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Research article

Dirichlet problem for two-dimensional quasilinear second-order elliptic equations in Besov spaces

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Abstract: Vekua established the existence and uniqueness of solutions to a quasilinear elliptic equation on the bounded domain Ω with a sufficiently smooth boundary, with Dirichlet boundary conditions in the Sobolev spaces $W_p^2(\Omega)$ for $p > 2$, the second-order derivatives of the solution belong to $L_p(\Omega)$, $p > 2$. The goal of this note is to extend his results: the corresponding solvability result in the Besov space $B_{p,1}^{2+\alpha}(\Omega) \hookrightarrow C^1(\overline{\Omega})$ for $1 < p < 2$ and $\alpha = 2/p - 1$ with the second-order derivatives of the solution belonging to $B_{p,1}^{\alpha}(\Omega) \hookrightarrow L_2(\Omega)$.

Keywords: dirichlet problem; Besov spaces; quasilinear elliptic equations; Green's function; existence and uniqueness
Mathematics Subject Classification: 35J25, 35J65

1. Introduction

Boundary value problems for differential equations have important applications in mathematical physics and applied mathematics. Such problems have been extensively studied in [1–3]. Dirichlet problems for elliptic equations are one of the most important parts of boundary value problems and have been widely investigated in Hölder spaces [4, 5] and Sobolev spaces [6, 7], as well as in nonlinear elliptic equations involving p-Laplacian, double-phase, and nonlocal Choquard terms [8–10]. Besov regularity of weak solutions of boundary value problems for elliptic equations and elliptic systems with rough coefficients in Besov spaces have been investigated in [11–13].

Recent developments in nonlinear and quasilinear elliptic problems include existence, multiplicity, a priori estimates, and compactness results for problems involving intrinsic operators, perturbations, nonstandard boundary conditions, and double-phase structures [14–16].

Let $\Omega \subset E$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Let L be the quasilinear elliptic operator defined as

$$Lu = a(x,y,u,u_x,u_y)u_{xx} + 2b(x,y,u,u_x,u_y)u_{xy} \\ + c(x,y,u,u_x,u_y)u_{yy} + d(x,y,u,u_x,u_y), \tag{1.1}$$

where $a(x,y,u,u_x,u_y)$, $b(x,y,u,u_x,u_y)$, and $c(x,y,u,u_x,u_y)$ are from the Besov space $B^{1+\alpha}_{p,1}(\Omega)$, $1 < p < 2$, $\alpha = 2/p - 1$; moreover, $d = d(x,y,u,u_x,u_y)$ is a measurable function, and $d(x,y,u,0,0) \in B^{\alpha}_{p,1}(\Omega)$.

This paper focuses on the Dirichlet problem for the equation

$$Lu = 0 \quad \text{in} \quad \Omega \tag{1.2}$$

with continuous boundary data

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{1.3}$$

Under suitable assumptions on the coefficients of Eq (1.1), Vekua studied the problem (1.1)–(1.3) and established the existence and uniqueness of solutions to Eq (1.1) with conditions (1.2) and (1.3) in the Sobolev spaces (see [7]):

$$u \in W^{2,p}(\Omega) \hookrightarrow C^1_{1-2/p}(\overline{\Omega}) \qquad \Longrightarrow \qquad D^2u \in L_p(\Omega) \quad \text{for} \quad p > 2,$$

where u denotes the solution of the corresponding Dirichlet problem (1.1)–(1.3). However, when $p \leq 2$, the space $W^{2,p}(\Omega)$ does not embed into $C^1(\overline{\Omega})$. Therefore, the result in [7] does not directly cover the low-integrability range $1 < p < 2$.

The purpose of this paper is to extend the corresponding solvability result in [7] to the range $1 < p < 2$ by using the Besov space $B^{2+\alpha}_{p,1}(\Omega)$, $\alpha = 2/p - 1$. This space is suitable because $B^{2+\alpha}_{p,1}(\Omega) \hookrightarrow C^1(\overline{\Omega})$. Moreover, the second-order derivatives of the solution satisfy $D^2u \in B^{\alpha}_{p,1}(\Omega) \hookrightarrow L_2(\Omega)$, and the space $B^{\alpha}_{p,1}(\Omega)$ does not embed into $L_q(\Omega)$ for $q > 2$. In this way, we cover the case $p = 2$, thus preserving the continuity of the solution.

