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Research article

Global boundedness and stability of solutions to a pest–natural enemy chemotaxis system

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Abstract: Motivated by the dynamics of codling moth pests and their natural enemies in fruit forestry, this paper develops a predator–prey chemotaxis system formulated through partial differential equations. For the proposed model incorporating a Holling type I functional response, the global boundedness of its solutions is proved via a combination of semigroup theory and energy estimates. Moreover, by constructing appropriate Lyapunov functionals, we demonstrate that under weak predation pressure, the prey-only equilibrium is globally asymptotically stable, whereas under strong predation pressure, sufficient conditions ensuring the global asymptotic stability of the positive constant equilibrium are established.

Keywords: predator–prey system; chemotaxis; global boundedness; local stability; asymptotic stability
Mathematics Subject Classification: 35K55, 35Q92

1. Introduction

In nature, many organisms exhibit chemotaxis—the ability to move directionally in response to chemical cues. Chemotactic behaviors are common in a variety of biological contexts; typical examples include insects releasing sex pheromones to attract mates and bees locating nectar by floral scent and color. In recent years, biological chemotaxis systems modeled by partial differential equations have found increasingly broad applications in the life sciences and related fields—for instance, chemotaxis models of virus transmission characterizing the formation of infection hot spots [1–3] and coupled chemotaxis–haptotaxis systems describing the infiltration of tumor cells into surrounding normal tissues [4–6]. In 1970, Keller and Segel [7] proposed the first biological

chemotaxis system, which is given by

$$\begin{cases} P_t = \Delta P - \nabla \cdot (P \nabla S), & x \in \Omega, t > 0, \\ S_t = \Delta S - S + P, & x \in \Omega, t > 0. \end{cases}$$

Consider a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $P(x,t)$ be the density of cells or bacteria and $S(x,t)$ the concentration of the chemical attractant. The term ΔP describes the random diffusion of the cells, while $-\nabla \cdot (P\nabla S)$ captures their tendency to move up the chemical gradient. Moreover, ΔS represents the diffusion of the chemical substance. There are also many scholars who have extensively studied the existence, boundedness, blow-up, and long-time behavior of solutions to such systems; see, e.g., [8–10].

Chemotaxis of predators holds considerable potential for the biological regulation of pest insects. In 1987, Kareiva and Odell [11] proposed a system describing the aggregation of predators in regions of high prey density to model the clustering phenomenon arising from the interaction between ladybirds and aphids. This system reads as follows:

$$\begin{cases} P_t = d_1 \Delta P - \nabla \cdot (\chi(P) \nabla S) - Ph(P) + bg(S)P, & x \in \Omega,\ t > 0, \\ S_t = d_2 \Delta S + k(S) - g(S)P, & x \in \Omega,\ t > 0, \end{cases}$$

where the interspecific interaction term is expressed as $g(S)P$. Moreover, the intraspecific interaction terms are represented by $Ph(P)$ and $k(S)$. In addition, $g(S)$ is the functional response function. The system captures the phenomenon that predators aggregate toward regions of high prey density, thereby locating prey more efficiently. Subsequently, numerous scholars have investigated and extended this system [12–15]. According to Ainseba [13], the system has global weak solutions. Under the dimension condition $n \leq 3$, Tao [14] showed that solutions exist globally. Lee et al. [15] explored pattern formation within the system and showed that the prey-taxis term contributes to system stabilization. Wu et al. [16] showed that regardless of the spatial dimension, the global existence and uniform boundedness of solutions hold whenever the chemotactic sensitivity coefficient is taken sufficiently small. Jin and Wang [17] also proved the boundedness and stability of global solutions for the system in a bounded domain when the spatial dimension satisfies $n \leq 2$. Furthermore, Ren and Liu [18] introduced rotational flux terms and established the existence of global generalized solutions in arbitrary dimensions. On this basis, scholars [19–21] further incorporated attraction-repulsion dual-taxis mechanisms and systematically proved the global existence and stability of solutions under various settings. Zhang et al. [22] extended this framework to a three-species food chain with alarm-taxis, obtaining global stability of multiple steady states in high dimensions under suitably large logistic sources. These works significantly enrich the mathematical theory of predator-prey taxis models and provide a more comprehensive description of ecological pattern formation.

In 2016, Tello and Wrzosek [23] introduced a predator–prey model incorporating an indirect prey-taxis term, which accounts for predator migration toward regions where a chemical signal emitted by the prey is more concentrated. This system reads as follows:

$$\begin{cases} P_t = d_P \Delta P - \nabla(\chi P \nabla S), & x \in \Omega,\ t > 0, \\ S_t = d_S \Delta S - \mu S + \alpha H F_0(P), & x \in \Omega,\ t > 0, \\ H_t = \lambda H \left(1 - \dfrac{H}{\kappa}\right) - H F_0(P), & x \in \Omega,\ t > 0. \end{cases}$$

Here, P and H represent the densities of predator and prey, respectively, while S stands for the concentration of the pheromone emitted by the prey. The parameters d_P and d_S are the diffusion rates of the predator and the chemoattractant, χ the chemotactic sensitivity, μ the degradation rate of the attractant, λ the prey's intrinsic growth rate, κ the environmental carrying capacity, and $F_0(P)$ the prey mortality induced by predation. Tello proved that the system admits global solutions. Moreover, when the prey's growth rate surpasses its predation-induced death rate, a positive steady state persists. Subsequently, they performed a linearization around the constant steady state and obtained conditions ensuring its local asymptotic stability. Furthermore, for more discussions on the existence and boundedness of global classical solutions and the asymptotic stability of steady states for multi-species chemotaxis–predator systems under suitable parameter conditions, we refer to [24, 25].

In fruit forestry, particularly in the fragrant pear (Korla pear) and apple industries, the codling moth causes extremely severe damage. It has been found that *Trichogramma*, a natural enemy that parasitizes codling moth eggs, can rapidly locate the eggs by following the pheromone released by them. The wasps crawl on the egg surface, continuously tap the eggshell, accurately identify the freshest codling moth eggs, and then oviposit and reproduce inside them. Based on the principal chemotactic interactions among codling moth eggs, *Trichogramma*, and the pheromone released by the eggs in fruit forestry pest control, this paper formulates the following biological chemotaxis system described by partial differential equations:

$$\begin{cases} P_t = d_1\Delta P - \nabla\cdot(\chi P\nabla S) + aPH - bP - lP^2, & x\in\Omega,\ t>0,\\ S_t = d_2\Delta S - \alpha S + \beta H, & x\in\Omega,\ t>0,\\ H_t = BH\left(1-\dfrac{H}{N}\right) - PH, & x\in\Omega,\ t>0,\\ \dfrac{\partial P}{\partial \nu} = \dfrac{\partial S}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0,\\ P(x,0) = P_0(x),\ S(x,0) = S_0(x),\ H(x,0) = H_0(x), & x\in\Omega. \end{cases} \tag{1.1}$$

Let $\Omega\subset\mathbb{R}^n$ (with $n\le 2$) be a bounded domain possessing a smooth boundary $\partial\Omega$. The functions $P(x,t)$ and $H(x,t)$ represent the population densities of the predator (*Trichogramma*) and the prey (codling moth eggs), respectively, while $S(x,t)$ denotes the concentration of a chemical attractant released by the eggs. The term $-\nabla\cdot(P\nabla S)$ captures the tendency of *Trichogramma* to migrate toward regions of higher attractant concentration. The positive parameters d_1 and d_2 are the diffusion rates of the predator and the attractant; χ is the chemotactic sensitivity. Moreover, a represents the predator's growth rate via predation, b is the natural death rate, and the term $-lP^2$ (with $l>0$) describes intraspecific competition among predators. The constants α and β are, respectively, the degradation and production rates of the chemoattractant. Finally, B denotes the intrinsic growth rate of the prey population, and N its environmental carrying capacity.

