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Research article

Stationary distribution of reaction-diffusion dengue fever model with mean-reverting white noise

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Abstract: In our previous study, we explored the stationary distribution of dengue fever models with Brownian motion, to investigate the differences between Brownian motion and the mean-reverting white noise. In this paper, we establish a dengue model with the mean-reverting white noise. First, we utilize the stochastic comparison theorem to prove solution boundedness, and construct a Lyapunov function to demonstrate the existence of the solution. Following this, sufficient conditions guaranteeing the stationary distribution of solutions are presented. To further validate the analytical derivations, numerical simulations are presented. Under the premise that the parameters remain constant, for case 1, compared with Brownian motion, the values of the system are more concentrated, and the histograms tend to be a normal distribution. For case 2, it was found that compared with Brownian motion, the model with the mean-reverting white noise shows an exponential decrease in the number of the infected population over time, and the distribution of the suspected population is closer to the normal distribution.

Keywords: dengue fever model; mean-reverting white noise; stationary distribution; reaction-diffusion
Mathematics Subject Classification: 35Q92, 37H05, 92B05

1. Introduction

Dengue fever represents an acute arthropod-borne infectious disease resulting from infection with mosquito-transmitted dengue virus. Due to its rapid spread and high incidence rate, it is currently one of the most widely distributed, most affected, and most harmful arbovirus diseases in the world. The dengue fever outbreak in 2025 was a mosquito-borne infectious disease event that broke out in many countries around the world that year. Aedes mosquitoes transmit the dengue virus to trigger this disease, which spreads chiefly throughout tropical and subtropical regions. For example, Brazil, the Philippines, Bangladesh, and Pacific Island countries were the main epidemic areas. As of

December 26, 2025, the number of suspected dengue fever cases reported in Brazil in 2025 decreased by 75% compared with the same period last year, still standing at 1,660,190. The World Health Organization said that the dengue fever epidemic in Sudan is severe, with more than 37,000 people infected. Entering 2026, the epidemic situation remains at a high level. As of early April 2026, Brazil reported 2,212 new dengue fever cases in a single week, topping the world in terms of the number of new cases. According to the monitoring data of the Ministry of Health of Malaysia and the World Health Organization, from January 4 to March 26, 2026, a total of 14,316 cases of dengue fever have been reported across the country, including six deaths. In the face of the high incidence of the epidemic, mathematical tools can effectively help us understand the dynamic behavior of dengue fever.

We have noticed that research on the dynamic behavior of dengue fever has been quite extensive. For example, Li et al. [1] studied the impact of temperature changes on the dengue fever model and the optimal control strategy and simultaneously revealed that intervention measures have a decisive influence on the effectiveness of epidemic control. Aprianti and Sonia [2] established the susceptible-exposed-infected-recovered aquatic-susceptible-exposed-infected (SEIR-ASEI) dengue fever model and conducted a stability analysis and sensitivity analysis of parameters. Zhu et al. [3] explored the dynamics of dengue fever transmission with non-local incidence and free boundaries. Guo et al. [4] built a delayed stochastic differential system model based on Wolbachia and illustrated the influence of environmental noise on virus dynamics through numerical simulation. Bounouiga et al. [5] studied the control measures of dengue fever through the basic reproduction number, equilibrium point, and stability. Cheng [6] confirmed the effectiveness of fused data augmentation, multi-scale feature engineering, and hybrid deep learning in dengue fever prediction. Li et al. [7] analyzed the impact of limited medical resources on the spread of dengue fever. Chen et al. [8] incorporated human mobility data into a deep learning-based dengue fever model to explore dynamic behavior. For more research on the dynamic behavior of dengue fever, please refer to the literature [9–11].

Additionally, mean reverting has been widely used in epidemiology [12–14] and financial economy [15–17] because it has better characteristics than the Brownian motion. It is not only found that the mean-reverting Ornstein-Uhlenbeck process can well describe the environmental variability in biological systems, but it is also closer to the reality in theory and biology. Wang et al. [18] proposed a new susceptible-infected-susceptible (SIS) model with the Ornstein-Uhlenbeck process of mean reversion and found that a smaller reversal rate or a larger fluctuation intensity can suppress the outbreak of the disease. Sun et al. [19] constructed an epidemic model with distributed delay and mean reversion Ornstein-Uhlenbeck processes. The existence of the sole traversal stationary distribution and the sufficient conditions for the eradication of infectious diseases were discussed. Liu and Chen [20] established a susceptible-infected-quarantined-recovered-susceptible (SIQRS) epidemic model with the Ornstein-Uhlenbeck process and obtained the threshold that determines the outbreak or extinction of the disease. Hao et al. [21] researched the combined influence of mean reduction white noise and spatial diffusion on the spread of diseases. We notice that the Ornstein-Uhlenbeck process is a Markov state process of a continuous orbit. In this paper, we consider the generalized colored driving noise of the Ornstein-Uhlenbeck process, namely, the mean-reverting white noise, which is often introduced into the equation as an external driving noise.

Based on the above analysis, we can see that there have been many studies on epidemic diseases caused by Brownian motion and the mean-reverting white noise. However, there is no comparison between the two at present. Therefore, in order to explore the influence of Brownian motion and the mean-reverting white noise on the dynamic behavior of dengue model. In this paper, a new dengue model formulated with spatial diffusion and the mean-reverting white noise is put forward, and the influences of Brownian motion and mean-reverting white noise on stationary distributions are analyzed. Three innovations are outlined below: (1) The inclusion of the mean-reverting white noise generalizes classical dengue models from existing literature; (2) we obtain the stationary distributions for the spatially diffusive stochastic dengue model and provide the sufficient conditions for its existence; (3) numerical examples verify our theoretical theorems and reveal differences between Brownian motion and the mean-reverting white noise in shaping stationary distributions.

The layout of this article is arranged as below. Preliminary theoretical foundations are provided, and a new model coupled with spatial diffusion and the mean-reverting white noise is formulated in the corresponding Section 2. We start Section 3 by establishing the well-posedness of solutions to the proposed dengue model, followed by the derivation of sufficient conditions for the existence of its stationary distributions. We perform numerical simulations in Section 4 to support the theoretical analysis, and Section 5 concludes this work.

2. Model

Aedes mosquitoes exhibit foraging behavior that tends toward human hosts, and their spatial distribution is significantly regulated by the spatial distribution of the population. Therefore, the cross-diffusion term is introduced to describe the directional migration behavior of Aedes mosquitoes caused by the host density gradient. On the basis of reference [22], we give the following model:

$$\begin{cases} \frac{\partial X_1}{\partial t} = d_1 \triangle X_1 + \mu_h N_H - \mu X_1 - \frac{\beta_X b}{N_H+m} X_1 Y_2, & x \in \Omega, t > 0, \\ \frac{\partial X_2}{\partial t} = d_2 \triangle X_2 + \frac{\beta_X b}{N_H+m} X_1 Y_2 - (\mu + \gamma_H) X_2, & x \in \Omega, t > 0, \\ \frac{\partial X_3}{\partial t} = d_3 \triangle X_3 + \gamma_H X_2 - \mu X_3, & x \in \Omega, t > 0, \\ \frac{\partial Y_1}{\partial t} = d_4 \triangle Y_1 + A - \nu Y_1 - \frac{\beta_Y b}{N_H+m} Y_1 X_2, & x \in \Omega, t > 0, \\ \frac{\partial Y_2}{\partial t} = d_5 \triangle Y_2 + \frac{\beta_Y b}{N_H+m} Y_1 X_2 - \nu Y_2, & x \in \Omega, t > 0, \end{cases} \tag{2.1}$$

with boundary condition

$$\frac{\partial X_1}{\partial \nu} = \frac{\partial X_2}{\partial \nu} = \frac{\partial X_3}{\partial \nu} = \frac{\partial Y_1}{\partial \nu} = \frac{\partial Y_2}{\partial \nu} = 0, \ x \in \partial\Omega, \ t > 0, \tag{2.2}$$

and initial condition

$$X_1(\cdot,0) = X_{1,0}(\cdot), X_2(\cdot,0) = X_{2,0}(\cdot), X_3(\cdot,0) = X_{3,0}(\cdot), Y_1(\cdot,0) = Y_{1,0}(\cdot), Y_2(\cdot,0) = Y_{2,0}(\cdot), x \in \Omega. \tag{2.3}$$

X_1, X_2, and X_3 represent the human population densities of susceptible, infectious, and recovered, at location x and time t. N_H is the human population and $N_H = X_1 + X_2 + X_3$. Y_1 and Y_2 represent the mosquitoes' population densities of susceptible and infectious, respectively. Ω is a bounded domain with smooth boundary. The meanings of the remaining parameters can be found in Reference [22].

We assume the death rate is a random process. Next, we introduce the mean-reverting white noise, which has the following form:

$$\begin{cases} d\mu(t) = \vartheta_1(\mu_e - \mu(t))dt + \xi_1 dB_1(t), \\ d\nu(t) = \vartheta_2(\nu_e - \nu(t))dt + \xi_2 dB_2(t), \end{cases} \tag{2.4}$$

where ϑ_i, ξ_i, and $B_i(t)$, $(i = 1, 2)$ represent the reversion rates, intensity of volatility, and Brownian motion, respectively. μ_e and ν_e denote the average level of mortality μ and ν. ϑ_i, ξ_i, μ_e, and ν_e are positive constants.

With the stochastic integral expression (2.4) for the arithmetic mean-reverting white noise, we arrive at the explicit solution below:

$$\begin{cases} \mu(t) = \mu_e + (\mu_0 - \mu_e)e^{-\vartheta_1 t} + \xi_1 \int_0^t e^{-\vartheta_1(t-s)} dB_1(s), \\ \nu(t) = \nu_e + (\nu_0 - \nu_e)e^{-\vartheta_2 t} + \xi_2 \int_0^t e^{-\vartheta_2(t-s)} dB_2(s), \end{cases} \tag{2.5}$$

and we formally rewrite the Itô integral as an integral with respect to white noise,

$$\xi_i \int_0^t e^{-\vartheta_1(t-s)} dB_i(s) = \xi_i \int_0^t e^{-\vartheta_i(t-s)} \delta(\cdot - s) ds \dot{B}_i(t).$$

$(\delta(\cdot - s)$ represents the Dirac δ generalized function). Let

$$\varepsilon_i(t) = \xi_i \int_0^t e^{-\vartheta_i(t-s)} \delta(\cdot - s) ds,$$

