



Research article**Coupling-induced Turing instability and $O(3)$ -equivariant mode interaction in a bulk–surface system with fractional surface diffusion****Duc Hau Nguyen***

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Abstract: We study a reaction–diffusion system in a three-dimensional ball and on its spherical boundary, coupled by species-dependent Robin exchange and fractional membrane transport. Both isolated compartments are linearly stable. A normalized spectral fractional Laplace–Beltrami operator fixes the damping of a reference spherical harmonic degree, allowing the fractional order to change spectral slope without merely rescaling diffusivity. Modewise boundary-response analysis reveals a coupling-induced instability band and a switch of the preferred degree from $\ell = 7$ to $\ell = 6$. At the codimension-two crossing, we derive the full cubic $O(3)$ -equivariant normal form on the 28-dimensional center space $V_6 \oplus V_7$, containing three independent quadratic and twenty independent cubic Gaunt channels. Independent spectral checks and exact-sphere simulations verify the reduction and show reproducible degree-six selection approaching saturation together with angularly resolved finite-time degree-seven selection at $t = 400$ from random multimode data. At $t = 800$, degree seven remains dominant, but its modal energy fraction still changes with cutoff refinement, so the calculations do not identify a stationary degree-seven state. Coefficient convergence, equivariance residuals, trajectory comparisons, perturbation tests, and mesh/cutoff studies are used to separate robust conclusions from numerical and geometric limitations. The results connect compartment exchange, anomalous surface transport, symmetry-constrained mode interaction, and finite-amplitude pattern selection.

Keywords: bulk–surface system; Turing instability; fractional diffusion; $O(3)$ symmetry; mode interaction; spherical harmonics

Mathematics Subject Classification: 35B32, 35K57, 35R11, 37G40, 65M60

1. Introduction

Diffusion-driven symmetry breaking, introduced by Turing [1], remains a basic mechanism for spontaneous pattern formation in chemical, biological, and material systems [2]. On curved surfaces, the admissible spatial scales are controlled by the spectrum of the Laplace–Beltrami operator;

on a sphere, this spectrum is organized by spherical harmonic degree and the associated $O(3)$ symmetry [3–5]. Classical equivariant bifurcation analyses show how spherical representations constrain normal forms and branch stability [6–8]. Recent equivariant Turing analysis on square domains has further demonstrated how multidimensional critical eigenspaces and symmetry-restricted normal forms organize competing spatial modes [9].

Many cellular and interfacial processes are not purely surface systems. Reactions and diffusion occur both in a volume and on its boundary, with exchange through the membrane. Coupled bulk–surface reaction–diffusion equations can generate patterns that are absent in either isolated compartment, induce boundary layers in the bulk, and support stationary or oscillatory bifurcations [10–12]. Recent work has established a weakly nonlinear theory for bulk–surface systems in a ball with classical surface diffusion and Robin coupling [13], and numerical methods for coupled bulk–surface equations have advanced in two and three dimensions [14–16]. Analytical well-posedness results for broad volume–surface systems are also available [17].

A second extension of classical Turing theory replaces ordinary diffusion by anomalous or nonlocal transport. On cellular membranes, departures from ordinary Brownian lateral diffusion have been associated with obstructed motion, transient trapping, compartmentalization, and heterogeneous membrane environments [18]. Such mechanisms motivate nonclassical surface transport models, although the spectral order α used below is a modelling parameter rather than a direct transcription of the anomalous exponent measured in particle-tracking experiments. Fractional reaction–diffusion systems can alter instability thresholds and reverse classical expectations concerning activator and inhibitor diffusivities [19, 20]. On a closed surface, spectral fractional powers of the Laplace–Beltrami operator are natural nonlocal transport operators and can be approximated by several numerical strategies [21]. Recent spatial-fractional reaction–diffusion studies continue to show that the fractional exponent can shift Turing thresholds and modify pattern selection [22].

The intersection of these directions remains less developed. The recent weakly nonlinear bulk–surface theory in a ball [13] provides a close reference point for symmetry-breaking bulk–surface bifurcations, but it assumes classical surface diffusion. Recent equivariant Turing analysis on planar symmetric domains [9] likewise demonstrates the importance of mode multiplicity and normal-form structure but not the spherical $O(3)$ multiplets considered here. Relative to these works, the present study has three central features: a normalized spectral fractional membrane operator that changes spectral slope while fixing a reference-mode damping; species-dependent Robin exchange that destabilizes two compartments that are separately stable; and a near-degenerate $V_6 \oplus V_7$ crossing for which the complete 28-dimensional cubic $O(3)$ field is constructed, verified, and connected to exact-sphere nonlinear mode selection.

Our main contributions are as follows:

- (i) A four-field model with species-dependent Robin exchange and a boundary-response determinant that reduces the coupled spectrum to independent radial problems for each spherical harmonic degree;
- (ii) A 275-point regime map and sensitivity study showing an open coupling-induced instability region in which the critical exchange strength increases with α and the preferred degree switches from $\ell = 7$ to $\ell = 6$;
- (iii) The full 28-dimensional cubic $O(3)$ -equivariant normal form on $V_6 \oplus V_7$, with three quadratic and twenty cubic Gaunt channels, dense coefficient tensors, and selected isotropy branch stability

calculations;

- (iv) Independent radial finite-element and Chebyshev verification together with a quantitative diagnosis of mode bias caused by straight-sided tetrahedral geometry, which can reorder closely competing angular modes;
- (v) Exact-sphere nonlinear simulations showing reproducible degree-six selection approaching saturation and angularly resolved finite-time degree-seven selection at $t = 400$ from random multimode data.

The nonlinear calculations provide different levels of numerical evidence: The tested degree-six trajectories approach similar near-saturated states, whereas the degree-seven computation is angularly resolved at $t = 400$ but remains cutoff-sensitive at $t = 800$.

The remainder of the paper is organized as follows. Section 2 develops the spectral reduction of the coupled bulk–surface problem, identifies the critical spherical harmonic degrees, and derives the cubic $O(3)$ -equivariant normal form governing their nonlinear interaction. These analytical steps connect the full PDE system to a tractable description of mode selection and competition. Section 3 then tests the resulting predictions computationally: Independent spectral calculations verify the linear mode selection, exact sphere simulations probe nonlinear mode competition in the full system, refinement and sensitivity studies assess finite-time numerical resolution and local parameter dependence, and the geometry comparison isolates angular-mode bias introduced by straight-sided tetrahedral discretization. Section 4 summarizes the conclusions supported jointly by the analytical and numerical evidence and delineates the limitations of the finite-time computations.

2. Materials and methods

2.1. Fractional bulk–surface model

2.1.1. Equations and common equilibrium

Let $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$, and $\Gamma = \partial\Omega = \mathbb{S}^2$. The bulk state is $\mathbf{U} = (U, V)^\top$, and the membrane state is $\mathbf{u} = (u, v)^\top$. With $\gamma\mathbf{U}$ denoting the trace on Γ , the dimensionless system is

$$\partial_t \mathbf{U} = \mathbf{D}_\Omega \Delta \mathbf{U} + \mathbf{R}_\Omega(\mathbf{U}), \quad x \in \Omega, \quad (2.1a)$$

$$\mathbf{D}_\Omega \partial_n \mathbf{U} = \mathbf{K}(\mathbf{u} - \gamma\mathbf{U}), \quad x \in \Gamma, \quad (2.1b)$$

$$\partial_t \mathbf{u} = \mathbf{R}_\Gamma(\mathbf{u}) - \mathbf{D}_{\Gamma,\alpha} \mathbf{u} + \mathbf{K}(\gamma\mathbf{U} - \mathbf{u}), \quad x \in \Gamma. \quad (2.1c)$$

Here, $\mathbf{D}_\Omega = \text{diag}(D_U, D_V)$, and $\mathbf{K} = \text{diag}(\kappa, 3\kappa)$. The base Schnakenberg vector is

$$\mathbf{q}(x, y) = \begin{pmatrix} a - x + x^2 y \\ b - x^2 y \end{pmatrix}.$$

We fix $a = 0.1$ and $b = 0.9$, and the compartment kinetics are positive row scalings,

$$\mathbf{R}_\Omega(\mathbf{z}) = \text{diag}(r_{\Omega,U}, r_{\Omega,V})\mathbf{q}(\mathbf{z}), \quad \mathbf{R}_\Gamma(\mathbf{z}) = \text{diag}(r_{\Gamma,u}, r_{\Gamma,v})\mathbf{q}(\mathbf{z}).$$

Both compartments therefore share the equilibrium

$$\mathbf{U}_* = \mathbf{u}_* = \begin{pmatrix} a + b \\ b/(a + b)^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.9 \end{pmatrix}.$$

The parameter values are listed in Table 1. At the common equilibrium, the two Jacobians are

$$\mathbf{J}_\Omega = \begin{pmatrix} 0.49890 & 0.62362 \\ -5.5249 & -3.0694 \end{pmatrix}, \quad (2.2)$$

$$\mathbf{J}_\Gamma = \begin{pmatrix} 0.54838 & 0.68547 \\ -2.4230 & -1.3461 \end{pmatrix}. \quad (2.3)$$

The bulk eigenvalues are $-1.2853 \pm 0.51211i$, whereas the membrane eigenvalues are $-0.39888 \pm 0.87386i$. Thus, both homogeneous kinetics are Hurwitz before spatial coupling is introduced.