Based on the chemotactic interactions among codling moth eggs, the pheromone released by the eggs, and the natural enemy *Trichogramma*, this paper formulates a predator–prey chemotaxis system. The global boundedness of solutions is established using semigroup theory and energy methods. By applying the linearization method, the number of equilibrium solutions and the conditions for their local stability are derived. Appropriate Lyapunov functionals are then established to demonstrate the global stability of the system and to derive criteria for the asymptotic stability of the equilibrium solutions. These conditions reveal that the stability of the semi-trivial constant equilibrium is affected

by the predation rate of *Trichogramma*. When the predation efficiency of *Trichogramma* is sufficiently high to suppress the population growth of codling moth eggs, a stable positive equilibrium exists, thereby preventing pest outbreaks. The chemoattractant released by codling moth eggs modulates the dynamics of the natural enemy *Trichogramma*. When the chemotactic sensitivity takes a small value or alternatively when the diffusion rates of *Trichogramma* and the chemoattractant are large, the constant equilibrium is asymptotically stable. This result offers a significant theoretical foundation for pest management in fruit orchards.

2. Global boundedness of solutions

Theorem 2.1. *Suppose that $\Omega \subset \mathbb{R}^n$ (with $n \leq 2$) is a bounded domain having a smooth boundary $\partial\Omega$. The nonnegative initial conditions are such that $P_0 \in C^0(\overline{\Omega})$, $S_0 \in C^1(\overline{\Omega})$, and $H_0 \in C^0(\overline{\Omega})$, and they satisfy*

$$P_0(x), S_0(x) \geq 0, H_0(x) > 0, \qquad P_0(x) \not\equiv 0,$$

then system (1.1) *on $\Omega \times (0, \infty)$ admits a unique global bounded classical solution*

$$\begin{aligned} P &\in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)), \\ S &\in C^0(\overline{\Omega} \times [0, \infty); W^{1,0}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)), \\ H &\in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{0,1}(\overline{\Omega} \times (0, \infty)). \end{aligned}$$

In order to prove Theorem 2.1, we require the following lemmas. We will now state and prove these lemmas.

Lemma 2.2. *(Local existence) Suppose $\Omega \subset \mathbb{R}^n$ (with $n \leq 2$) is a bounded domain with a smooth boundary. The initial data are taken to satisfy $P_0 \in C^0(\overline{\Omega})$, $S_0 \in C^1(\overline{\Omega})$, and $H_0 \in C^0(\overline{\Omega})$, together with*

$$P_0(x), S_0(x) \geq 0, H_0(x) > 0, \qquad P_0(x) \not\equiv 0. \tag{2.1}$$

Then there exists a constant $T_{\max} \in (0, \infty]$ such that problem (1.1) *has a unique classical solution*

$$\begin{aligned} P &\in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)), \\ S &\in C^0(\overline{\Omega} \times [0, \infty); W^{1,0}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)), \\ H &\in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{0,1}(\overline{\Omega} \times (0, \infty)), \end{aligned}$$

with

$$P, S > 0 \quad \text{and} \quad 0 < H \leq m := \max\{\|H_0\|_{L^\infty(\Omega)}, N\}, \quad \text{for all } t > 0. \tag{2.2}$$

Moreover, if $T_{\max} < \infty$, then

$$\|P(\cdot, t)\|_{L^\infty(\Omega)} + \|\nabla S(\cdot, t)\|_{L^\theta(\Omega)} \to \infty, \quad \text{as } t \to T_{\max},$$

where $\theta > n$.

Proof. We first apply the Banach fixed point theorem to establish the local existence and extensibility of solutions (see [26]). Then, the maximum principle is used to deduce (2.2). Finally, we address

the positivity of P and H, which is essential for the well-definedness of the logarithmic Lyapunov functionals used later. Following the argument in the proof of Lemma 2.1 of [17], we rewrite the first equation of (1.1) as

$$P_t - d_1\Delta P + \chi\nabla S \cdot \nabla P + (\chi\Delta S - aH + b + lP)P = 0,$$

which is a linear parabolic equation for P with bounded coefficients on any finite time interval, due to the smoothness of solutions from the local existence result. Since $P_0 \geq 0$ and $P_0 \not\equiv 0$, the strong maximum principle implies $P(x,t) > 0$ for all $x \in \Omega$ and $t > 0$. For H, as noted above, the assumption $H_0(x) > 0$ directly ensures $H(x,t) > 0$. Hence, the logarithmic terms in the subsequent Lyapunov functionals are well defined. □

Lemma 2.3. *Let (P, S, H) be a solution of system* (1.1)*. Then there exists a positive constant A_1 such that*

$$\|P(\cdot,t)\|_{L^1(\Omega)} + \|S(\cdot,t)\|_{L^1(\Omega)} + \|H(\cdot,t)\|_{L^1(\Omega)} \leq A_1, \qquad \forall\, t \in (0, T_{\max}). \tag{2.3}$$

Moreover,

$$\int_t^{t+\tau}\int_\Omega P^2\,dx\,ds \leq A_2, \qquad \int_t^{t+\tau}\int_\Omega H^2\,dx\,ds \leq A_3, \qquad \forall\, t \in (0, \tilde{T}_{\max}), \tag{2.4}$$

where $\tilde{T}_{\max} \in (0, T_{\max} - \tau)$, $\tau = \min\{1, \frac{1}{2}T_{\max}\}$, and $A_2, A_3 > 0$ are constants.