The proof is based on Green function representations and Calderón–Zygmund singular integral operators. After transferring the problem to the unit disk and introducing complex derivatives, the original differential equation is rewritten as a complex elliptic equation. This approach reduces the quasilinear Dirichlet problem to a nonlinear singular integral equation in Besov spaces and avoids a preliminary reduction of the elliptic equation to canonical form.

To avoid confusion, we provide explicit notations for the function spaces that we will use frequently. Let $E = E^2$ be a two-dimensional Euclidean space with points $x = (x_1, x_2)$. All measures and integrals below are understood in the Lebesgue sense.

Let Ω be a bounded domain in E with sufficiently smooth boundary $\partial\Omega$. The space $C(\overline{\Omega})$ consists of all continuous real or complex valued functions on $\overline{\Omega}$, with the norm

$$\|f\|_{C(\overline{\Omega})} = \sup_{x\in\overline{\Omega}} |f(x)|.$$

For an integer $l \geq 0$, the space $C^l(\overline{\Omega})$ consists of functions whose partial derivatives up to order l are continuous on $\overline{\Omega}$. The space $C^l(\overline{\Omega})$ is a Banach space with the norm

$$\|f\|_{C^l(\overline{\Omega})} = \sum_{k=0}^{l} \sum_{k_1+k_2=k} \left\| \frac{\partial^k f}{\partial x_1^{k_1} \partial x_2^{k_2}} \right\|_{C(\overline{\Omega})}.$$

For $0 < \beta \le 1$, let $C_\beta(\overline{\Omega})$ denote the space of functions satisfying the Hölder condition

$$|f(x) - f(y)| \le H|x - y|^\beta, \qquad x, y \in \overline{\Omega}.$$

The Hölder seminorm is

$$H_\beta(f) = \sup_{x,y\in\overline{\Omega}} \frac{|f(x) - f(y)|}{|x - y|^\beta},$$

and the corresponding norm is

$$\|f\|_{C_\beta(\overline{\Omega})} = \|f\|_{C(\overline{\Omega})} + H_\beta(f).$$

For an integer $m \ge 0$, the space $C^m_\beta(\overline{\Omega})$ consists of all $f \in C^m(\overline{\Omega})$ such that

$$\frac{\partial^m f}{\partial x_1^{m_1} \partial x_2^{m_2}} \in C_\beta(\overline{\Omega}), \qquad m_1 + m_2 = m, \qquad 0 < \beta \le 1.$$

It is equipped with the norm

$$\|f\|_{C^m_\beta(\overline{\Omega})} = \|f\|_{C^m(\overline{\Omega})} + \sum_{m_1+m_2=m} H_\beta\left(\frac{\partial^m f}{\partial x_1^{m_1} \partial x_2^{m_2}}\right).$$

For $l = k + \beta$, where $k \ge 0$ is an integer and where $0 < \beta < 1$, we also write $C^l(\overline{\Omega}) = C^k_\beta(\overline{\Omega})$.

For $1 \le p < \infty$, the Lebesgue space $L_p(\Omega)$ consists of measurable functions for which

$$\|f\|_{L_p(\Omega)} = \left(\int_\Omega |f(x)|^p \, \mathrm{d}x\right)^{1/p} < \infty.$$

The space $L_\infty(\Omega)$ is equipped with the norm

$$\|f\|_{L_\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)|.$$

The Sobolev space $W^l_p(\Omega)$ consists of all functions $f \in L_p(\Omega)$ which have generalized derivatives of all forms up to and including the integral order $l \ge 1$ and the pth power of which are summable over Ω. The norm in $W^l_p(\Omega)$ is defined by

$$\|f\|_{W^l_p(\Omega)} = \sum_{k=0}^{l} \sum_{k_1+k_2=k} \left\| \frac{\partial^k f}{\partial x_1^{k_1} \partial x_2^{k_2}} \right\|_{L_p(\Omega)}.$$

Besov space. We use the complex variable $z = x + \mathrm{i}y$ on the plane $\mathbb{C} \equiv E$. The function $f(x, y) \equiv f(z)$ belongs to the Besov space $B^\alpha_{p,1}(\Omega)$, $1 \le p < \infty$, $0 < \alpha < 1$ if the following hold:

1) $f \in L_p(\Omega)$;
2) $\|f\|_{b^\alpha_{p,1}(\Omega)} \equiv \iint\limits_{|h|<\varepsilon} |h|^{-(2+\alpha)} \|\Delta_h f\|_{L_p(\Omega)} \mathrm{d}h_1 \mathrm{d}h_2 < \infty,$

where $\Delta_h f(z) = f(z+h) - f(z)$ is the difference with $h = h_1 + \mathrm{i}h_2$, $|h| < \varepsilon$, $\varepsilon > 0$. The difference $\Delta_h f(z)$ is defined for $z \in \Omega$ such that $[z,\ z+h] \subset \Omega$, if $z + h \notin \Omega$; we set $\Delta_h f(z) = 0$. The norm in $B^{\alpha}_{p,1}(\Omega)$ is defined by

$$\|f\|_{B^{\alpha}_{p,1}(\Omega)} = \|f\|_{L_p(\Omega)} + \|f\|_{b^{\alpha}_{p,1}(\Omega)}.$$

For an integer $r \geq 1$, we denote by $B^{r+\alpha}_{p,1}(\Omega)$ the space of functions $u \in L_p(\Omega)$, generalized Sobolev derivatives of order r,

$$f^{(r)}_j = \frac{\partial^r u}{\partial z^j \partial \overline{z}^{r-j}}, \qquad j = 0, 1, \ldots, r,$$

belonging to $B^{\alpha}_{p,1}(\Omega)$. The norm in this space is given by

$$\|f\|_{B^{r+\alpha}_{p,1}(\Omega)} = \|f\|_{L_p(\Omega)} + \sum_{j=1}^{r} \|f^{(r)}_j\|_{b^{\alpha}_{p,1}(\Omega)} < \infty.$$

For more details on Besov spaces (see [17]), equivalent norm characterizations can be found in [18,19].

2. Auxiliary results

In this section, we establish several auxiliary results that will be used in the proof of the main theorem.

Lemma 2.1. *Let* $0 < \alpha < 1$, $1 \leq p < \infty$, *and* $\Omega \subset E$ *be bounded with sufficiently smooth boundary* $\partial\Omega$. *If* $f \in B^{\alpha}_{p,1}(\Omega)$, *then* $\bar{z}f \in B^{\alpha}_{p,1}(\Omega)$, *and*

$$\|\bar{z} f\|_{B^{\alpha}_{p,1}(\Omega)} \leq C\|f\|_{B^{\alpha}_{p,1}(\Omega)}, \tag{2.1}$$

where C is a positive constant and not depending on f.

Proof: Because Ω is bounded, we have $M = \sup\limits_{z\in\Omega} |z| < \infty$. Then, $|\bar{z} f(z)| \leq M|f(z)|$, so it follows that

$$\|\bar{z} f\|_{L_p(\Omega)} \leq M\|f\|_{L_p(\Omega)}. \tag{2.2}$$

Moreover, for $|h| < \varepsilon$, we have $|\overline{z+h}| \leq M + \varepsilon$, and

$$\Delta_h \bar{z} f(z) = \overline{z+h}\, f(z+h) - \bar{z}\, f(z) = \overline{z+h}\, \Delta_h f(z) + \bar{h}\, f(z).$$

Applying the definition of the Besov seminorm, we have

$$\|\bar{z}f\|_{b^{\alpha}_{p,1}(\Omega)} \leq (M+\varepsilon)\|f\|_{b^{\alpha}_{p,1}(\Omega)} + \frac{2\pi}{1-\alpha}\varepsilon^{1-\alpha}\|f\|_{L_p(\Omega)}.$$

Combining these estimates, we obtain

$$\|\bar{z}f\|_{B^{\alpha}_{p,1}(\Omega)} \leq C\|f\|_{B^{\alpha}_{p,1}(\Omega)},$$

where, because ε is fixed, $C = C(\Omega, \alpha, \varepsilon)$, which is independent of f. Therefore, $\bar{z}f \in B^{\alpha}_{p,1}(\Omega)$. □

Consider the integral operators

$$Tf(z) = -\frac{1}{\pi}\iint_{\Omega} \frac{f(\tau)}{\tau - z}\,\mathrm{d}\Omega_\tau, \qquad \mathrm{d}\Omega_\tau = \mathrm{d}x\mathrm{d}y, \tag{2.3}$$

$$Kf(z) = -\frac{1}{\pi}\iint_{\Omega} \frac{f(\tau)}{1 - \bar{\tau}z}\,\mathrm{d}\Omega_\tau. \tag{2.4}$$

We now recall some fundamental result, which will be used in the sequel. The following special case follows from Theorem 1.1 in [20].