$(\varepsilon_i(t)$ is the generalized random amplitude), and (2.5) can be almost surely (a.s.) rewritten as

$$\begin{cases} \mu(t) = \mu_e + (\mu_0 - \mu_e)e^{-\vartheta_1 t} + \varepsilon_1(t)\dot{B}_1(t), \\ \nu(t) = \nu_e + (\nu_0 - \nu_e)e^{-\vartheta_2 t} + \varepsilon_2(t)\dot{B}_2(t), \end{cases} \tag{2.6}$$

where $\mu_0 := \mu(0) > 0$, $\nu_0 := \nu(0) > 0$, and $\varepsilon_i = \frac{\xi_i}{2\vartheta_i}\sqrt{1 - e^{-2\vartheta_i t}}, (i = 1, 2)$. Upon substituting (2.6) into system (1) (where $\dot{B}_i(t)$ is white noise, equivalent to the Itô differential form $\varepsilon_i(t)\dot{B}_i(t) = \varepsilon_i(t)dB_i(t)$), we obtain the subsequent stochastic system as shown below:

$$\begin{cases} dX_1 = [d_1 \triangle X_1 + \mu_h N_H - \mu_e X_1 - (\mu_0 - \mu_e)e^{-\vartheta_1 t} X_1 - \frac{\beta_X b}{N_H + m} X_1 Y_2]dt - \varepsilon_1(t) X_1 dB_1(t), \\ dX_2 = [d_2 \triangle X_2 + \frac{\beta_X b}{N_H + m} X_1 Y_2 - \gamma_H X_2 - \mu_e X_2 - (\mu_0 - \mu_e)e^{-\vartheta_1 t} X_2]dt - \varepsilon_1(t) X_2 dB_1(t), \\ dX_3 = [d_3 \triangle X_3 + \gamma_H X_2 - \mu_e X_3 - (\mu_0 - \mu_e)e^{-\vartheta_1 t} X_3]dt - \varepsilon_1(t) X_3 dB_1(t), \\ dY_1 = [d_4 \triangle Y_1 + A - \nu_e Y_1 - (\nu_0 - \nu_e)e^{-\vartheta_2 t} Y_1 - \frac{\beta_Y b}{N_H + m} Y_1 X_2]dt - \varepsilon_2(t) Y_1 dB_2(t), \\ dY_2 = [d_5 \triangle Y_2 + \frac{\beta_Y b}{N_H + m} Y_1 X_2 - \nu_e Y_2 - (\nu_0 - \nu_e)e^{-\vartheta_2 t} Y_2]dt - \varepsilon_2(t) Y_2 dB_2(t), \end{cases} \tag{2.7}$$

with boundary condition (ν indicates the outer normal direction)

$$\frac{\partial X_1}{\partial \nu} = \frac{\partial X_2}{\partial \nu} = \frac{\partial X_3}{\partial \nu} = \frac{\partial Y_1}{\partial \nu} = \frac{\partial Y_2}{\partial \nu} = 0, \ x \in \partial\Omega, \ t > 0,$$

and initial condition

$$X_1(\cdot, 0) = X_{1,0}(\cdot), X_2(\cdot, 0) = X_{2,0}(\cdot), X_3(\cdot, 0) = X_{3,0}(\cdot), Y_1(\cdot, 0) = Y_{1,0}(\cdot), Y_2(\cdot, 0) = Y_{2,0}(\cdot), x \in \Omega.$$

Let

$$V = \mathcal{H}^1(\Omega) = \{\phi \mid \phi \in L^2(\Omega), \frac{\partial \phi}{\partial x} \in L^2(\Omega), \text{ where } \frac{\partial \phi}{\partial x} \text{ are generalized partial derivatives}\}.$$

$V' = \mathcal{H}^{-1}(\Omega)$ is the dual space of V, $\|\cdot\|$ denotes the norm in V, and $\langle \cdot, \cdot \rangle$ represents the duality product between V and V'. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete probability space and $B_i(t), (i = 1, 2)$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, $R_+^5 = (x_1, x_2, x_3, x_4, x_5) \in R^5$, and $x_i > 0, (i = 1, 2, 3, 4, 5)$.

3. Stationary distribution

An auxiliary lemma is first provided to support the derivation of our main conclusions.

Lemma 3.1. *For any initial data* $(X_{1,0}, X_{2,0}, X_{3,0}, Y_{1,0}, Y_{2,0})$*, the solution* $\mathcal{W}(x,t) = (X_1(\cdot,t), X_2(\cdot,t), X_3(\cdot,t), Y_1(\cdot,t), Y_2(\cdot,t))$ *of system (2.7) satisfies that*

$$\limsup_{t\to\infty}(X_1(\cdot,t) + X_2(\cdot,t) + X_3(\cdot,t) + Y_1(\cdot,t) + Y_2(\cdot,t)) < \infty.$$

Proof. Let

$$\mathbb{N}(t) = \int_\Omega [X_1(\cdot,t) + X_2(\cdot,t) + X_3(\cdot,t) + Y_1(\cdot,t) + Y_2(\cdot,t)]dx.$$

By (2.7), we have

$$\begin{aligned}
\frac{\partial \mathbb{N}(t)}{\partial t} &= \int_\Omega [\frac{\partial}{\partial t}X_1(\cdot,t) + \frac{\partial}{\partial t}X_2(\cdot,t) + \frac{\partial}{\partial t}X_3(\cdot,t) + \frac{\partial}{\partial t}Y_1(\cdot,t) + \frac{\partial}{\partial t}Y_2(\cdot,t)]dx \\
&= \int_\Omega [d_1\triangle X_1 + \mu_h N_H - \mu_e X_1 - (\mu_0 - \mu_e)e^{-\vartheta_1 t}X_1 - \frac{\beta_X b}{N_H + m}X_1Y_2 - \varepsilon_1(t)X_1\dot{B}_1(t) \\
&\quad + d_2\triangle X_2 + \frac{\beta_X b}{N_H + m}X_1Y_2 - \gamma_H X_2 - \mu_e X_2 - (\mu_0 - \mu_e)e^{-\vartheta_1 t}X_2 - \varepsilon_1(t)X_2\dot{B}_1(t) \\
&\quad + d_3\triangle X_3 + \gamma_H X_2 - \mu_e X_3 - (\mu_0 - \mu_e)e^{-\vartheta_1 t}X_3 - \varepsilon_1(t)X_3\dot{B}_1(t) + d_4\triangle Y_1 + A \\
&\quad - \nu_e Y_1 - (\nu_0 - \nu_e)e^{-\vartheta_2 t}Y_1 - \frac{\beta_Y b}{N_H + m}Y_1X_2 - \varepsilon_2(t)Y_1\dot{B}_2(t) + d_5\triangle Y_2 + \frac{\beta_Y b}{N_H + m}Y_1X_2 \\
&\quad - \nu_e Y_2 - (\nu_0 - \nu_e)e^{-\vartheta_2 t}Y_2 - \varepsilon_2(t)Y_2\dot{B}_2(t)]dx.
\end{aligned}$$

Next, we continue our process in both cases.

Case 1: $\mu_0 > \mu_e, \nu_0 > \nu_e$.

$$\begin{aligned}
\frac{\partial \mathbb{N}(t)}{\partial t} \le\ & d_1\int_{\partial\Omega}(\frac{\partial}{\partial \nu}X_1(\cdot,t))dx + d_2\int_{\partial\Omega}(\frac{\partial}{\partial \nu}X_2(\cdot,t))dx + X_3\int_{\partial\Omega}(\frac{\partial}{\partial \nu}X_3(\cdot,t))dx \\
& + d_4\int_{\partial\Omega}(\frac{\partial}{\partial \nu}Y_1(\cdot,t))dx + d_5\int_{\partial\Omega}(\frac{\partial}{\partial \nu}Y_2(\cdot,t))dx + \int_\Omega(\mu_h N_H + A - \mu_e X_1(\cdot,t) - \mu_e \\
& \times X_2(\cdot,t) - \mu_e X_3(\cdot,t)) - \nu_e Y_1(\cdot,t) - \nu_e Y_2(\cdot,t) - \varepsilon_1(t)X_1\dot{B}_1(t) - \varepsilon_1(t)X_2\dot{B}_1(t) \\
& - \varepsilon_1(t)X_3\dot{B}_1(t) - \varepsilon_2(t)Y_1\dot{B}_2(t) - \varepsilon_2(t)Y_2\dot{B}_2(t))dx \\
=\ & (\mu_h N_H + A)|\Omega| - min\{\mu_e, \nu_e\}\mathbb{N}(t) - \int_\Omega(\varepsilon_1(t)X_1\dot{B}_1(t) + \varepsilon_1(t)X_2\dot{B}_1(t) \\
& + \varepsilon_1(t)X_3\dot{B}_1(t) + \varepsilon_2(t)Y_1\dot{B}_2(t) + \varepsilon_2(t)Y_2\dot{B}_2(t))dx,
\end{aligned}$$

where $|\Omega|$ denotes the volume of Ω. The following stochastic differential equation admits the solution $\mathbb{X}(t)$:

$$\begin{cases}
d\mathbb{X}(t) = [(\mu_h N_H + A) - min\{\mu_e, \nu_e\}\mathbb{X}(t)]dt - \int_\Omega \varepsilon_1(s)X_1(\cdot,s)dxdB_1(s) - \int_\Omega \varepsilon_1(s)X_2(\cdot,s)dxdB_1(s) \\
\qquad - \int_\Omega \varepsilon_1(s)X_3(\cdot,s)dxdB_1(s) - \int_\Omega \varepsilon_2(s)Y_1(\cdot,s)dxdB_2(s) - \int_\Omega \varepsilon_2(s)Y_2(\cdot,s)dxdB_2(s), \\
\mathbb{X}(0) = \mathbb{N}(0).
\end{cases} \tag{3.1}$$

We get the solution to Eq (3.1) in the following form:

$$\mathbb{X}(T)=\frac{\mu_h N_H+A}{min\{\mu_e,\nu_e\}}+(\mathbb{X}(0)-\frac{\mu_h N_H+A}{min\{\mu_e,\nu_e\}})e^{-min\{\mu_e,\nu_e\}t}+\mathbb{G}(t),$$

where

$$\begin{aligned}\mathbb{G}(t)=&-\int_0^t e^{-min\{\mu_e,\nu_e\}(t-s)}\int_\Omega \varepsilon_1(s)X_1(\cdot,s)dxdB_1(s)-\int_0^t e^{-min\{\mu_e,\nu_e\}(t-s)}\int_\Omega \varepsilon_2(s)X_2(\cdot,s)dxdB_1(s)\\&-\int_0^t e^{-min\{\mu_e,\nu_e\}(t-s)}\int_\Omega \varepsilon_1(s)X_3(\cdot,s)dxdB_1(s)-\int_0^t e^{-min\{\mu_e,\nu_e\}(t-s)}\int_\Omega \varepsilon_2(s)Y_1(\cdot,s)dxdB_2(s)\\&-\int_0^t e^{-min\{\mu_e,\nu_e\}(t-s)}\int_\Omega \varepsilon_1(s)Y_2(\cdot,s)dxdB_2(s),\end{aligned}$$

and $\mathbb{G}(t)$ is a continuous local martingale. The diffusion terms of $\mathbb{N}(t)$ and $\mathbb{X}(t)$ are exactly the same and share the same set of Brownian motions $B_1(t)$ and $B_2(t)$, satisfying the condition of the stochastic comparison theorem that the diffusion coefficients are of equal origin. By virtue of the stochastic comparison theorem, we obtain $\mathbb{N}(t)\le\mathbb{X}(t)$, a.s.

Define

$$\begin{aligned}\mathbb{X}(t)&=\mathbb{X}(0)+\mathbb{K}(t)-\mathbb{J}(t)+\mathbb{G}(t),\\ \mathbb{K}(t)&=\frac{\mu_h N_H+A}{min\{\mu_e,\nu_e\}}(1-e^{-min\{\mu_e,\nu_e\}t}),\end{aligned}$$

and

$$\mathbb{J}(t)=\mathbb{X}(0)(1-e^{-min\{\mu_e,\nu_e\}t}).$$

Obviously, $\mathbb{K}$(t)and $\mathbb{J}(t)$ are continuous adapted increasing processes on $t\ge 0$ with $\mathbb{K}(0)=\mathbb{J}(0)=0$. By virtue of the nonnegative semimartingale convergence theorem [23], we have $\lim_{t\to\infty}X(t)<\infty$ a.s. Therefore, $\limsup_{t\to\infty}\mathbb{N}(t)<\infty$, a.s.