Table 1. Fixed dimensionless model parameters. The threshold study varies only α and κ .

Parameter block	Dimensionless value
Kinetics (a, b)	(0.1, 0.9)
Bulk diffusion (D_U, D_V)	(0.02904, 0.7618)
Membrane diffusion (d_u, d_v)	(0.001423, 0.001699)
Bulk row rates ($r_{\Omega,U}, r_{\Omega,V}$)	(0.6236, 3.069)
Membrane row rates ($r_{\Gamma,u}, r_{\Gamma,v}$)	(0.6855, 1.346)
Exchange ratio ρ_κ	3
Fractional normalization ℓ_{ref}	2

Lemma 2.1 (Mass conservation under Robin exchange). *If the reaction terms are removed, then each species has a conserved total bulk–surface mass:*

$$\frac{d}{dt} \left(\int_\Omega U \, dx + \int_\Gamma u \, ds \right) = 0, \quad \frac{d}{dt} \left(\int_\Omega V \, dx + \int_\Gamma v \, ds \right) = 0.$$

Proof. Integrating the bulk equation and using the divergence theorem converts bulk diffusion to the boundary flux. The Robin condition gives $\int_\Gamma \kappa(u - U) \, ds$ for the first species and $\int_\Gamma \rho_\kappa \kappa(v - V) \, ds$ for the second. The surface diffusion operators annihilate constants, and the exchange terms in (2.1c) have the opposite sign. The terms cancel species by species. $\square$

2.1.2. Normalized fractional membrane transport

Let $Y_{\ell m}$ be real orthonormal spherical harmonics satisfying $-\Delta_\Gamma Y_{\ell m} = \lambda_\ell Y_{\ell m}$, where $\lambda_\ell = \ell(\ell + 1)$. The membrane diffusion operator acts diagonally in this basis. The inhibitor is classical, and the activator uses

$$\mathcal{D}_{u,\alpha} = d_u \lambda_{\text{ref}}^{1-\alpha/2} (-\Delta_\Gamma)^{\alpha/2}, \quad \lambda_{\text{ref}} = \ell_{\text{ref}}(\ell_{\text{ref}} + 1), \quad \ell_{\text{ref}} = 2. \quad (2.4)$$

Thus,

$$\mathbf{D}_{\Gamma,\alpha} Y_{\ell m} = \text{diag}(d_u \lambda_{\text{ref}}^{1-\alpha/2} \lambda_\ell^{\alpha/2}, d_v \lambda_\ell) Y_{\ell m}.$$

Proposition 2.2 (Reference mode normalization). *For every $\alpha \in (0, 2]$, the activator damping at $\ell = \ell_{\text{ref}}$ equals $d_u \lambda_{\text{ref}}$. Hence, varying α changes the spectral slope while fixing the transport scale at the reference mode.*

Proof. Substituting $\lambda_\ell = \lambda_{\text{ref}}$ into (2.4) gives $d_u \lambda_{\text{ref}}^{1-\alpha/2} \lambda_{\text{ref}}^{\alpha/2} = d_u \lambda_{\text{ref}}$. $\square$

In the present spectral model, $\alpha = 2$ recovers classical Laplace–Beltrami diffusion, whereas $0 < \alpha < 2$ reduces the relative damping of higher spherical harmonic degrees and provides a phenomenological nonlocal surface transport law. Because the dimensionless model is not calibrated to a particular membrane system, α is treated here as a continuation parameter rather than assigned a universal experimental membrane range. The choice $\ell_{\text{ref}} = 2$ is a normalization convention: It fixes the damping of one low nonconstant mode while α changes the spectral slope; no biological wavelength preference is attached to that degree.

This normalization is essential for interpretation. Without it, decreasing α simultaneously changes both the shape and the absolute scale of the diffusion symbol. We therefore use the normalized sweep for the main claims and retain the unnormalized sweep only as a control.

2.1.3. Fixed parameter family and scope

The coefficients in Table 1 define a dimensionless parameter family rather than a calibration to one biochemical system. They were obtained in a bounded exploratory search subject to four constraints: positive diffusion and reaction rate factors, a common homogeneous equilibrium, Hurwitz bulk and membrane Jacobians, and stability of the isolated spatial subsystems. The selected coefficient vector was then frozen before the reported threshold, sensitivity, and nonlinear studies. After this family was fixed, the threshold and mode-selection studies varied only α and κ ; no coefficient was retuned along the reported curves. One-at-a-time and simultaneous perturbation tests, summarized in Section 3.1, are used to assess the local sensitivity of the selected instability and mode switch.

Figure 1 summarizes the model, the degree-six/degree-seven critical space, and the verification pathway used throughout the study.

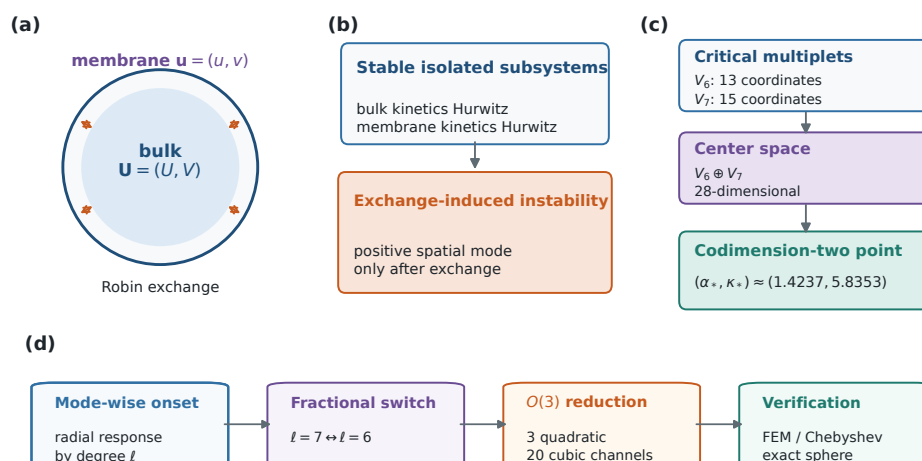


Figure 1. Model, critical symmetry and verification pathway. (a) Bulk–membrane Robin exchange. (b) and (c) Instability mechanism and the $V_6 \oplus V_7$ center space. (d) Computational reduction and verification sequence.

2.2. Modewise linear stability and independent discretizations

2.2.1. Spherical harmonic reduction

Linearize (2.1) at $(\mathbf{U}_*, \mathbf{u}_*)$, and denote the Jacobians by $\mathbf{J}_\Omega$ and $\mathbf{J}_\Gamma$. For a perturbation of degree ℓ and order m , write

$$\delta \mathbf{U} = e^{\sigma t} \mathbf{A}_\ell(r) Y_{\ell m}(\omega), \quad \delta \mathbf{u} = e^{\sigma t} \mathbf{a}_\ell Y_{\ell m}(\omega).$$

The radial amplitudes satisfy

$$\sigma \mathbf{A}_\ell = \mathbf{D}_\Omega \mathcal{L}_\ell \mathbf{A}_\ell + \mathbf{J}_\Omega \mathbf{A}_\ell, \quad 0 < r < 1, \quad (2.5a)$$

$$\mathbf{D}_\Omega \mathbf{A}'_\ell(1) = \mathbf{K}(\mathbf{a}_\ell - \mathbf{A}_\ell(1)), \quad (2.5b)$$

$$\sigma \mathbf{a}_\ell = (\mathbf{J}_\Gamma - \mathbf{M}_{\Gamma, \ell}^{(\alpha)} - \mathbf{K}) \mathbf{a}_\ell + \mathbf{K} \mathbf{A}_\ell(1), \quad (2.5c)$$

where

$$\mathcal{L}_\ell = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{\ell(\ell+1)}{r^2}, \quad \mathbf{M}_{\Gamma, \ell}^{(\alpha)} = \text{diag}(d_u \lambda_{\text{ref}}^{1-\alpha/2} \lambda_\ell^{\alpha/2}, d_v \lambda_\ell).$$

Regularity at $r = 0$ is imposed in the usual way: Degree-zero amplitudes have vanishing radial derivative, whereas positive-degree amplitudes vanish like r^ℓ .

Proposition 2.3 (Modal decoupling). *The linearized bulk–surface operator leaves each subspace generated by $\{Y_{\ell m} : -\ell \leq m \leq \ell\}$ invariant. For a fixed ℓ , the spectrum of (2.5) is independent of m and therefore has angular multiplicity $2\ell + 1$.*

Proof. The ball, the spherical boundary, the diffusion operators, the homogeneous equilibrium, and the exchange coefficients are rotationally invariant. Orthogonality of spherical harmonics removes coupling between distinct (ℓ, m) pairs. Because the Laplace–Beltrami eigenvalue depends only on ℓ , every order m of a fixed degree yields the same radial system (2.5). $\square$

2.2.2. Boundary response dispersion determinant

For a fixed degree ℓ and spectral parameter σ , define $\mathbf{Q}_\Omega(\sigma) = \mathbf{D}_\Omega^{-1}(\sigma I - \mathbf{J}_\Omega)$. Let $\Phi_\ell(r; \sigma)$ be the regular 2×2 fundamental matrix satisfying

$$\mathcal{L}_\ell \Phi_\ell = \mathbf{Q}_\Omega(\sigma) \Phi_\ell, \quad \lim_{r \downarrow 0} r^{-\ell} \Phi_\ell(r; \sigma) = I,$$

and whenever the required inverses exist, set

$$\mathbf{P}_\ell(\sigma) = \Phi'_\ell(1; \sigma) \Phi_\ell(1; \sigma)^{-1}, \quad \mathbf{H}_\ell(\sigma) = (\mathbf{D}_\Omega \mathbf{P}_\ell(\sigma) + \mathbf{K})^{-1} \mathbf{K}. \quad (2.6)$$

Here, $\mathbf{H}_\ell$ maps a membrane amplitude to the associated bulk trace.

