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Research article

A_α-spectral radius and path-factor covered graphs

Sizhong Zhou[1,*], **Hongxia Liu**[2] **and Qiuxiang Bian**[1]

[1] School of Science, Jiangsu University of Science and Technology, Zhenjiang, Jiangsu 212100, China

[2] School of Mathematics and Information Science, Yantai University, Yantai, Shandong 264005, China

* **Correspondence:** Email: zsz_cumt@163.com.

Abstract: Let $\alpha \in [0, 1)$, and let G be a connected graph of order n with $n \geq f(\alpha)$, where $f(\alpha) = 14$ for $\alpha \in [0, \frac{1}{2}]$, $f(\alpha) = 17$ for $\alpha \in (\frac{1}{2}, \frac{2}{3}]$, $f(\alpha) = 20$ for $\alpha \in (\frac{2}{3}, \frac{3}{4}]$ and $f(\alpha) = \frac{5}{1-\alpha} + 1$ for $\alpha \in (\frac{3}{4}, 1)$. A path factor is a spanning subgraph F of G such that every component of F is a path with at least two vertices. Let $k \geq 2$ be an integer. A $P_{\geq k}$-factor means a path-factor with each component being a path of order at least k. A graph G is called a $P_{\geq k}$-factor covered graph if G has a $P_{\geq k}$-factor containing e for any $e \in E(G)$. Let $A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G)$, where $D(G)$ denotes the diagonal matrix of vertex degrees of G and $A(G)$ denotes the adjacency matrix of G. The largest eigenvalue of $A_\alpha(G)$ is called the A_α-spectral radius of G, which is denoted by $\rho_\alpha(G)$. In this paper, it is proved that G is a $P_{\geq 2}$-factor covered graph if $\rho_\alpha(G) > \eta(n)$, where $\eta(n)$ is the largest root of $x^3 - ((\alpha+1)n+\alpha-4)x^2 + (\alpha n^2 + (\alpha^2 - 2\alpha - 1)n - 2\alpha + 1)x - \alpha^2 n^2 + (5\alpha^2 - 3\alpha + 2)n - 10\alpha^2 + 15\alpha - 8 = 0$. Furthermore, we provide a graph to show that the bound on A_α-spectral radius is optimal.

Keywords: graph; A_α-spectral radius; $P_{\geq 2}$-factor; $P_{\geq 2}$-factor covered graph
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1. Introduction

In this paper, we only consider finite and undirected graphs which have neither loops nor multiple edges. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ is the vertex set of G and $E(G)$ is the edge set of G. The order of G is denoted by $|V(G)| = n$. A graph G is called trivial if $|V(G)| = 1$. Let $i(G)$ denote the number of isolated vertices in G. For any $v \in V(G)$, the neighborhood $N_G(v)$ of v in G is defined by $\{u \in V(G) : uv \in E(G)\}$. The cardinality of $N_G(v)$ is called the degree of v in G, written as $d_G(v)$. For a vertex subset S of G, the induced subgraph induced by S, denoted by $G[S]$, is the subgraph of G whose vertex set is S and whose edge set consists of all edges of G which have both ends in S. We write $G-S$ for $G[V(G)\setminus S]$. A subset $S\subseteq V(G)$ is called an independent set if $G[S]$ has no edges. The complement of a graph G is a graph $\overline{G}$ with the same vertex set as G, in which any two distinct vertices are adjacent if and only if they are nonadjacent in G. Let P_n and K_n denote the path and the complete graph of order n, respectively. Let G_1 and G_2 be two disjoint graphs. The union of G_1 and G_2, denoted by $G_1\cup G_2$, is the graph with vertex set $V(G_1)\cup V(G_2)$ and edge set $E(G_1)\cup E(G_2)$. The join of G_1 and G_2, denoted by $G_1\vee G_2$, is the graph with vertex set $V(G_1)\cup V(G_2)$ and edge set $E(G_1)\cup E(G_2)\cup\{uv: u\in V(G_1), v\in V(G_2)\}$. Let $k\geq 3$ be an integer. The sequential join of graphs $G_1, G_2, \cdots, G_k$, denoted by $G_1\vee G_2\vee\cdots\vee G_k$, is the graph with vertex set $V(G_1)\cup V(G_2)\cup\cdots\cup V(G_k)$ and edge set $E(G_1)\cup E(G_2)\cup\cdots\cup E(G_k)\cup\{e=x_ix_{i+1}: x_i\in V(G_i), x_{i+1}\in V(G_{i+1}), 1\leq i\leq k-1\}$.

Given a graph G with vertex set $V(G)=\{v_1, v_2, \cdots, v_n\}$, the adjacency matrix of G is defined as $A(G)=(a_{ij})$, where $a_{ij}=1$ if $v_iv_j\in E(G)$, and $a_{ij}=0$ otherwise. Let $D(G)$ denote the diagonal matrix of vertex degrees of G. The signless Laplacian matrix $Q(G)$ of G is defined by $Q(G)=D(G)+A(G)$. For any $\alpha\in[0,1)$, Nikiforov [12] defined the A_α-matrix of G as

$$A_\alpha(G)=\alpha D(G)+(1-\alpha)A(G).$$

Obviously, $A_0(G)=A(G)$ and $A_{\frac{1}{2}}(G)=\frac{1}{2}Q(G)$. Thus, $A_\alpha(G)$ generalizes both the adjacency matrix and the signless Laplacian matrix of G. Note that $A_\alpha(G)$ is a real symmetric nonnegative matrix. Hence, the eigenvalues of $A_\alpha(G)$ are real. The largest eigenvalue of $A_\alpha(G)$ is called the A_α-spectral radius of G, which is denoted by $\rho_\alpha(G)$. More contents on A_α-matrix can be found in [1, 8, 13, 16, 24, 37].

A subset $M\subseteq E(G)$ is called a matching if any two edges of M have no common vertices. M is called a perfect matching if each vertex of G is incident to exactly one edge of M. A path factor is a spanning subgraph F of G such that every component of F is a path with at least two vertices. A $P_{\geq k}$-factor means a path-factor with every component having order at least k, where $k\geq 2$ is an integer. In fact, a graph G has a perfect matching if and only if G has a $\{P_2\}$-factor. A graph G is called a $P_{\geq k}$-factor covered graph if G has a $P_{\geq k}$-factor containing e for any $e\in E(G)$.

Suil [14] obtained an adjacency spectral radius condition for the existence of a perfect matching in a graph. Liu, Pan and Li [9] established a relationship between a perfect matching and a signless Laplacian spectral radius of a graph. Zhang and Lin [25] proved an upper bound for the distance spectral radius to guarantee the existence of a perfect matching in a graph. Zhao, Huang and Wang [26] presented an A_α-spectral radius condition for a graph to contain a perfect matching. For the relationship between spectral radius and the existence of certain spanning subgraphs in graphs, one may be referred to [3, 18, 27, 29, 30, 33, 35] and the references cited in. Egawa and Furuya [2] got a sufficient condition for a graph to have a $\{P_2, P_5\}$-factor. Las Vergnas [7] showed a characterization for a graph with a $P_{\geq 2}$-factor. Zhou, Sun and Liu [34], Zhou, Bian and Sun [32], Pan and Zhou [15] claimed some sufficient conditions for a graph with a $P_{\geq 2}$-factor. Kaneko [5] provided a necessary and sufficient condition for a graph to contain a $P_{\geq 3}$-factor. Liu and Pan [11] gave independence number and minimum degree conditions for the existence of a $P_{\geq 2}$-factor and a $P_{\geq 3}$-factor in a graph, respectively. Kano, Lu and Yu [6] established a connection between the number of isolated vertices and $P_{\geq 3}$-factors in graphs. Liu [10] gave some sufficient conditions for graphs to contain $P_{\geq 3}$-factors with given properties. More results on graph factors, we refer the reader to [17, 19–21, 28, 31].