Case 2: $\mu_0<\mu_e,\nu_0<\nu_e$.

$$\begin{aligned}\frac{\partial\mathbb{N}(t)}{\partial t}\le\;& d_1\int_{\partial\Omega}(\frac{\partial}{\partial\nu}X_1(\cdot,t))dx+d_2\int_{\partial\Omega}(\frac{\partial}{\partial\nu}X_2(\cdot,t))dx+X_3\int_{\partial\Omega}(\frac{\partial}{\partial\nu}X_3(\cdot,t))dx\\&+d_4\int_{\partial\Omega}(\frac{\partial}{\partial\nu}Y_1(\cdot,t))dx+d_5\int_{\partial\Omega}(\frac{\partial}{\partial\nu}Y_2(\cdot,t))dx+\int_\Omega(\mu_h N_H+A-\mu_0X_1(\cdot,t)-\mu_0\\&\times X_2(\cdot,t)-\mu_0X_3(\cdot,t))-\nu_0Y_1(\cdot,t)-\nu_0Y_2(\cdot,t)-\varepsilon_1(t)X_1\dot{B}_1(t)-\varepsilon_1(t)X_1\dot{B}_1(t)\\&-\varepsilon_1(t)X_3\dot{B}_1(t)-\varepsilon_2(t)Y_1\dot{B}_2(t)-\varepsilon_2(t)Y_2\dot{B}_2(t))dx\\=\;&(\mu_h N_H+A)|\Omega|-min\{\mu_0,\nu_0\}\mathbb{N}(t)-\int_\Omega(\varepsilon_1(t)X_1\dot{B}_1(t)+\varepsilon_1(t)X_2\dot{B}_1(t)+\varepsilon_1(t)X_3\dot{B}_1(t)\\&+\varepsilon_2(t)Y_1\dot{B}_2(t)+\varepsilon_2(t)Y_2\dot{B}_2(t))dx.\end{aligned}$$

By a similar method, we acquire $\limsup_{t\to\infty}\mathbb{N}(t)<\infty$, a.s. Proof complete. □

We then demonstrate that system (2.7) admits a unique solution.

Theorem 3.2. *For any initial data* $(X_{1,0},X_{2,0},X_{3,0},Y_{1,0},Y_{2,0})>0$*, system (2.7) exists a unique solution for* $t>0$ *on* Ω*.*

Proof. The local Lipschitz condition holds for the coefficients of system (2.7), implying a unique local solution on $t \in [0, \tau_e)$, where τ_e is the explosion time. Let $l_0 > 1$ be sufficiently large for

$$\frac{1}{l_0} \leq \min_{0<t<\tau_e} |\mathcal{W}(x,t)| \leq \max_{0<t<\tau_e} |\mathcal{W}(x,t)| \leq l_0.$$

For each integer $l > l_0$, define the stopping time

$$\begin{aligned}\tau_l = inf\{t \in [0, \tau_e] : min(X_1(\cdot,t), X_2(\cdot,t), X_3(\cdot,t), Y_1(\cdot,t), Y_2(\cdot,t)) \leq \frac{1}{l}\\ or \;\; max(X_1(\cdot,t), X_2(\cdot,t), X_3(\cdot,t), Y_1(\cdot,t), Y_2(\cdot,t)) \geq l\}.\end{aligned}$$

We assign $inf\emptyset = \infty$, with $\emptyset$ representing the empty set. Since τ_l forms an increasing sequence for $l \to \infty$, define $\tau_\infty = \lim_{l\to\infty} \tau_l$, such that $\tau_\infty < \tau_e$ holds, a.s. It remains to demonstrate that $\tau_\infty = \infty$, a.s. Hence, invoking Itô's formula yields

$$\begin{aligned}
&d(\|X_1(\cdot,t)\|^2 + \|X_2(\cdot,t)\|^2 + \|X_3(\cdot,t)\|^2 + \|Y_1(\cdot,t)\|^2 + \|Y_2(\cdot,t)\|^2)\\
= \;&\{2\langle X_1(\cdot,t), d_1\triangle X_1 + \mu_h N_H - \mu_e X_1 - (\mu_0-\mu_e)e^{-\vartheta_1 t}X_1 - \frac{\beta_X b}{N_H+m}X_1Y_2\rangle\\
&+ 2\langle X_2(\cdot,t), d_2\triangle X_2 + \frac{\beta_X b}{N_H+m}X_1Y_2 - \gamma_H X_2 - \mu_e X_2 - (\mu_0-\mu_e)e^{-\vartheta_1 t}X_2\rangle\\
&+ 2\langle X_3(\cdot,t), d_3\triangle X_3 + \gamma_H X_2 - \mu_e X_3 - (\mu_0-\mu_e)e^{-\vartheta_1 t}X_3\rangle + 2\langle Y_1(\cdot,t), d_4\triangle Y_1\\
&+ A - \nu_e Y_1 - (\nu_0-\nu_e)e^{-\vartheta_2 t}Y_1 - \frac{\beta_Y b}{N_H+m}Y_1X_2\rangle + 2\langle Y_2(\cdot,t), d_5\triangle Y_2\\
&+ \frac{\beta_Y b}{N_H+m}Y_1X_2 - \nu_e Y_2 - (\nu_0-\nu_e)e^{-\vartheta_2 t}Y_2\rangle + \varepsilon_1^2(t)\|X_1(\cdot,t)\|^2 + \varepsilon_1^2(t)\|X_2(\cdot,t)\|^2\\
&+ \varepsilon_1^2(t)\|X_3(\cdot,t)\|^2 + \varepsilon_2^2(t)\|Y_1(\cdot,t)\|^2 + \varepsilon_2^2(t)\|Y_2(\cdot,t)\|^2\}dt + 2\langle X_1(\cdot,t), -\varepsilon_1 X_1(\cdot,t)\cdot dB_1(t)\rangle\\
&+ 2\langle X_2(\cdot,t), -\varepsilon_1 X_2(\cdot,t)dB_1(t)\rangle + 2\langle X_3(\cdot,t), -\varepsilon_1 X_3(\cdot,t)dB_1(t)\rangle + 2\langle Y_1(\cdot,t), -\varepsilon_2 Y_1(\cdot,t)dB_2(t)\rangle\\
&+ 2\langle Y_2(\cdot,t), -\varepsilon_2 Y_2(\cdot,t)dB_2(t)\rangle.
\end{aligned} \tag{3.2}$$

Suppose $l > l_0$ and $T > 0$. Integrating (3.2) from 0 to $\tau_l \wedge T$ and taking expectations yields

$$\begin{aligned}
&E[\|X_1(\cdot,\tau_l\wedge T)\|^2 + \|X_2(\cdot,\tau_l\wedge T)\|^2 + \|X_3(\cdot,\tau_l\wedge T)\|^2 + \|Y_1(\cdot,\tau_l\wedge T)\|^2\\
&+ \|Y_2(\cdot,\tau_l\wedge T)\|^2] - (\|X_{1,0}\|^2 + \|X_{2,0}\|^2 + \|X_{3,0}\|^2 + \|Y_{1,0}\|^2 + \|Y_{2,0}\|^2)\\
\leq \;& E\int_0^{\tau_l\wedge T}\{-2\langle\nabla X_1(\cdot,s), d_1\nabla X_1(\cdot,s)\rangle + 2\langle\mu_h N_H, X_1(\cdot,s)\rangle - 2\langle\nabla X_2(\cdot,s), d_2X_2(\cdot,s)\rangle\\
&+ 2\langle X_2(\cdot,s), \frac{\beta_X b}{N_H+m}X_1Y_2\rangle - 2\langle\nabla X_3(\cdot,s), d_3\nabla X_3(\cdot,s)\rangle + 2\langle X_3(\cdot,s), \gamma_H X_2\rangle\\
&- 2\langle\nabla Y_1(\cdot,s), d_4\nabla Y_1(\cdot,s)\rangle + 2\langle Y_1(\cdot,s), A\rangle - 2\langle\nabla Y_2(\cdot,s), d_5Y_2(\cdot,s)\rangle\\
&+ 2\langle Y_2(\cdot,s), \frac{\beta_Y b}{N_H+m}Y_1X_2\rangle + \varepsilon_1^2\|X_1(\cdot,s)\|^2 + \varepsilon_1^2X_2(\cdot,s) + \varepsilon_1^2X_3(\cdot,s) + \varepsilon_2^2Y_1(\cdot,s) + \varepsilon_2^2Y_2(\cdot,s)\}ds
\end{aligned}$$

$$
\begin{aligned}
&\leq\ E\int_0^{\tau_l\wedge T}\{-2d_1\lambda_0\|X_1(\cdot,s)\|^2+2\langle\mu_h N_H,X_1(\cdot,s)\rangle-2d_2\lambda_0\|X_2(\cdot,s)\|^2\\
&\quad+2\langle X_2(\cdot,s),\frac{\beta_X b}{N_H+m}X_1Y_2\rangle-2d_3\lambda_0\|X_3(\cdot,s)\|^2+2\langle X_3(\cdot,s),\gamma_H X_2\rangle-2d_4\lambda_0\|Y_1(\cdot,s)\|^2\\
&\quad+2\langle Y_1(\cdot,s),A\rangle-2d_5\lambda_0\|Y_2(\cdot,s)\|^2+2\langle Y_2(\cdot,s),\frac{\beta_Y b}{N_H+m}Y_1X_2\rangle+\varepsilon_1^2\|X_1(\cdot,s)\|^2\\
&\quad+\varepsilon_1^2\|X_2(\cdot,s)\|^2+\varepsilon_1^2\|X_3(\cdot,s)\|^2+\varepsilon_2^2\|Y_1(\cdot,s)\|^2+\varepsilon_2^2\|Y_2(\cdot,s)\|^2\}ds,
\end{aligned}
$$

where $\lambda_0=\inf_{u\in\mathcal{H}}\|\nabla u(x,s)\|^2/\|u(x,s)\|^2$ [24].

Then, applying Lemma 3.1 along with the fundamental inequality yields

$$
\begin{aligned}
&E[\|X_1(\cdot,\tau_l\wedge T)\|^2+\|X_2(\cdot,\tau_l\wedge T)\|^2+\|X_3(\cdot,\tau_l\wedge T)\|^2+\|Y_1(\cdot,\tau_l\wedge T)\|^2+\|Y_2(\cdot,\tau_l\wedge T)\|^2]\\
\leq\ &(\|X_{1,0}\|^2+\|X_{2,0}\|^2+\|X_{3,0}\|^2+\|Y_{1,0}\|^2+\|Y_{2,0}\|^2)+E\int_0^{\tau_l\wedge T}\{-2d_1\lambda_0\|X_1(\cdot,s)\|^2+\mu_h^2N_H^2\\
&+\|X_1(\cdot,s)\|^2-2d_2\lambda_0\|X_2(\cdot,s)\|^2+\|X_2(\cdot,s)\|^2+\frac{\beta_X^2b^2}{(N_H+m)^2}M_1^2\|Y_2\|^2-2d_3\lambda_0\|X_3(\cdot,s))\|^2\\
&+\|X_3(\cdot,s)\|^2+\gamma_H^2\|X_2\|^2-2d_4\lambda_0\|Y_1(\cdot,s)\|^2+A^2+\|Y_1(\cdot,s)\|^2-2d_5\lambda_0\|Y_2(\cdot,s)\|^2\\
&+\|Y_2(\cdot,s)\|^2+\frac{\beta_Y^2b^2}{(N_H+m)^2}M_1^2\|X_2\|^2+\varepsilon_1^2\|X_1(\cdot,s)\|^2+\varepsilon_1^2X_2(\cdot,s)+\varepsilon_1^2X_3(\cdot,s)\\
&+\varepsilon_2^2Y_1(\cdot,s)+\varepsilon_2^2Y_2(\cdot,s)\}ds\\
\leq\ &M_2+M_3E\int_0^{\tau_l\wedge T}\{\|X_1(\cdot,s)\|^2+\|X_2(\cdot,s)\|^2+\|X_3(\cdot,s)\|^2+\|Y_1(\cdot,s)\|^2+\|Y_2(\cdot,s)\|^2\},
\end{aligned}
$$

where M_1 represents the upper bound of the system solution,

$$M_2=\|X_{1,0}\|^2+\|X_{2,0}\|^2+\|X_{3,0}\|^2+\|Y_{1,0}\|^2+\|Y_{2,0}\|^2+(\mu_h^2N_H^2+A^2)\tau_l,$$

and

$$
\begin{aligned}
M_3=&max\{(1-2d_1\lambda_0+\varepsilon_1^2),(1-2d_2\lambda_0+\gamma_H^2+\varepsilon_1^2+\frac{\beta_X^2b^2M_1^2}{(N_H+m)^2}),(1+\varepsilon_1^2),(1-2d_4\lambda_0\\
&+\varepsilon_2^2),(1-2d_5\lambda_0+\varepsilon_2^2+\frac{\beta_Y^2b^2M_1^2}{(N_H+m)^2})\}.
\end{aligned}
$$

