mathbf{M}_\varphi\|$. Since ξ is arbitrary, it follows that $\varphi \in \mathbf{H}^\infty$.

Now consider the case $p = q + n + 1$. Since $\frac{2}{1-\langle\eta,\xi\rangle} \neq 0$ and $\frac{2}{1-\langle\eta,\xi\rangle} \neq 1$ for $\eta, \xi \in \mathbb{B}$, we define

$$f_\xi(\eta) = \left(\log\frac{2}{1-\langle\eta,\xi\rangle}\right)^{1+\frac{2}{p}}\left(\log\frac{2}{1-|\xi|^2}\right)^{-\frac{2}{p}}. \tag{4.14}$$

By applying Lemmas 2.4, 2.6, and [8, Theorem 1.4.10], we obtain

$$\|f_\xi\|_{Q_K(p,q)} \lesssim 1.$$

Therefore, for every $x \in \mathbb{B}$, we have

$$\int_{\mathbb{B}}|\nabla\varphi(\eta)f_\xi(\eta)|^p(1-|\eta|^2)^qK\left(1-|\Phi_x(\eta)|^2\right)d\lambda(\eta) \lesssim \|\mathbf{M}_\varphi\|^p. \tag{4.15}$$

A straightforward calculation reveals that

$$\begin{aligned}&\int_{\mathbb{B}}|f_\xi(\eta)\nabla\varphi(\eta)|^p(1-|\eta|^2)^qK\left(1-|\Phi_x(\eta)|^2\right)d\lambda(\eta)\\ \lesssim\;& \|\mathbf{M}_\varphi\|^p\|\varphi\|_\infty^p\|f_\xi\|^p_{Q_K(p,q)} \lesssim \|\mathbf{M}_\varphi\|^p\|\varphi\|_\infty^p.\end{aligned} \tag{4.16}$$

Consequently, from (4.9) and (4.16), combined with [19, Lemma 2.24] at $\eta = \xi$, we have

$$\begin{aligned}\left|f_\xi(\xi)\nabla\varphi(\xi)\right|^p &\lesssim \frac{1}{(1-|\xi|^2)^{n+1}}\int_{\mathbb{E}(x,1/2)}\left|f_\xi(\eta)\nabla\varphi(\eta)\right|^p d\lambda(\eta)\\ &\approx \frac{1}{K(3/4)(1-|\xi|^2)^{q+n+1}}\\ &\quad\times\int_{\mathbb{E}(x,1/2)}\left|f_\xi(\eta)\nabla\varphi(\eta)\right|^p(1-|\eta|^2)^qK\left(1-|\Phi_x(\eta)|^2\right)d\lambda(\eta)\\ &\lesssim \frac{1}{K(3/4)(1-|\xi|^2)^{q+n+1}}\left(\|\mathbf{M}_\varphi\|^p + \|\varphi\|_\infty^p\right).\end{aligned} \tag{4.17}$$

Since $p = q + n + 1$, the estimate above gives

$$\left\{(1-|\xi|^2)|\nabla\varphi(\xi)|\log\frac{2}{1-|\xi|^2}\right\}^p \lesssim \|\mathbf{M}_\varphi\|^p + \|\varphi\|_\infty^p. \tag{4.18}$$

Thus,

$$\sup_{\xi\in\mathbb{B}}\left\{(1-|\xi|^2)|\nabla\varphi(\xi)|\log\frac{2}{1-|\xi|^2}\right\} < \infty, \tag{4.19}$$

which is exactly Condition (4.2).

To establish sufficiency, we consider the four cases in Theorem 4.1 separately.

Case (1). $q + n + 1 < p$ and $\varphi \in Q_K(p, q)$.