Zhou, Zhang and Sun [36] provided an A_α-spectral radius condition for a graph with a $P_{\geq 2}$-factor, which is shown in the following.

Theorem 1.1. (Zhou, Zhang and Sun [36]). Let $\alpha\in[0,1)$, and let G be a connected graph of order n

with $n \geq f(\alpha)$, where

$$f(\alpha) = \begin{cases} 14, & if\ \alpha \in [0, \frac{1}{2}]; \\ 17, & if\ \alpha \in (\frac{1}{2}, \frac{2}{3}]; \\ 20, & if\ \alpha \in (\frac{2}{3}, \frac{3}{4}]; \\ \frac{5}{1-\alpha} + 1, & if\ \alpha \in (\frac{3}{4}, 1). \end{cases}$$

If $\rho_\alpha(G) > \theta(n)$, then G contains a $P_{\geq 2}$-factor, where $\theta(n)$ is the largest root of $x^3 - ((\alpha + 1)n + \alpha - 5)x^2 + (\alpha n^2 + (\alpha^2 - 3\alpha - 1)n - 2\alpha + 1)x - \alpha^2 n^2 + (7\alpha^2 - 5\alpha + 3)n - 18\alpha^2 + 29\alpha - 15 = 0$.

In this paper, we extend Theorem 1.1 to $P_{\geq 2}$-factor covered graphs, and obtain a result on the existence of a $P_{\geq 2}$-factor covered graph by using A_α-spectral radius.

Theorem 1.2. Let $\alpha \in [0, 1)$, and let G be a connected graph of order n with $n \geq f(\alpha)$, where $f(\alpha)$ is defined as in Theorem 1.1. If $\rho_\alpha(G) > \eta(n)$, then G is a $P_{\geq 2}$-factor covered graph, where $\eta(n)$ is the largest root of $x^3 - ((\alpha + 1)n + \alpha - 4)x^2 + (\alpha n^2 + (\alpha^2 - 2\alpha - 1)n - 2\alpha + 1)x - \alpha^2 n^2 + (5\alpha^2 - 3\alpha + 2)n - 10\alpha^2 + 15\alpha - 8 = 0$.

2. Preliminaries

In this section, we provide several necessary preliminary results, which will be used to verify Theorem 1.2. Las Vergnas [7] posed a necessary and sufficient condition for graphs to possess $P_{\geq 2}$-factors.

Lemma 2.1. (Las Vergnas [7]). A graph G has a $P_{\geq 2}$-factor if and only if

$$i(G - S) \leq 2|S|$$

for any vertex subset S of G.

Zhang and Zhou [23] obtained a necessary and sufficient condition for the existence of a $P_{\geq 2}$-factor covered graph.

Lemma 2.2. (Zhang and Zhou [23]). Let G be a connected graph. Then G is a $P_{\geq 2}$-factor covered graph if and only if the following three conditions hold for any $S \subseteq V(G)$:

(i) $i(G - S) \leq 2|S|$;

(ii) If S is nonempty and there exists a connected component of $G - S$ with at least two vertices, then $i(G - S) \leq 2|S| - 1$;

(iii) If S is not an independent set, then $i(G - S) \leq 2|S| - 2$.

Lemma 2.3. (Nikiforov [12]). For a complete graph K_n of order n, we have

$$\rho_\alpha(K_n) = n - 1.$$

Lemma 2.4. (Nikiforov [12]). Let G be a connected graph, and H be a proper subgraph of G. Then

$$\rho_\alpha(G) > \rho_\alpha(H).$$

Let M be a real symmetric matrix whose rows and columns are indexed by $V = \{1, 2, \cdots, n\}$. Assume that M, with respect to the partition $\pi : V = V_1 \cup V_2 \cup \cdots \cup V_r$, can be written as

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1,r} \\ \vdots & \ddots & \vdots \\ M_{r1} & \cdots & M_{r,r} \end{pmatrix},$$

where M_{ij} denotes the submatrix (block) of M formed by rows in V_i and columns in V_j. Let q_{ij} denote the average row sum of M_{ij}. Then the matrix $M_\pi = (q_{ij})$ is called the quotient matrix of M. If the row sum of each block M_{ij} is a constant, then the partition is equitable.

Lemma 2.5. ([22]). Let M be a real symmetric matrix with an equitable partition π, and let M_π be the corresponding quotient matrix. Then every eigenvalue of M_π is an eigenvalue of M. Furthermore, if M is nonnegative, then the largest eigenvalues of M and M_π are equal.

3. The proof of Theorem 1.2

Proof of Theorem 1.2. Suppose to the contrary that G is not $P_{\geq 2}$-factor covered. Choose a connected graph G of order n such that its A_α-spectral radius is as large as possible.

Let $h(x) = x^3 - ((\alpha+1)n + \alpha - 5)x^2 + (\alpha n^2 + (\alpha^2 - 3\alpha - 1)n - 2\alpha + 1)x - \alpha^2 n^2 + (7\alpha^2 - 5\alpha + 3)n - 18\alpha^2 + 29\alpha - 15$. By the condition of Theorem 1.1, $\theta(n)$ is the largest root of $h(x) = 0$. Let $\varphi(x) = x^3 - ((\alpha+1)n+\alpha-4)x^2 + (\alpha n^2 + (\alpha^2 - 2\alpha - 1)n - 2\alpha + 1)x - \alpha^2 n^2 + (5\alpha^2 - 3\alpha + 2)n - 10\alpha^2 + 15\alpha - 8$. According to the condition of Theorem 1.2, $\eta(n)$ is the largest root of $\varphi(x) = 0$. We verify Theorem 1.2 by the next two possible cases.

Case 1. G has no $P_{\geq 2}$-factor.

In terms of Theorem 1.1, we have $\rho_\alpha(G) \leq \theta(n)$ for $n \geq f(\alpha)$, where

$$f(\alpha) = \begin{cases} 14, & if\ \alpha \in [0, \frac{1}{2}]; \\ 17, & if\ \alpha \in (\frac{1}{2}, \frac{2}{3}]; \\ 20, & if\ \alpha \in (\frac{2}{3}, \frac{3}{4}]; \\ \frac{5}{1-\alpha} + 1, & if\ \alpha \in (\frac{3}{4}, 1). \end{cases}$$

It suffices to show $\theta := \theta(n) \leq \eta(n)$. Notice that $h(\theta) = 0$. Plugging θ into x of $\varphi(x) - h(x)$ gives

$$\varphi(\theta) = \varphi(\theta) - h(\theta) = -\theta^2 + \alpha n\theta - (2\alpha^2 - 2\alpha + 1)n + 8\alpha^2 - 14\alpha + 7.$$

Claim 1. $\theta > n - 4$.

Proof. Let $H = K_1 \vee (K_{n-4} \cup 3K_1)$. We consider the partition $V(H) = V(3K_1) \cup V(K_{n-4}) \cup V(K_1)$. The corresponding quotient matrix of $A_\alpha(H)$ equals

$$B_0 = \begin{pmatrix} \alpha & 0 & 1-\alpha \\ 0 & n+\alpha-5 & 1-\alpha \\ 3(1-\alpha) & (1-\alpha)(n-4) & \alpha n - \alpha \end{pmatrix}.$$

By a direct computation, the characteristic polynomial of B_0 is

$$h(x) = x^3 - ((\alpha+1)n + \alpha - 5)x^2 + (\alpha n^2 + (\alpha^2 - 3\alpha - 1)n - 2\alpha + 1)x$$

$$-\alpha^2n^2+(7\alpha^2-5\alpha+3)n-18\alpha^2+29\alpha-15.$$

Notice that the partition $V(H)=V(3K_1)\cup V(K_{n-4})\cup V(K_1)$ is equitable. According to Lemma 2.5, the largest root θ of $h(x)=0$ equals $\rho_\alpha(H)$, that is, $\rho_\alpha(H)=\theta$.