By virtue of the Gronwall inequality,

$$E[\|X_1(\cdot,\tau_l\wedge T)\|^2+\|X_2(\cdot,\tau_l\wedge T)\|^2+\|X_3(\cdot,\tau_l\wedge T)\|^2+\|Y_1(\cdot,\tau_l\wedge T)\|^2+\|Y_2(\cdot,\tau_l\wedge T)\|^2]\leq M_2e^{M_3T}.\quad(3.3)$$

Define

$$\lambda_l=\inf_{\|W(t)\|>l,0<t<\infty}(\|X_1(\cdot,t)\|^2+\|X_2(\cdot,t)\|^2+\|X_3(\cdot,t)\|^2+\|Y_1(\cdot,t)\|^2+\|Y_2(\cdot,t)\|^2),\ for\ any\ l>l_0.\quad(3.4)$$

Combine (3.3) and (3.4) to get

$$\lambda_lP(\tau_l\leq T)\leq M_2e^{M_3T}.\quad(3.5)$$

Since $\lim_{l\to\infty} \lambda_l = \infty$, in the above inequality, and let $l \to \infty$, we can get $P(\tau_\infty \le T) = 0$, namely,

$$P(\tau_l \ge T) = 1.$$

By (3.5), $l \to \infty$ means that

$$E[\|X_1(\cdot,T)\|^2 + \|X_2(\cdot,T)\|^2 + \|X_3(\cdot,T)\|^2 + \|Y_1(\cdot,T)\|^2 + \|Y_2(\cdot,T)\|^2] \le M_2 e^{M_3 T}.$$

The proof is thus concluded. As stated in the preceding theorem, system (2.7) possesses a unique global solution. □

Definition 3.3. *[25] A probability measure $\lambda \in \mathbb{P}(\mathcal{H})$ is called a stationary distribution for $\mathcal{W}(x,t)$, $t \ge 0$ of system (2.7) if it satisfies*

$$\lambda(\mho) = \lambda(\mathbb{P}_t\mho), t > 0.$$

Here

$$\lambda(\mho) := \int_{\mathcal{H}} \mho(\chi)\lambda(d\chi), P_t\mho(\chi) := E\mho(\mathcal{W}(x,t,\chi)), \mho \in C_b(\mathcal{H}).$$

For $\lambda_1, \lambda_2 \in P(\mathcal{H})$, define a metric on $P(\mathcal{H})$ by

$$d(\lambda_1,\lambda_2) = \sup_{\mho\in\mathcal{A}} |\int_{\mathcal{H}} \mho(\chi)\lambda_1(d\chi) - \mho(\varrho)\lambda_2(d\varrho)|,$$

where

$$\mathcal{A} := \{\mho : \mathcal{H} \to R, |\mho(\chi) - \mho(\varrho)| \le |\chi - \varrho|_{\mathcal{H}},\ \chi,\ \varrho \in \mathcal{H} \text{ and } |\mho(\cdot)| \le 1\}.$$

$P(\mathcal{H})$ is complete under the metric $d(\cdot,\cdot)$.

Theorem 3.4. *For any bounded subset B of $\mathcal{H}$, $m \ge 1$, we have*

$$(1) \lim_{t\to\infty} \sup_{\Xi,\Theta\in B} E\|\mathcal{W}(x,t,\Xi) - \mathcal{W}(x,t,\Theta)\|^m_{\mathcal{H}} = 0;$$
$$(2)\ \eta > 0,\ k > 1\ satisfying\ \eta + M_5 < 0,$$

then, there exists a unique stationary distribution.

Proof. We take into account the difference between a pair of mild solutions of system (2.7) generated by separate initial data $\Xi, \Theta \in \Omega$:

$$\zeta(\cdot,t) = \begin{pmatrix} \zeta_1(\cdot,t,\Xi,\Theta) \\ \zeta_2(\cdot,t,\Xi,\Theta) \\ \zeta_3(\cdot,t,\Xi,\Theta) \\ \zeta_4(\cdot,t,\Xi,\Theta) \\ \zeta_5(\cdot,t,\Xi,\Theta) \end{pmatrix} = \begin{pmatrix} X_1(\cdot,t,\Xi) - X_1(\cdot,t,\Theta) \\ X_2(\cdot,t,\Xi) - X_2(\cdot,t,\Theta) \\ X_3(\cdot,t,\Xi) - X_3(\cdot,t,\Theta) \\ Y_1(\cdot,t,\Xi) - Y_1(\cdot,t,\Theta) \\ Y_2(\cdot,t,\Xi) - Y_2(\cdot,t,\Theta) \end{pmatrix}, \tag{3.6}$$

with

$$\|\zeta(\cdot,t,\Xi,\Theta)\|^\kappa = \|\zeta_1(\cdot,t,\Xi,\Theta)\|^\kappa + \|\zeta_2(\cdot,t,\Xi,\Theta)\|^\kappa + \|\zeta_3(\cdot,t,\Xi,\Theta)\|^\kappa + \|\zeta_4(\cdot,t,\Xi,\Theta)\|^\kappa + \|\zeta_5(\cdot,t,\Xi,\Theta)\|^\kappa.$$

By Lemma 3.1 and Itô formula, we have

$$
\begin{aligned}
& d(\zeta^{\eta t}\|\zeta(\cdot,t,\Xi,\Theta)\|^{\kappa}) \\
= \ & \eta\zeta^{\eta t}\|\zeta(\cdot,t,\Xi,\Theta)\|^{\kappa}dt + \zeta^{\eta t}\{\kappa\|\zeta_1(\cdot,t,\Xi,\Theta)\|^{\kappa-2}\langle\zeta_1(\cdot,t,\Xi,\Theta), d_1\triangle\zeta_1(\cdot,t,\Xi,\Theta) - \mu_e\zeta_1(\cdot,t,\Xi.\Theta) \\
& - (\mu_0-\mu_e)e^{-\vartheta_1 t}\zeta_1(\cdot,t,\Xi,\Theta) - \frac{\beta_X b}{N_H+m}(X_1(\cdot,t,\Xi)Y_2(\cdot,t,\Xi) - X_1(\cdot,t,\Theta)Y_2(\cdot,t,\Theta))\rangle dt \\
& + \kappa\|\zeta_2(\cdot,t,\Xi,\Theta)\|^{\kappa-2}\langle\zeta_2(\cdot,t,\Xi,\Theta), d_2\triangle\zeta_2(\cdot,t,\Xi,\Theta) + \frac{\beta_X b}{N_H+m}(X_1(\cdot,t,\Xi)Y_2(\cdot,t,\Xi) \\
& - X_1(\cdot,t,\Theta)\cdot Y_2(\cdot,t,\Theta)) - \mu_e\zeta_2(\cdot.t,\Xi,\Theta) - (\mu_0-\mu_e)e^{-\vartheta_1 t}\zeta_2(\cdot,t,\Xi,\Theta)\rangle dt \\
& + \kappa\|\zeta_3(\cdot,t,\Xi,\Theta)\|^{\kappa-2}\langle\zeta_3(\cdot,t,\Xi,\Theta), d_3\cdot\triangle\zeta_3(\cdot,t,\Xi,\Theta) - \mu_e\zeta_3(\cdot,t,\Xi,\Theta) \\
& - (\mu_0-\mu_e)e^{-\vartheta_1 t}\zeta_3(\cdot,t,\Xi,\Theta)\rangle d_4\triangle\zeta_4(\cdot,t,\Xi,\Theta) - \nu_e\zeta_4(\cdot,t,\Xi,\Theta) - (\nu_0-\nu_e)e^{-\vartheta_2 t}\zeta_4(\cdot,t,\Xi,\Theta) \\
& - \frac{\beta_Y b}{N_H+m}(Y_1(\cdot,t,\Xi)X_2(\cdot,t,\Xi) - Y_1(\cdot,t,\Theta)X_2(\cdot,t,\Theta))\rangle dt \\
& + \kappa\|\zeta_5(\cdot,t,\Xi,\Theta)\|^{\kappa-2}\langle\zeta_5(\cdot,t,\Xi,\Theta), d_5\triangle\zeta_5(\cdot,t,\Xi,\Theta) + \frac{\beta_Y b}{N_H+m}(Y_1(\cdot,t,\Xi)\cdot X_2(\cdot,t,\Xi) \\
& - Y_1(\cdot,t,\Theta)X_2(\cdot,t,\Theta)) - \nu_e\zeta_5(\cdot.t,\Xi,\Theta) - (\nu_0-\nu_e)e^{-\vartheta_2 t}\zeta_5(\cdot,t,\Xi,\Theta)\rangle dt \\
& + \frac{1}{2}\kappa(\kappa-1)\varepsilon_1^2(t)\cdot\|\zeta_1(\cdot,t,\Xi,\Theta)\|^{k}dt + \frac{1}{2}\kappa(\kappa-1)\varepsilon_1^2(t)\|\zeta_2(\cdot,t,\Xi,\Theta)\|^{k}dt \\
& + \frac{1}{2}\kappa(\kappa-1)\varepsilon_1^2(t)\|\zeta_3(\cdot,t,\Xi,\Theta)\|^{k}dt + \frac{1}{2}\kappa(\kappa-1)\cdot\varepsilon_2^2(t)\|\zeta_4(\cdot,t,\Xi,\Theta)\|^{k}dt \\
& + \frac{1}{2}\kappa(\kappa-1)\varepsilon_2^2(t)\|\zeta_5(\cdot,t,\Xi,\Theta)\|^{k}dt + \kappa\|\zeta_1(\cdot,t,\Xi,\Theta)\|^{\kappa-2}\langle\zeta_1(\cdot,t,\Xi,\Theta), -\varepsilon_1(t)\cdot\triangle\zeta_1(\cdot,t,\Xi,\Theta)dB_1(t)\rangle \\
& + \kappa\|\zeta_2(\cdot,t,\Xi,\Theta)\|^{\kappa-2}\langle\zeta_2(\cdot,t,\Xi,\Theta), -\varepsilon_1(t)\triangle\zeta_2(\cdot,t,\Xi,\Theta)dB_1(t)\rangle \\
& + \kappa\cdot\|\zeta_3(\cdot,t,\Xi,\Theta)\|^{\kappa-2}\cdot\langle\zeta_3(\cdot,t,\Xi,\Theta), -\varepsilon_1(t)\triangle\zeta_3(\cdot,t,\Xi,\Theta)dB_1(t)\rangle\} \\
& + \kappa\|\zeta_4(\cdot,t,\Xi,\Theta)\|^{\kappa-2}\langle\zeta_4(\cdot,t,\Xi,\Theta), -\varepsilon_2(t)\triangle\zeta_4(\cdot,t,\Xi,\Theta)dB_2(t)\rangle \\
& + \kappa\|\zeta_5(\cdot,t,\Xi,\Theta)\|^{\kappa-2}\langle\zeta_5(\cdot,t,\Xi,\Theta), -\varepsilon_2(t)\triangle\zeta_5(\cdot,t,\Xi,\Theta)dB_2(t)\rangle.
\end{aligned}
$$