Proof. We integrate the equations in (1.1) over Ω to obtain

$$\frac{d}{dt}\int_\Omega P = a\int_\Omega PH - b\int_\Omega P - l\int_\Omega P^2, \tag{2.5}$$

$$\frac{d}{dt}\int_\Omega S = -\alpha\int_\Omega S + \beta\int_\Omega H, \tag{2.6}$$

$$a\frac{d}{dt}\int_\Omega H = aB\int_\Omega H - \frac{aB}{N}\int_\Omega H^2 - a\int_\Omega PH. \tag{2.7}$$

By Young's inequality, a simple calculation yields

$$\begin{aligned}\frac{d}{dt}\left(\int_\Omega P + \int_\Omega S + a\int_\Omega H\right) + b\int_\Omega P + \alpha\int_\Omega S + a\int_\Omega H \\ = -l\int_\Omega P^2 + (a + \beta + aB)\int_\Omega H - \frac{aB}{N}\int_\Omega H^2 \\ \leq -l\int_\Omega P^2 - \frac{aB}{2N}\int_\Omega H^2 + c_1,\end{aligned} \tag{2.8}$$

where c_1 is a positive constant. Using Lemma 3.4 of Reference [27], we obtain (2.3). Integrating (2.8) over $(t, t+\tau)$ for $t \in (0, \tilde{T}_{\max})$ implies (2.4). □

Lemma 2.4. *Let (P, S, H) be a solution of system* (1.1)*. There exist positive constants A_4, A_5 such that*

$$\|\nabla S\|_{L^2(\Omega)} \leq A_4, \qquad \textit{for all } t \in (0, T_{\max}), \tag{2.9}$$

and

$$\int_t^{t+\tau}\int_\Omega |\Delta S|^2 \leq A_5, \qquad \textit{for all } t \in (0, \tilde{T}_{\max}), \tag{2.10}$$

where $\tilde{T}_{\max} \in (0, T_{\max} - \tau)$ and $\tau = \min\left\{1, \frac{1}{2}T_{\max}\right\}$.

Proof. The second equation of (1.1) is multiplied by $-2\Delta S$ and integrated by parts. Applying Young's inequality to the resulting expression yields

$$\frac{d}{dt}\int_\Omega |\nabla S|^2 + d_2\int_\Omega |\Delta S|^2 + 2\alpha\int_\Omega |\nabla S|^2 \le \frac{\beta^2}{d_2}\int_\Omega H^2. \tag{2.11}$$

Using Lemma 2.3 of Reference [28], we derive (2.9). Furthermore, integrating (2.11) with $(t, t+\tau)$ for $t \in (0, \tilde{T}_{\max})$ completes the proof of Lemma 2.4. □

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^n$ ($n \le 2$) be a bounded domain having a smooth boundary $\partial\Omega$ and (P, S, H) be a solution of system* (1.1)*. Then there exists a constant $A_6 > 0$ such that*

$$\|P(\cdot, t)\|_{L^2(\Omega)} \le A_6.$$

Proof. Multiplying the first equation of system (1.1) by P, integrating over Ω, and using (2.2) together with integration by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega P^2 &= -d_1\int_\Omega |\nabla P|^2 + \chi\int_\Omega P\nabla P\cdot\nabla S + a\int_\Omega P^2 H - b\int_\Omega P^2 - l\int_\Omega P^3 \\ &\le -d_1\int_\Omega |\nabla P|^2 + \chi\int_\Omega P\nabla P\cdot\nabla S + (am-b)\int_\Omega P^2 - l\int_\Omega P^3.\end{aligned} \tag{2.12}$$

By Young's inequality,

$$\chi\int_\Omega P\nabla P\cdot\nabla S \le \frac{d_1}{2}\int_\Omega |\nabla P|^2 + \frac{\chi^2}{2d_1}\int_\Omega P^2|\nabla S|^2. \tag{2.13}$$

Combining (2.12) and (2.13) yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega P^2 + \frac{d_1}{2}\int_\Omega |\nabla P|^2 \le \frac{\chi^2}{2d_1}\int_\Omega P^2|\nabla S|^2 + (am-b)\int_\Omega P^2.$$

To control the last integral, Hölder's inequality gives

$$\int_\Omega P^2|\nabla S|^2 \le \|P\|^2_{L^4(\Omega)}\|\nabla S\|^2_{L^4(\Omega)}.$$

Since $n \le 2$, the Gagliardo–Nirenberg inequality provides

$$\|P\|^2_{L^4(\Omega)} \le c_2\left(\|\nabla P\|_{L^2(\Omega)}\|P\|_{L^2(\Omega)} + \|P\|^2_{L^2(\Omega)}\right),$$

and

$$\|\nabla S\|^2_{L^4(\Omega)} \le c_3\left(\|D^2 S\|_{L^2(\Omega)}\|\nabla S\|_{L^2(\Omega)} + \|\nabla S\|^2_{L^2(\Omega)}\right).$$

Using elliptic regularity under Neumann boundary conditions together with (9), we have

$$\|D^2 S\|_{L^2(\Omega)} \le c_4\left(\|\Delta S\|_{L^2(\Omega)} + \|\nabla S\|_{L^2(\Omega)}\right) \le c_5\left(\|\Delta S\|_{L^2(\Omega)} + 1\right).$$

Consequently,

$$\int_\Omega P^2|\nabla S|^2 \le c_6\left(\|\nabla P\|_{L^2(\Omega)}\|P\|_{L^2(\Omega)} + \|P\|^2_{L^2(\Omega)}\right)\left(\|\Delta S\|_{L^2(\Omega)} + 1\right). \tag{2.14}$$

Applying Young's inequality to (2.14), we obtain

$$\int_\Omega P^2|\nabla S|^2 \le \frac{d_1}{2\chi^2}\|\nabla P\|^2_{L^2(\Omega)} + c_7\|P\|^2_{L^2(\Omega)}\left(\|\Delta S\|_{L^2(\Omega)} + 1\right).$$

Inserting this into the previous differential inequality yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega P^2 + \frac{d_1}{4}\int_\Omega |\nabla P|^2 \le c_8\left((am-b) + (\|\Delta S\|^2_{L^2(\Omega)} + 1)\right)\int_\Omega P^2. \tag{2.15}$$

Set $y(t) = \|P(\cdot,t)\|^2_{L^2(\Omega)}$. Then the above estimate can be rewritten as

$$y'(t) \le g(t)y(t),$$

where

$$g(t) = 2c_8\left((am-b) + (\|\Delta S\|^2_{L^2(\Omega)} + 1)\right).$$

By Lemma 2.3 of Reference [28], there exists $t_0 \in [t-\tau, t]$ such that

$$y(t_0) \le c_9,$$

with $\tau = \min\{1, T_{\max}/2\}$. From (2.10), we infer

$$\int_{t_0}^t g(s)\,ds \le 2c_8(A_5+1) + 2|am-b| := A_7,$$

where A_7 is a positive constant. Finally, applying Gronwall's inequality to $[t_0, t]$ gives

$$y(t) \le y(t_0)\exp\left(\int_{t_0}^t g(s)\,ds\right) \le A_6,$$

which completes the proof of Lemma 2.5. □

Lemma 2.6. *Let the same assumptions as in Lemma 2.2 hold. Then for all*

$$\theta \in \begin{cases}(1, \frac{nr}{n-r}), & r \le n,\\ [1,\infty], & r > n,\end{cases}$$

there exists a positive constant C such that

$$\|\nabla S(\cdot,t)\|_{L^\theta(\Omega)} \le C. \tag{2.16}$$

Proof. From (2.2), we have $0 < H \le m$, which immediately implies

$$|H(\cdot,t)\|_{L^r(\Omega)} \le m|\Omega|^{1/r}$$

for any $r \ge 1$. Applying the variation-of-constants formula to the second equation in (1.1) yields

$$S(\cdot,t) = e^{t(d_2\Delta-\alpha)}S(\cdot,t_0) + \beta\int_0^t e^{(t-s)(d_2\Delta-\alpha)}H(\cdot,s)\,ds. \tag{2.17}$$