Lemma 2.2. *Let $0 < \alpha < 1$, $1 < p < \infty$, and $\Omega \subset E$ be bounded with sufficiently smooth boundary $\partial\Omega$. Then, Tf in* (2.3) *is a bounded operator from $B^{\alpha}_{p,1}(\Omega)$ to $B^{1+\alpha}_{p,1}(\Omega)$; that is one has the inequality*

$$\|Tf\|_{B^{1+\alpha}_{p,1}(\Omega)} \le C\|f\|_{B^{\alpha}_{p,1}(\Omega)}, \tag{2.5}$$

where C is a positive constant and is not dependent on f.

The following lemma is obtained from Lemma 1.1 in [20].

Lemma 2.3. *Let Ω be the unit disk $|z| < 1$ and $0 < \alpha < 1$, $1 < p < \infty$. Then, $Kf(z)$ in* (2.4) *belongs to $B^{1+\alpha}_{p,1}(\Omega)$, and*

$$\|Kf\|_{B^{1+\alpha}_{p,1}(\Omega)} \le C\|f\|_{B^{\alpha}_{p,1}(\Omega)}, \tag{2.6}$$

where C is a positive constant and is not dependent on f.

We will prove the following lemma.

Lemma 2.4. *Let Ω be the unit disk $|z| < 1$, and*

$$\mathcal{G}f(z) = \iint_{\Omega} g_0(z,\tau)\, f(\tau)\mathrm{d}\Omega_\tau, \qquad g_0(z,\tau) = \frac{2}{\pi}\log\Big|\frac{z-\tau}{1-z\bar{\tau}}\Big|. \tag{2.7}$$

If $f \in B^{\alpha}_{p,1}(\Omega)$, $1 < p < \infty$, $0 < \alpha < 1$, then $\mathcal{G}f \in B^{2+\alpha}_{p,1}(\Omega)$.

Proof: Because $g_0(z,\tau) = 1/\pi\Big(\log|z-\tau|^2 - \log|1 - z\bar{\tau}|^2\Big)$, we write $u = u_1 - u_2$, where

$$\mathcal{G}_1 f(z) = \frac{1}{\pi}\iint_{\Omega} \log|z-\tau|^2\, f(\tau)\mathrm{d}\Omega_\tau, \quad \mathcal{G}_2 f(z) = \frac{1}{\pi}\iint_{\Omega} \log|1 - z\bar{\tau}|^2\, f(\tau)\mathrm{d}\Omega_\tau.$$

Therefore, we obtain

$$\partial_z \mathcal{G}f(z) = \frac{1}{\pi}\iint_{\Omega} \frac{f(\tau)}{z-\tau}\mathrm{d}\Omega_\tau + \frac{1}{\pi}\iint_{\Omega} \frac{\bar{\tau}\,f(\tau)}{1 - z\bar{\tau}}\mathrm{d}\Omega_\tau = Tf(z) - K(\bar{\tau}f)(z).$$

By Lemma 2.2, we have $\|Tf\|_{B^{1+\alpha}_{p,1}(\Omega)} \le C\|f\|_{B^{\alpha}_{p,1}(\Omega)}$. Moreover, from Lemma 2.1, we get $\|\bar{\tau}f\|_{B^{\alpha}_{p,1}(\Omega)} \le C\|f\|_{B^{\alpha}_{p,1}(\Omega)}$, then so it follows from Lemma 2.3 that $\|K(\bar{\tau}f)\|_{B^{1+\alpha}_{p,1}(\Omega)} \le C\|f\|_{B^{\alpha}_{p,1}(\Omega)}$, which implies $\|\partial_z \mathcal{G}f\|_{B^{1+\alpha}_{p,1}(\Omega)} \le C\|f\|_{B^{\alpha}_{p,1}(\Omega)}$. Applying the same to $\partial_{\bar{z}}\mathcal{G}f$, we obtain $\|\partial_{\bar{z}}\mathcal{G}f\|_{B^{1+\alpha}_{p,1}(\Omega)} \le C\|f\|_{B^{\alpha}_{p,1}(\Omega)}$. By definition of a Besov space, if $v \in B^{1+\alpha}_{p,1}(\Omega)$, then $\partial_z v$, $\partial_{\bar{z}} v \in B^{\alpha}_{p,1}(\Omega)$. Applying this to $v = \partial_z \mathcal{G}f$, we obtain $\partial_z\partial_z u \in B^{\alpha}_{p,1}(\Omega)$, $\partial_{\bar{z}}\partial_z \mathcal{G}f \in B^{\alpha}_{p,1}(\Omega)$. Applying it to $v = \partial_{\bar{z}}\mathcal{G}f$, we obtain $\partial_z\partial_{\bar{z}} u \in B^{\alpha}_{p,1}(\Omega)$, $\partial_{\bar{z}}\partial_{\bar{z}}\mathcal{G}f \in B^{\alpha}_{p,1}(\Omega)$, which is equivalent to $\mathcal{G}f \in B^{2+\alpha}_{p,1}(\Omega)$. Moreover,

$$\|\mathcal{G}f\|_{B^{2+\alpha}_{p,1}(\Omega)} \le C\|f\|_{B^{\alpha}_{p,1}(\Omega)}.$$

The proof is complete. □

3. Main result

We first state the assumptions on the coefficients of Eq (1.1).