Next, we have a further transformation to

$$(X_1(\cdot,t,\Xi)Y_2(\cdot,t,\Xi) - X_1(\cdot,t,\Theta)Y_2(\cdot,t,\Theta))$$

and

$$(Y_1(\cdot,t,\Xi)X_2(\cdot,t,\Xi) - Y_1(\cdot,t,\Theta)X_2(\cdot,t,\Theta)),$$

and we get the following equality:

$$
\begin{aligned}
& d(\zeta^{\eta t}\|\zeta(\cdot,t,\Xi,\Theta)\|^{\kappa}) \\
= & \eta\zeta^{\eta t}\|\zeta(\cdot,t,\Xi,\Theta)\|^{\kappa}dt + \zeta^{\eta t}\{\kappa\|\zeta_1(\cdot,t,\Xi,\Theta)\|^{\kappa-2}\langle\zeta_1(\cdot,t,\Xi,\Theta), d_1\triangle\zeta_1(\cdot,t,\Xi,\Theta) - \mu_e\zeta_1(\cdot,t,\Xi.\Theta) \\
& - (\mu_0-\mu_e)e^{-\vartheta_1 t}\zeta_1(\cdot,t,\Xi,\Theta) - \frac{\beta_X b}{N_H+m}(X_1(\cdot,t,\Xi)\zeta_5(\cdot,t,\Xi,\Theta) + \zeta_1(\cdot,t,\Xi,\Theta)Y_2(\cdot,t,\Theta))\rangle dt \\
& + \kappa\|\zeta_2(\cdot,t,\Xi,\Theta)\|^{\kappa-2}\langle\zeta_2(\cdot,t,\Xi,\Theta), d_2\triangle\zeta_2(\cdot,t,\Xi,\Theta) + \frac{\beta_X b}{N_H+m}(X_1(\cdot,t,\Xi)\zeta_5(\cdot,t,\Xi,\Theta) \\
& + \zeta_1(\cdot,t,\Xi,\Theta)\cdot Y_2(\cdot,t,\Theta)) - \mu_e\zeta_2(\cdot.t,\Xi,\Theta) - (\mu_0-\mu_e)e^{-\vartheta_1 t}\zeta_2(\cdot,t,\Xi,\Theta)\rangle dt \\
& + \kappa\|\zeta_3(\cdot,t,\Xi,\Theta)\|^{\kappa-2}\langle\zeta_3(\cdot,t,\Xi,\Theta), d_3\triangle\zeta_3(\cdot,t,\Xi,\Theta) - \mu_e\zeta_3(\cdot,t,\Xi,\Theta) - (\mu_0-\mu_e)e^{-\vartheta_1 t}\zeta_3(\cdot,t,\Xi,\Theta)\rangle dt \\
& + \kappa\|\zeta_4(\cdot,t,\Xi,\Theta)\|^{\kappa-2}\langle\zeta_4(\cdot,t,\Xi,\Theta), d_4\triangle\zeta_4(\cdot,t,\Xi,\Theta) - \nu_e\zeta_4(\cdot,t,\Xi,\Theta)
\end{aligned}
$$

$$
\begin{aligned}
&-(\nu_0-\nu_e)e^{-\vartheta_2 t}\zeta_4(\cdot,t,\Xi,\Theta)-\frac{\beta_Y b}{N_H+m}(Y_1(\cdot,t,\Xi)\zeta_2(\cdot,t,\Xi,\Theta)+\zeta_4(\cdot,t,\Xi,\Theta)X_2(\cdot,t,\Theta))\rangle dt\\
&+\kappa\|\zeta_5(\cdot,t,\Xi,\Theta)\|^{\kappa-2}\langle\zeta_5(\cdot,t,\Xi,\Theta),d_5\triangle\zeta_5(\cdot,t,\Xi,\Theta)+\frac{\beta_Y b}{N_H+m}\cdot(Y_1(\cdot,t,\Xi)\zeta_2(\cdot,t,\Xi,\Theta)\\
&+\zeta_4(\cdot,t,\Xi,\Theta)X_2(\cdot,t,\Theta))-\nu_e\zeta_5(\cdot.t,\Xi,\Theta)-(\nu_0-\nu_e)e^{-\vartheta_2 t}\zeta_5(\cdot,t,\Xi,\Theta)\rangle dt\\
&+\frac{1}{2}\kappa(\kappa-1)\varepsilon_1^2(t)\|\zeta_1(\cdot,t,\Xi,\Theta)\|^k dt+\frac{1}{2}\kappa(\kappa-1)\varepsilon_1^2(t)\|\zeta_2(\cdot,t,\Xi,\Theta)\|^k dt\\
&+\frac{1}{2}\kappa(\kappa-1)\varepsilon_1^2(t)\|\zeta_3(\cdot,t,\Xi,\Theta)\|^k dt+\frac{1}{2}\kappa(\kappa-1)\varepsilon_2^2(t)\|\zeta_4(\cdot,t,\Xi,\Theta)\|^k dt+\frac{1}{2}\kappa(\kappa-1)\varepsilon_2^2(t)\|\zeta_5(\cdot,t,\Xi,\Theta)\|^k dt\\
&+\kappa\|\zeta_1(\cdot,t,\Xi,\Theta)\|^{\kappa-2}\langle\zeta_1(\cdot,t,\Xi,\Theta),-\varepsilon_1(t)\triangle\zeta_1(\cdot,t,\Xi,\Theta)dB_1(t)\rangle\\
&+\kappa\|\zeta_2(\cdot,t,\Xi,\Theta)\|^{\kappa-2}\cdot\langle\zeta_2(\cdot,t,\Xi,\Theta),-\varepsilon_1(t)\triangle\zeta_2(\cdot,t,\Xi,\Theta)dB_1(t)\rangle\\
&+\kappa\|\zeta_3(\cdot,t,\Xi,\Theta)\|^{\kappa-2}\langle\zeta_3(\cdot,t,\Xi,\Theta),-\varepsilon_1(t)\triangle\zeta_3(\cdot,t,\Xi,\Theta)dB_1(t)\rangle\}\\
&+\kappa\|\zeta_4(\cdot,t,\Xi,\Theta)\|^{\kappa-2}\langle\zeta_4(\cdot,t,\Xi,\Theta),-\varepsilon_2(t)\triangle\zeta_4(\cdot,t,\Xi,\Theta)dB_2(t)\rangle\\
&+\kappa\|\zeta_5(\cdot,t,\Xi,\Theta)\|^{\kappa-2}\cdot\langle\zeta_5(\cdot,t,\Xi,\Theta),-\varepsilon_2(t)\triangle\zeta_5(\cdot,t,\Xi,\Theta)dB_2(t)\rangle.
\end{aligned}
$$

Upon rearranging terms in the foregoing equality, we arrive at

$$
\begin{aligned}
&d(\zeta^{\eta t}\|\zeta(\cdot,t,\Xi,\Theta)\|^{\kappa})\\
\le&\eta\zeta^{\eta t}\|\zeta(\cdot,t,\Xi,\Theta)\|^{\kappa}dt+\kappa\zeta^{\eta t}\{-\|\zeta_1(\cdot,t,\Xi,\Theta)\|^{k-2}\langle\nabla\zeta_1(\cdot,t,\Xi,\Theta),d_1\nabla\zeta_1(\cdot,t,\Xi,\Theta)\rangle\\
&-\|\zeta_2(\cdot,t,\Xi,\Theta)\|^{k-2}\langle\nabla\zeta_2(\cdot,t,\Xi,\Theta),d_2\nabla\zeta_2(\cdot,t,\Xi,\Theta)\rangle\\
&+\frac{\beta_X b}{N_H+m}\|\zeta_2(\cdot,t,\Xi,\Theta)\|^{k-2}\langle\zeta_2(\cdot,t,\Xi,\Theta),X_1(\cdot,t,\Xi)\zeta_5(\cdot,t,\Xi,\Theta)+\zeta_1(\cdot,t,\Xi,\Theta)Y_2(\cdot,t,\Theta)\rangle\\
&-\|\zeta_3(\cdot,t,\Xi,\Theta)\|^{k-2}\langle\nabla\zeta_3(\cdot,t,\Xi,\Theta),d_3\nabla\zeta_3(\cdot,t,\Xi,\Theta)\rangle-\|\zeta_4(\cdot,t,\Xi,\Theta)\|^{k-2}\langle\nabla\zeta_4(\cdot,t,\Xi,\Theta),d_4\nabla\zeta_4(\cdot,t,\Xi,\Theta)\rangle\\
&-\|\zeta_5(\cdot,t,\Xi,\Theta)\|^{k-2}\langle\nabla\zeta_5(\cdot,t,\Xi,\Theta),d_5\nabla\zeta_5(\cdot,t,\Xi,\Theta)\rangle\\
&+\frac{\beta_Y b}{N_H+m}\cdot\|\zeta_5(\cdot,t,\Xi,\Theta)\|^{k-2}\langle\zeta_5(\cdot,t,\Xi,\Theta),Y_1(\cdot,t,\Xi)\zeta_2(\cdot,t,\Xi,\Theta)+\zeta_4(\cdot,t,\Xi,\Theta)X_2(\cdot,t,\Theta)\rangle\}dt\\
&+\frac{1}{2}\kappa(\kappa-1)\varepsilon_1^2(t)\|\zeta_1(\cdot,t,\Xi,\Theta)\|^k dt+\frac{1}{2}\kappa(\kappa-1)\varepsilon_1^2(t)\|\zeta_2(\cdot,t,\Xi,\Theta)\|^k dt+\frac{1}{2}\kappa(\kappa-1)\varepsilon_1^2(t)\|\zeta_3(\cdot,t,\Xi,\Theta)\|^K dt\\
&+\frac{1}{2}\kappa(\kappa-1)\varepsilon_2^2(t)\|\zeta_4(\cdot,t,\Xi,\Theta)\|^k dt+\frac{1}{2}\kappa(\kappa-1)\varepsilon_2^2(t)\|\zeta_5(\cdot,t,\Xi,\Theta)\|^k dt\\
\le&\eta\zeta^{\eta t}\|\zeta(\cdot,t,\Xi,\Theta)\|^{\kappa}dt+\kappa\zeta^{\eta t}\{-d_1\lambda_0\|\zeta_1(\cdot,t,\Xi,\Theta)\|^k-d_2\lambda_0\|\zeta_2(\cdot,t,\Xi,\Theta)\|^k\\
&+\frac{\beta_X bM_1}{N_H+m}\|\zeta_2(\cdot,t,\Xi,\Theta)\|^{k-1}(\|\zeta_5(\cdot,t,\Xi,\Theta)\|+\|\zeta_1(\cdot,t,\Xi,\Theta)\|)-d_3\lambda_0\|\zeta_3(\cdot,t,\Xi,\Theta)\|^k-d_4\lambda_0\|\zeta_4(\cdot,t,\Xi,\Theta)\|^k\\
&+\frac{\beta_Y bM_1}{N_H+m}\|\zeta_5(\cdot,t,\Xi,\Theta)\|^{k-1}(\|\zeta_2(\cdot,t,\Xi,\Theta)\|+\|\zeta_4(\cdot,t,\Xi,\Theta)\|)\}dt+\frac{1}{2}\kappa(\kappa-1)\varepsilon_1^2(t)\|\zeta_1(\cdot,t,\Xi,\Theta)\|^k dt\\
&+\frac{1}{2}\kappa(\kappa-1)\varepsilon_1^2(t)\|\zeta_2(\cdot,t,\Xi,\Theta)\|^k dt+\frac{1}{2}\kappa(\kappa-1)\varepsilon_1^2(t)\|\zeta_3(\cdot,t,\Xi,\Theta)\|^K dt\\
&+\frac{1}{2}\kappa(\kappa-1)\varepsilon_2^2(t)\|\zeta_4(\cdot,t,\Xi,\Theta)\|^k dt+\frac{1}{2}\kappa(\kappa-1)\varepsilon_2^2(t)\|\zeta_5(\cdot,t,\Xi,\Theta)\|^k dt.
\end{aligned}
$$