Let $h \in Q_K(p, q)$. By Lemmas 2.1 and 2.5, we have

$$Q_K(p,q) \subseteq \mathcal{B}^\gamma \subset \mathbf{H}^\infty, \quad \gamma = \frac{q+n+1}{p},$$

so, in particular, $h, \varphi \in \mathbf{H}^\infty$. For any $x \in \mathbb{B}$ we estimate

$$\begin{aligned}
&\int_{\mathbb{B}} |\nabla[\varphi(\eta)h(\eta)]|^p (1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \\
\leq\ & 2^p \int_{\mathbb{B}} |\varphi(\eta)|^p |\nabla h(\eta)|^p (1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \\
&+2^p \int_{\mathbb{B}} |h(\eta)|^p |\nabla \varphi(\eta)|^p (1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \\
\lesssim\ & \|\varphi\|_\infty^p \|h\|_{Q_K(p,q)}^p + \|h\|_\infty^p \|\varphi\|_{Q_K(p,q)}^p.
\end{aligned} \tag{4.20}$$

Since $h \in Q_K(p,q)$ was arbitrary, it follows that $\varphi h \in Q_K(p,q)$ for all h, i.e., $\varphi \in \mathbf{PM}\{Q_K(p,q)\}$ in Case (1).

Case (2). $0 < p < q + n + 1$ and $\varphi \in \mathbf{H}^\infty$.

Let $h \in Q_K(p, q)$. By Theorem 3.1 and Corollary 3.1, when $0 < p < q + 1$, we have

$$\begin{aligned}
&\int_{\mathbb{B}} |h(\eta)|^p (1-|\eta|^2)^{q-p} K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \\
\lesssim\ & |h(0)|^p + \sup_{x\in\mathbb{B}} \int_{\mathbb{B}} |\nabla h(\eta)|^p (1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \\
\lesssim\ & \|h\|_{Q_K(p,q)}^p.
\end{aligned} \tag{4.21}$$

Since $\varphi \in \mathbf{H}^\infty \subset \mathcal{B}^1$, Lemma 2.5 gives

$$|\nabla \varphi(\eta)| \lesssim \frac{\|\varphi\|_{\mathcal{B}^1}}{1-|\eta|^2}. \tag{4.22}$$

Therefore

$$\begin{aligned}
&\int_{\mathbb{B}} |\nabla[\varphi(\eta)h(\eta)]|^p (1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \\
\leq\ & 2^p \|\varphi\|_\infty^p \int_{\mathbb{B}} |\nabla h(\eta)|^p (1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \\
&+2^p \int_{\mathbb{B}} |h(\eta)|^p |\nabla \varphi(\eta)|^p (1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \\
\leq\ & 2^p \|\varphi\|_\infty^p \|h\|_{Q_K(p,q)}^p \\
&+2^p \|\varphi\|_{\mathcal{B}^1}^p \int_{\mathbb{B}} |h(\eta)|^p (1-|\eta|^2)^{q-p} K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta)
\end{aligned}$$

$$\lesssim \quad \|\varphi\|_{\infty}^{p}\|h\|_{Q_K(p,q)}^{p} + \|\varphi\|_{\mathcal{B}^1}^{p}\|h\|_{Q_K(p,q)}^{p}. \tag{4.23}$$

Hence $\varphi h \in Q_K(p,q)$ for all h, so $\varphi \in \mathbf{PM}\{Q_K(p,q)\}$ in Case (2).

Case (3). $q+n+1 > p \geq q+1$, $\varphi \in \mathbf{H}^{\infty}$, and (4.1) holds.

Let $h \in Q_K(p,q)$. By Lemmas 2.1 and 2.5, we have

$$|h(\eta)| \lesssim (1-|\eta|^2)^{1-\frac{q+n+1}{p}}\|h\|_{Q_K(p,q)}. \tag{4.24}$$

Since $\varphi \in \mathbf{H}^{\infty} \subset \mathcal{B}^1$, we also have

$$|\nabla\varphi(\eta)| \lesssim (1-|\eta|^2)^{-1}\|\varphi\|_{\mathcal{B}^1},$$

and thus, for all $k \geq 1$,

$$\sum_{|j|=k}\left|\frac{\partial^j\varphi}{\partial\zeta^j}(\eta)\right| \lesssim (1-|\eta|^2)^{-k}\|\varphi\|_{\mathcal{B}^1}. \tag{4.25}$$