Assumptions on the coefficients.

Condition 3.1. *Assume that the coefficients*

$$a(x, y, u, u_x, u_y), \quad b(x, y, u, u_x, u_y), \quad c(x, y, u, u_x, u_y)$$

in (1.1) belong to the Besov space $B^{1+\alpha}_{p,1}(\Omega)$. Furthermore, we assume that $u^2+u_x^2+u_y^2 \le \Lambda$, where $\Lambda > 0$ is an arbitrarily fixed constant. For the same values of the arguments, the condition $ac - b^2 \ge \Lambda_0 > 0$ is satisfied, where $\Lambda_0 = \Lambda_0(\Lambda)$ is a positive constant.

For convenience, we set

$$w := u_x = \frac{\partial u}{\partial x}, \qquad \phi := u_y = \frac{\partial u}{\partial y}.$$

Thus, the coefficient functions $a(x, y, u, w, \phi)$, $b(x, y, u, w, \phi)$, and $c(x, y, u, w, \phi)$ are evaluated at $(w, \phi) = (u_x, u_y)$. In addition, the functions a, b, and c satisfy the Lipschitz condition with respect to the variables u, w, and φ*:*

$$|f(u_1, w_1, \varphi_1) - f(u_2, w_2, \varphi_2)| \le \Lambda_1(|u_1 - u_2| + |w_1 - w_2| + |\varphi_1 - \varphi_2|).$$

Condition 3.2. *Assume that the function $d(x, y, u, w, \varphi)$ satisfies the condition*

$$|d(x, y, u_1, w_1, \varphi_1) - d(x, y, u_2, w_2, \varphi_2)| \le d_0(x, y)(|u_1 - u_2| + |w_1 - w_2| + |\varphi_1 - \varphi_2|),$$

where $d_0(x, y) \in B^{\alpha}_{p,1}(\Omega)$.

Under these assumptions, we obtain the following main result.

Theorem 3.1. *Let Conditions 3.1 and 3.2 be satisfied and $\Omega \subset E$ be bounded with sufficiently smooth boundary $\partial\Omega$. Then, the problem* (1.1)–(1.3) *has a unique solution in $B^{2+\alpha}_{p,1}(\Omega) \hookrightarrow C^1(\overline{\Omega})$, $\alpha = 2/p - 1$, $1 < p < 2$.*

Proof: Let $\xi = \varphi(x, y)$, $t = \psi(x, y)$, be a conformal mapping of the unit disk $\xi^2 + t^2 < 1$ onto the domain Ω. The unit circle $\xi^2 + t^2 = 1$ corresponds to $\partial\Omega$. We assume that φ and ψ are from the space $B^{2+\alpha}_{p,1}(\Omega)$. Consequently, the functions φ and ψ are continuous in the closure; that is, $\varphi, \psi \in C^1(\overline{\Omega})$ [17].

We keep the same notation and assumption, without loss of generality, that Ω is the unit disk $x^2+y^2 < 1$. We reformulate the variable as a function of the complex coordinate $z = x + \mathrm{i}\,y$, $\overline{z} = x - \mathrm{i}\,y$, and introduce the complex derivatives $\partial_z = 1/2\,(\partial/\partial x - \mathrm{i}\partial/\partial y)$, $\partial_{\overline{z}} = 1/2\,(\partial/\partial x + \mathrm{i}\partial/\partial y)$. Then, the Eq (1.2) can be written in the form

$$\partial_z \partial_{\overline{z}} u + \mathrm{Re}[A(z, u, \partial_z u)\partial_z \partial_z u] + B(z, u, \partial_z u) = 0, \tag{3.1}$$

where $A(z, u, \partial_z u) = (a - c + 2\,\mathrm{i}\,b)/(a + c)$, $\quad B(z, u, \partial_z u) = d/(a + c)$.