After integrating the foregoing inequality and computing expectations, an application of Young's inequality yields

$$
\begin{aligned}
& E[\zeta^{\eta t}||\zeta(\cdot, t, \Xi, \Theta)||^{\kappa}] \\
\leq \ & ||\zeta(\cdot, 0, \Xi, \Theta)||^{\kappa} + E\int_0^t \zeta^{\eta s}\{\eta||\zeta(\cdot, s, \Xi, \Theta)||^{\kappa} \\
& + (-kd_1\lambda_0 + \frac{(\beta_X bM_1)^k}{(N_H + m)^k} + \frac{1}{2}\kappa(\kappa-1)\varepsilon_1^2)||\zeta_1(\cdot, t, \Xi, \Theta)||^{\kappa} \\
& + (-kd_2\lambda_0 + 2k - 2 + \frac{(\beta_Y bM_1)^k}{(N_H + m)^k} + \frac{1}{2}\kappa(\kappa-1)\varepsilon_1^2)||\zeta_2(\cdot, t, \Xi, \Theta)||^{\kappa} \\
& + (-kd_3\lambda_0 + \frac{1}{2}\kappa(\kappa-1)\varepsilon_1^2)||\zeta_3(\cdot, t, \Xi, \Theta)||^{\kappa} + (-kd_4\lambda_0 + \frac{(\beta_Y bM_1)^k}{(N_H + m)^k} + \frac{1}{2}\kappa(\kappa-1)\varepsilon_2^2)||\zeta_4(\cdot, t, \Xi, \Theta)||^{\kappa} \\
& + (-kd_5\lambda_0 + 2k - 2 + \frac{(\beta_X bM_1)^k}{(N_H + m)^k} + \frac{1}{2}\kappa(\kappa-1)\varepsilon_2^2)||\zeta_5(\cdot, t, \Xi, \Theta)||^{\kappa}.
\end{aligned}
$$

Subsequently, we take the supremum of each side of the inequality presented above.

$$
E\sup_{0\leq t\leq T}[\zeta^{\eta t}||\zeta(\cdot, t, \Xi, \Theta)||^{\kappa}] \leq ||\zeta(\cdot, 0, \Xi, \Theta)||^{\kappa} + E\sup_{0\leq t\leq T}(\eta + M_5)\int_0^t \zeta^{\eta s}||\zeta(\cdot, s, \Xi, \Theta)||^{\kappa} ds. \tag{3.7}
$$

Here

$$
\begin{aligned}
M_5 = \ & max\{-kd_1\lambda_0 + \frac{(\beta_X bM_1)^k}{(N_H + m)^k} + \frac{1}{2}\kappa(\kappa-1)\varepsilon_1^2, -kd_2\lambda_0 + 2k - 2 + \frac{(\beta_Y bM_1)^k}{(N_H + m)^k} + \frac{1}{2}\kappa(\kappa-1)\varepsilon_1^2 \\
& - kd_3\lambda_0 + \frac{1}{2}\kappa(\kappa-1)\varepsilon_1^2, -kd_4\lambda_0 + \frac{(\beta_Y bM_1)^k}{(N_H + m)^k} + \frac{1}{2}\kappa(\kappa-1)\varepsilon_2^2 \\
& - kd_5\lambda_0 + 2k - 2 + \frac{(\beta_X bM_1)^k}{(N_H + m)^k} + \frac{1}{2}\kappa(\kappa-1)\varepsilon_2^2\}.
\end{aligned}
$$

Based on the Gronwall inequality, we obtain

$$
||\zeta(\cdot, t, \Xi, \Theta)||^{\kappa} \leq ||\zeta(\cdot, 0, \Xi, \Theta)||^{\kappa} e^{(\eta + M_5)t}.
$$

As $\eta + M_5 < 0$, the exponential moment decay property of the solution can be rigorously derived, namely

$$
\lim_{t\to\infty} E||\zeta(\cdot, t, \Xi, \Theta)||^{\kappa} = 0.
$$

In addition, let $\bar{\lambda}$ denote an alternative stationary distribution associated with $\mathcal{W}(x, t)$. For a suitable constant $M > 0$, we obtain the following relation:

$$
|\lambda(\mho) - \bar{\lambda}(\mho)| \leq \int_{\mathcal{H}\times\mathcal{H}} |P_t\mho(\Xi) - P_t\mho(\Theta)|\lambda(d\Xi)\bar{\lambda}(d\Theta) \leq M\zeta^{-\eta t}.
$$

When $t \to \infty$, we conclude that the stationary distribution is unique. □

4. Numerical simulations

In this section, we conduct numerical simulations from two cases, where $\vartheta_1 = \vartheta_2 = 2$, $\varepsilon_1 = \varepsilon_2 = 0.01$. Case 1: $\mu_0 > \mu_e, \nu_0 > \nu_e$. Case 2: $\mu_0 < \mu_e, \nu_0 < \nu_e$. For case 1, let $\mu_0 = 0.125$, $\mu_e = 0.1$, $\nu_0 = 0.8$, and $\nu_e = 0.6$, and other parameters remain consistent with those in Reference [22]. The numerical simulation results are shown in Figure 1. All the variables in the model tend to stabilize over time, where, the evolution path of X_1 for case 1 is depicted in Figure 1(a), the evolution path of X_2 for case 1 is depicted in Figure 1(b), the evolution path of X_3 for case 1 is depicted in Figure 1(c), the evolution path of Y_1 for case 1 is depicted in Figure 1(d), and the evolution path of Y_2 for case 1 is depicted in Figure 1(e). Figure 2 is the histogram corresponding to Figure 1, and the all variables show a normal distribution (all the results were compared with Figure 2 in Reference [22]. For the sake of comparison, Figure 2 of Reference [22] is incorporated into the present article, namely Figure 3, where, $S_H,\ I_H, R_H,\ S_V,$ and I_V correspond respectively to $X_1,\ X_2,\ X_3,\ Y_1,$ and Y_2).

For case 2, let $\mu_0 = 0.125$, $\mu_e = 0.3$, $\nu_0 = 0.8$, and $\nu_e = 0.9$, and other parameters remain consistent with those in Reference [22]. The numerical simulation results are shown in Figure 4, where the evolution path of X_1 for case 2 is depicted in Figure 4(a), the evolution path of X_2 for case 2 is depicted in Figure 4(b), the evolution path of X_3 for case 2 is depicted in Figure 4(c), the evolution path of Y_1 for case 2 is depicted in Figure 4(d), and the evolution path of Y_2 for case 2 is depicted in Figure 4(e). We can see that the numbers of susceptible humans and susceptible mosquitoes are steadily distributed over time, and the infected population gradually reaches region 0. Figure 5 is the histogram corresponding to Figure 4, and the susceptible group shows a normal distribution (all the results were compared with Figure 2 in Reference [22]. For the sake of comparison, Figure 2 of Reference [22] is incorporated into the present article, namely Figure 3, where, $S_H,\ I_H, R_H,\ S_V,$ and I_V correspond respectively to $X_1,\ X_2,\ X_3,\ Y_1,\ and\ Y_2$).

From the above analysis, the following conclusion can be drawn: when $\mu_0 < \mu_e, \nu_0 < \nu_e$, we cannot well observe that the infected population follows a normal distribution; conversely, as $\mu_0 > \mu_e, \nu_0 > \nu_e$, the system as a whole well presents a normal distribution. Meanwhile, compared with Brownian motion [22], the mean-reverting white noise tends to a normal distribution better.

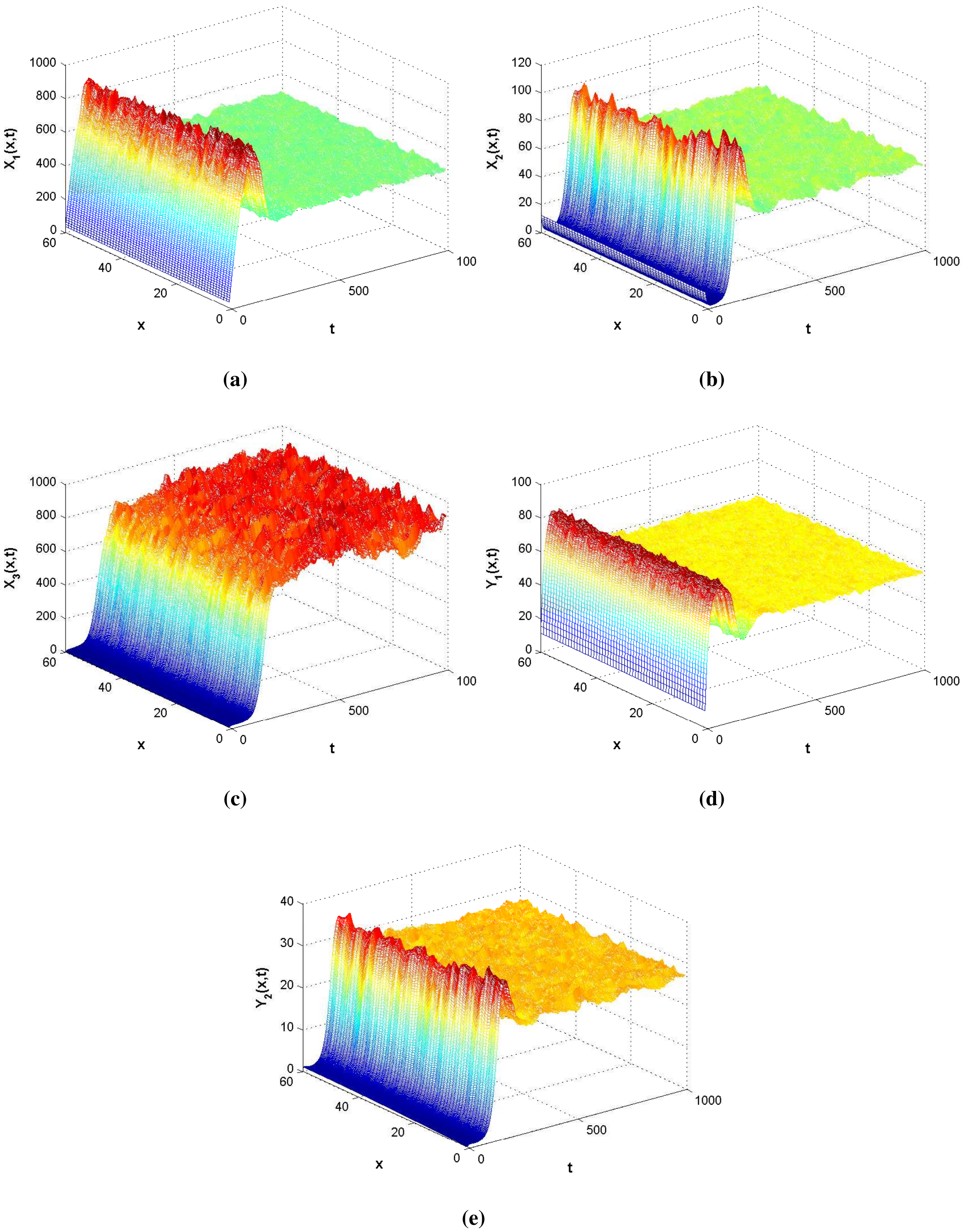


Figure 1. The evolution path for case 1.