Obviously, K_{n-3} is a proper subgraph of $H=K_1\vee(K_{n-4}\cup 3K_1)$. Using Lemmas 2.3 and 2.4, we conclude

$$\theta=\rho_\alpha(H)=\rho_\alpha(K_1\vee(K_{n-4}\cup 3K_1))>\rho_\alpha(K_{n-3})=n-4.$$

Claim 1 is true. □

By virtue of Claim 1, we get

$$\frac{\alpha n}{2}<n-4<\theta.$$

Then

$$\begin{aligned}\varphi(\theta)&=-\theta^2+\alpha n\theta-(2\alpha^2-2\alpha+1)n+8\alpha^2-14\alpha+7\\&<-(n-4)^2+\alpha n(n-4)-(2\alpha^2-2\alpha+1)n+8\alpha^2-14\alpha+7\\&=-(1-\alpha)n^2+(7-2\alpha-2\alpha^2)n+8\alpha^2-14\alpha-9.\end{aligned}$$

If $0\le\alpha\le\frac{3}{4}$, then $\frac{7-2\alpha-2\alpha^2}{2-2\alpha}<14\le f(\alpha)\le n$. Thus, we possess

$$\begin{aligned}\varphi(\theta)&<-(1-\alpha)n^2+(7-2\alpha-2\alpha^2)n+8\alpha^2-14\alpha-9\\&\le-196(1-\alpha)+14(7-2\alpha-2\alpha^2)+8\alpha^2-14\alpha-9\\&=-20\alpha^2+154\alpha-107\\&<0.\end{aligned}$$

If $\frac{3}{4}<\alpha<1$, then $\frac{7-2\alpha-2\alpha^2}{2-2\alpha}<\frac{5}{1-\alpha}+1=f(\alpha)\le n$. Thus, we obtain

$$\begin{aligned}\varphi(\theta)&<-(1-\alpha)n^2+(7-2\alpha-2\alpha^2)n+8\alpha^2-14\alpha-9\\&\le-(1-\alpha)\left(\frac{5}{1-\alpha}+1\right)^2+(7-2\alpha-2\alpha^2)\left(\frac{5}{1-\alpha}+1\right)+8\alpha^2-14\alpha-9\\&=\frac{1}{1-\alpha}(-6\alpha^3+11\alpha^2-12\alpha-3)\\&<0.\end{aligned}$$

From the discussion above, we always have $\varphi(\theta)<0$, which implies $\theta(n)=\theta<\eta(n)$. Together with $\rho_\alpha(G)\le\theta(n)$ for $n\ge f(\alpha)$, we conclude $\rho_\alpha(G)\le\theta(n)<\eta(n)$ for $n\ge f(\alpha)$, which is a contradiction to $\rho_\alpha(G)>\eta(n)$.

Case 2. G has a $P_{\ge2}$-factor.

According to Lemma 2.1, we obtain $i(G-S)\le 2|S|$ for any subset S of $V(G)$. Since G is not $P_{\ge2}$-factor covered, using Lemma 2.2, there exists a subset S of $V(G)$ so that either S is nonempty and some connected component of $G-S$ is nontrivial, or S is not an independent set. Otherwise, it follows from Lemma 2.2 that G is $P_{\ge2}$-factor covered, a contradiction. Furthermore, at least one of the following statements holds:

(a) S is nonempty, one connected component of $G-S$ is nontrivial, and $i(G-S)\ge 2|S|$;

(b) S is not an independent set, and $i(G-S) \geq 2|S|-1$.

Subcase 2.1. Statement (a) holds.

In this subcase, we have $|S| \geq 1$ and $i(G-S) = 2|S|$. In view of Lemma 2.4 and the choice of G, we conclude that

- the induced subgraph $G[S]$ and every connected component of $G-S$ are complete graphs;
- $G = G[S] \vee (G-S)$;
- $G-S$ has only one nontrivial connected component, say G_1.

Let $|S| = s$ and $|V(G_1)| = n_1 \geq 2$. Recall that $i(G-S) = 2|S|$. Then $G = K_s \vee (K_{n_1} \cup 2sK_1)$, where $n_1 = n-3s \geq 2$. It is obvious that the partition $V(G) = V(K_s) \cup V(K_{n_1}) \cup V(2sK_1)$ is an equitable partition of G, and the corresponding quotient matrix of $A_\alpha(G)$ is

$$B_1 = \begin{pmatrix} \alpha n - \alpha s + s - 1 & (1-\alpha)(n-3s) & 2s(1-\alpha) \\ (1-\alpha)s & n + \alpha s - 3s - 1 & 0 \\ (1-\alpha)s & 0 & \alpha s \end{pmatrix}.$$

By a simple computation, the characteristic polynomial of B_1 is given by

$$\begin{aligned} \varphi_1(x) =& x^3 - ((\alpha+1)n + (\alpha-2)s - 2)x^2 \\ &+ (\alpha n^2 + (\alpha^2-\alpha)sn - (\alpha+1)n - 2s^2 - (2\alpha-2)s + 1)x \\ &- \alpha^2 sn^2 + (4\alpha^2 - 4\alpha + 2)s^2 n + (\alpha^2+\alpha)sn \\ &- (8\alpha^2 - 14\alpha + 6)s^3 - (2\alpha^2 - 2\alpha + 2)s^2 - \alpha s. \end{aligned} \tag{3.1}$$

By virtue of Lemma 2.5, $\rho_\alpha(G)$ is the largest root of $\varphi_1(x) = 0$. Let $\eta_1 = \rho_\alpha(G) \geq \eta_2 \geq \eta_3$ be the three roots of $\varphi_1(x) = 0$ and $R_1 = diag(s, n-3s, 2s)$. One checks that $R_1^{\frac{1}{2}} B_1 R_1^{-\frac{1}{2}}$ is symmetric, and also contains

$$\begin{pmatrix} n + \alpha s - 3s - 1 & 0 \\ 0 & \alpha s \end{pmatrix}$$

as its submatrix. Since $R_1^{\frac{1}{2}} B_1 R_1^{-\frac{1}{2}}$ and B_1 possess the same eigenvalues, the Cauchy interlacing theorem (cf. [4]) implies that $\eta_2 \leq n + \alpha s - 3s - 1 < n-3$.

If $s=1$, then $G = K_1 \vee (K_{n-3} \cup 2K_1)$ and $\varphi_1(x) = \varphi(x)$. Recall that $\eta(n)$ is the largest root of $\varphi(x) = 0$ and $\rho_\alpha(G)$ is the largest root of $\varphi_1(x) = 0$. Thus, we conclude $\rho_\alpha(G) = \eta(n)$, which is a contradiction to $\rho_\alpha(G) > \eta(n)$. In what follows, we are to consider $s \geq 2$.

From the discussion above, we easily see $\eta(n) = \rho_\alpha(K_1 \vee (K_{n-3} \cup 2K_1))$. Note that K_{n-2} is a proper subgraph of $K_1 \vee (K_{n-3} \cup 2K_1)$. From Lemmas 2.3 and 2.4, we have $\rho_\alpha(K_1 \vee (K_{n-3} \cup 2K_1)) > \rho_\alpha(K_{n-2}) = n-3$. Thus, we obtain

$$\eta(n) = \rho_\alpha(K_1 \vee (K_{n-3} \cup 2K_1)) > \rho_\alpha(K_{n-2}) = n - 3 > \eta_2. \tag{3.2}$$

Subcase 2.1.1. $0 \leq \alpha \leq \frac{3}{4}$.