Proposition 2.4 (Boundary response dispersion relation). *Away from poles of the bulk response, every coupled eigenvalue with nonzero membrane amplitude is a zero of*

$$\mathcal{F}_\ell(\sigma) = \det[\sigma I - \mathbf{J}_\Gamma + \mathbf{M}_{\Gamma, \ell}^{(\alpha)} + \mathbf{K} - \mathbf{K} \mathbf{H}_\ell(\sigma)]. \quad (2.7)$$

Conversely, every zero of (2.7) produces a regular mode of (2.5).

Proof. Write the boundary trace as $\mathbf{b}_\ell = \mathbf{A}_\ell(1)$. Regularity gives $\mathbf{A}'_\ell(1) = \mathbf{P}_\ell(\sigma)\mathbf{b}_\ell$. The Robin condition then yields $(\mathbf{D}_\Omega \mathbf{P}_\ell + \mathbf{K})\mathbf{b}_\ell = \mathbf{K}\mathbf{a}_\ell$, and hence, $\mathbf{b}_\ell = \mathbf{H}_\ell \mathbf{a}_\ell$. Substitution into the membrane equation gives the homogeneous 2×2 system in (2.7); a nonzero membrane amplitude exists exactly when its determinant vanishes. Reversing these steps reconstructs the regular bulk amplitude. Exceptional values at which a response matrix is singular are handled directly by the generalized eigenvalue discretization. $\square$

Equation (2.7) separates the three mechanisms in the linear problem: membrane kinetics and fractional damping, direct exchange loss, and the frequency-dependent feedback returned through the bulk. In particular, $\mathbf{KH}_\ell(\sigma)$ is the bulk-mediated return term: It measures how a membrane perturbation is transmitted through the radial bulk problem and fed back at the boundary. Thus, a zero of $\mathcal{F}_\ell$ marks a balance among membrane response, direct exchange, and bulk-return feedback for degree ℓ . The numerical eigenproblems below discretize the full system (2.5), and the response form provides a compact analytical interpretation and an independent determinant check.

Let $\sigma_\ell^{\max} = \max\{\Re \sigma : \sigma \text{ solves (2.5)}\}$. The homogeneous/radial sector $\ell = 0$ is monitored separately. A Turing mode requires $\ell \geq 1$ and $\sigma_\ell^{\max} > 0$.

Definition 2.5 (Coupling-induced instability). *The isolated bulk system is obtained by setting $\mathbf{K} = 0$ and imposing homogeneous Neumann conditions; the isolated membrane system is obtained by setting $\mathbf{K} = 0$ in the surface equation. A parameter point is called coupling-induced if both isolated spatial systems are linearly stable, but the coupled system has $\max_{\ell \geq 1} \sigma_\ell^{\max} > 0$ while the $\ell = 0$ sector remains stable.*

2.2.3. Independent discretizations

We use two independent discretizations of (2.5). The first is a weak-form continuous P1 finite-element method on $r \in [0, 1]$. Its degree-dependent stiffness form is

$$K_\ell(\phi, \psi) = \int_0^1 r^2 \phi' \psi' \, dr + \ell(\ell + 1) \int_0^1 \phi \psi \, dr.$$

The second removes the origin singularity analytically for each scalar radial component by writing $f_\ell(r) = r^\ell w_\ell(s)$ with $s = r^2$. Then,

$$\mathcal{L}_\ell[r^\ell w_\ell(r^2)] = r^\ell (4s w_\ell'' + (4\ell + 6)w_\ell'),$$

and Chebyshev–Lobatto collocation is applied to w_ℓ on $[0, 1]$. Agreement of the weak and strong formulations is used as the reference verification.

2.3. Exact-sphere nonlinear implementation and diagnostics

2.3.1. Spectral–radial representation

To avoid geometric contamination of angular modes, nonlinear calculations are performed directly on the exact sphere. For real orthonormal spherical harmonics,

$$U(r, \omega, t) = U_* + \sum_{\ell=0}^L \sum_{m=-\ell}^{\ell} U_{\ell m}(r, t) Y_{\ell m}(\omega),$$

with analogous expansions for V, u, v . The bulk radial amplitudes use continuous P1 finite elements. For each degree ℓ , the linear block is identical for all $2\ell + 1$ orders, so one factorization advances the complete multiplet.

The nonlinear Schnakenberg reactions are evaluated pseudospectrally on a tensor quadrature grid. Radial Gauss points are combined with Gauss–Legendre points in $\mu = \cos \theta$ and uniform longitude points. Projected nonlinear loads take the form

$$N_{i,\ell m}^U = \int_0^1 r^2 \varphi_i(r) \int_{\mathbb{S}^2} N_U(r, \omega) Y_{\ell m}(\omega) \, d\omega \, dr.$$

Time integration uses a second-order Crank–Nicolson/Adams–Bashforth scheme,

$$\left(M_\ell - \frac{\Delta t}{2} A_\ell\right) z_\ell^{n+1} = \left(M_\ell + \frac{\Delta t}{2} A_\ell\right) z_\ell^n + \Delta t \left(\frac{3}{2} N_\ell^n - \frac{1}{2} N_\ell^{n-1}\right). \quad (2.8)$$

Checkpoint files store the complete spherical harmonic basis and run configuration, making restart invariant under rotations within degenerate eigenspaces.

2.3.2. Resolution diagnostics

For the membrane perturbation, let E_ℓ be the sum of squared modal coefficients over all orders and both membrane species. We monitor

$$P_{\ell_c}(t) = \frac{E_{\ell_c}(t)}{\sum_{j=1}^L E_j(t)}, \quad T_L(t) = \frac{E_L(t)}{\sum_{j=1}^L E_j(t)}.$$

The first quantity is the dominant-mode purity. The second is a truncation indicator. A snapshot is called angularly resolved when $T_L < 10^{-3}$. This criterion is more informative than dominant-mode identification alone because nonlinear interactions can transfer energy to the truncation boundary while the same leading degree remains dominant.

2.4. Computational workflows

The three computational workflows (Algorithms 1–3) summarized below were implemented in Python. They describe the operations used to generate the onset diagrams, the full cubic reduction and the nonlinear validation, and the resolution and consistency checks used in interpreting the computations.

Algorithm 1. Mode-resolved onset and codimension-two detection

Input: fixed model parameters; ranges of α and κ ; maximum degree $\ell_{\max}$; radial resolution.

Output: regime map, neutral curve, selected degree, and the $\ell = 6/\ell = 7$ codimension-two point.

1. Compute the isolated bulk and membrane spectra. If either isolated subsystem is unstable, exclude the point from the coupling-induced classification.
 2. For each sampled (α, κ) , loop over $0 \leq \ell \leq \ell_{\max}$, assemble the radial coupled problem, and compute the rightmost eigenvalue σ_ℓ .
 3. Set $\sigma_{\max} = \max_\ell \Re \sigma_\ell$ and record the maximizing degree. Classify the point as coupling-induced only if the isolated spectra and the $\ell = 0$ sector are stable while $\sigma_{\max} > 0$ for some $\ell \geq 1$.
 4. For each α , bracket and bisection in κ to locate the neutral threshold.
 5. Solve $\sigma_6(\alpha, \kappa) = \sigma_7(\alpha, \kappa) = 0$ for the codimension-two crossing.
 6. Repeat the reference points with radial P1 finite elements and Chebyshev collocation, and reject a reported threshold if the independent discretizations or spectral-tail checks fail.
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Algorithm 2. Full cubic $O(3)$ -equivariant reduction

Input: codimension-two parameter point; critical left/right eigenvectors for V_6 and V_7 ; radial resolution.

Output: quadratic/cubic Gaunt channel coefficients, dense reduced tensors, and verification diagnostics.

1. Compute and biorthogonally normalize the critical left and right eigenvectors.
 2. Generate all symmetry-allowed quadratic and cubic Gaunt channels consistent with the parity of degrees six and seven.
 3. For each quadratic source, solve the stable-complement equation for the quadratic center manifold correction.
 4. Project $2\mathcal{N}_2(x, h_2) + \mathcal{N}_3(x, x, x)$ onto $V_6 \oplus V_7$, and fit the independent Gaunt channel coefficients.
 5. Reconstruct the dense coefficient tensors from the fitted channel representation.
 6. Verify radial convergence, random rotation equivariance, inversion symmetry, tensor reconstruction, and stable solve residuals before using the reduced field for branch/trajectory calculations.
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Algorithm 3. Exact-sphere nonlinear selection and validation

Input: model parameters and (α, κ) ; angular cutoff L ; radial grid; time step; initial perturbation.

Output: modal energy history, resolution diagnostics, selected degree, and (for near-critical tests) PDE/normal-form trajectory comparison.