By [29, Lemma 1.3], we derive that

$$\begin{aligned}\|\nabla S(\cdot,t)\|_{L^\theta(\Omega)} &\le \left\|\nabla e^{(t-s)(d_2\Delta-\alpha)}S(\cdot,t_0)\right\|_{L^\theta(\Omega)} + \beta\int_0^t \left\|\nabla e^{(t-s)(d_2\Delta-\alpha)}H(\cdot,s)\right\|_{L^r(\Omega)} ds\\ &\le c_{10}\|\nabla S(\cdot,t_0)\|_{L^\theta(\Omega)} + c_{11}\int_0^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{\theta})}\right)e^{-\lambda_1(t-s)}\|H(\cdot,s)\|_{L^r(\Omega)}ds\\ &\le c_{10}\|\nabla S(\cdot,t_0)\|_{L^\theta(\Omega)} + c_{12}\int_0^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{\theta})}\right)e^{-\lambda_1(t-s)}ds\\ &\le C,\end{aligned}$$

where c_i $(i = 10, 11, 12)$ are positive constants. □

Lemma 2.7. *Let (P, S, H) be a solution of system* (1.1)*. Then there exists a positive constant C such that*

$$\|P(\cdot,t)\|_{L^\infty(\Omega)} \le C, \qquad \text{for all } t \in (0, T_{\max}).$$

Furthermore, there exists $\sigma \in (0,1)$ such that

$$\|P\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\overline{\Omega}\times[t,t+1])} + \|S\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\overline{\Omega}\times[t,t+1])} \le C, \qquad \text{for all } t \in (0, T_{\max}). \tag{2.18}$$

Proof. First, we show the boundedness of $\|P(\cdot,t)\|_{L^3(\Omega)}$, which is required for the proof of the subsequent conclusions.

The first equation of system (1.1) is multiplied by P^2 and integrated by parts; Young's inequality is then applied, yielding

$$\begin{aligned}\frac{1}{3}\frac{d}{dt}\int_\Omega P^3 + \int_\Omega P^3 &= \int_\Omega (d_1\Delta P - \chi\nabla\cdot(P\nabla S) + aPH - bP - lP^2)P^2 + \int_\Omega P^3\\ &= -2d_1\int_\Omega |\nabla P|^2 P + 2\chi\int_\Omega P^2\nabla P\cdot\nabla S + a\int_\Omega P^3 H\\ &\quad - b\int_\Omega P^3 - l\int_\Omega P^4 + \int_\Omega P^3\\ &\le -d_1\int_\Omega |\nabla P|^2 P + \frac{\chi^2}{d_1}\int_\Omega P^3|\nabla S|^2\\ &\quad + a\int_\Omega P^3 H + (1-b)\int_\Omega P^3 - l\int_\Omega P^4.\end{aligned} \tag{2.19}$$

By Hölder's inequality and Young's inequality, together with (2.16), we derive

$$\frac{\chi^2}{d_1}\int_\Omega P^3|\nabla S|^2 \le \frac{\chi^2}{d_1}\left(\int_\Omega P^4\right)^{\frac{3}{4}}\left(\int_\Omega |\nabla S|^8\right)^{\frac{1}{4}} \le \frac{l}{4}\int_\Omega P^4 + c_{13}. \tag{2.20}$$

Combining (2.2) with Young's inequality leads to

$$a\int_\Omega P^3 H \le \frac{l}{4}\int_\Omega P^4 + c_{14}. \tag{2.21}$$

Using (2.19)–(2.21) together with Young's inequality gives

$$\frac{1}{3}\frac{d}{dt}\int_\Omega P^3 + \int_\Omega P^3 \le -d_1\int_\Omega |\nabla P|^2 P + (1-b)\int_\Omega P^3 - \frac{l}{2}\int_\Omega P^4 + c_{15} \le c_{16}.$$

Hence, we derive that

$$\|P(\cdot,t)\|_{L^3(\Omega)} \le c_{17}, \tag{2.22}$$

where c_i ($i = 13, 14, 15, 16, 17$) are positive constants.

Next, we prove the boundedness of $\|P(\cdot,t)\|_{L^\infty(\Omega)}$.

For any given $T \in (0, T_{\max})$, we suppose

$$M(T) := \sup_{0<t<T} \|P(\cdot,t)\|_{L^\infty(\Omega)}. \tag{2.23}$$

Invoking the associated variation-of-constants formula for the first equation of system (1.1), it follows that

$$P(\cdot,t) \le e^{(t-t_0)d_1\Delta}P(\cdot,t_0) - \int_{t_0}^t e^{(t-s)d_1\Delta}\nabla\cdot(P\nabla S)\,ds + a\int_{t_0}^t e^{(t-s)d_1\Delta}PH\,ds,$$

where $t \in (0,T]$. Then from [29, Lemma 1.3], we have

$$\begin{aligned}\|P(\cdot,t)\|_{L^\infty(\Omega)} \le{}& \|e^{(t-t_0)d_1\Delta}P(\cdot,t_0)\|_{L^\infty(\Omega)}\\ &+ \int_{t_0}^t \|e^{(t-s)d_1\Delta}\nabla\cdot(P\nabla S)\|_{L^\infty(\Omega)}\,ds\\ &+ a\int_{t_0}^t \|e^{(t-s)d_1\Delta}PH\|_{L^\infty(\Omega)}\,ds,\end{aligned} \tag{2.24}$$

where $t_0 := (t-1)_+$. In the case $0 \le t \le 1$, using the maximum principle for the heat equation, we obtain

$$\|e^{(t-t_0)d_1\Delta}P(\cdot,t_0)\|_{L^\infty(\Omega)} \le c_{18}\|P(\cdot,t_0)\|_{L^\infty(\Omega)}. \tag{2.25}$$

In the case $t > 1$, by [29, Lemma 1.3] and Lemma 2.3, we obtain

$$\|e^{(t-t_0)d_1\Delta}P(\cdot,t_0)\|_{L^\infty(\Omega)} \le c_{19}((t-t_0)^{-\frac{n}{2}})\|P(\cdot,t)\|_{L^1(\Omega)} \le c_{20}. \tag{2.26}$$

Combining Lemma 2.6, Lemma 2.3, and Hölder's inequality, we obtain

$$\begin{aligned}\int_{t_0}^t \left\|e^{(t-s)d_1\Delta}\nabla\cdot(P\nabla S)\right\|_{L^\infty(\Omega)}ds &\le c_{21}\int_{t_0}^t (1+(t-s)^{-\frac{1}{2}-\frac{n}{2}\cdot\frac{1}{p}})\|P\nabla S\|_{L^p(\Omega)}\,ds.\\ &\le c_{22}\int_{t_0}^t (1+(t-s)^{-\frac{1}{2}-\frac{n}{2}\cdot\frac{1}{p}})\|P\|_{(\Omega)}^{\frac{pq}{q-p}}\|\nabla S\|_{L^q(\Omega)}\,ds\\ &\le c_{23}\int_{t_0}^t (1+(t-s)^{-\frac{1}{2}-\frac{n}{2}\cdot\frac{1}{p}})\|P\|_{L^\infty(\Omega)}^{\lambda}\|P\|_{L^1(\Omega)}^{1-\lambda}\|\nabla S\|_{L^q(\Omega)}\,ds,\end{aligned}$$

where $n < p < q$, $q \in \left(n, \frac{nr}{(n-r)_+}\right)$ and $\lambda = 1 - \frac{q-p}{pq} \in (0,1)$.