Write $\eta := \eta(n)$. Recall that $\varphi(\eta) = 0$. A directed calculation yields that

$$\varphi_1(\eta) = \varphi_1(\eta) - \varphi(\eta) = (s-1)g_1(\eta), \tag{3.3}$$

where $g_1(\eta) = (2-\alpha)\eta^2 - ((\alpha-\alpha^2)n + 2s + 2\alpha)\eta - \alpha^2 n^2 + (4\alpha^2 - 4\alpha + 2)sn + (5\alpha^2 - 3\alpha + 2)n - (8\alpha^2 - 14\alpha + 6)s^2 - (10\alpha^2 - 16\alpha + 8)s - 10\alpha^2 + 15\alpha - 8$. Note that

$$\frac{(\alpha-\alpha^2)n + 2s + 2\alpha}{2(2-\alpha)} < n - 3 < \eta$$

by (3.2), $s \geq 2$ and $n \geq 3s + 2$. Thus, we obtain

$$\begin{aligned}
g_1(\eta) >& g_1(n-3) \\
=& (2-\alpha)(n-3)^2 - ((\alpha-\alpha^2)n + 2s + 2\alpha)(n-3) - \alpha^2 n^2 \\
&+ (4\alpha^2 - 4\alpha + 2)sn + (5\alpha^2 - 3\alpha + 2)n - (8\alpha^2 - 14\alpha + 6)s^2 \\
&- (10\alpha^2 - 16\alpha + 8)s - 10\alpha^2 + 15\alpha - 8 \\
=& (2-2\alpha)n^2 + ((4\alpha^2 - 4\alpha)s + 2\alpha^2 + 4\alpha - 10)n - (8\alpha^2 - 14\alpha + 6)s^2 \\
&- (10\alpha^2 - 16\alpha + 2)s - 10\alpha^2 + 12\alpha + 10. \qquad (3.4)
\end{aligned}$$

Let $l_1(s,n) = (2-2\alpha)n^2 + ((4\alpha^2 - 4\alpha)s + 2\alpha^2 + 4\alpha - 10)n - (8\alpha^2 - 14\alpha + 6)s^2 - (10\alpha^2 - 16\alpha + 2)s - 10\alpha^2 + 12\alpha + 10$. Notice that

$$-\frac{(4\alpha^2 - 4\alpha)s + 2\alpha^2 + 4\alpha - 10}{2(2-2\alpha)} < 3s + 2 \leq n.$$

Then

$$\begin{aligned}
l_1(s,n) \geq& l_1(s, 3s+2) \\
=& (2-2\alpha)(3s+2)^2 + ((4\alpha^2 - 4\alpha)s + 2\alpha^2 + 4\alpha - 10)(3s+2) \\
&- (8\alpha^2 - 14\alpha + 6)s^2 - (10\alpha^2 - 16\alpha + 2)s - 10\alpha^2 + 12\alpha + 10 \\
=& (4s^2 + 4s - 6)\alpha^2 - (16s^2 + 4s - 12)\alpha + 12s^2 - 8s - 2 \\
\geq& \frac{9}{16}(4s^2 + 4s - 6) - \frac{3}{4}(16s^2 + 4s - 12) + 12s^2 - 8s - 2 \\
=& \frac{1}{8}(18s^2 - 70s + 29) \\
>& 0,
\end{aligned}$$

where the last two inequalities hold from $\frac{16s^2+4s-12}{2(4s^2+4s-6)} > \frac{3}{4} \geq \alpha$ and $s \geq 4$, respectively.

If $s = 3$, then $-\frac{(4\alpha^2-4\alpha)s+2\alpha^2+4\alpha-10}{2(2-2\alpha)} = -\frac{7\alpha^2-4\alpha-5}{2-2\alpha} < 14 \leq f(\alpha) \leq n$ due to $0 \leq \alpha \leq \frac{3}{4}$. Consequently, for $0 \leq \alpha \leq \frac{3}{4}$ and $n \geq f(\alpha) \geq 14$, we obtain

$$l_1(3,n) \geq l_1(3,14) = 84\alpha^2 - 318\alpha + 202 > 0.$$

If $s = 2$, then $-\frac{(4\alpha^2-4\alpha)s+2\alpha^2+4\alpha-10}{2(2-2\alpha)} = -\frac{5\alpha^2-2\alpha-5}{2-2\alpha} < 14 \leq f(\alpha) \leq n$ due to $0 \leq \alpha \leq \frac{3}{4}$. Hence, for $0 \leq \alpha \leq \frac{3}{4}$ and $n \geq f(\alpha) \geq 14$, we conclude

$$l_1(2,n) \geq l_1(2,14) = 78\alpha^2 - 328\alpha + 234 > 0.$$

From the discussion above, we have $l_1(s,n) > 0$ for $2 \leq s \leq \frac{n-2}{3}$. Together with (3.3), (3.4) and $s \geq 2$, we get

$$\varphi_1(\eta) = (s-1)g_1(\eta) > (s-1)g_1(n-3) = (s-1)l_1(s,n) > 0.$$

Recall that $\eta = \eta(n)$ and $\rho_\alpha(G)$ is the largest root of $\varphi_1(x) = 0$. As $\eta(n) = \rho_\alpha(K_1 \vee (K_{n-3} \cup 2K_1)) > n-3 > \eta_2$ (see (3.2)), we infer $\rho_\alpha(G) < \eta(n)$ for $2 \le s \le \frac{n-2}{3}$, which is a contradiction to $\rho_\alpha(G) > \eta(n)$.

Subcase 2.1.2. $\frac{3}{4} < \alpha < 1$.

According to (3.1), we have

$$\begin{aligned}\varphi_1(n-3) =&(n-3)^3 - ((\alpha+1)n + (\alpha-2)s - 2)(n-3)^2\\ &+ (\alpha n^2 + (\alpha^2-\alpha)sn - (\alpha+1)n - 2s^2 - (2\alpha-2)s + 1)(n-3)\\ &- \alpha^2 sn^2 + (4\alpha^2 - 4\alpha + 2)s^2 n + (\alpha^2+\alpha)sn\\ &- (8\alpha^2 - 14\alpha + 6)s^3 - (2\alpha^2 - 2\alpha + 2)s^2 - \alpha s\\ =&2(1-\alpha)(4\alpha-3)s^3 + ((4\alpha^2-4\alpha)n - 2\alpha^2 + 2\alpha + 4)s^2\\ &+ ((2-2\alpha)n^2 - (2\alpha^2 - 8\alpha + 10)n - 4\alpha + 12)s\\ &+ (2\alpha-2)n^2 - (6\alpha-10)n - 12\\ =:&\Phi_1(s,n).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\frac{\partial\Phi_1(s,n)}{\partial s} =&6(1-\alpha)(4\alpha-3)s^2 + 2((4\alpha^2-4\alpha)n - 2\alpha^2 + 2\alpha + 4)s\\ &+ (2-2\alpha)n^2 - (2\alpha^2 - 8\alpha + 10)n - 4\alpha + 12.\end{aligned}$$

Recall that $\frac{3}{4} < \alpha < 1$ and $n \ge f(\alpha) = \frac{5}{1-\alpha} + 1 > \frac{5}{1-\alpha}$. By a direct calculation, we have

$$\begin{aligned}\frac{\partial\Phi_1(s,n)}{\partial s}\Big|_{s=2} =&(2-2\alpha)n^2 + (14\alpha^2 - 8\alpha - 10)n - 104\alpha^2 + 172\alpha - 44\\ >&(2-2\alpha)\left(\frac{5}{1-\alpha}\right)^2 + (14\alpha^2 - 8\alpha - 10)\left(\frac{5}{1-\alpha}\right) - 104\alpha^2 + 172\alpha - 44\\ =&\frac{1}{1-\alpha}(104\alpha^3 - 206\alpha^2 + 176\alpha - 44)\\ >&0,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\Phi_1(s,n)}{\partial s}\Big|_{s=\frac{n-2}{3}} =&\frac{1}{3}((6\alpha^2 - 12\alpha + 2)n - 24\alpha^2 + 36\alpha - 4)\\ <&\frac{1}{3}\left((6\alpha^2 - 12\alpha + 2)\left(\frac{5}{1-\alpha}\right) - 24\alpha^2 + 36\alpha - 4\right)\\ =&\frac{1}{3(1-\alpha)}(24\alpha^3 - 30\alpha^2 - 20\alpha + 6)\\ <&0.\end{aligned}$$