1. Initialize either a prescribed critical mode or a random multimode perturbation in the exact-sphere spectral–radial representation.
 2. For each time step, evaluate the nonlinear reaction terms pseudospectrally, and advance the modal system with the CNAB2 update.
 3. At each checkpoint, compute E_ℓ , dominant mode purity, positivity diagnostics, and T_L .
 4. If a near-critical comparison is requested, project the PDE state onto $V_6 \oplus V_7$, integrate the cubic normal form from the same projected state, and compute the full 28D trajectory discrepancy.
 5. Treat a snapshot as angularly resolved only if the stated tail criterion is satisfied, and the selected degree is stable under the required refinement; otherwise, report it only as a broadening/long-time diagnostic.
 6. Compare the winning degree with the exact spherical spectrum before using any tetrahedral run as continuum mode-selection evidence.
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3. Results and discussion

3.1. Coupling-induced onset, normalization, and sensitivity

3.1.1. Robin exchange opens a coupling-induced instability band

For the parameter family in Table 1, the leading isolated growth rates are $\sigma_\Omega^{\text{iso}} = -0.12627$ and $\sigma_\Gamma^{\text{iso}} = -0.39888$. Thus, neither the bulk nor the membrane can form a pattern on its own. The situation changes after Robin exchange is activated. The 275-point sweep in Figure 2 contains 182 coupling-induced unstable states and 93 stable states. The unstable set forms a connected band on the sampled grid, and every unstable point selects either degree six or degree seven. This is the first central observation of the study: Exchange does not merely shift an instability already present in one compartment; it creates the instability of the combined system.

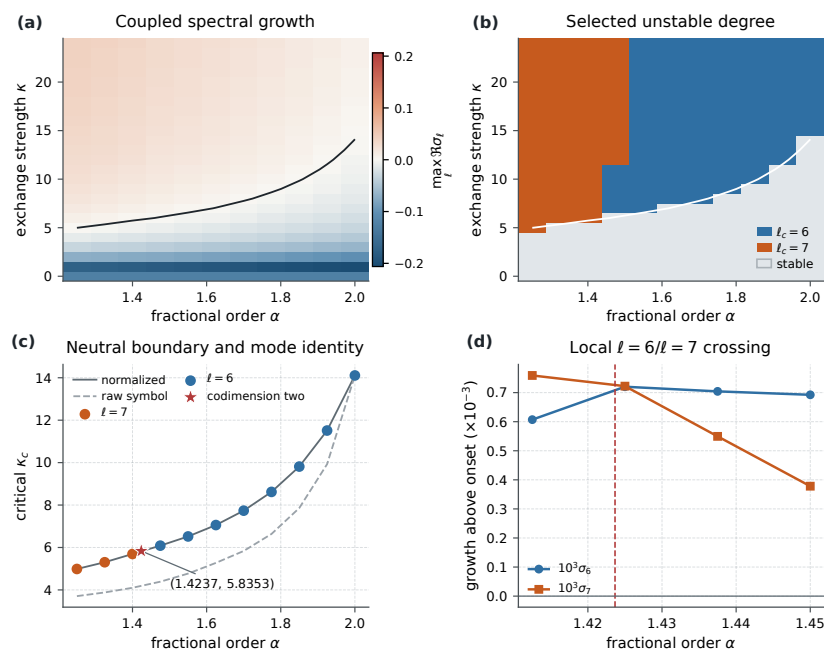


Figure 2. Coupling-induced onset and fractional mode switching. (a) and (b) Coupled growth and selected-degree maps. (c) Normalized and raw neutral curves. (d) Local degree-six/degree-seven crossing.

Two illustrative points within the coupling-induced region are

$$\begin{aligned} (\alpha, \kappa) &= (1.5, 10) : \quad \sigma_{\max} = 0.01544, \quad \ell_c = 6, \\ (\alpha, \kappa) &= (1.25, 5.5) : \quad \sigma_{\max} = 0.004760, \quad \ell_c = 7. \end{aligned}$$

In both cases, the homogeneous sector remains stable. The positive eigenvalue is therefore associated with an angular pattern rather than a spatially uniform drift.

3.1.2. Fractional order controls both onset and angular scale

The refined neutral curve and the crossing of the two leading angular degrees are shown together in Figure 2. The critical exchange increases monotonically from $\kappa_c \approx 4.987$ at $\alpha = 1.25$ to $\kappa_c \approx 14.108$ in the classical limit $\alpha = 2$. At the same time, the preferred degree changes from seven to six near $\alpha \approx 1.425$. Hence, the fractional order controls not only whether a pattern appears but also its characteristic angular scale.

3.1.3. Normalization separates spectral slope from trivial rescaling

To separate a genuine change in spectral slope from a simultaneous change in the overall damping scale, we compare the normalized fractional operator with the corresponding raw fractional symbol. Table 2 reports the two neutral thresholds at matched values of α and at the same numerical resolution.

Table 2. Neutral thresholds for the normalized fractional operator and the corresponding raw symbol control at matched values of α .

α	κ_c^{norm}	ℓ_c^{norm}	κ_c^{raw}	ℓ_c^{raw}
1.250	4.9868	7	3.7075	7
1.400	5.6938	7	4.1011	7
1.475	6.0864	6	4.3872	7
1.700	7.7358	6	5.8354	6
1.850	9.8140	6	7.8599	6
2.000	14.108	6	14.108	6

The two conventions coincide at $\alpha = 2$, as required by the reference mode normalization. For $\alpha < 2$, however, the raw symbol shifts the neutral threshold substantially and, over part of the range, also changes the selected angular degree. The normalization in Proposition 2.2 therefore isolates the redistribution of damping across angular scales from a simultaneous rescaling of the transport magnitude.

The modewise dispersion curves in Figure 3 provide a direct spectral view. Both isolated subsystems remain below the imaginary axis, whereas the coupled branch crosses into positive growth over a finite interval of angular degrees. This confirms that the unstable band is generated by the bulk–surface feedback encoded in the boundary response determinant.

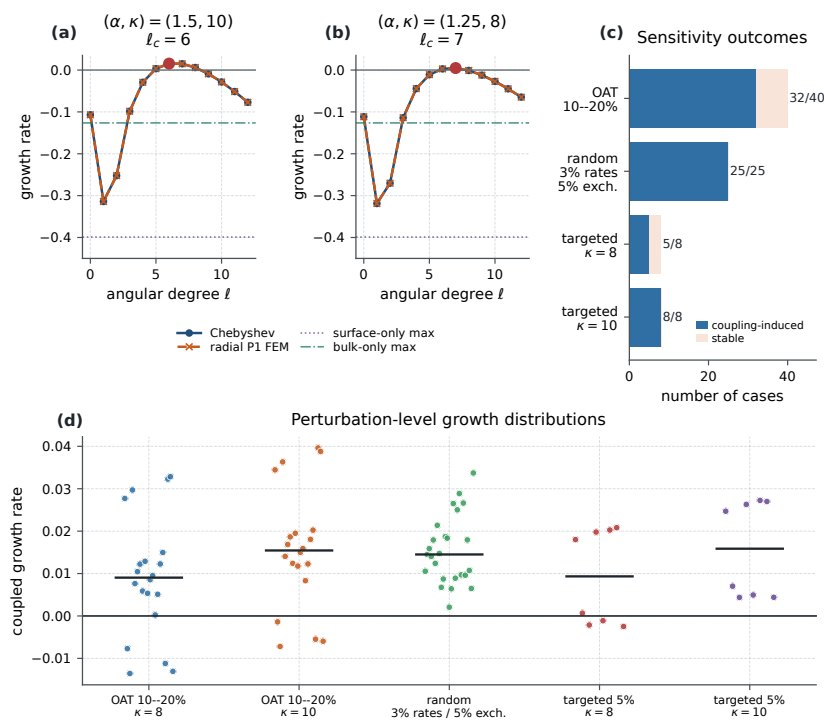


Figure 3. Spectral mechanism and parameter sensitivity. (a) and (b) Independent radial discretizations compared with the isolated bulk and membrane maxima. (c) Sensitivity tests. (d) Perturbation-level growth distributions.

3.1.4. Sensitivity of the coupling-induced band to parameter perturbations

The regime map establishes existence; the perturbation experiments test how sensitive the result is to moderate parameter changes. The outcomes in Figure 3 distinguish a near-onset point, $\kappa = 8$, from the deeper point $\kappa = 10$. The one-at-a-time dataset contains 40 perturbations, 20 at each value of κ , together with an unperturbed baseline at each point. Of the 40 perturbed cases, 32 remain coupling-induced, and 8 return to stability. In the 25 simultaneous trials at $(\alpha, \kappa) = (1.5, 10)$, the eight rate/diffusion coefficients are varied independently by up to 3% and the exchange ratio by up to 5%; all 25 remain coupling-induced, with a smallest growth rate of 2.07×10^{-3} . The targeted five-percent tests show the same distance-to-threshold effect: Three of eight variations at $\kappa = 8$ return to stability, whereas all eight corresponding variations at $\kappa = 10$ remain unstable.

As a separate significant-digit check, all eight diffusion and row rate coefficients were rounded to approximately three significant digits: $(D_U, D_V) = (0.0290, 0.762)$, $(d_u, d_v) = (0.00142, 0.00170)$, $(r_{\Omega,U}, r_{\Omega,V}) = (0.624, 3.07)$, and $(r_{\Gamma,u}, r_{\Gamma,v}) = (0.685, 1.35)$. At $n_r = 80$, the isolated systems remain stable, the codimension-two point moves only to $(1.42949, 5.92897)$, and the two representative selections remain $\ell = 6$ at $(1.50, 10)$ and $\ell = 7$ at $(1.25, 8)$; see Table 3. This representation check shows that the reported crossing and representative mode selections are insensitive to rounding the printed pilot coefficients to three significant digits.