Under the condition $p(N+1) > n$, Theorem 3.2 together with (4.1), (4.24), and (4.25) yield

$$\begin{aligned}
&\int_{\mathbb{B}}|\nabla[\varphi(\eta)h(\eta)]|^p(1-|\eta|^2)^qK\left(1-|\Phi_x(\eta)|^2\right)d\lambda(\eta)\\
\lesssim\quad & \sum_{|l|\leq N}\left|\frac{\partial^l\varphi}{\partial\zeta^l}(0)\right|^p\\
&+\sup_{x\in\mathbb{B}}\sum_{|l|=N+1}\int_{\mathbb{B}}\left|\frac{\partial^l(\varphi h)}{\partial\zeta^l}(\eta)\right|^p(1-|\eta|^2)^{q+pN}K\left(1-|\Phi_x(\eta)|^2\right)d\lambda(\eta)\\
\lesssim\quad & \|\varphi h\|_{Q_K(p,q)}^p\\
&+\sup_{x\in\mathbb{B}}\sum_{|j|=0}^{N+1}\sum_{|i|=N+1-|j|}\int_{\mathbb{B}}\left|\frac{\partial^j\varphi}{\partial\zeta^j}(\eta)\right|^p\left|\frac{\partial^i h}{\partial\zeta^i}(\eta)\right|^p\frac{K\left(1-|\Phi_x(\eta)|^2\right)}{(1-|\eta|^2)^{-q-pN}}d\lambda(\eta)\\
\lesssim\quad & \|\varphi h\|_{Q_K(p,q)}^p+\|\varphi\|_{\mathcal{B}^1}^p\sup_{x\in\mathbb{B}}\sum_{|i|=1}^{N+1}\int_{\mathbb{B}}\left|\frac{\partial^i h}{\partial\zeta^i}(\eta)\right|^p\frac{K\left(1-|\Phi_x(\eta)|^2\right)}{(1-|\eta|^2)^{-q-p(|i|-1)}}d\lambda(\eta)\\
&+\|\varphi\|_{\mathcal{B}^1}^p\sup_{x\in\mathbb{B}}\sum_{|l|=N+1}\int_{\mathbb{B}}\left|\frac{\partial^l\varphi}{\partial\zeta^l}(\eta)\right|^p|h(\eta)|^p\frac{K\left(1-|\Phi_x(\eta)|^2\right)}{(1-|\eta|^2)^{-q-pN}}d\lambda(\eta)\\
\lesssim\quad & \|\mathbf{M}_\varphi\|^p\|h\|_{Q_K(p,q)}^p+\|\varphi\|_{\mathcal{B}^1}^p\|h\|_{Q_K(p,q)}^p\\
&+\|h\|_{Q_K(p,q)}^p\sup_{x\in\mathbb{B}}\sum_{|l|=N+1}\int_{\mathbb{B}}\left|\frac{\partial^l\varphi}{\partial\zeta^l}(\eta)\right|^p\frac{K\left(1-|\Phi_x(\eta)|^2\right)}{(1-|\eta|^2)^{-p(N+1)}}\frac{d\lambda(\eta)}{(1-|\eta|^2)^{n+1}}\\
\lesssim\quad & \|\mathbf{M}_\varphi\|^p\|h\|_{Q_K(p,q)}^p+\|\varphi\|_{\mathcal{B}^1}^p\|h\|_{Q_K(p,q)}^p+\|h\|_{Q_K(p,q)}^p.
\end{aligned} \tag{4.26}$$

Consequently, for every $h \in Q_K(p,q)$, we have $\varphi h \in Q_K(p,q)$, meaning that $\varphi \in \mathbf{PM}\{Q_K(p,q)\}$ in Case (3).

Case (4). $p = q+n+1$, $\varphi \in \mathbf{H}^{\infty}$, and (4.3) holds.