This implies that $\varphi_1(n-3) = \Phi_1(s,n) \ge \min\left\{\Phi_1(2,n), \Phi_1\left(\frac{n-2}{3}, n\right)\right\}$ because the leading coefficient of $\Phi_1(s,n)$ (view as a cubic polynomial of s) is positive, and $2 \le s \le \frac{n-2}{3}$. By virtue of $\frac{3}{4} < \alpha < 1$ and $n \ge f(\alpha) = \frac{5}{1-\alpha} + 1 > \frac{5}{1-\alpha}$, we obtain

$$\Phi_1(2,n) =(2-2\alpha)n^2 + (12\alpha^2 - 6\alpha - 10)n - 72\alpha^2 + 112\alpha - 20$$

$$
\begin{aligned}
&>(2-2\alpha)\left(\frac{5}{1-\alpha}\right)^2+(12\alpha^2-6\alpha-10)\left(\frac{5}{1-\alpha}\right)-72\alpha^2+112\alpha-20\\
&=\frac{1}{1-\alpha}(72\alpha^3-124\alpha^2+102\alpha-20)\\
&>0,
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi_1\left(\frac{n-2}{3},n\right)=&\frac{1}{27}((4\alpha^2-16\alpha+12)n^3+(-24\alpha^2+132\alpha-132)n^2\\
&+(12\alpha^2-246\alpha+438)n+40\alpha^2-16\alpha-444)\\
>&\frac{1}{27}((4\alpha^2-16\alpha+12)\left(\frac{5}{1-\alpha}\right)^3+(-24\alpha^2+132\alpha-132)\left(\frac{5}{1-\alpha}\right)^2\\
&+(12\alpha^2-246\alpha+438)\left(\frac{5}{1-\alpha}\right)+40\alpha^2-16\alpha-444)\\
=&\frac{1}{27(1-\alpha)^2}(40\alpha^4-156\alpha^3+318\alpha^2+252\alpha-54)\\
>&0.
\end{aligned}
$$

Consequently, we conclude $\varphi_1(n-3)\geq\min\left\{\Phi_1(2,n),\Phi_1\left(\frac{n-2}{3},n\right)\right\}>0$ for $2\leq s\leq\frac{n-2}{3}$. As $\eta(n)=\rho_\alpha(K_1\vee(K_{n-3}\cup 2K_1))>n-3>\eta_2$ (see (3.2)), we deduce $\rho_\alpha(G)<\eta(n)$ for $2\leq s\leq\frac{n-2}{3}$, which is a contradiction to $\rho_\alpha(G)>\eta(n)$.

Subcase 2.2. Statement (b) holds.

In this subcase, we infer $|S|\geq 2$ and $i(G-S)=2|S|-1$ or $2|S|$. Similarly, one has

- the induced subgraph $G[S]$ and every connected component of $G-S$ are complete graphs;
- $G=G[S]\vee(G-S)$;
- $G-S$ has at most one nontrivial connected component G_1.

We also let $|S|=s\geq 2$ and $|V(G_1)|=n_1$. Notice that $n_1=0$ or $n_1\geq 2$.

Subcase 2.2.1. $i(G-S)=2s-1$.

Subcase 2.2.1.1. $n_1\geq 2$.

In this subcase, $G=K_s\vee(K_{n_1}\cup(2s-1)K_1)$ and $n=3s-1+n_1\geq 3s+1$. It is clear that the partition $V(G)=V(K_s)\cup V(K_{n_1})\cup V((2s-1)K_1)$ is an equitable partition of G, and the corresponding quotient matrix of $A_\alpha(G)$ equals

$$
B_2=\begin{pmatrix}
\alpha n-\alpha s+s-1 & (1-\alpha)(n-3s+1) & (1-\alpha)(2s-1)\\
(1-\alpha)s & n+\alpha s-3s & 0\\
(1-\alpha)s & 0 & \alpha s
\end{pmatrix}.
$$

By a direct computation, we obtain the characteristic polynomial of B_2 as

$$
\begin{aligned}
\varphi_2(x)=&x^3-((\alpha+1)n+(\alpha-2)s-1)x^2\\
&+(\alpha n^2+(\alpha^2-\alpha)sn-n-2s^2-(2\alpha-3)s)x\\
&-\alpha^2sn^2+(4\alpha^2-4\alpha+2)s^2n-(\alpha^2-3\alpha+1)sn
\end{aligned}
$$

$$-(8\alpha^2-14\alpha+6)s^3+(4\alpha^2-9\alpha+3)s^2. \tag{3.5}$$

According to Lemma 2.5, $\rho_\alpha(G)$ is the largest root of $\varphi_2(x)=0$. Let $\beta_1=\rho_\alpha(G)\geq\beta_2\geq\beta_3$ be the three roots of $\varphi_2(x)=0$ and $R_2=diag(s,n-3s+1,2s-1)$. One checks that $R_2^{\frac{1}{2}}B_2R_2^{-\frac{1}{2}}$ is symmetric, and also contains

$$\begin{pmatrix} n+\alpha s-3s & 0 \\ 0 & \alpha s \end{pmatrix}$$

as its submatrix. Since $R_2^{\frac{1}{2}}B_2R_2^{-\frac{1}{2}}$ and B_2 have the same eigenvalues, the Cauchy interlacing theorem (cf. [4]) yields that $\beta_2\leq n+\alpha s-3s<n-2s\leq n-4$.

Recall that $\eta=\eta(n)=\rho_\alpha(K_1\vee(K_{n-3}\cup 2K_1))$. Since K_{n-2} is a proper subgraph of $K_1\vee(K_{n-3}\cup 2K_1)$, it follows from Lemmas 2.3 and 2.4 that

$$\eta=\eta(n)=\rho_\alpha(K_1\vee(K_{n-3}\cup 2K_1))>\rho_\alpha(K_{n-2})=n-3>n-4>\beta_2. \tag{3.6}$$

First we discuss $0\leq\alpha\leq\frac{3}{4}$. Recall that $\varphi(\eta)=0$. A simple computation yields that

$$\begin{aligned}
\varphi_2(\eta)=&\varphi_2(\eta)-\varphi(\eta)\\
=&((2-\alpha)s+\alpha-3)\eta^2+(((\alpha^2-\alpha)s-\alpha^2+2\alpha)n-2s^2-(2\alpha-3)s+2\alpha-1)\eta\\
&-\alpha^2(s-1)n^2+(4\alpha^2-4\alpha+2)s^2n-(\alpha^2-3\alpha+1)sn-(5\alpha^2-3\alpha+2)n\\
&-(8\alpha^2-14\alpha+6)s^3+(4\alpha^2-9\alpha+3)s^2+10\alpha^2-15\alpha+8.
\end{aligned}$$