Table 3. Parameter-rounding check at $n_r = 80$. Growth rates are the leading coupled values at the two representative points.

Parameter set	α_*	κ_*	(ℓ, σ) at $(1.50, 10)$	(ℓ, σ) at $(1.25, 8)$
Full precision	1.42334	5.83377	(6, 0.01544)	(7, 0.01944)
Three significant digits	1.42949	5.92897	(6, 0.01500)	(7, 0.01902)

We also continued the $\ell = 6/\ell = 7$ codimension-two point under 20 moderate one-at-a-time coefficient perturbations. In all cases, the isolated bulk and surface spectra remained stable, and degrees six and seven remained the critical pair. The crossing location varied substantially, spanning $1.14379 \leq \alpha_* \leq 1.65710$, and $3.26865 \leq \kappa_* \leq 25.62023$ at $n_r = 80$. Thus, the same critical pair is recovered across these 20 continuation tests, whereas the crossing location and the local mode-switching boundary vary substantially.

Because the degree-six and degree-seven growth rates are close near the switching boundary, the angular geometry must also be resolved accurately. Section 3.4 shows that a straight-sided tetrahedral approximation can reorder this pair; this motivates the exact-sphere representation used for the mode-selection results below.

3.2. Full $O(3)$ -equivariant interaction near the degree-six/degree-seven crossing

3.2.1. Codimension-two point and representation structure

The mode switch on the neutral curve is a genuine codimension-two point. Solving $\sigma_6(\alpha_*, \kappa_*) = \sigma_7(\alpha_*, \kappa_*) = 0$ with 120 radial P1 elements gives

$$(\alpha_*, \kappa_*) = (1.4237, 5.8353). \quad (3.1)$$

The independent Chebyshev discretization gives (1.4239, 5.8364). Excluding the two critical multiplets, the rightmost eigenvalue is -8.056×10^{-3} , so the center spectrum is separated from the stable spectrum. For $\delta\alpha = \alpha - \alpha_*$ and $\delta\kappa = \kappa - \kappa_*$, central differences yield

$$\mu_6 = -0.03362 \delta\alpha + 0.007323 \delta\kappa + O(|\delta|^2), \quad (3.2a)$$

$$\mu_7 = -0.04705 \delta\alpha + 0.007552 \delta\kappa + O(|\delta|^2). \quad (3.2b)$$

The larger magnitude of the α -slope for degree seven explains why decreasing the fractional order preferentially promotes the finer angular mode.

The center space is the direct sum $E_c = V_6 \oplus V_7$ with $\dim E_c = 13 + 15 = 28$, where V_ℓ denotes the real irreducible $O(3)$ representation spanned by the degree- ℓ spherical harmonics. We write the center coordinates as $A = (A_{-6}, \dots, A_6) \in V_6$ and $B = (B_{-7}, \dots, B_7) \in V_7$. The detunings μ_6 and μ_7 in (3.2) measure the linear departures of the degree-six and degree-seven critical multiplets, and the nonlinear interactions between A and B are organized by the Gaunt constructions introduced below. The right radial eigenvectors are normalized by a unit membrane-activator component, and the adjoint vectors satisfy $p_i^\top M q_j = \delta_{ij}$.

For real orthonormal spherical harmonics, we define the Gaunt product map

$$[\mathcal{G}_{\ell_1 \ell_2}^J(x, y)]_M = \sum_{m_1, m_2} \left(\int_{\mathbb{S}^2} Y_{JM} Y_{\ell_1 m_1} Y_{\ell_2 m_2} d\omega \right) x_{m_1} y_{m_2}. \quad (3.3)$$

A cubic channel with intermediate degree L is

$$C_{\ell_1 \ell_2 \ell_3}^{JL}(x, y, z) = \mathcal{G}_{L \ell_3}^J(\mathcal{G}_{\ell_1 \ell_2}^L(x, y), z), \quad (3.4)$$

with $\mathcal{SC}$ denoting symmetrization over identical arguments. These maps are $O(3)$ -equivariant by construction and are the real-basis counterparts of Clebsch–Gordan tensor products [4, 6, 13].

Proposition 3.1 (Full cubic $O(3)$ normal form). *At (3.1), the center manifold dynamics through total degree three can be written as*

$$\begin{aligned} \dot{A} = & \mu_6 A + q_{66} \mathcal{G}_{66}^6(A, A) + q_{77} \mathcal{G}_{77}^6(B, B) \\ & + \sum_{L \in \{0, 2, 6\}} c_{AAA, L}^A \mathcal{SC}_{666}^{6, L}(A, A, A) + \sum_{L=0, 2, \dots, 12} c_{ABB, L}^A \mathcal{C}_{776}^{6, L}(B, B, A) + O_4, \end{aligned} \quad (3.5a)$$

$$\begin{aligned} \dot{B} = & \mu_7 B + q_{67} \mathcal{G}_{67}^7(A, B) \\ & + \sum_{L=0, 2, \dots, 12} c_{AAB, L}^B \mathcal{C}_{667}^{7, L}(A, A, B) + \sum_{L \in \{0, 2, 6\}} c_{BBB, L}^B \mathcal{SC}_{777}^{7, L}(B, B, B) + O_4. \end{aligned} \quad (3.5b)$$

Here, $O_4 = O((\|A\| + \|B\|)^4 + |\delta|(\|A\| + \|B\|)^2)$. The quadratic part has three independent equivariant channels, and the cubic part has twenty: three in $\text{Hom}_{O(3)}(\text{Sym}^3 V_6, V_6)$, seven in $\text{Hom}_{O(3)}(V_6 \otimes \text{Sym}^2 V_7, V_6)$, seven in $\text{Hom}_{O(3)}(\text{Sym}^2 V_6 \otimes V_7, V_7)$, and three in $\text{Hom}_{O(3)}(\text{Sym}^3 V_7, V_7)$.

Proof. The spectral gap and the semilinear parabolic structure give a 28-dimensional local center manifold tangent to $E_c = V_6 \oplus V_7$. Quantitative comparison of PDE and center manifold dynamics

is standard in symmetric dissipative problems [23]. For integer spherical representations, the Clebsch–Gordan rule and exchange symmetry give

$$\mathrm{Sym}^2 V_\ell = \bigoplus_{j=0,2,\dots,2\ell} V_j, \quad V_6 \otimes V_7 = \bigoplus_{j=1}^{13} V_j.$$

Consequently, V_6 occurs once in each of $\mathrm{Sym}^2 V_6$ and $\mathrm{Sym}^2 V_7$, and V_7 occurs once in $V_6 \otimes V_7$. These are precisely the three quadratic maps displayed in (3.5). The omitted quadratic maps are forbidden by inversion: Because $Y_{\ell m}(-\omega) = (-1)^\ell Y_{\ell m}(\omega)$, inversion acts on the center variables as $(A, B) \mapsto (A, -B)$.

For the cubic terms, the number of independent equivariants is the character inner product $\dim \mathrm{Hom}_{O(3)}(W, V_J) = \langle \chi_W, \chi_J \rangle_{O(3)}$. Applying this count to the four inversion-compatible cubic domains gives the following multiplicities, in agreement with the character/Molien construction for symmetric bifurcation problems [8].

Cubic sector	Candidate intermediate degrees L	Channel rank	Basis retained
$\mathrm{Sym}^3 V_6 \rightarrow V_6$	$0, 2, \dots, 12$	3	0, 2, 6
$V_6 \otimes \mathrm{Sym}^2 V_7 \rightarrow V_6$	$0, 2, \dots, 12$	7	all seven
$\mathrm{Sym}^2 V_6 \otimes V_7 \rightarrow V_7$	$0, 2, \dots, 12$	7	all seven
$\mathrm{Sym}^3 V_7 \rightarrow V_7$	$0, 2, \dots, 14$	3	0, 2, 6

The mixed sectors are already independent in the displayed seven-channel family. Symmetrization makes the pure-sector families overcomplete; a rank-revealing pivoted QR selects the stated three-channel bases, and direct dense tensor reconstruction verifies that no cubic equivariant is lost.

Finally, let P_c and Q_s denote the center and stable projections, and write the nonlinearity as $\mathcal{N}_2 + \mathcal{N}_3$. The quadratic center manifold graph solves

$$\mathcal{L}_s h_2(x, x) = -Q_s \mathcal{N}_2(x, x), \quad P_c h_2 = 0.$$

Projecting the vector field restricted to $x + h_2(x, x)$ gives the cubic contribution

$$P_c \{2\mathcal{N}_2(x, h_2(x, x)) + \mathcal{N}_3(x, x, x)\},$$

which, expanded in the full-rank Gaunt bases above, yields (3.5). $\square$

The three quadratic coefficients are $q_{66} = -0.07489$, $q_{77} = -0.02227$, and $q_{67} = -0.09950$. The twenty cubic coefficients in the Gaunt channel convention of (3.4) are summarized in Table 4. The apparently different scale from the two-amplitude coefficients below is expected: The latter include the values of the Gaunt tensors on a particular fixed direction.