When $p > n$, we have

$$\int_{t_0}^{t} \left(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2}\cdot\frac{1}{p}}\right) ds < \infty.$$

Rearranging yields

$$\int_{t_0}^{t} \left\| e^{(t-s)d_1\Delta} \nabla \cdot (P\nabla S) \right\|_{L^\infty(\Omega)} ds \le c_{24} M^\lambda(T). \tag{2.27}$$

A combination of (2.2) and (2.22) yields

$$\begin{aligned} a\int_{t_0}^{t} \|e^{(t-s)\Delta} PH\|_{L^\infty(\Omega)} ds &\le c_{25} \int_{t_0}^{t} \left(1 + (t-s)^{-\frac{n}{6}} e^{-\lambda_1(t-s)} \|PH\|_{L^3(\Omega)}\right) ds \\ &\le c_{26} \|P(\cdot,t)\|_{L^3(\Omega)} \int_{t_0}^{t} \left(1 + (t-s)^{-\frac{n}{6}}\right) ds \\ &\le c_{27}. \end{aligned} \tag{2.28}$$

Combining (2.25)–(2.28), we have

$$\|P(\cdot,t)\|_{L^\infty(\Omega)} \le c_{28} + c_{24} M^\lambda(T), \qquad \text{for all } t \in (0,T),$$

which implies

$$M(T) \le c_{28} + c_{24} M^\lambda(T), \qquad \text{for all } t \in (0, T_{\max}),$$

hence we derive that

$$M(T) \le \max\left\{1,\ (c_{28} + c_{24})^{\frac{1}{1-\lambda}}\right\}, \qquad \text{for all } t \in (0, T_{\max}),$$

where c_i ($i = 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28$) are positive constants.

From the boundedness of P, H and interior Schauder estimates, we deduce $S \in C^{2+\sigma, 1+\frac{\sigma}{2}}$. Rewriting the first equation of the system as

$$P_t = \Delta P - \nabla S \cdot \nabla P - P\Delta S + aPH - bP - lP^2$$

and using the Hölder regularity of $\nabla S, \Delta S$ and the bounded nonlinear terms, parabolic Schauder estimates give $P \in C^{2+\sigma, 1+\sigma/2}$. Finally, the ordinary differential equation (ODE) for H with Hölder continuous coefficients implies $H \in C^{\sigma, \sigma/2}$. Thus, Lemma 2.7 is proved.

Letting $\theta = \infty$ in Lemma 2.6, we obtain $\|\nabla S(\cdot,t)\|_{L^\infty(\Omega)} \le C$, and by Lemma 2.2, we obtain $T_{\max} = \infty$. Thus, the proof of Theorem 2.1 is complete. □

3. Linear stability and global stability of solutions

In this section, following the approach in [30], we consider the local stability and global asymptotic stability of system (1.1).

There are three constant steady states of the system (1.1): the trivial one $E_0(0,0,0)$, the semi-trivial one $E_1(0, \frac{\beta N}{\alpha}, N)$, and, whenever $aN > b$, a unique positive steady state $E_2(P^*, S^*, H^*)$ given by

$$P^* = \frac{B(aN-b)}{aN+lB}, \qquad S^* = \frac{\beta N(b+lB)}{\alpha(aN+lB)}, \qquad H^* = \frac{N(b+lB)}{aN+lB}.$$

According to [30], the linearized system of (1.1) at the constant steady state $(\overline{P}, \overline{S}, \overline{H})$ can be expressed as

$$\begin{pmatrix} \phi_t \\ \varphi_t \\ \psi_t \end{pmatrix} = (D\Delta + J_{(\bar{P},\bar{S},\bar{H})}) \begin{pmatrix} \phi \\ \varphi \\ \psi \end{pmatrix} \tag{3.1}$$

where

$$D = \begin{pmatrix} d_1 & -\chi\bar{P} & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad J_{(\bar{P},\bar{S},\bar{H})} = \begin{pmatrix} a\bar{H} - b - 2l\bar{P} & 0 & a\bar{P} \\ 0 & -\alpha & \beta \\ -\bar{H} & 0 & B - \frac{2B\bar{H}}{N} - \bar{P} \end{pmatrix}.$$

Then the linearized stability of $(\bar{P}, \bar{S}, \bar{H})$ is determined by the eigenvalue problem:

$$\begin{cases} d_1\Delta\phi - \chi\bar{P}\Delta\varphi + (a\bar{H} - b - 2l\bar{P})\phi + a\bar{P}\psi = \lambda\phi, & x \in \Omega, \\ d_2\Delta\varphi - \alpha\varphi + \beta\psi = \lambda\varphi, & x \in \Omega, \\ -\bar{H}\phi + \left(B - \dfrac{2B\bar{H}}{N} - \bar{P}\right)\psi = \lambda\psi, & x \in \Omega, \\ \dfrac{\partial\phi}{\partial\nu} = \dfrac{\partial\varphi}{\partial\nu} = \dfrac{\partial\psi}{\partial\nu} = 0, & x \in \partial\Omega, \end{cases} \tag{3.2}$$

Let $0 = \mu_1 < \mu_2 \le \mu_3 \le \cdots \le \mu_i \le \cdots$ $(i \in \mathbb{N})$ be the eigenvalues of the operator $-\Delta$ on Ω with homogeneous Neumann boundary conditions, let y_i be the corresponding eigenfunctions, and let $E(\bar{P}, \bar{S}, \bar{H})$ be the constant steady state. Then the linearization of model (1.1) at the steady state $(\bar{P}, \bar{S}, \bar{H})$ leads to the following matrix:

$$\Gamma(\mu_i, E) = \begin{pmatrix} -d_1\mu_i + a\bar{H} - b - 2l\bar{P} & \chi\bar{P}\mu_i & a\bar{P} \\ 0 & -d_2\mu_i - \alpha & \beta \\ -\bar{H} & 0 & B - \dfrac{2B}{N}\bar{H} - \bar{P} \end{pmatrix}. \tag{3.3}$$

For the local stability of nonnegative constant steady states, we have the following theorem.