Note that

$$-\frac{((\alpha^2-\alpha)s-\alpha^2+2\alpha)n-2s^2-(2\alpha-3)s+2\alpha-1}{2((2-\alpha)s+\alpha-3)}<n-3<\eta$$

by (3.6), $s\geq 2$ and $n\geq 3s+1$. Hence, we obtain

$$\begin{aligned}
\varphi_2(\eta)>&((2-\alpha)s+\alpha-3)(n-3)^2+(((\alpha^2-\alpha)s-\alpha^2+2\alpha)n-2s^2-(2\alpha-3)s+2\alpha-1)(n-3)\\
&-\alpha^2(s-1)n^2+(4\alpha^2-4\alpha+2)s^2n-(\alpha^2-3\alpha+1)sn-(5\alpha^2-3\alpha+2)n\\
&-(8\alpha^2-14\alpha+6)s^3+(4\alpha^2-9\alpha+3)s^2+10\alpha^2-15\alpha+8\\
=&((2-2\alpha)s+3\alpha-3)n^2+((4\alpha^2-4\alpha)s^2-(4\alpha^2-10\alpha+10)s-2\alpha^2-7\alpha+15)n\\
&-(8\alpha^2-14\alpha+6)s^3+(4\alpha^2-9\alpha+9)s^2-3(\alpha-3)s+10\alpha^2-12\alpha-16.
\end{aligned} \tag{3.7}$$

Let $l_2(s,n)=((2-2\alpha)s+3\alpha-3)n^2+((4\alpha^2-4\alpha)s^2-(4\alpha^2-10\alpha+10)s-2\alpha^2-7\alpha+15)n-(8\alpha^2-14\alpha+6)s^3+(4\alpha^2-9\alpha+9)s^2-3(\alpha-3)s+10\alpha^2-12\alpha-16$. Note that $(2-2\alpha)s+3\alpha-3>0$ and

$$-\frac{(4\alpha^2-4\alpha)s^2-(4\alpha^2-10\alpha+10)s-2\alpha^2-7\alpha+15}{2((2-2\alpha)s+3\alpha-3)}<3s+1\leq n.$$

Then

$$\begin{aligned}
l_2(s,n)\geq& l_2(s,3s+1)\\
=&((2-2\alpha)s+3\alpha-3)(3s+1)^2\\
&+((4\alpha^2-4\alpha)s^2-(4\alpha^2-10\alpha+10)s-2\alpha^2-7\alpha+15)(3s+1)
\end{aligned}$$

$$
\begin{aligned}
&-(8\alpha^2-14\alpha+6)s^3+(4\alpha^2-9\alpha+9)s^2-3(\alpha-3)s+10\alpha^2-12\alpha-16\\
=&(4s^3-4s^2-10s+8)\alpha^2-(16s^3-32s^2-2s+16)\alpha+12s^3-36s^2+28s-4\\
\geq&\frac{9}{16}(4s^3-4s^2-10s+8)-\frac{3}{4}(16s^3-32s^2-2s+16)+12s^3-36s^2+28s-4\\
=&\frac{1}{8}(18s^3-114s^2+191s-92)\\
\geq&0,
\end{aligned}
$$

where the last two inequalities hold from $\frac{16s^2+4s-12}{2(4s^2+4s-6)} > \frac{3}{4} \geq \alpha$ and $s \geq 4$, respectively.

If $s = 3$, then $-\frac{(4\alpha^2-4\alpha)s^2-(4\alpha^2-10\alpha+10)s-2\alpha^2-7\alpha+15}{2((2-2\alpha)s+3\alpha-3)} = -\frac{22\alpha^2-13\alpha-15}{6-6\alpha} < 14 \leq f(\alpha) \leq n$ due to $0 \leq \alpha \leq \frac{3}{4}$. Hence, for $0 \leq \alpha \leq \frac{3}{4}$ and $n \geq f(\alpha) \geq 14$, we get

$$l_2(3,n) \geq l_2(3,14) = 138\alpha^2 - 494\alpha + 308 > 0.$$

If $s = 2$, then $-\frac{(4\alpha^2-4\alpha)s^2-(4\alpha^2-10\alpha+10)s-2\alpha^2-7\alpha+15}{2((2-2\alpha)s+3\alpha-3)} = -\frac{6\alpha^2-3\alpha-5}{2-2\alpha} < 14 \leq f(\alpha) \leq n$ due to $0 \leq \alpha \leq \frac{3}{4}$. Hence, for $0 \leq \alpha \leq \frac{3}{4}$ and $n \geq f(\alpha) \geq 14$, we have

$$l_2(2,n) \geq l_2(2,14) = 46\alpha^2 - 180\alpha + 115 > 0.$$

From the discussion above, we conclude $l_2(s,n) \geq 0$ for $2 \leq s \leq \frac{n-1}{3}$. Combining this with (3.7) and $s \geq 2$, we obtain

$$\varphi_2(\eta) > l_2(s,n) \geq 0.$$

Recall that $\rho_\alpha(G)$ is the largest root of $\varphi_2(x) = 0$. As $\eta = \eta(n) = \rho_\alpha(K_1 \vee (K_{n-3} \cup 2K_1)) > n-3 > \beta_2$ (see (3.6)), we deduce $\rho_\alpha(G) < \eta(n)$ for $2 \leq s \leq \frac{n-1}{3}$, which is a contradiction to $\rho_\alpha(G) > \eta(n)$. In what follows, we consider $\frac{3}{4} < \alpha < 1$.

In terms of (3.5), we get

$$
\begin{aligned}
\varphi_2(n-3) =&(n-3)^3-((\alpha+1)n+(\alpha-2)s-1)(n-3)^2\\
&+(\alpha n^2+(\alpha^2-\alpha)sn-n-2s^2-(2\alpha-3)s)(n-3)\\
&-\alpha^2sn^2+(4\alpha^2-4\alpha+2)s^2n-(\alpha^2-3\alpha+1)sn\\
&-(8\alpha^2-14\alpha+6)s^3+(4\alpha^2-9\alpha+3)s^2\\
=&2(1-\alpha)(4\alpha-3)s^3+((4\alpha^2-4\alpha)n+4\alpha^2-9\alpha+9)s^2\\
&+((2-2\alpha)n^2-(4\alpha^2-10\alpha+10)n-3\alpha+9)s\\
&+(3\alpha-3)n^2-(9\alpha-15)n-18\\
=:&\Phi_2(s,n).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\frac{\partial\Phi_2(s,n)}{\partial s} =&6(1-\alpha)(4\alpha-3)s^2+2((4\alpha^2-4\alpha)n+4\alpha^2-9\alpha+9)s\\
&+(2-2\alpha)n^2-(4\alpha^2-10\alpha+10)n-3\alpha+9.
\end{aligned}
$$

Recall that $\frac{3}{4} < \alpha < 1$ and $n \geq f(\alpha) = \frac{5}{1-\alpha} + 1 > \frac{5}{1-\alpha}$. By a simple calculation, we obtain

$$\left.\frac{\partial\Phi_2(s,n)}{\partial s}\right|_{s=2} =(2-2\alpha)n^2+(12\alpha^2-6\alpha-10)n-80\alpha^2+129\alpha-27$$

$$
\begin{aligned}
&>(2-2\alpha)\left(\frac{5}{1-\alpha}\right)^2+(12\alpha^2-6\alpha-10)\left(\frac{5}{1-\alpha}\right)-80\alpha^2+129\alpha-27\\
&=\frac{1}{1-\alpha}(80\alpha^3-149\alpha^2+126\alpha-27)\\
&>0,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial\Phi_2(s,n)}{\partial s}\Big|_{s=\frac{n-1}{3}} &=\frac{1}{3}((4\alpha^2-8\alpha)n-16\alpha^2+23\alpha+3)\\
&<\frac{1}{3}\left((4\alpha^2-8\alpha)\left(\frac{5}{1-\alpha}\right)-16\alpha^2+23\alpha+3\right)\\
&=\frac{1}{3(1-\alpha)}(16\alpha^3-19\alpha^2-20\alpha+3)\\
&<0.
\end{aligned}
$$