Let $h \in Q_K(p,q)$. Then $h \in \mathcal{B}^1$, and by Lemmas 2.1 and 2.5

$$|h(\eta)| \lesssim \log \frac{2}{1-|\eta|^2} \|h\|_{\mathcal{B}^1}. \tag{4.27}$$

In view of (4.3), it is also necessary that $\varphi \in Q_K(p,q)$. A direct calculation yields

$$\begin{aligned}
&\int_{\mathbb{B}} |\nabla[\varphi(\eta)h(\eta)]|^p (1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \\
\leq\ & 2^p \int_{\mathbb{B}} |\varphi(\eta)|^p |\nabla h(\eta)|^p (1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \\
&+2^p\|h\|_{\mathcal{B}^1}^p \int_{\mathbb{B}} |\nabla\varphi(\eta)|^p \left(\log\frac{2}{1-|\eta|^2}\right)^p (1-|\eta|^2)^q K\left(1-|\Phi_x(\eta)|^2\right) d\lambda(\eta) \\
\lesssim\ & \|\varphi\|_\infty^p \|h\|_{Q_K(p,q)}^p + \|\varphi\|_{Q_K(p,q)}^p \|h\|_{\mathcal{B}^1}^p + \|h\|_{\mathcal{B}^1}^p.
\end{aligned} \tag{4.28}$$

Therefore, for all $h \in Q_K(p,q)$, we have $\varphi h \in Q_K(p,q)$, which shows that $\varphi \in \mathbf{PM}\{Q_K(p,q)\}$ in Case (4).

Combining the four cases, we obtain the sufficiency part of Theorem 4.1. Together with the necessity already established, the proof is complete. □

The following corollary is a direct consequence of Theorem 4.1 (cf. Corollary 4.2 in [15]).

Corollary 4.1. *(a) When $0 < \gamma < 1$, we have $\varphi \in \mathbf{PM}\{\mathcal{B}^\gamma\} \Longleftrightarrow \varphi \in \mathcal{B}^\gamma$.*
(b) When $\gamma > 1$, we have $\varphi \in \mathbf{PM}\{\mathcal{B}^\gamma\} \Longleftrightarrow \varphi \in \mathbf{H}^\infty$.
(c) When $\gamma > -1$, we have $\varphi \in \mathbf{PM}\{\mathcal{N}_K(p,\gamma)\} \Longleftrightarrow \varphi \in \mathbf{H}^\infty$.
(d) When $\gamma = 1$, we have $\varphi \in \mathbf{PM}\{\mathcal{B}^1\} \Longleftrightarrow \varphi \in \mathbf{H}^\infty$ and

$$\sup_{\eta\in\mathbb{B}} (1-|\eta|^2)\, |\nabla\varphi(\eta)| \log\frac{2}{1-|\eta|^2} < \infty. \tag{4.29}$$

Proof. In Theorem 4.1, set $\gamma = (q+n+1)/p$. Under Condition (1.1), Lemma 2.1 ensures that

$$Q_K(p,q) = \mathcal{B}^\gamma,$$

and an application of Case (1) of Theorem 4.1 establishes Conclusion (a).

Under Condition (1.1) with $0 < p < q+n+1$, Case (2) of Theorem 4.1 applies and yields Conclusion (b).

Since $\mathcal{N}_K(p,\gamma) = Q_K(p, p+\gamma)$, this falls under Case (2) of Theorem 4.1, which gives Conclusion (c).

Finally, under Condition (1.1) with $p = q+n+1$, by [8, Theorem 1.4.10], if Condition (4.2) holds then Condition (4.3) also holds, and thus Conclusion (d) of the corollary follows. □

It is worth noting that Corollary 4.1 was established by a different approach in [15, Corollary 4.2].

Remark 4.1. Determining sharp necessary and sufficient conditions for Cases (3) and (4) of Theorem 4.1, particularly a clean sufficient condition, remains an open problem.

5. Spectrum of multiplication operators on $Q_K(p,q)$-type spaces

Recall that for a bounded linear operator $\mathcal{T}$ on a function space U, the spectrum of $\mathcal{T}$ is the closed set

$$\sigma(\mathcal{T}) = \{z \in \mathbb{C} : \mathcal{T} - zI \text{ is not invertible}\},$$

where I denotes the identity operator on U. Based on this definition, the spectrum of multiplication operators on $Q_K(p,q)$-type spaces admits an explicit characterization, as we establish below.