Theorem 3.1. *(Local stability). Let $\Omega \subset \mathbb{R}^n$ $(n \le 2)$ be a bounded domain with a smooth boundary.*

(1) The trivial constant steady state $(0, 0, 0)$ *of system* (1.1) *is unstable.*

(2) When $aN < b$, the semi-trivial constant steady state $\left(0, \dfrac{\beta N}{\alpha}, N\right)$ *of system* (1.1) *is linearly stable.*

(3) When $aN > b$ and $0 < \chi < \dfrac{b_1 b_2 - b_3}{P^* H^* \beta \mu_i} := \chi(i)$, the positive constant steady state (P^*, S^*, H^*) of system (1.1) *is linearly stable, where*

$$P^* = \frac{B(aN - b)}{aN + lB}, \qquad S^* = \frac{\beta N(b + lB)}{\alpha(aN + lB)}, \qquad H^* = \frac{N(b + lB)}{aN + lB}.$$

Remark 1. $(0, 0, 0)$ *represents the extinction state. When the predation effect of Trichogramma is weak, the system reaches the steady state* $(0, \frac{\beta N}{\alpha}, N)$ *that depends only on the eggs of the codling moth; when the predation effect is strong, the system reaches the steady state of coexistence* (P^*, S^*, H^*).

Proof of Theorem 3.1. **(1)** The Jacobi matrix at $E_0(0,0,0)$ is

$$\Gamma(\mu_i, E_0) = \begin{pmatrix} -d_1\mu_i - b & 0 & 0 \\ 0 & -d_2\mu_i - \alpha & \beta \\ 0 & 0 & B \end{pmatrix}.$$

Thus if $i = 1$, then $B > 0$, and we get that E_0 is unstable by the Routh–Hurwitz criterion.

(2) The Jacobi matrix at $E_1\left(0, \frac{\beta N}{\alpha}, N\right)$ is

$$\Gamma(\mu_i, E_1) = \begin{pmatrix} -d_1\mu_i + aN - b & 0 & 0 \\ 0 & -d_2\mu_i - \alpha & \beta \\ -N & 0 & -B \end{pmatrix}.$$

The corresponding characteristic polynomial is

$$\begin{aligned} \det(\lambda I - \Gamma(\mu_i, E_1)) &= \begin{vmatrix} \lambda + d_1\mu_i - aN + b & 0 & 0 \\ 0 & \lambda + d_2\mu_i + \alpha & -\beta \\ N & 0 & \lambda + B \end{vmatrix} \\ &= (\lambda + d_1\mu_i - aN + b)(\lambda + d_2\mu_i + \alpha)(\lambda + B). \end{aligned}$$

Thus if $aN < b$, E_1 is linearly stable; if $aN > b$, E_1 is unstable.

(3) The Jacobi matrix at $E_2(P^*, S^*, H^*)$ is

$$\begin{aligned} \Gamma(\mu_i, E_2) &= \begin{pmatrix} -d_1\mu_i - lP^* & \chi P^*\mu_i & aP^* \\ 0 & -d_2\mu_i - \alpha & \beta \\ -H^* & 0 & -\dfrac{B}{N}H^* \end{pmatrix} \\ &= \begin{pmatrix} -d_1\mu_i - B_1 & \chi P^*\mu_i & aP^* \\ 0 & -d_2\mu_i - \alpha & \beta \\ -H^* & 0 & -B_2 \end{pmatrix}, \end{aligned}$$

where

$$B_1 = \frac{lB(aN - b)}{aN + lB}, \qquad B_2 = \frac{B(b + lB)}{aN + lB}.$$

The corresponding characteristic polynomial is

$$\det(\lambda I - \Gamma(\mu_i, E_2)) = \begin{vmatrix} \lambda + d_1\mu_i + B_1 & -\chi P^*\mu_i & -aP^* \\ 0 & \lambda + d_2\mu_i + \alpha & -\beta \\ H^* & 0 & \lambda + B_2 \end{vmatrix}$$

$$= (\lambda + d_1\mu_i + B_1)(\lambda + d_2\mu_i + \alpha)(\lambda + B_2) + \chi P^*\mu_i\beta H^* + aP^*(\lambda + d_2\mu_i + \alpha)H^*.$$

Thus the characteristic polynomial takes the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where

$$
\begin{aligned}
a_1 &= (d_1 + d_2)\mu_i + B_1 + B_2 + \alpha,\\
a_2 &= d_1 d_2 \mu_i^2 + ((d_1 + d_2)B_2 + d_1\alpha + B_1 d_2)\mu_i + \alpha(B_1 + B_2) + B_1 B_2 + aP^* H^*,\\
a_3 &= d_1 d_2 B_2 \mu_i^2 + ((d_1\alpha + d_2 B_1)B_2 + ad_2 P^* H^*)\mu_i + aP^* H^* \alpha + B_1 B_2 \alpha + \chi P^* H^* \beta \mu_i.
\end{aligned}
$$

Since $aN > b$, we have $a_1 > 0$ and $a_3 > 0$. Now set

$$
a_1 = b_1, \qquad a_2 = b_2, \qquad a_3 = b_3 + \chi P^* H^* \beta \mu_i.
$$

Then the condition $a_1 a_2 - a_3 > 0$ becomes

$$
b_1 b_2 - b_3 - \chi P^* H^* \beta \mu_i > 0,
$$

which is equivalent to

$$
0 < \chi < \frac{b_1 b_2 - b_3}{P^* H^* \beta \mu_i} := \chi(i).
$$

Therefore, by the Routh–Hurwitz criterion, if $aN > b$ and $0 < \chi < \dfrac{b_1 b_2 - b_3}{P^* H^* \beta \mu_i} := \chi(i)$, then the positive steady state (P^*, S^*, H^*) is linearly stable. □

Remark 2. *Theorem 3.1 shows that the positive equilibrium E_2 is locally asymptotically stable when $aN > b$ and χ is sufficiently small. If parameters vary continuously such that $a_1 a_2 = a_3$, the characteristic equation admits a pair of purely imaginary eigenvalues, indicating the possible occurrence of a Hopf bifurcation. Moreover, when $aN = b$, the positive equilibrium E_2 coalesces with the semi-trivial equilibrium E_1, suggesting a saddle-node bifurcation at this threshold. Similar bifurcation phenomena in predator–prey models have been studied in, e.g., [31], where a delayed Leslie–Gower fractional-order model with fear effect and prey refuge was considered. A rigorous bifurcation analysis of system (1.1) with respect to χ, a, N, and other parameters is left for future work.*

Theorem 3.2. *(Global stability) Assume that the conditions in Theorem 2.1 hold. Then the following conclusions hold:*

(1) If $aN < b$, $\dfrac{aB}{N} > \dfrac{\beta^2}{2\alpha}$, then

$$
\|P\|_{L^\infty(\Omega)} + \left\|S - \frac{\beta N}{\alpha}\right\|_{L^\infty(\Omega)} + \|H - N\|_{L^\infty(\Omega)} \to 0, \quad \text{as } t \to \infty.
$$

(2) If $aN > b$, $\chi^2 < \dfrac{4d_1 d_2}{P^*}$, and $\dfrac{aB}{N} > \dfrac{\beta^2}{2\alpha}$, then

$$
\|P - P^*\|_{L^\infty(\Omega)} + \|S - S^*\|_{L^\infty(\Omega)} + \|H - H^*\|_{L^\infty(\Omega)} \to 0, \quad \text{as } t \to \infty.
$$

The proof of Theorem 3.2 will be carried out by constructing suitable Lyapunov functionals, drawing upon the boundedness of solutions to system (1.1) proven in Theorem 2.1. As a first step, we will show the stability of the semi-trivial constant steady state.