This implies that $\varphi_2(n-3)=\Phi_2(s,n)\geq\min\left\{\Phi_2(2,n),\Phi_2\left(\frac{n-1}{3},n\right)\right\}$ because the leading coefficient of $\Phi_2(s,n)$ (view as a cubic polynomial of s) is positive, and $2\leq s\leq\frac{n-1}{3}$. In light of $\frac{3}{4}<\alpha<1$ and $n\geq f(\alpha)=\frac{5}{1-\alpha}+1>\frac{5}{1-\alpha}$, we have

$$
\begin{aligned}
\Phi_2(2,n) &=(1-\alpha)n^2+(8\alpha^2-4\alpha-5)n-48\alpha^2+70\alpha-12\\
&>(1-\alpha)\left(\frac{5}{1-\alpha}\right)^2+(8\alpha^2-4\alpha-5)\left(\frac{5}{1-\alpha}\right)-48\alpha^2+70\alpha-12\\
&=\frac{1}{1-\alpha}(48\alpha^3-78\alpha^2+62\alpha-12)\\
&>0,
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi_2\left(\frac{n-1}{3},n\right) &=\frac{1}{27}((4\alpha^2-16\alpha+12)n^3+(-24\alpha^2+144\alpha-144)n^2\\
&\quad+(-249\alpha+504)n+20\alpha^2-14\alpha-534)\\
&>\frac{1}{27}((4\alpha^2-16\alpha+12)\left(\frac{5}{1-\alpha}\right)^3+(-24\alpha^2+144\alpha-144)\left(\frac{5}{1-\alpha}\right)^2\\
&\quad+(-249\alpha+504)\left(\frac{5}{1-\alpha}\right)+20\alpha^2-14\alpha-534)\\
&=\frac{1}{27(1-\alpha)^2}(20\alpha^4-54\alpha^3+159\alpha^2+389\alpha-114)\\
&>0.
\end{aligned}
$$

Hence, we have $\varphi_2(n-3)\geq\min\left\{\Phi_2(2,n),\Phi_2\left(\frac{n-1}{3},n\right)\right\}>0$ for $2\leq s\leq\frac{n-1}{3}$. As $\eta=\eta(n)=\rho_\alpha(K_1\vee(K_{n-3}\cup 2K_1))>n-3>\beta_2$ (see (3.6)), we conclude $\rho_\alpha(G)<\eta(n)$ for $2\leq s\leq\frac{n-1}{3}$, which is a contradiction to $\rho_\alpha(G)>\eta(n)$.

Subcase 2.2.1.2. $n_1 = 0$.

In this subcase, $G = K_s \vee (2s-1)K_1$ and $n = 3s-1$. The quotient matrix of $A_\alpha(G)$ by the partition $V(G) = V(K_s) \cup V((2s-1)K_1)$ equals

$$B_3 = \begin{pmatrix} 2\alpha s + s - \alpha - 1 & (1-\alpha)(2s-1) \\ (1-\alpha)s & \alpha s \end{pmatrix}.$$

By a direct computation, we get the characteristic polynomial of B_3 as

$$\varphi_3(x) = x^2 - (3\alpha s + s - \alpha - 1)x + 5\alpha s^2 - 2s^2 - 3\alpha s + s.$$

Since the partition $V(G) = V(K_s) \cup V((2s-1)K_1)$ is equitable, it follows from Lemma 2.5 that $\rho_\alpha(G)$ is the largest root of $\varphi_3(x) = 0$. Note that $n = 3s-1$. By a direct computation, we obtain

$$\varphi_3(n-3) = \varphi_3(3s-4) = (4-4\alpha)s^2 + (12\alpha - 16)s - 4\alpha + 12.$$

If $0 \leq \alpha \leq \frac{1}{2}$, then $n = 3s-1 \geq f(\alpha) = 14$. Thus, we infer $s \geq 5$, and so

$$\begin{aligned} \varphi_3(n-3) =& (4-4\alpha)s^2 + (12\alpha - 16)s - 4\alpha + 12 \\ \geq& 5(4-4\alpha)s + (12\alpha - 16)s - 4\alpha + 12 \\ =& (4-8\alpha)s - 4\alpha + 12 \\ \geq& 5(4-8\alpha) - 4\alpha + 12 \\ =& 32 - 44\alpha \\ >& 0. \end{aligned}$$

If $\frac{1}{2} < \alpha \leq \frac{2}{3}$, then $n = 3s-1 \geq f(\alpha) = 17$. Thus, we have $s \geq 6$, and so

$$\begin{aligned} \varphi_3(n-3) =& (4-4\alpha)s^2 + (12\alpha - 16)s - 4\alpha + 12 \\ \geq& 6(4-4\alpha)s + (12\alpha - 16)s - 4\alpha + 12 \\ =& (8-12\alpha)s - 4\alpha + 12 \\ \geq& 6(8-12\alpha) - 4\alpha + 12 \\ =& 60 - 76\alpha \\ >& 0. \end{aligned}$$

If $\frac{2}{3} < \alpha \leq \frac{3}{4}$, then $n = 3s-1 \geq f(\alpha) = 20$. Thus, we get $s \geq 7$, and so

$$\begin{aligned} \varphi_3(n-3) =& (4-4\alpha)s^2 + (12\alpha - 16)s - 4\alpha + 12 \\ \geq& 7(4-4\alpha)s + (12\alpha - 16)s - 4\alpha + 12 \\ =& (12-16\alpha)s - 4\alpha + 12 \\ \geq& 7(12-16\alpha) - 4\alpha + 12 \\ =& 96 - 116\alpha \\ >& 0. \end{aligned}$$

If $\frac{3}{4} < \alpha < 1$, then $n = 3s - 1 \geq f(\alpha) = \frac{5}{1-\alpha} + 1$. Thus, we deduce $s \geq \frac{5}{3(1-\alpha)} + \frac{2}{3}$, and so

$$\begin{aligned}\varphi_3(n-3) =& (4-4\alpha)s^2 + (12\alpha - 16)s - 4\alpha + 12\\ \geq& (4-4\alpha)\left(\frac{5}{3(1-\alpha)} + \frac{2}{3}\right)s + (12\alpha - 16)s - 4\alpha + 12\\ =& \frac{1}{3}((28\alpha - 20)s - 12\alpha + 36)\\ \geq& \frac{1}{3}\left((28\alpha - 20)\left(\frac{5}{3(1-\alpha)} + \frac{2}{3}\right) - 12\alpha + 36\right)\\ =& \frac{1}{9(1-\alpha)}(-20\alpha^2 + 92\alpha - 32)\\ >& 0.\end{aligned}$$

From the discussion above, we conclude $s \geq 5$ and $\varphi_3(n-3) > 0$. Next, we consider the derivative of $\varphi_3(x)$, and we have $\varphi_3'(x) = 2x - (3\alpha s + s - \alpha - 1)$. It follows from $s \geq 5$ and $n = 3s - 1$ that

$$\begin{aligned}\varphi_3'(n-3) =& 2(n-3) - (3\alpha s + s - \alpha - 1)\\ =& (5-3\alpha)s + \alpha - 7\\ \geq& 5(5-3\alpha) + \alpha - 7\\ =& 18 - 14\alpha\\ >& 0.\end{aligned}$$

Combining this with $\varphi_3(n-3) > 0$ and $\eta(n) > n-3$, we conclude $\rho_\alpha(G) < n-3 < \eta(n)$ for $s \geq 5$, which contradicts $\rho_\alpha(G) > \eta(n)$.

Subcase 2.2.2. $i(G-S) = 2s$.

In this subcase, $G = K_s \vee (K_{n_1} \cup 2sK_1)$. If $n_1 \geq 2$, then $n = 3s + n_1 \geq 3s + 2$. By a similar proof as that of Subcase 2.1 above, we have $\rho_\alpha(G) < \eta(n)$ for $2 \leq s \leq \frac{n-2}{3}$, which contradicts $\rho_\alpha(G) > \eta(n)$.