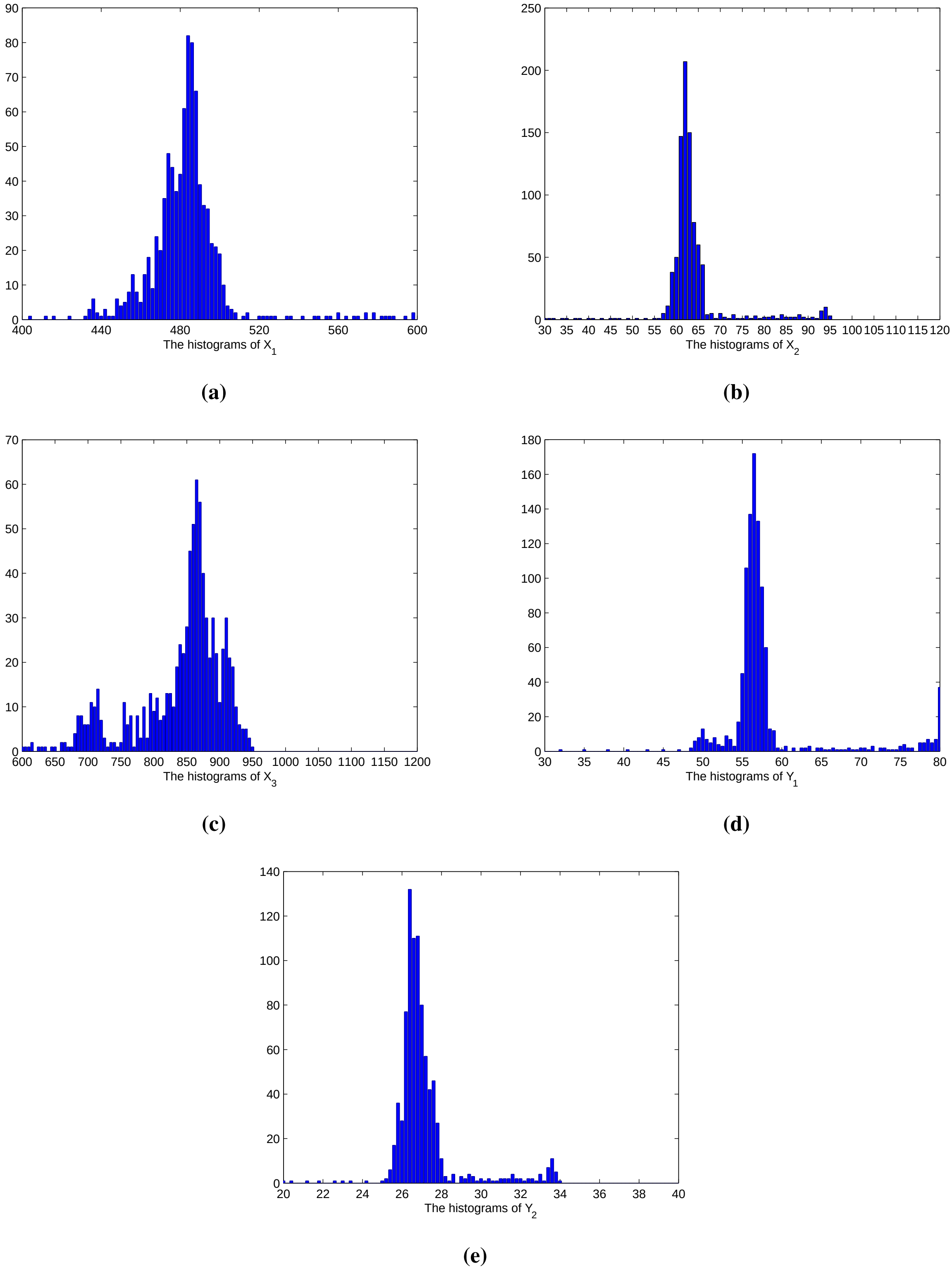
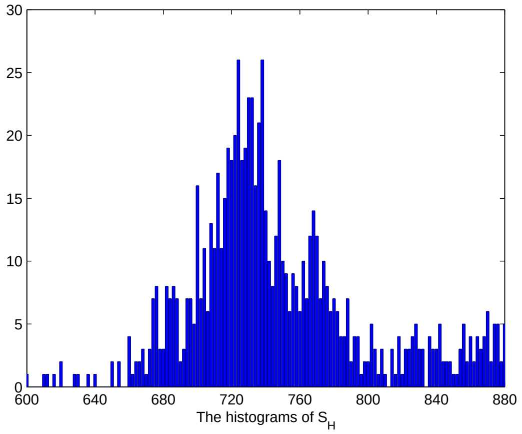
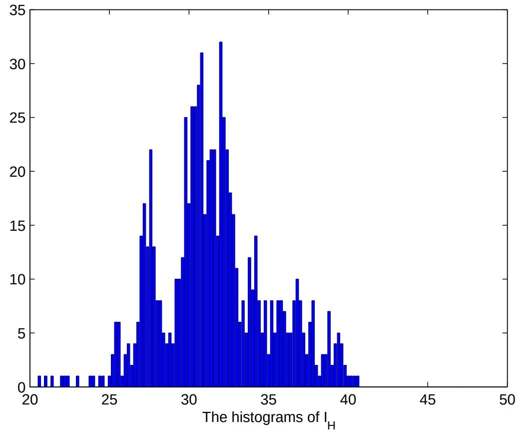


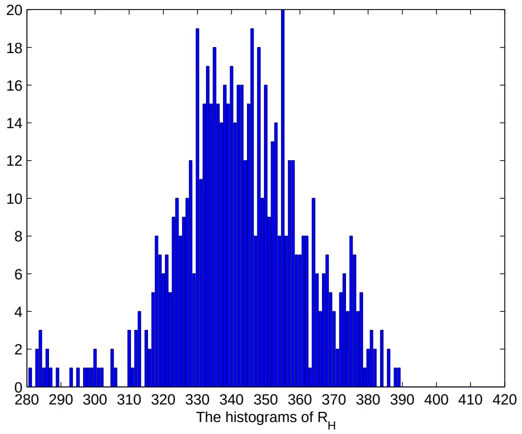
Figure 2. The histograms for case 1.



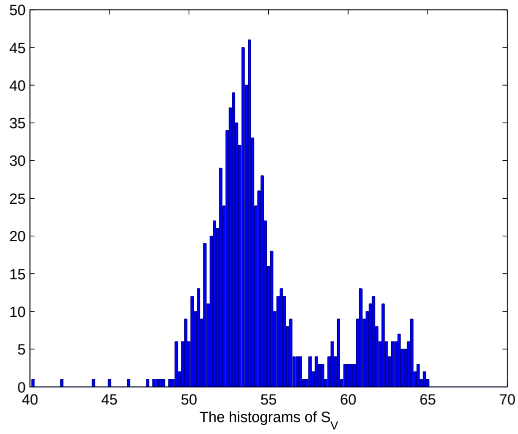
(a)



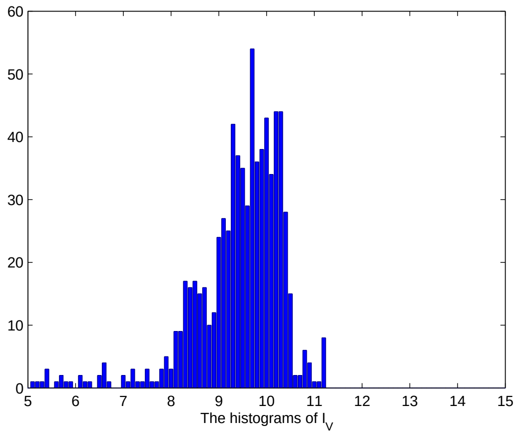
(b)



(c)

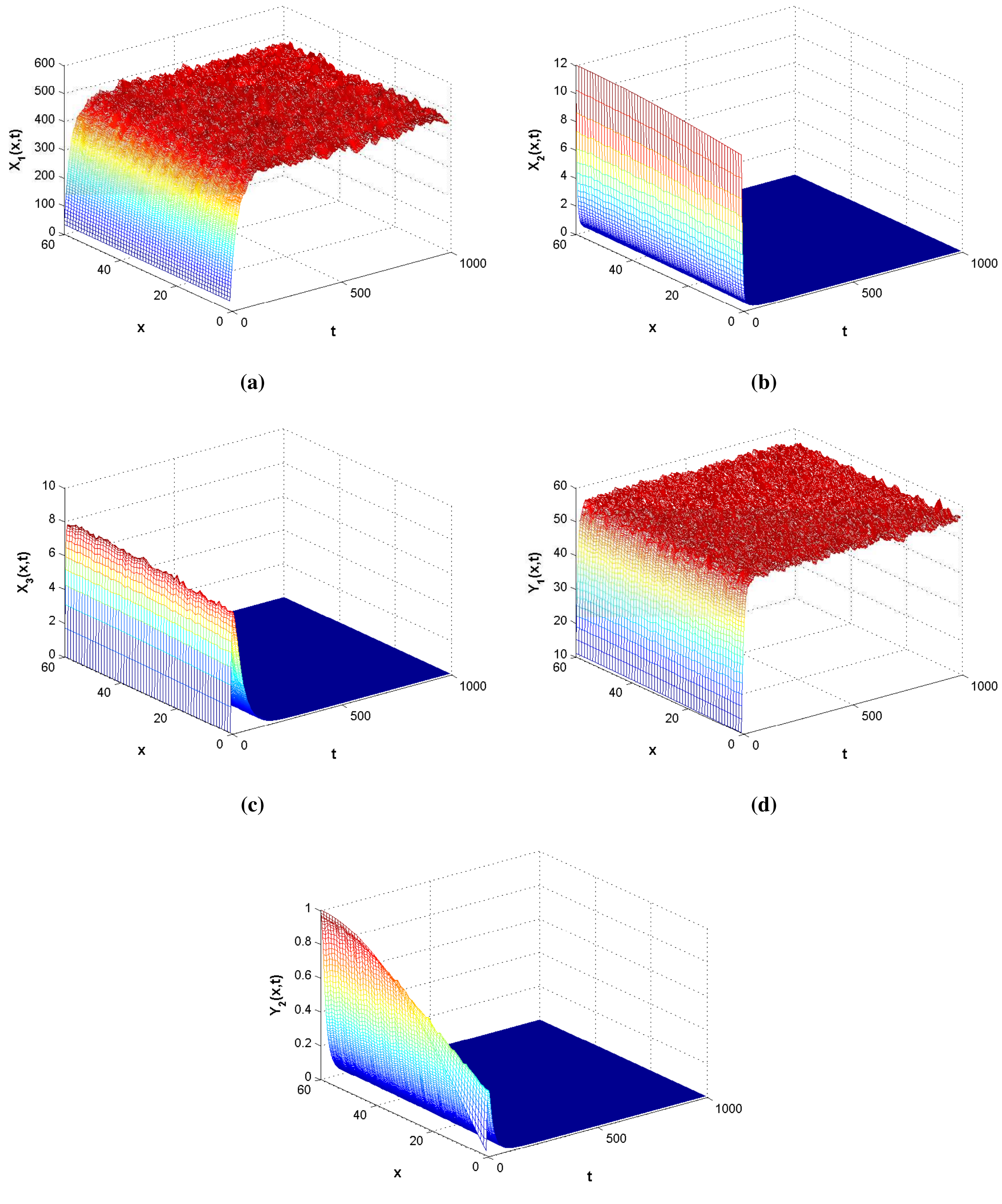


(d)



(e)

Figure 3. The histograms for Brownian motion [22, Figure 2].



(a) (b) (c) (d) (e)

Figure 4. The evolution path for case 2.