Lemma 3.3. *Let (P, S, H) be a solution of system* (1.1) *and let $aN < b$. Define the following functional:*

$$E_1(t) = \int_\Omega P\,dx + \frac{1}{2}\int_\Omega \left(S - \frac{\beta N}{\alpha}\right)^2 dx + a\int_\Omega \left(H - N - N\ln\frac{H}{N}\right)dx.$$

If the parameters satisfy $\dfrac{aB}{N} > \dfrac{\beta^2}{2\alpha}$, then

$$\lim_{t\to\infty}\left(\|P\|_{L^\infty(\Omega)} + \left\|S - \frac{\beta N}{\alpha}\right\|_{L^\infty(\Omega)} + \|H - N\|_{L^\infty(\Omega)}\right) = 0.$$

Proof.

$$\begin{aligned}\frac{dE_1(t)}{dt} &= \int_\Omega P_t\,dx + \int_\Omega \left(S - \frac{\beta N}{\alpha}\right)S_t\,dx + a\int_\Omega \left(\frac{H-N}{H}\right)H_t\,dx\\ &= a\int_\Omega PH\,dx - b\int_\Omega P\,dx - l\int_\Omega P^2\,dx - d_2\int_\Omega |\nabla S|^2\,dx\\ &\quad - \int_\Omega \alpha\left(S - \frac{\beta N}{\alpha}\right)S\,dx + \beta\int_\Omega \left(S - \frac{\beta N}{\alpha}\right)H\,dx\\ &\quad + a\int_\Omega \frac{H-N}{H}\left(BH\left(1 - \frac{H}{N}\right) - PH\right)dx. \end{aligned} \tag{3.4}$$

Let

$$I_1(t) = -\alpha\int_\Omega \left(S - \frac{\beta N}{\alpha}\right)S\,dx + \beta\int_\Omega \left(S - \frac{\beta N}{\alpha}\right)H\,dx,$$

$$I_2(t) = a\int_\Omega \frac{H-N}{H}\left(BH\left(1 - \frac{H}{N}\right) - PH\right)dx.$$

Then

$$\begin{aligned} I_1(t) &= -\alpha\int_\Omega \left(S - \frac{\beta N}{\alpha}\right)^2 dx - \beta N\int_\Omega \left(S - \frac{\beta N}{\alpha}\right)dx\\ &\quad + \beta\int_\Omega \left(S - \frac{\beta N}{\alpha}\right)(H - N)dx + \beta N\int_\Omega \left(S - \frac{\beta N}{\alpha}\right)dx\\ &= -\alpha\int_\Omega \left(S - \frac{\beta N}{\alpha}\right)^2 dx + \beta\int_\Omega \left(S - \frac{\beta N}{\alpha}\right)(H - N)dx, \end{aligned} \tag{3.5}$$

$$\begin{aligned} I_2(t) &= a\int_\Omega \frac{H-N}{H}\left(BH\left(1 - \frac{H}{N}\right) - PH\right)dx\\ &= -a\int_\Omega (H - N)P\,dx - \frac{aB}{N}\int_\Omega (H - N)^2 dx. \end{aligned} \tag{3.6}$$

Combining (3.4)–(3.6), we obtain

$$\begin{aligned}\frac{dE_1(t)}{dt} &= a\int_\Omega P(H - N)\,dx + (aN - b)\int_\Omega P\,dx - l\int_\Omega P^2\,dx - d_2\int_\Omega |\nabla S|^2\,dx\\ &\quad - \alpha\int_\Omega \left(S - \frac{\beta N}{\alpha}\right)^2 dx + \beta\int_\Omega \left(S - \frac{\beta N}{\alpha}\right)(H - N)\,dx\end{aligned}$$

$$
\begin{aligned}
&- a\int_\Omega (H-N)P\,dx - \frac{aB}{N}\int_\Omega (H-N)^2 dx \\
&= (aN-b)\int_\Omega P\,dx - l\int_\Omega P^2\,dx - d_2\int_\Omega |\nabla S|^2\,dx - \alpha\int_\Omega \left(S-\frac{\beta N}{\alpha}\right)^2 dx \\
&\quad + \beta\int_\Omega \left(S-\frac{\beta N}{\alpha}\right)(H-N)\,dx - \frac{aB}{N}\int_\Omega (H-N)^2 dx \\
&\le (aN-b)\int_\Omega P\,dx - l\int_\Omega P^2\,dx - \frac{\alpha}{2}\int_\Omega \left(S-\frac{\beta N}{\alpha}\right)^2 dx \\
&\quad + \left(\frac{\beta^2}{2\alpha}-\frac{aB}{N}\right)\int_\Omega (H-N)^2 dx.
\end{aligned}
$$

If $aN < b$ and $\frac{aB}{N} > \frac{\beta^2}{2\alpha}$, then $\frac{dE_1(t)}{dt} \le 0$ for $t > 0$. Since $(P,S,H) = \left(0, \frac{\beta N}{\alpha}, N\right)$ is a solution of system (1.1), when $\frac{dE_1(t)}{dt} = 0$, we have $(P,S,H) = \left(0, \frac{\beta N}{\alpha}, N\right)$. Finally, applying LaSalle's invariance principle, we obtain that (P,S,H) converges to $\left(0, \frac{\beta N}{\alpha}, N\right)$ in the space $L^\infty(\Omega)$ as $t \to \infty$. Hence, conclusion (1) of Theorem 3.2 follows. □

Next, we further investigate the convergence of the system toward its positive constant steady state.

Lemma 3.4. *Let (P,S,H) be a solution of system* (1.1) *and let $aN > b$. Define the following functional:*

$$
\begin{aligned}
E_2(t) = &\int_\Omega \left(P - P^* - P^* \ln\frac{P}{P^*}\right)dx + \frac{1}{2}\int_\Omega (S-S^*)^2 dx \\
&+ a\int_\Omega \left(H - H^* - H^* \ln\frac{H}{H^*}\right)dx.
\end{aligned}
$$

If $\chi^2 < \frac{4d_1d_2}{P^*}$ and $\frac{aB}{N} > \frac{\beta^2}{2\alpha}$, then

$$
\lim_{t\to\infty}\left(\|P-P^*\|_{L^\infty(\Omega)} + \|S-S^*\|_{L^\infty(\Omega)} + \|H-H^*\|_{L^\infty(\Omega)}\right) = 0.
$$

Proof.