If $n_1 = 0$, then $G = K_s \vee 2sK_1$ and $n = 3s$. Consider the partition $V(G) = V(K_s) \cup V(2sK_1)$. The corresponding quotient matrix of $A_\alpha(G)$ equals

$$B_4 = \begin{pmatrix} 2\alpha s + s - 1 & 2s(1-\alpha) \\ (1-\alpha)s & \alpha s \end{pmatrix}.$$

The characteristic polynomial of B_4 is

$$\varphi_4(x) = x^2 - (3\alpha s + s - 1)x + 5\alpha s^2 - 2s^2 - \alpha s.$$

Since the partition $V(G) = V(K_s) \cup V(2sK_1)$ is equitable, it follows from Lemma 2.5 that $\rho_\alpha(G)$ is the largest root of $\varphi_4(x) = 0$. Notice that $n = 3s$. By a simple calculation, we get

$$\varphi_4(n-3) = \varphi_4(3s-3) = (4-4\alpha)s^2 + (8\alpha - 12)s + 6.$$

If $0 \leq \alpha \leq \frac{2}{3}$, then $n = 3s \geq f(\alpha) \geq 14$. Thus, we have $s \geq 5$, and so

$$\varphi_4(n-3) = (4-4\alpha)s^2 + (8\alpha - 12)s + 6$$

$$
\begin{aligned}
&\geq 5(4-4\alpha)s+(8\alpha-12)s+6\\
&=(8-12\alpha)s+6\\
&\geq 5(8-12\alpha)+6\\
&=46-60\alpha\\
&>0.
\end{aligned}
$$

If $\frac{2}{3}<\alpha\leq\frac{3}{4}$, then $n=3s\geq f(\alpha)=20$. Thus, we conclude $s\geq 7$, and so

$$
\begin{aligned}
\varphi_4(n-3)=&(4-4\alpha)s^2+(8\alpha-12)s+6\\
\geq&7(4-4\alpha)s+(8\alpha-12)s+6\\
=&(16-20\alpha)s+6\\
\geq&7(16-20\alpha)+6\\
=&118-140\alpha\\
>&0.
\end{aligned}
$$

If $\frac{3}{4}<\alpha<1$, then $n=3s\geq f(\alpha)=\frac{5}{1-\alpha}+1$. Thus, we get $s\geq\frac{5}{3(1-\alpha)}+\frac{1}{3}$, and so

$$
\begin{aligned}
\varphi_4(n-3)=&(4-4\alpha)s^2+(8\alpha-12)s+6\\
\geq&(4-4\alpha)\left(\frac{5}{3(1-\alpha)}+\frac{1}{3}\right)s+(8\alpha-12)s+6\\
=&\frac{1}{3}((20\alpha-12)s+18)\\
\geq&\frac{1}{3}\left((20\alpha-12)\left(\frac{5}{3(1-\alpha)}+\frac{1}{3}\right)+18\right)\\
=&\frac{1}{9(1-\alpha)}(-20\alpha^2+78\alpha-18)\\
>&0.
\end{aligned}
$$

From the discussion above, we obtain $s\geq 5$ and $\varphi_4(n-3)>0$. In what follows, we consider the derivative of $\varphi_4(x)$, and we get $\varphi_4'(x)=2x-(3\alpha s+s-1)$. According to $s\geq 5$ and $n=3s$, we obtain

$$
\begin{aligned}
\varphi_4'(n-3)=&2(n-3)-(3\alpha s+s-1)\\
=&(5-3\alpha)s-5\\
\geq&5(5-3\alpha)-5\\
=&20-15\alpha\\
>&0.
\end{aligned}
$$

Combining this with $\varphi_4(n-3)>0$ and $\eta(n)>n-3$, we deduce $\rho_\alpha(G)<n-3<\eta(n)$ for $s\geq 5$, which is a contradiction to $\rho_\alpha(G)>\eta(n)$. This completes the proof of Theorem 1.2. □

4. Extremal graphs

In this section, we create a graph to claim that the bound on A_α-spectral radius established in Theorem 1.1 is sharp.

Theorem 4.1. Let $\alpha \in [0,1)$, and let $\eta(n)$ be the largest root of $x^3-((\alpha+1)n+\alpha-4)x^2+(\alpha n^2+(\alpha^2-2\alpha-1)n-2\alpha+1)x-\alpha^2n^2+(5\alpha^2-3\alpha+2)n-10\alpha^2+15\alpha-8=0$. Then $\rho_\alpha(K_{n-3}\vee K_1\vee \overline{K_2})=\eta(n)$ and $K_{n-3}\vee K_1\vee \overline{K_2}$ is not a $P_{\geq 2}$-factor covered graph if $n\geq f(\alpha)$, where

$$f(\alpha)=\begin{cases} 14, & if\ \alpha\in[0,\frac{1}{2}];\\ 17, & if\ \alpha\in(\frac{1}{2},\frac{2}{3}];\\ 20, & if\ \alpha\in(\frac{2}{3},\frac{3}{4}];\\ \frac{5}{1-\alpha}+1, & if\ \alpha\in(\frac{3}{4},1).\end{cases}$$

Proof. We partition the vertex set of the graph $K_{n-3}\vee K_1\vee \overline{K_2}$ as $V(K_{n-3})\cup V(K_1)\cup V(\overline{K_2})$. Then the quotient matrix of $A_\alpha(K_{n-3}\vee K_1\vee \overline{K_2})$ in terms of the partition $V(K_{n-3})\cup V(K_1)\cup V(\overline{K_2})$ can be written as

$$B(K_{n-3}\vee K_1\vee \overline{K_2})=\begin{pmatrix} \alpha(n-1) & (1-\alpha)(n-3) & 2(1-\alpha)\\ 1-\alpha & n+\alpha-4 & 0\\ 1-\alpha & 0 & \alpha\end{pmatrix},$$

whose characteristic polynomial is $x^3-((\alpha+1)n+\alpha-4)x^2+(\alpha n^2+(\alpha^2-2\alpha-1)n-2\alpha+1)x-\alpha^2n^2+(5\alpha^2-3\alpha+2)n-10\alpha^2+15\alpha-8$. Since the partition is equitable, it follows from Lemma 2.5 that the largest root $\eta(n)$ of $x^3-((\alpha+1)n+\alpha-4)x^2+(\alpha n^2+(\alpha^2-2\alpha-1)n-2\alpha+1)x-\alpha^2n^2+(5\alpha^2-3\alpha+2)n-10\alpha^2+15\alpha-8=0$ is equal to the A_α-spectral radius of the graph $K_{n-3}\vee K_1\vee \overline{K_2}$, that is to say, $\rho_\alpha(K_{n-3}\vee K_1\vee \overline{K_2})=\eta(n)$. Write $S=V(K_1)$. Then $i(K_{n-3}\vee K_1\vee \overline{K_2}-S)=2>1=2|S|-1$. According to Lemma 2.2 (ii), we know that the graph $K_{n-3}\vee K_1\vee \overline{K_2}$ is not a $P_{\geq 2}$-factor covered graph. Theorem 4.1 is proved. □

5. Conclusions

In this paper, we establish a connection between the A_α-spectral radius and $P_{\geq 2}$-factor covered graphs and obtain a sufficient condition for a graph G to be $P_{\geq 2}$-factor covered with respect to the A_α-spectral radius of G. Inspired by the work above, it is natural and interesting to give some other types of spectral radius conditions to ensure that a graph G is $P_{\geq 2}$-factor covered.

Author contributions

Sizhong Zhou: Methodology, formal analysis, investigation, writing-original draft preparation, funding acquisition; Hongxia Liu: Conceptualization, methodology, validation, writing review and editing; Qiuxiang Bian: Conceptualization, methodology, validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest to this work.

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