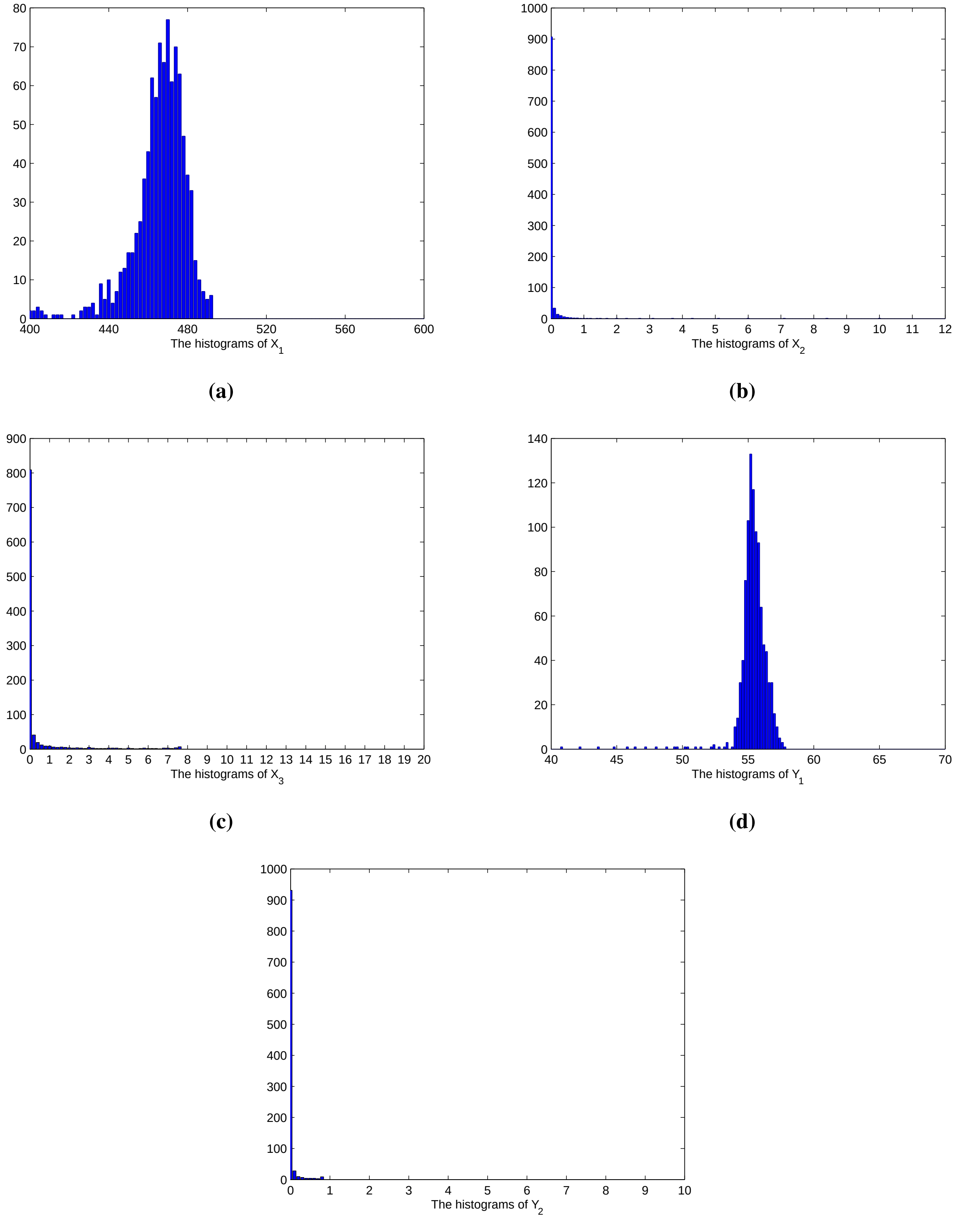


Figure 5. The histograms for case 2.

5. Conclusions

Dengue fever is prevalent at a high rate every year. Since June 13, 2026, the dengue fever epidemic in Brazil has remained at a high level, with a total of 609,000 cases reported, ranking first in the world in terms of the number of cases. As of June 5, 2026, Sri Lanka has reported a total of 36,168 confirmed cases of dengue fever, with 20 deaths (including three children), resulting in a fatality rate of 0.06%. Compared with the same period in 2025, the number of cases has risen significantly by 30% to 40%, with approximately 10,000 new cases. The epidemic is showing a trend of early outbreak and rapid spread. As of May 31, 2026, Samoa reported 157 new dengue fever cases in a single week, a slight decrease of 6% compared to the previous week. However, the epidemic remains highly active, and both DENV-1 and DENV-2 serotypes coexist, with children accounting for over 70%. Therefore, it is of practical significance to understand the dynamic behavior of disease transmission through mathematical models.

In this paper, to better understand the stationary distribution of the dengue fever model, we introduced a dengue fever model with the mean-reverting white noise and compared the effects of Brownian motion and the mean-reverting white noise on the stationary distribution. First, by virtue of the stochastic comparison theorem and the nonnegative semimartingale convergence theorem, the boundedness of the solution was proved. Moreover, on the basis of the Lyapunov function, we proved the uniqueness of the existence of the solution. Next, the stationary distribution of the solution was obtained through the constructor. Finally, the stationary distribution of the solution was verified through numerical simulation. Meanwhile, on the premise of maintaining consistent parameters, the effects of the mean-reverting white noise and Brownian motion on the dengue fever model were compared. We can observe that the model with the mean-reverting white noise better achieves a stationary distribution, and the distribution of the histograms are more concentrated.

In the future, we will consider other dynamic behaviors with Ornstein-Uhlenbeck processes. Meanwhile, we will also attempt to introduce Lévy noise and Markov chains into the model to explore its dynamic behaviors.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that he has no conflict of interest.

References

1. D. Li, W. Li, S. Hu, Temperature effect on the periodic dengue epidemic model and the optimal control strategy, *Int. J. Biomath.*, 2025, 2550152. https://doi.org/10.1142/S1793524525501529
2. E. Aprianti, S. Sonia, Stability and sensitivity analysis of parameters in the SEIR-ASEI model for the transmission of dengue fever, *Jambura J. Biomath.*, **6** (2025), 340–349. https://doi.org/10.37905/jjbm.v6i4.32754
3. M. Zhu, Z. Lin, L. Zhang, Spatial-temporal risk index and transmission of a nonlocal dengue model, *Nonlinear Anal. Real World Appl.*, **53** (2020), 103076. https://doi.org/10.1016/j.nonrwa.2019.103076
4. W. Guo, B. Zheng, J. Yu, Modeling the dengue control dynamics based on a delay stochastic differential system, *Comp. Appl. Math.,* **43** (2024), 166. https://doi.org/10.1007/s40314-024-02674-x
5. S. Bounouiga, B. Basti, N. Benhamidouche, Insights into dengue fever dynamics through advanced mathematical modeling and simulation, *J. Math. Biol.,* **92** (2026), 56. https://doi.org/10.1007/s00285-026-02367-5
6. Y. Cheng, R. Cheng, T. Xu, X. Tan, Y. Bai, Dengue fever prediction based on meteorological features and deep learning models, *Infect. Dis. Model.*, **11** (2026), 683–700. https://doi.org/10.1016/j.idm.2025.12.010
7. D. Li, W. Li, J. Zhou, Analyzing dengue fever with constrained medical resources: A dynamical systems approach, *Chaos*, **36** (2026), 023103. https://doi.org/10.1063/5.0308575
8. X. Chen, P. Moraga, Dengue forecasting and outbreak detection in Brazil using LSTM: Integrating human mobility and climate factors, *Infect. Dis. Model.*, **11** (2026), 338–354. https://doi.org/10.1016/j.idm.2025.11.002
9. S. Guo, J. Lu, Z. Zheng, P. D. N. Silva, A dengue fever epidemiological model with vaccination and vaccine inefficacy, *Math. Meth. Appl. Sci.*, **48** (2025), 14785–14797. https://doi.org/10.1002/mma.11213
10. A. Abdelrazec, J. Blair, C. Shan, H. Zhu, Modeling the spread and control of dengue with limited public health resources, *Math Biosci.*, **271** (2016), 136–145. https://doi.org/10.1016/j.mbs.2015.11.004
11. H. S. Rodrigues, M. T. T. Monteiro, D. F. M. Torres, Seasonality effects on dengue: Basic reproduction number, sensitivity analysis and optimal control, *Math. Meth. Appl. Sci.*, **39** (2016), 4671–4679. https://doi.org/10.1002/mma.3319
12. E. Allen, Environmental variability and mean-reverting processes, *Discrete Contin. Dyn. Syst. Ser. B*, **21** (2016), 2073–2089. https://doi.org/10.3934/dcdsb.2016037
13. Y. Hao, Y. Luo, J. Huang, L. Zhang, Z. Teng, Analysis of a stochastic HIV/AIDS model with commercial heterosexual activity and Ornstein-Uhlenbeck process, *Math. Comput. Simulation*, **234** (2025), 50–72. https://doi.org/10.1016/j.matcom.2025.02.020
14. H. Cao, X. Liu, L. Nie, Extinction and stationary distribution of a novel SIRS epidemic model with general incidence rate and Ornstein-Uhlenbeck process, *Adv. Cont. Discr. Mod.*, **2024** (2024), 24. https://doi.org/10.1186/s13662-024-03821-8

15. F. Wu, X. Mao, K. Chen, A highly sensitive mean-reverting process in finance and the Euler-Maruyama approximations, *J. Math. Anal. Appl.*, **348** (2008), 540–554. https://doi.org/10.1016/j.jmaa.2008.07.069
16. Y. Jia, K. Zhang, J. Xie, Y. Sun, L. Hong, Z. Wang, Option pricing formulas of uncertain mean-reverting stock model with symmetry analysis, *Symmetry*, **17** (2025), 1830. https://doi.org/10.3390/sym17111830
17. Y. Imamura, T. Kato, R. Miura, J. Yen, Occupation time, quantile and rank of the Ornstein-Uhlenbeck process and their applications to mathematical finance, *Front. Math. Finance*, **5** (2025), 121–139. https://doi.org/10.3934/fmf.2025007
18. W. Wang, Y. Cai, Z. Ding, Z. Gui, A stochastic differential equation SIS epidemic model incorporating Ornstein-Uhlenbeck process, *Phys. A*, **509** (2018), 921–936. https://doi.org/10.1016/j.physa.2018.06.099
19. Y. Sun, C. Liu, L. Cheung, Weak persistence and extinction of a stochastic epidemic model with distributed delay and Ornstein-Uhlenbeck process, *Adv. Cont. Discr. Mod.*, **2025** (2025), 114. https://doi.org/10.1186/s13662-025-03972-2
20. Q. Liu, Q. Chen, Threshold of a stochastic SIQRS epidemic model with nonlinear incidence rate and lognorma Ornstein-Uhlenbeck process, *Chaos*, **36** (2026), 033132. https://doi.org/10.1063/5.0307391
21. Y. Hao, Y. Luo, J. Huang, L. Zhang, Z. Teng, Dynamics of a stochastic reaction-diffusion Kawasaki disease model with Ornstein-Uhlenbeck process, *Nonlinear Dyn.*, **113** (2025), 15619–15643. https://doi.org/ 10.1007/s11071-025-10886-x
22. K. Chang, Q. Zhang, H. Yuan, Stationary distribution and control strategy of a stochastic dengue model with spatial diffusion, *J. Appl. Anal. Comput.*, **12** (2022), 153–178. https://doi.org/10.11948/20210094
23. X. Mao, *Stochastic differential equations and applications*, 2 Eds., Woodhead Publishing, 2008.
24. K. Liu, X. Mao, Exponential stability of non-linear stochastic evolution equations, *Stochastic Process. Appl.*, **78** (1998), 173–193. https://doi.org/10.1016/S0304-4149(98)00048-9
25. K. Liu, Stationary distributions of second order stochastic evolution equations with memory in Hilbert spaces, *Stochastic Process. Appl.*, **130** (2020), 366–393. https://doi.org/10.1016/j.spa.2019.03.015