Table 4. Independent cubic Gaunt channel coefficients at $n_r = 120$, arranged by target equation and intermediate degree L . A dash indicates that the corresponding channel is absent at that degree.

L	V_6 equation		V_7 equation	
	$AAA \rightarrow V_6$	$ABB \rightarrow V_6$	$BBB \rightarrow V_7$	$AAB \rightarrow V_7$
0	-0.04901	-0.55434	-0.55645	-0.36455
2	-0.34103	-0.39768	-0.18178	-0.27120
4	–	-0.87960	–	-0.64166
6	-0.71952	-0.78081	-0.04460	-0.86877
8	–	-0.14680	–	-0.002459
10	–	-1.0353	–	-1.2435
12	–	-0.15205	–	-0.10194

Within the V_6 equation, $AAA \rightarrow V_6$ is the self-interaction channel and $ABB \rightarrow V_6$ the mixed channel; within the V_7 equation, the corresponding channels are $BBB \rightarrow V_7$ and $AAB \rightarrow V_7$. Their combined contribution determines the cubic reduced vector field, and branch and transverse stability are assessed from the resulting 28-dimensional Jacobian.

3.2.2. Coefficient convergence and equivariance checks

The full coefficient set was recomputed with 40, 60, 80, and 120 radial elements. All channels of moderate magnitude are effectively converged by $n_r = 80$ –120; the largest relative change occurs in the very small AAB , $L = 8$ coefficient, whose absolute value is only 2.46×10^{-3} . The fitted Gaunt representation reproduces the directly projected center manifold vector field with relative error 2.53×10^{-14} . Random-rotation tests give errors 5.48×10^{-14} for the direct reduced field and 5.43×10^{-14} for the fitted channel form. Inversion symmetry is satisfied to machine zero, the constrained stable-solve residual is 7.30×10^{-14} , and dense coefficient tensors reconstructed from the channel form agree to 8.85×10^{-15} .

The trajectory comparison was repeated with the norm of each initial critical multiplet reduced from 0.004 to 0.002, using the same random direction, detuning, radial grid, and time step. The maximum absolute discrepancy decreases from 1.34×10^{-4} to 6.33×10^{-5} , and the relative discrepancy remains between 1.46% and 1.52% (Table 5). Thus, the reported agreement is not tied to one initial amplitude. The nearly unchanged relative error is consistent with the fixed discretization error contributing alongside the omitted higher-order normal-form terms.

Table 5. Two-amplitude exact-sphere/28D normal-form comparison at $(\alpha, \kappa) = (1.43162, 5.92575)$, $n_r = 40$, $\Delta t = 5$, and $t_{\max} = 1200$.

Initial $\ A\ = \ B\ $	Maximum absolute error	Maximum relative error	Final PDE norm	Final reduced norm
0.004	1.336×10^{-4}	1.519%	0.008657	0.008790
0.002	6.326×10^{-5}	1.460%	0.004271	0.004334

Figure 4 collects the channel structure, coefficient convergence, algebraic consistency checks, and a representative exact-sphere/reduced-trajectory comparison.

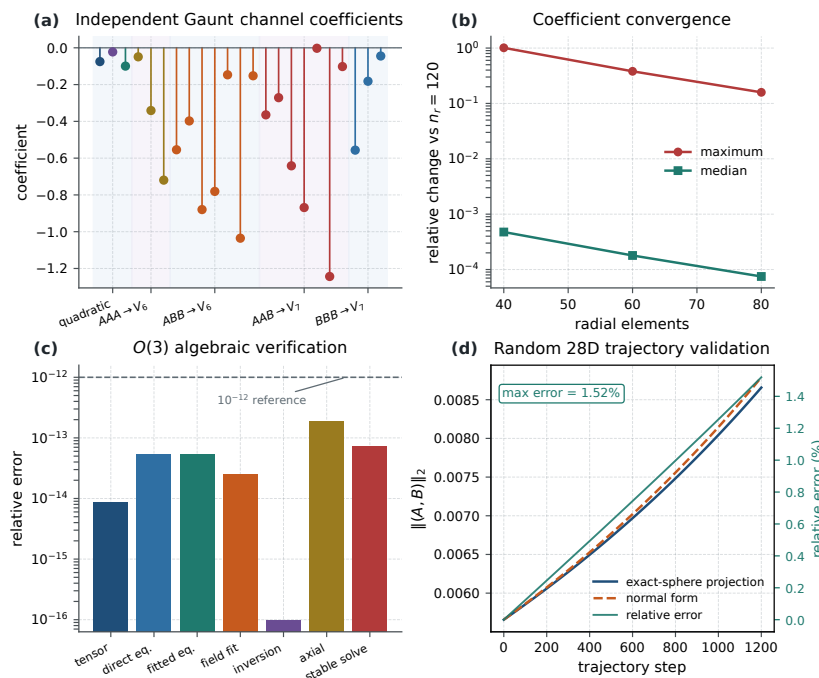


Figure 4. Construction and verification of the full cubic $O(3)$ normal form. (a) Independent Gaunt channels. (b) Radial coefficient convergence. (c) Algebraic consistency checks. (d) Random non-axisymmetric exact-sphere trajectory compared with the 28-dimensional reduction.

3.2.3. Axisymmetric diagnostic slice and parity-forced mixing

The restriction below is used as a diagnostic slice of the full $O(3)$ -equivariant normal form. Stability within this invariant subspace does not imply stability to nonaxisymmetric perturbations; full-space stability is assessed from the complete 28-dimensional Jacobian.

The axisymmetric fixed space is obtained by setting $A = A_0 e_{60}$ and $B = B_0 e_{70}$. Restricting (3.5) gives

$$\dot{A}_0 = \mu_6 A_0 + a_{20} A_0^2 + a_{02} B_0^2 + a_{30} A_0^3 + a_{12} A_0 B_0^2 + O_4, \quad (3.6a)$$

$$\dot{B}_0 = B_0 (\mu_7 + b_{11} A_0 + b_{21} A_0^2 + b_{03} B_0^2) + O_4, \quad (3.6b)$$

with $(a_{20}, a_{02}, a_{30}, a_{12}) = (-0.008575, -0.002452, -0.02213, -0.09924)$ and $(b_{11}, b_{21}, b_{03}) = (-0.01096, -0.07800, -0.04947)$. Thus, $B_0 = 0$ is invariant, whereas a nonzero degree-seven amplitude immediately forces an even degree-six component through $a_{02} B_0^2$. The degree-seven-dominated state is therefore generically mixed. Within this diagnostic slice, the cubic truncation contains a degree-six branch and a narrow region with two locally stable slice equilibria.

3.2.4. Selected isotropy branches and transverse stability

The full tensor representation permits systematic restriction to fixed-point spaces. The dimensions for several standard rotational isotropy groups are

Group	axial	T	O	I	D_4	D_6	D_7
$\dim \text{Fix}_{V_6}$	1	2	1	1	2	2	1
$\dim \text{Fix}_{V_7}$	1	1	0	0	1	1	1

The octahedral and icosahedral spaces therefore give scalar pure- V_6 equations $\dot{a} = \mu_6 a + q_G a^2 + c_G a^3$, with $(q_O, c_O) = (-1.21271 \times 10^{-3}, -4.08849 \times 10^{-3})$ and $(q_I, c_I) = (1.56423 \times 10^{-2}, -3.52922 \times 10^{-2})$. The D_7 fixed space is two-dimensional and has the same parity structure as (3.6) but orientation-dependent coefficients; notably, its $B^2 \rightarrow A$ coefficient is $+1.75350 \times 10^{-3}$ rather than the negative axial value.

At the nearby weakly nonlinear point $(\alpha, \kappa) = (1.4247, 5.8453)$, corresponding to $(\mu_6, \mu_7) = (3.961 \times 10^{-5}, 2.848 \times 10^{-5})$, the selected fixed spaces contain the small equilibria listed in Table 6. The full 28-dimensional Jacobian, rather than only the restricted Jacobian, is used for classification. The neutral eigenvalue counts agree with the dimensions of the rotational group orbits: two for an axial state and three for finite isotropy. Every displayed small branch has positive transverse eigenvalues. Consequently, the bistability seen in the axisymmetric map is bistability only inside that invariant slice; it is not an orbitally stable coexistence result for the unrestricted spherical problem.

Table 6. Selected small-amplitude equilibria of the full cubic normal form. The Morse index is the number of eigenvalues of the full 28-dimensional Jacobian with positive real part. Neutral orbit modes are symmetry-generated zero (numerically near-zero) eigenvalues associated with continuous rotations of the same pattern and are listed separately; they are not counted in the Morse index.

Branch family	Amplitude norm	Full Morse index	Neutral orbit modes
Axial pure degree six	4.566×10^{-3}	18	2
Axial mixed degree six/seven	4.784×10^{-3}	21	2
Octahedral degree six	2.969×10^{-2}	12	3
Icosahedral degree six	2.518×10^{-3}	21	3
D_7 mixed degree six/seven	1.039×10^{-2}	14	3

Figure 5 summarizes the reduced axisymmetric organization, representative isotropy-fixed pattern families, and their full-space transverse stability diagnostics.

The present calculation therefore completes the cubic $O(3)$ normal form and resolves orientation-dependent interactions within both critical multiplets. It does not attempt an exhaustive classification of the entire isotropy lattice or of all secondary and heteroclinic dynamics; that substantially larger classification problem is separated from the derivation and validation of the normal form itself.