Because $B^{1+\alpha}_{p,1}(\Omega)$ is a Banach algebra [20] and by Condition 3.1, we have

$$|a + c|^2 = (a - c)^2 + 4ac \ge 4\Lambda_0,$$

and hence, $|a + c| \geq 2\sqrt{\Lambda_0} > 0$. Therefore, the composition property of Besov spaces gives

$$(a + c)^{-1} \in B^{1+\alpha}_{p,1}(\Omega).$$

Because $a - c + 2ib \in B^{1+\alpha}_{p,1}(\Omega)$, the algebra property yields

$$A(z, u, u_z) = (a - c + 2ib)(a + c)^{-1} \in B^{1+\alpha}_{p,1}(\Omega).$$

For every fixed $\Lambda > 0$, there exists a constant $\Lambda_2 = \Lambda_2(\Lambda) < 1$ such that

$$|A(z, u, v)| \leq \Lambda_2, \qquad z \in \partial\Omega, \quad |u| + |v| \leq \Lambda. \tag{3.2}$$

Moreover, it satisfies the Lipschitz condition with respect to z:

$$|A(z, u_1, v_1) - A(z, u_2, v_2)| \leq \Lambda_1(|u_1 - u_2| + |v_1 - v_2|), \tag{3.3}$$

where $\Lambda_1 > 0$ is independent of z. The function $B(z, u, v)$ is measurable in z, and $B(z, 0, 0) \in B^{\alpha}_{p,1}(\Omega)$, and

$$|B(z, u_1, v_1) - B(z, u_2, v_2)| \leq B_0(z)(|u_1 - u_2| + |v_1 - v_2|), \tag{3.4}$$

where $B_0(z) \in B^{\alpha}_{p,1}(\Omega)$.

To reformulate the boundary value problem as an equation for a function in $B^{\alpha}_{p,1}(\Omega)$, we use the operator $\mathcal{G}$ introduced in (2.7). We set

$$f := \partial_z \partial_{\bar{z}} u = \frac{1}{4}\Delta u,$$

where Δ is the Laplace operator, and we represent the solution as

$$u = \mathcal{G}f.$$

This representation automatically satisfies the homogeneous boundary condition because $\mathcal{G}f = 0$ on $\partial\Omega$. If $f \in B^{\alpha}_{p,1}(\Omega)$, then Lemma 2.4 gives $\mathcal{G}f \in B^{2+\alpha}_{p,1}(\Omega)$. Furthermore, $\partial_z \partial_{\bar{z}} \mathcal{G}f = f$.

Next, we extend f, defined on the unit disk $\Omega = \{z \in E : |z| < 1\}$, to the entire complex plane by the inversion formula

$$f^*(z) = \begin{cases} f(z), & |z| < 1, \\ -\dfrac{1}{|z|^4} f\left(\dfrac{1}{\bar{z}}\right), & |z| \geq 1. \end{cases}$$

This definition is well defined because when $|z| \geq 1$, the point $1/\bar{z}$ belongs to the closed unit disk $\overline{\Omega}$. Thus, f^* is an inversion extension of f rather than its extension by zero. Using this extension, we obtain

$$\partial_z G f = T f^*,$$

where T is the operator defined in (2.3).

By the extension result in [21], the inversion extension defined above preserves the required Besov regularity. Therefore,

$$f \in B^{\alpha}_{p,1}(\Omega) \quad \Longrightarrow \quad f^* \in B^{\alpha}_{p,1}(E).$$

It is known [22, 23] that if $f^* \in L_p(E),\ p > 1$, then Tf_* admits a derivative with respect to z, which is expressed by the singular integral

$$Sf(z) = \frac{\partial T f^*(z)}{\partial z} = -\frac{1}{\pi}\iint_E \frac{f^*(\tau)}{(\tau - z)^2}\,\mathrm{d}\tau. \tag{3.5}$$

Moreover, it is known (see [24]) that the operator S is bounded on $L^p(E)$ for every $1 < p < \infty$, so there exists a constant $\Lambda_p = \|S\|_{L^p(E)\to L^p(E)}$ such that

$$\|Sf^*\|_{L^p(E)} \leq \Lambda_p \|f^*\|_{L^p(E)}.$$

Applying the difference operator to the singular integral operator (3.5) for $h \in \mathbb{C}$, we have

$$\begin{aligned}\Delta_h Sf(z) &= -\frac{1}{\pi}\iint_E \frac{f^*(\tau)}{(\tau - z - h)^2}\mathrm{d}\tau + \frac{1}{\pi}\iint_E \frac{f^*(\tau)}{(\tau - z)^2}\mathrm{d}\tau\\ &= -\frac{1}{\pi}\iint_E f^*(\tau)\left[\frac{1}{(\tau - z - h)^2} - \frac{1}{(\tau - z)^2}\right]\mathrm{d}\tau.\end{aligned}$$