Theorem 5.1. *Let $q > -1$ and $0 < p < q + n + 1$, and suppose that $K \in \mathbf{RC}^+$ satisfies (1.2). Let φ be the symbol of a bounded multiplication operator $\mathbf{M}_\varphi$ on $Q_K(p,q)$. Then $\sigma(\mathbf{M}_\varphi) = \overline{\varphi(\mathbb{B})}$, which is a compact subset of $\mathbb{C}$ contained in $\overline{\mathbb{B}}$.*

Proof. For $\omega \in \mathbb{C}$, observe that

$$\mathbf{M}_\varphi - \omega I = \mathbf{M}_{\varphi-\omega},$$

so $\omega \in \sigma(\mathbf{M}_\varphi)$ if and only if $\mathbf{M}_{\varphi-\omega}$ fails to be invertible. Indeed, whenever $\mathbf{M}_{\varphi-\omega}^{-1}$ exists, it necessarily coincides with $\mathbf{M}_{(\varphi-\omega)^{-1}}$.

Now let $\eta \in \varphi(\mathbb{B})$. Then $\zeta \in \mathbb{B}$ exists with $\varphi(\zeta) = \eta$, so $(\varphi(\zeta) - \eta)^{-1}$ is not defined at ζ, and hence the operator $\mathbf{M}_{\varphi-\eta}$ cannot be invertible on $Q_K(p,q)$. Since $\sigma(\mathbf{M}_\varphi)$ is closed, it follows that

$$\overline{\varphi(\mathbb{B})} \subseteq \sigma(\mathbf{M}_\varphi).$$

Conversely, suppose $\eta \notin \overline{\varphi(\mathbb{B})}$. Then a constant $C > 0$ exists such that

$$\inf_{\zeta\in\mathbb{B}} |\varphi(\zeta) - \eta| \geq C,$$

which ensures that

$$g(\zeta) = \frac{1}{\varphi(\zeta) - \eta}$$

belongs to $\mathbf{H}^\infty$. For any $h \in Q_K(p,q)$ and any $x \in \mathbb{B}$, we estimate

$$\begin{aligned}
&\int_{\mathbb{B}} |\nabla[g(\zeta)h(\zeta)]|^p (1-|\zeta|^2)^q K\left(1-|\Phi_x(\zeta)|^2\right) d\lambda(\zeta) \\
\leq\ & 2^p \int_{\mathbb{B}} \left(\frac{|\nabla h(\zeta)|^p}{|\varphi(\zeta)-\eta|^p} + |h(\zeta)|^p \frac{|\nabla\varphi(\zeta)|^p}{|\varphi(\zeta)-\eta|^{2p}}\right)(1-|\zeta|^2)^q K\left(1-|\Phi_x(\zeta)|^2\right) d\lambda(\zeta) \\
\lesssim\ & \int_{\mathbb{B}} |\nabla h(\zeta)|^p (1-|\zeta|^2)^q K\left(1-|\Phi_x(\zeta)|^2\right) d\lambda(\zeta) \\
&+ \int_{\mathbb{B}} |h(\zeta)|^p |\nabla\varphi(\zeta)|^p (1-|\zeta|^2)^q K\left(1-|\Phi_x(\zeta)|^2\right) d\lambda(\zeta) \\
\lesssim\ & \|h\|^p_{Q_K(p,q)} + \int_{\mathbb{B}} |h(\zeta)|^p |\nabla\varphi(\zeta)|^p (1-|\zeta|^2)^q K\left(1-|\Phi_x(\zeta)|^2\right) d\lambda(\zeta).
\end{aligned} \tag{5.1}$$