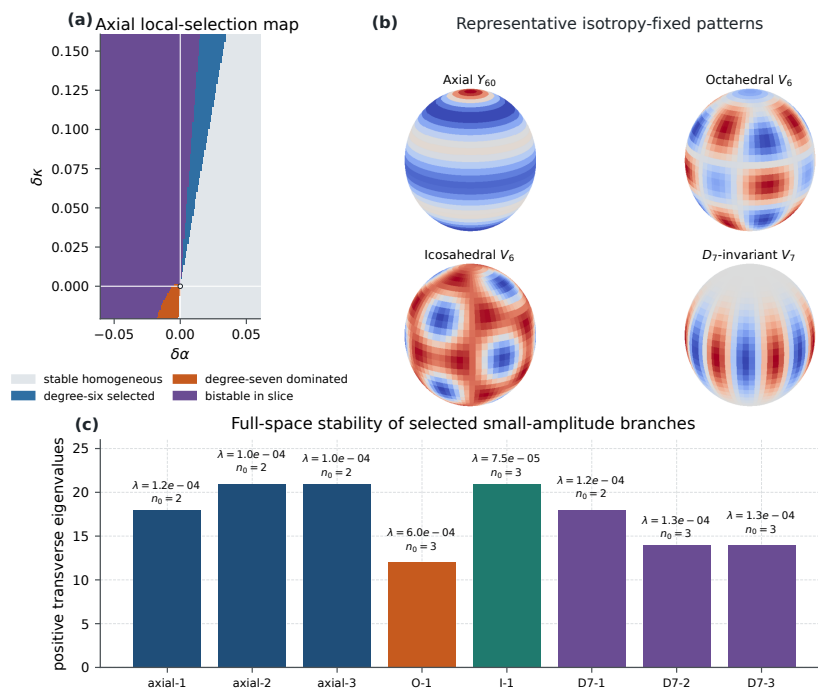


Figure 5. Reduced nonlinear organization. (a) Axisymmetric diagnostic slice and its local-selection map; the coexistence region refers only to the restricted equations. (b) Normalized representatives of four isotropy-fixed pattern families. (c) Full-space transverse Morse indices and neutral orbit-mode counts.

3.3. Nonlinear mode selection on the exact sphere

Throughout this section, angular resolution refers to the cutoff criterion above at the reported time; it is distinct from convergence of the trajectory to a stationary state.

3.3.1. Reproducible degree-six selection across the tested initial perturbations

At $(\alpha, \kappa) = (1.5, 10)$, one simulation was initialized with the leading eigenmode, and two were initialized with independent random perturbations. All three trajectories select degree six and approach similar finite-amplitude states. At $t = 1600$, the degree-six energy fractions are 0.99920, 0.99655, and 0.99784, and every concentration remains positive. The small late-time logarithmic growth rates indicate that these states are close to saturation over the simulated interval rather than merely passing through a long linear transient. Figure 6 summarizes these degree-six runs together with the degree-seven field and cutoff diagnostics discussed next.

The agreement of the three norm histories is as important as the final mode label. A single eigenmode initialization could artificially preserve a symmetry subspace, but the random starts populate many spherical harmonics. Their consistent selection of the same degree-six branch therefore provides a genuine nonlinear selection test.

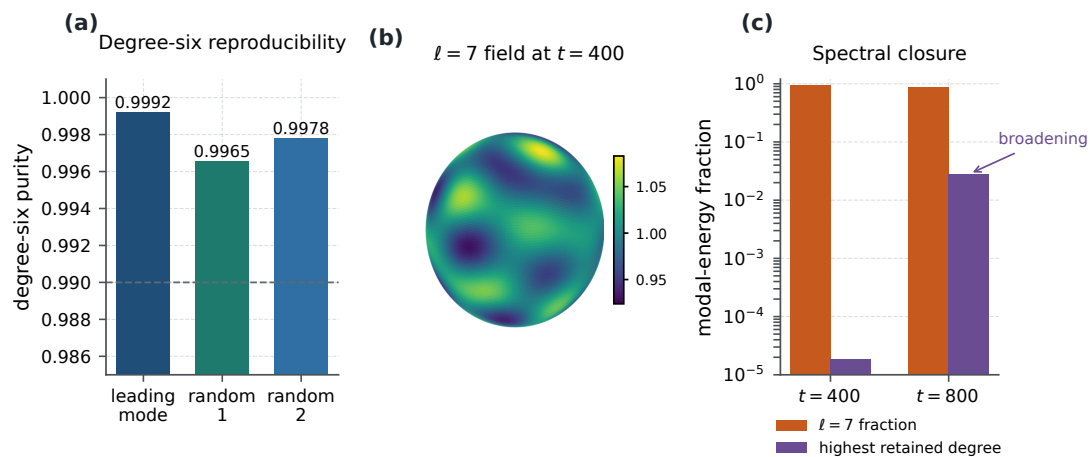


Figure 6. Nonlinear mode selection on the exact sphere. (a) Reproducible degree-six selection from three independent initial perturbations. (b) Degree-seven activator field at $t = 400$. (c) Degree-seven energy and cutoff tail diagnostics at $t = 400$ and $t = 800$.

3.3.2. Degree seven emergence from random multimode data

For $(\alpha, \kappa) = (1.25, 8)$, exact-sphere linear theory predicts $\sigma_{\text{lin}} = 0.01926$ and $\ell_c = 7$. The initial random perturbation contains all unstable degrees and initially assigns more total energy to degree eight because that eigenspace has larger dimension. Degree seven nevertheless overtakes degree eight at approximately $t = 35$, as shown in Figure 7. A fit over $300 \leq t \leq 400$ gives $\sigma_{\text{fit}} = 0.01885$, within about 2.1% of the linear prediction.

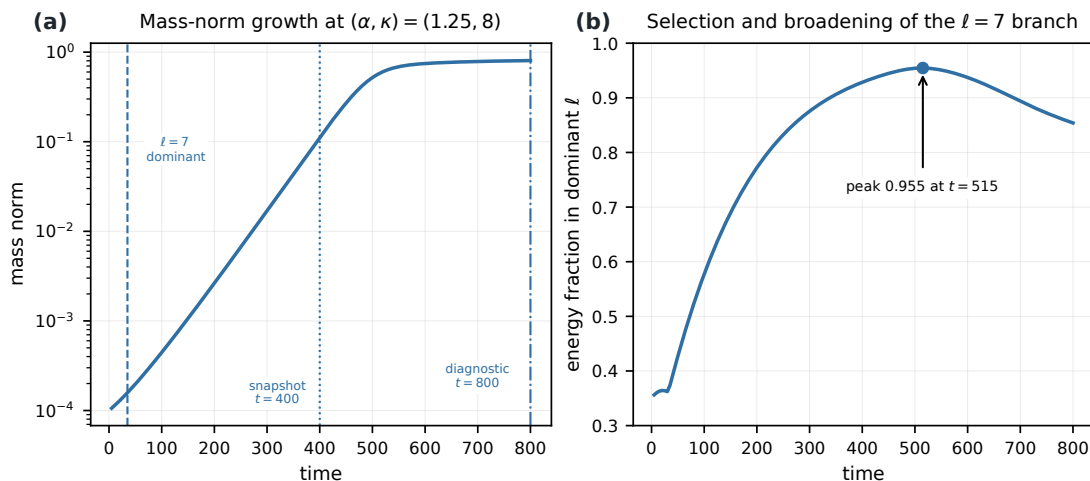


Figure 7. Exact-sphere degree-seven dynamics. (a) Mass-norm growth and the three diagnostic times. (b) Dominant-degree energy fraction, peaking at 0.955 near $t = 515$ before nonlinear spectral broadening develops.

At $t = 400$, degree seven carries 92.85% of the membrane modal energy in the baseline $L = 9$ calculation. Higher angular cutoffs give essentially the same selection: $P_7 = 0.928503$, 0.928536 , and 0.928567 for $L = 9, 12, 14$, respectively, and the highest-retained-degree fraction decreases from 1.82×10^{-5} to 1.67×10^{-7} . At $t = 400$, the degree-seven selection is therefore angularly resolved under

this cutoff refinement.

At $t = 800$, the baseline $L = 9$ run places 2.79% of the membrane energy at the truncation boundary, whereas this fraction falls below 3.1×10^{-4} for $L = 12$ and $L = 14$. Thus, the large $L = 9$ tail is a low-cutoff truncation effect rather than persistent pile-up at the retained boundary. Degree seven remains dominant at the higher cutoffs, but its energy fraction still changes with angular refinement, so the $t = 800$ state is not spectrally converged under the present refinement. The resolved nonlinear result is therefore the finite-time degree-seven selection at $t = 400$.

3.4. Numerical cross-verification and geometric mode bias

A conforming tetrahedral bulk–surface discretization was developed as an independent three-dimensional implementation check. It reproduces the degree-six reference point under refinement, but it also reveals a limitation that is easy to miss: When degrees six and seven are close, the straight-sided approximation of the sphere shifts the angular spectrum enough to alter the winning degree. The comparison in Figure 8 and Table 7 separates ordinary radial convergence from this geometric error.