We may write

$$S(\Delta_h f)(z) = -\frac{1}{\pi}\iint_E \frac{f^*(\tau + h) - f^*(\tau)}{(\tau - z)^2}\mathrm{d}\tau.$$

The change of variables by $t = \tau + h$. Then,

$$S(\Delta_h f)(z) = -\frac{1}{\pi}\iint_E \frac{f^*(t)}{(t - z - h)^2}\mathrm{d}t + \frac{1}{\pi}\iint_E \frac{f^*(t)}{(t - z)^2}\mathrm{d}t.$$

Thus, for $h \in E$,

$$\Delta_h(Sf^*)(z) = S(\Delta_h f^*)(z).$$

Applying the L^p-boundedness of S to $\Delta_h f^*$ gives

$$\|\Delta_h(Sf^*)\|_{L^p(E)} \leq \Lambda_p \|\Delta_h f^*\|_{L^p(E)}.$$

Multiplying this inequality by $|h|^{-(2+\alpha)}$ and integrating over $|h| < \varepsilon$, we obtain

$$\begin{aligned}\|Sf^*\|_{b^\alpha_{p,1}(E)} &= \int_{|h|<\varepsilon} |h|^{-(2+\alpha)} \|\Delta_h(Sf^*)\|_{L^p(E)}\,\mathrm{d}h\\ &\leq \Lambda_p \int_{|h|<\varepsilon} |h|^{-(2+\alpha)} \|\Delta_h f^*\|_{L^p(\mathbb{C})}\,\mathrm{d}h\\ &= \Lambda_p \|f^*\|_{b^\alpha_{p,1}(E)}.\end{aligned}$$

Combining this estimate with the L^p estimate yields

$$\|Sf^*\|_{B^\alpha_{p,1}(E)} \leq \Lambda_p \|f^*\|_{B^\alpha_{p,1}(E)}.$$

Because the extension operator is bounded, there exists a constant $C_1 > 0$ such that

$$\|f^*\|_{B^\alpha_{p,1}(E)} \leq C_1 \|f\|_{B^\alpha_{p,1}(\Omega)}.$$

Therefore, by restriction to Ω,

$$\|Sf\|_{B^{\alpha}_{p,1}(\Omega)} \le C\|f\|_{B^{\alpha}_{p,1}(\Omega)},$$

where $C = \Lambda_p C_1$. Hence, S is a bounded linear operator from $B^{\alpha}_{p,1}(\Omega)$ into itself. Thus, using

$$u = \mathcal{G}f, \qquad \partial_z\partial_{\bar{z}}u = f, \qquad \partial_z u = Tf, \qquad \partial_z\partial_z u = Sf,$$

Equation (3.1) becomes

$$f(z) + \operatorname{Re}\left[A(z,\mathcal{G}f,Tf)Sf\right] + B(z,\mathcal{G}f,Tf) = 0. \tag{3.6}$$

Consider the operator

$$\mathcal{L}f = -\operatorname{Re}\left[A(z,\mathcal{G}f,Tf)Sf\right] - B(z,\mathcal{G}f,Tf). \tag{3.7}$$

By Lemmas 2.1–2.4, the operator $\mathcal{L}$ defined by (3.7) is a bounded mapping in $B^{\alpha}_{p,1}(\Omega)$. Moreover, if $f_1, f_2 \in B^{\alpha}_{p,1}(\Omega)$, we have

$$\mathcal{L}f_1 - \mathcal{L}f_2 = -\operatorname{Re}\Big[(A_1 - A_2)Sf_1 + A_2(Sf_1 - Sf_2)\Big] - (B_1 - B_2),$$

where $A_j = A(z,\mathcal{G}f_j,Tf_j)$, $B_j = B(z,\mathcal{G}f_j,Tf_j)$, $j = 1,2$, and $|\mathcal{L}f_1 - \mathcal{L}f_2| \le |(A_1 - A_2)Sf_1| + |A_2(Sf_1 - Sf_2)| + |B_1 - B_2|$. By Lemmas 2.1–2.4, and by Conditions (3.2)–(3.4), we obtain

$$|\mathcal{L}f_1 - \mathcal{L}f_2| \le \Lambda_2|S(f_1 - f_2)| + (2\Lambda_1 C_0|Sf_1| + 2C_0B_0(z))\|f_1 - f_2\|_{B^{\alpha}_{p,1}(\Omega)}. \tag{3.8}$$