Since $0 < p < q + n + 1$, an argument analogous to the proof of Theorem 4.1 shows that

$$\mathbf{PM}\{Q_K(p,q)\} \subseteq \mathbf{H}^\infty.$$

In particular, because $\varphi \in \mathbf{PM}\{Q_K(p,q)\}$, we have $\varphi \in \mathbf{H}^\infty$ and $\varphi h \in Q_K(p,q)$ for every $h \in Q_K(p,q)$. It follows that

$$\begin{aligned}
&\int_{\mathbb{B}} |h(\zeta)|^p |\nabla\varphi(\zeta)|^p (1-|\zeta|^2)^q K\left(1-|\Phi_x(\zeta)|^2\right) d\lambda(\zeta) \\
\lesssim\ & \int_{\mathbb{B}} |\nabla[h(\zeta)\varphi(\zeta)]|^p (1-|\zeta|^2)^q K\left(1-|\Phi_x(\zeta)|^2\right) d\lambda(\zeta) \\
&+ \int_{\mathbb{B}} |\nabla h(\zeta)|^p |\varphi(\zeta)|^p (1-|\zeta|^2)^q K\left(1-|\Phi_x(\zeta)|^2\right) d\lambda(\zeta) \\
\lesssim\ & \|\varphi h\|^p_{Q_K(p,q)} + \|\varphi\|^p_\infty \|h\|^p_{Q_K(p,q)}. \qquad (5.2)
\end{aligned}$$

Combining (5.1) and (5.2), we see that

$$\|gh\|_{Q_K(p,q)} \lesssim \|h\|_{Q_K(p,q)}$$

for every $h \in Q_K(p,q)$, which shows that $gh \in Q_K(p,q)$ and hence

$$g \in \mathbf{PM}\{Q_K(p,q)\}.$$

In particular, the multiplication operator $\mathbf{M}_g$ is bounded on $Q_K(p,q)$ and serves as the inverse of $\mathbf{M}_\varphi - \eta I$; that is

$$(\mathbf{M}_\varphi - \eta I)\mathbf{M}_g = \mathbf{M}_g(\mathbf{M}_\varphi - \eta I) = I.$$

Thus $\eta \notin \sigma(\mathbf{M}_\varphi)$.

Since we already know that $\overline{\varphi(\mathbb{B})} \subseteq \sigma(\mathbf{M}_\varphi)$, the argument above shows that the reverse inclusion also holds. Therefore

$$\sigma(\mathbf{M}_\varphi) = \overline{\varphi(\mathbb{B})},$$

as claimed. □

6. Conclusions

In the present work, we have investigated $Q_K(p,q)$-type spaces on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ from several complementary perspectives. We established equivalent norm characterizations of these spaces, providing new descriptions in terms of higher-order derivatives that extend and unify previously known results in the literature. These characterizations not only clarify the internal structure of $Q_K(p,q)$-type spaces, but also reveal their connections with several classical function spaces, including Bloch-type spaces, Besov spaces, and $BMOA(\mathbb{B})$.

As an application of the equivalent norm characterizations obtained above, we have analyzed pointwise multipliers on $Q_K(p,q)$-type spaces and established explicit conditions for boundedness. In addition, the spectrum of the pointwise multiplication operator $\mathbf{M}_\varphi$ on $Q_K(p,q)$ was fully determined, yielding the identity

$$\sigma(\mathbf{M}_\varphi) = \overline{\varphi(\mathbb{B})}.$$

These findings extend corresponding results on Bloch-type spaces, Besov spaces, $BMOA$, and classical Q_K spaces, due to Zhang, Ortega, and Li–Wulan, to the more general framework of $Q_K(p,q)$-type spaces.

The results of this paper suggest several directions for future research, including the study of Volterra-type operators, composition operators, and weighted differentiation operators on $Q_K(p,q)$-type spaces, as well as further spectral-theoretic investigations in this framework.

Author contributions

M. A. Bakhit, A. N. A. Koam, and H. Gissy contributed equally to the conceptualization, methodology, validation, preparation of the original draft, and review and editing of the manuscript. All authors read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

During the preparation of this work, the authors used Perplexity to support the organization of the manuscript and to refine the style, clarity, and readability of the English text. The tool was not used to generate scientific ideas, results, or interpretations. After using this tool, the authors carefully reviewed and edited all AI-assisted content to ensure its accuracy, originality, and consistency with the scientific content and objectives of the paper. The authors take full responsibility for the content of the published article.

Conflict of interest

The authors declare that they have no conflicts of interest.

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