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega \left(P - P^* - P^*\ln\frac{P}{P^*}\right)dx &= \int_\Omega \left(\frac{P-P^*}{P}\right)P_t\,dx \\
&= \int_\Omega \left(\frac{P-P^*}{P}\right)(d_1\Delta P - \nabla\cdot(\chi P\nabla S))dx \\
&\quad + \int_\Omega \left(\frac{P-P^*}{P}\right)(aPH - bP - lP^2)dx \qquad (3.7)\\
&= -d_1P^*\int_\Omega \frac{|\nabla P|^2}{P^2}dx + P^*\chi\int_\Omega \left|\frac{\nabla P}{P}\cdot\nabla S\right|dx \\
&\quad + a\int_\Omega (P-P^*)(H-H^*)dx - l\int_\Omega (P-P^*)^2 dx.
\end{aligned}
$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega (S-S^*)^2dx &= \int_\Omega (S-S^*)S_t\,dx\\ &= \int_\Omega (S-S^*)(d_2\Delta S-\alpha S+\beta H)\,dx\\ &= -d_2\int_\Omega |\nabla S|^2dx-\alpha\int_\Omega (S-S^*)^2dx\\ &\quad -\alpha S^*\int_\Omega (S-S^*)dx+\beta\int_\Omega (S-S^*)(H-H^*)dx\\ &\quad +\beta H^*\int_\Omega (S-S^*)dx\\ &= -d_2\int_\Omega |\nabla S|^2dx-\alpha\int_\Omega (S-S^*)^2dx\\ &\quad +\beta\int_\Omega (S-S^*)(H-H^*)dx.\end{aligned} \tag{3.8}$$

$$\begin{aligned}&a\frac{d}{dt}\int_\Omega \Big(H-H^*-H^*\ln\frac{H}{H^*}\Big)dx\\ &= a\int_\Omega \frac{H-H^*}{H}H_t\,dx\\ &= a\int_\Omega \frac{H-H^*}{H}\Big(BH\Big(1-\frac{H}{N}\Big)-PH\Big)dx\\ &= -\int_\Omega a(H-H^*)(P-P^*)\,dx-a\int_\Omega P^*(H-H^*)\,dx\\ &\quad +a\int_\Omega (H-H^*)\Big(B\Big(1-\frac{H}{N}\Big)-P^*\Big)dx\\ &= -a\int_\Omega (H-H^*)(P-P^*)\,dx-\frac{aB}{N}\int_\Omega (H-H^*)^2\,dx.\end{aligned} \tag{3.9}$$

Rearranging, we derive

$$\begin{aligned}E_2'(t) &= -d_1P^*\int_\Omega \frac{|\nabla P|^2}{P^2}\,dx+P^*\chi\int_\Omega \Big|\frac{\nabla P}{P}\cdot\nabla S\Big|dx-l\int_\Omega (P-P^*)^2dx\\ &\quad -d_2\int_\Omega |\nabla S|^2dx-\alpha\int_\Omega (S-S^*)^2dx+\beta\int_\Omega (S-S^*)(H-H^*)dx\\ &\quad -\frac{aB}{N}\int_\Omega (H-H^*)^2dx\\ &\le -\Big(d_1P^*-\frac{P^{*2}\chi^2}{4d_2}\Big)\int_\Omega \frac{|\nabla P|^2}{P^2}\,dx-l\int_\Omega (P-P^*)^2dx\\ &\quad -\frac{\alpha}{2}\int_\Omega (S-S^*)^2dx-\Big(\frac{aB}{N}-\frac{\beta^2}{2\alpha}\Big)\int_\Omega (H-H^*)^2dx.\end{aligned} \tag{3.10}$$

Therefore, when $\chi^2<\frac{4d_1d_2}{P^*}$ and $\frac{aB}{N}>\frac{\beta^2}{2\alpha}$, we have $\frac{dE(t)}{dt}\le 0$ for $t>0$. Applying LaSalle's invariance principle, we obtain that (P,S,H) converges to (P^*,S^*,H^*) in the space $L^\infty(\Omega)$ as $t\to\infty$. Therefore, the

positive constant steady state of system (1.1) is globally asymptotically stable [27]. Hence, conclusion (2) of Theorem 3.2 is proved. □

Remark 3. *When $aN < b$ and $\frac{aB}{N} > \frac{\beta^2}{2\alpha}$, the predator tends to extinction because its predation gain cannot compensate for its natural mortality, and the prey recovers to its environmental carrying capacity, indicating failure of biological control. When $aN > b$, $\frac{aB}{N} > \frac{\beta^2}{2\alpha}$, and $\chi^2 < \frac{4d_1d_2}{P^*}$, the predator can effectively suppress the prey population; meanwhile, the chemotactic sensitivity is constrained by the diffusion coefficients and the steady-state predator density, and the chemical signal system remains stable. These three factors together ensure the long-term stable coexistence of predator and prey, so that biological control achieves success. If the chemotactic coefficient exceeds the critical threshold, over-aggregation of the natural enemy will destroy the system stability. In pest biological control, it is necessary to select or regulate natural enemy species with moderate chemotactic responses, so as to avoid violent population oscillations and ensure long-term control effectiveness.*

4. Conclusions

In this paper, we have studied a predator-prey chemotaxis system motivated by biological pest control in fruit orchards. The main theoretical results can be summarized as follows. First, in the spatial dimension $n \leq 2$, by combining semigroup theory with energy estimates, we established the global boundedness of classical solutions to system (1.1). The proof relies on a series of a priori estimates, including L^1-boundedness, L^2-boundedness, and higher-order regularity estimates for the chemoattractant and the predator density. Second, through linearized stability analysis, we identified three constant equilibria—the trivial equilibrium $E_0(0, 0, 0)$, the semi-trivial equilibrium $E_1(0, \beta N/\alpha, N)$, and the positive equilibrium $E_2(P^*, S^*, H^*)$, which exists only when $aN > b$—and gave the local stability criteria for these equilibria via the Routh-Hurwitz criterion. Third, by constructing suitable Lyapunov functionals, we proved the global asymptotic stability of the semi-trivial equilibrium under weak predation pressure, and the global asymptotic stability of the positive equilibrium under strong predation pressure (i.e., $aN > b$) provided that the chemotactic sensitivity χ is sufficiently small relative to the diffusion coefficients.

Compared with classical prey-taxis models, our model explicitly included the production and degradation of the chemoattractant through an additional equation for S. This indirect prey-taxis structure allowed us to capture the delay effect of chemical signal diffusion on predator movement. In contrast with the density-dependent prey-taxis framework studied in [17], we obtained explicit stability conditions, in particular $\chi^2 < \frac{4d_1d_2}{P^*}$, which quantitatively link the chemotactic sensitivity to the diffusion coefficients and the steady-state predator density. However, several open problems remain. First, the global boundedness proof relies on the condition $n \leq 2$; the extension to higher dimensions is still pending. Second, our analysis was carried out only for the Holling type I functional response; although the framework can be extended to other bounded functional responses such as Holling types II and III, the corresponding stability conditions require separate verification. Finally, numerical simulations and related application issues are left for future investigation.

Author contributions

Lulu Liu: Funding acquisition, Conceptualization, Writing–original draft, Writing–review and editing; Liu Han and Dingjiang Wang: Writing–review and editing. All authors have reviewed the final manuscript and consented to its submission.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors state that they have no conflicts of interest.

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