Because $B^{\alpha}_{p,1}(\Omega)$ is a Banach algebra [20], using Conditions (3.1) and (3.2), and the boundedness of $\mathcal{G}$, T, and S, we obtain from (3.8)

$$\|\mathcal{L}f_1 - \mathcal{L}f_2\|_{B^{\alpha}_{p,1}(\Omega)} \le C_p\|f_1 - f_2\|_{B^{\alpha}_{p,1}(\Omega)}, \tag{3.9}$$

where $C_p = \Lambda_2\,\Lambda_p + 2\Lambda_1C_0\Lambda_p\|f_1\|_{B^{\alpha}_{p,1}(\Omega)} + 2C_0\|B_0\|_{B^{\alpha}_{p,1}(\Omega)}$.

Because $\Lambda_p = \|S\|_{L_p}$ depends on p [24], and $\|S\|_{L_2} = 1$ ([7], Chapter 4 §9), it follows that for any fixed $\Lambda > 0$, there exists $\epsilon = \epsilon(\Lambda) > 0$ such that

$$\Lambda_2(\Lambda)\Lambda_p < 1 \quad \text{if} \quad 0 < 2 - p \le \epsilon(\Lambda). \tag{3.10}$$

For any p satisfying (3.10), we choose $\rho > 0$, $\delta > 0$ such that $2K\rho < \Lambda$, and therefore,

$$\|B_0\|_{B^{\alpha}_{p,1}(\Omega)} \le \delta, \quad \text{and} \quad \mathcal{S}(0,\rho) : \|f\|_{B^{\alpha}_{p,1}(\Omega)} \le \rho. \tag{3.11}$$

From (3.9) and (3.11), we obtain

$$\|\mathcal{L}f_1 - \mathcal{L}f_2\|_{B^{\alpha}_{p,1}(\Omega)} \le \beta\|f_1 - f_2\|_{B^{\alpha}_{p,1}(\Omega)}, \quad C_p \le \beta < 1, \tag{3.12}$$

where $\beta = \Lambda_2(\Lambda)\Lambda_p + 2\Lambda_1\rho C_0\Lambda_p + 2C_0\delta < 1$. Taking into account that $f_0 = 0$, let us assume that $\|\mathcal{L}f_0\|_{B^{\alpha}_{p,1}(\Omega)} = \|B(z,0,0)\|_{B^{\alpha}_{p,1}(\Omega)} < (1-\beta)\rho$. It follows that the operator $\mathcal{L}$ maps the closed ball $\overline{\mathcal{S}}(0,\rho)$ into itself, and inequality (3.12) holds within this ball, and that $\mathcal{L}$ is a contraction mapping in $\overline{\mathcal{S}}(0,\rho)$. Therefore, the contraction mapping principle is applicable in $\overline{\mathcal{S}}(0,\rho)$, which implies that $f - \mathcal{L}f = 0$ has a unique solution $f \in B^{\alpha}_{p,1}(\Omega)$ that belongs to this ball, $\|f\|_{B^{\alpha}_{p,1}(\Omega)} \le \rho$. If $B(z,0,0) = 0$, then the solution of $f - \mathcal{L}f = 0$ is $f = 0$. Thus, if the functions $B(z,0,0)$ and $B_0(z)$ are sufficiently small in the $B^{\alpha}_{p,1}(\Omega)$-norm, the quasilinear Eq (3.1) with boundary condition (1.3) has a solution. This solution is given explicitly by Formula (2.7) and therefore belongs to the space $B^{\alpha+2}_{p,1}(\Omega)$. Because $B^{2+\alpha}_{p,1}(\Omega) \hookrightarrow C^1(\overline{\Omega})$, the solution and its first-order derivatives are continuous on $\overline{\Omega}$. This completes the proof. □

4. Conclusions

In this work, the assumptions in Conditions 3.1 and 3.2 are based on the method, which investigate the quasilinear elliptic equation without reducing to canonical form in [7, Chapter 9, page 333]. Our main contribution is the extension of Vekua's result to a broader functional space while preserving the original assumptions on the coefficients.

Author contributions

This work was carried out in collaboration with two authors. Nazarbay Bliev: designed the study and guided the research. Nurlan Yerkinbayev: performed the analysis and wrote the first draft of the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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