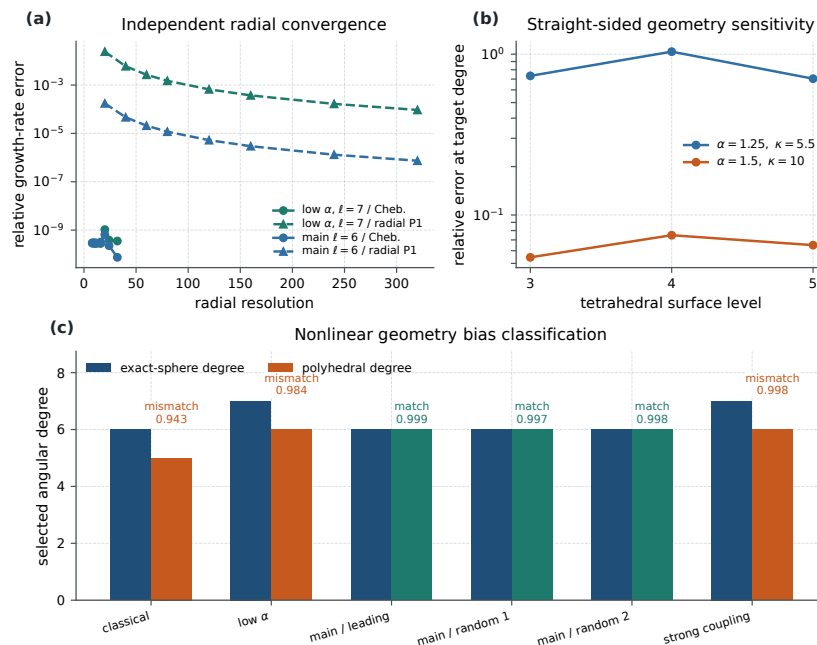


Figure 8. Numerical cross-verification and geometry bias. (a) Radial finite-element and Chebyshev limits. (b) Straight-sided geometric error. (c) Agreement or mismatch in the selected angular degree.

Table 7. Cross-verification and nonlinear geometry classification. Relative linear errors use the Chebyshev value as reference.

(a) Linear reference points.

Case	Method	Resolution	Selected ℓ	Growth	Relative error
$\ell = 6$	Chebyshev in $s = r^2$	24	6	0.01544415	2.3×10^{-10}
$\ell = 6$	Radial P1 FEM	320	6	0.01544417	7.4×10^{-7}
$\ell = 6$	Tetrahedral FEM	level 5	6	0.01444119	6.50×10^{-2}
$\ell = 7$	Chebyshev in $s = r^2$	24	7	0.00476043	3.0×10^{-10}
$\ell = 7$	Radial P1 FEM	320	7	0.00475998	9.34×10^{-5}
$\ell = 7$	Tetrahedral FEM	level 5	6	0.00140651	7.04×10^{-1}

(b) Six-run polyhedral nonlinear comparison.

Case	α	κ	Exact ℓ	Poly. ℓ	Purity	Interpretation
Main, leading mode	1.50	10	6	6	0.99920	Exact-sphere consistent
Main, random seed 1	1.50	10	6	6	0.99655	Exact-sphere consistent
Main, random seed 2	1.50	10	6	6	0.99784	Exact-sphere consistent
Low fractional order	1.25	8	7	6	0.98413	Geometry diagnostic only
Classical membrane diffusion	2.00	20	6	5	0.94319	Geometry diagnostic only
Strong coupling	1.50	25	7	6	0.99847	Geometry diagnostic only

This is not a cosmetic discrepancy. In a near-degenerate problem, a solver can reproduce a plausible positive growth rate and a saturated-looking pattern while selecting the wrong angular degree. The six-run polyhedral comparison in Table 7 makes that distinction explicit. The three degree-six runs agree with the exact-sphere prediction and support the same continuum mode selection. The low- α , classical, and strong-coupling runs select neighboring degrees and are retained only as diagnostics of geometric bias.

The practical lesson is twofold. First, continuum mode-selection claims on a sphere should be checked against exact spherical eigenvalues or a geometry-consistent high-order method. Second, agreement in aggregate quantities is insufficient when neighboring angular modes are nearly degenerate. The exact-sphere spectral–radial method is therefore used for the nonlinear degree-seven claim, whereas the tetrahedral method is retained for conservation, implementation, and geometry-error diagnostics. This separation is consistent with surface finite-element theory, in which geometry approximation is part of the discretization error [24].

3.5. Interpretation, scope, and limitations

The calculations reveal a coherent sequence from linear onset to nonlinear selection. Robin exchange first creates an instability that neither compartment possesses in isolation. The fractional order then reshapes the relative damping of spherical harmonic degrees, moving the neutral curve and switching the preferred angular scale. Finally, exact-sphere simulations show that the degrees predicted by the linear problem can emerge from random multimode data and persist into the nonlinear regime.

The normalization of the fractional operator is essential to this interpretation. Without it, changing α simultaneously alters the spectral slope and the absolute damping scale, so threshold shifts can be

mistaken for a fractional-order effect. By fixing one reference degree, the present comparison isolates how anomalous transport redistributes damping between angular scales. The contrast between the normalized and raw mechanism maps in Figure 2 shows that this is not a minor technical choice.

Species-dependent exchange is equally important in this parameter family. Symmetric exchange was predominantly dissipative in preliminary searches, whereas $k_v/k_u = 3$ produced a broad coupling-induced band. Here, $k_v/k_u = 3$ represents differential permeability or attachment–detachment kinetics, allowing the bulk reservoir and membrane reactions to feed back at different rates.

The nonlinear calculations provide different levels of numerical evidence. For degree six, the three tested initializations approach similar finite-amplitude states with small late-time growth rates, consistent with near saturation over the simulated interval. For degree seven, the exact-sphere calculation gives an angularly resolved finite-time selection at $t = 400$. At $t = 800$, higher cutoffs remove the truncation-boundary pile-up seen in the baseline $L = 9$ run, but the degree-seven energy fraction still changes with refinement. Establishing a stationary degree-seven state would require longer integrations with simultaneous angular and radial refinement together with continuation and stability analysis of the corresponding stationary branch.

The parameter family is dimensionless rather than calibrated to a specific membrane–cytosol network. The perturbation and continuation tests therefore describe local sensitivity within this parameter family: Some one-at-a-time changes remove the instability, and the codimension-two location and the local mode-switching boundary can shift substantially. Larger changes can leave the isolated-stability regime or move the crossing beyond the continuation window considered here. The fractional order α is treated as a deterministic continuation parameter; probabilistic uncertainty propagation in α or in the remaining coefficients is beyond the present study. Alternative kinetic laws are likewise not tested here because they change both the modewise Jacobians and the nonlinear coefficients entering the equivariant reduction and therefore require a separate stability and normal-form analysis.

The exchange-driven spectral mechanism is not intrinsically restricted to a sphere: On a general smooth bulk–surface geometry, Robin coupling can reorganize the joint spectrum and may destabilize a coupled mode even when the corresponding isolated compartments remain stable. The detailed $V_6 \oplus V_7$ interaction is nevertheless sphere-specific because the $(2\ell + 1)$ -fold degeneracy and the irreducible $O(3)$ multiplets rely on exact spherical symmetry. On nonspherical domains, these multiplets generally split, and the relevant symmetry group changes. The polyhedral calculations above approximate the same sphere and should therefore be interpreted as geometry discretization diagnostics rather than as simulations of a physically nonspherical domain.

Only selected isotropy-fixed branches of the complete cubic $O(3)$ normal form have been classified. A full isotropy lattice, secondary bifurcations, and possible heteroclinic dynamics would require a separate global analysis of the 28-dimensional polynomial system. Natural extensions include uncertainty-aware continuation of neutral and mode-switching boundaries in multidimensional parameter space, probabilistic parameter studies beyond one-at-a-time perturbations, alternative kinetics, distinct fractional orders for the two membrane species, and nonspherical or evolving boundaries.

4. Conclusions

Species-dependent Robin exchange can destabilize a bulk–surface reaction–diffusion system even when the bulk and membrane are separately stable. Across the full 275-point sweep, normalized fractional membrane transport changes the critical exchange strength by almost a factor of three and switches the preferred spherical harmonic degree from seven to six. At the crossing, the full cubic $O(3)$ reduction resolves all 28 center coordinates through three quadratic and twenty cubic Gaunt channels. Opposite parity makes quadratic interactions essential: Degree-seven content forces an even degree-six response, but the resulting branch stability depends on orientation within the complete multiplets. In particular, the apparent bistability of the axial two-amplitude restriction is not orbitally stable at the representative full-space detuning examined here. The 28-dimensional reduced trajectory agrees with a random exact-sphere calculation to within 1.52%. Independent radial discretizations verify the spectrum, and exact-sphere simulations show reproducible degree-six selection approaching saturation and angularly resolved finite-time degree-seven selection at $t = 400$ from random multimode initial data. At $t = 800$, degree seven remains dominant, but the modal energy fraction remains cutoff-sensitive, and no stationary degree-seven state is identified. Some one-at-a-time perturbations remove the instability, and the same $\ell = 6/\ell = 7$ critical pair is recovered across the 20 continuation tests with a substantially shifted crossing location. The polyhedral comparison shows that low-order geometry can bias closely competing angular modes. Together, these results connect anomalous membrane transport and compartment exchange to linear onset, symmetry-constrained mode interaction, and finite-amplitude pattern selection.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no competing interests.

Data availability

No external experimental data were used. All governing equations, parameter values, numerical procedures, resolution criteria, and quantitative summaries supporting the conclusions are contained in the article.

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