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Research article

Algebraic properties of complex intuitionistic fuzzy submodules over $\mathbb{R}$-modules

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Abstract: Complex intuitionistic fuzzy sets ($\mathbb{CIFS}$s) extend intuitionistic fuzzy sets ($\mathbb{IFS}$s) by allowing membership and non-membership grades to take complex values, thus providing a more flexible framework for handling uncertainty. Motivated by the importance of module theory in modern algebra, this paper introduces the concept of complex intuitionistic fuzzy submodules of $\mathbb{R}$-modules ($\mathbb{CIFSM}$s$\mathbb{R}$) and investigates their structural properties. Fundamental notions including complex intuitionistic fuzzy quotient modules ($\mathbb{CIFQM}$), level submodules, and supports are defined and characterized. Moreover, the behavior of complex intuitionistic fuzzy submodules under module homomorphisms ($\mathbb{CIFSMH}$) is studied, and complex intuitionistic fuzzy versions of the fundamental homomorphism theorems for modules are established. The results extend existing work on intuitionistic fuzzy algebraic structures and provide a foundation for further study of complex fuzzy algebraic systems.

Keywords: complex intuitionistic fuzzy submodules over $\mathbb{R}$-modules; support and level set of $\mathbb{CIFSM}$s$\mathbb{R}$; quotient of $\mathbb{CIFSM}$s$\mathbb{R}$; Homomorphism of $\mathbb{CIFSM}$s$\mathbb{R}$
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1. Introduction

Module theory is a fundamental area of abstract algebra that generalizes vector spaces by allowing scalars from rings instead of fields. Since the pioneering work of Noether [1], modules have become essential tools for studying algebraic structures, while Cartan and Eilenberg [2] established their importance in homological algebra. The concept of submodules plays a key role in describing module structures through quotient modules and homomorphisms [3]. Due to their structural importance, modules have found applications in representation theory, coding theory, and various engineering fields [4, 5]. In recent decades, application of fuzzy set theory in algebra has become attractive technique for representing the vagueness and resulting in the construction of fuzzy modules and fuzzy submodules.

The idea of fuzzy set ($\mathbb{FS}$) theory was initiated in 1965 by Zadeh [6] as a tool of expressing uncertainty. The concept was first introduced to investigate the uncertainty and indeterminacy in the human thought and its speech. Each element in FS is given a membership value that represents its level of set membership and falls within the interval $[0, 1]$. Rosenfeld [7] presented the concept of fuzzy subgroups and many of the fundamental features of group theory was discussed using this idea. Liu et al. [8, 9] introduced fuzzy subring and various important properties were investigated. Niya et al. [10] defined semi-ring and fuzzy sub semi-ring on a nexus. Some properties and applications were studied. Negoita and Relescu [11] introduced fuzzy submodule of module. Fuzzy submodule is an essential component in the discipline of fuzzy logic and has a great significance in various fields like engineering and science. Fuzzy modules are important because they allow for the creation of intelligent systems that can process and analyze complex data sets and make decisions based on that data. Lopez-Permouth and Malik [12] discussed the various aspects of fuzzy modules in details. Gulistan [13] analyzed the ideals using the notation of anti fuzzy ideals on modules. Hyper groups extend classical groups by allowing the product of two elements to be a nonempty subset. Jun et al. [14] developed hyperfuzzy subalgebras of BCK/BCI-algebras and discussed their fundamental attributes. Song et al. [15] introduced hyperfuzzy ideals and investigated their properties and relationships with hyperfuzzy subalgebras. Moreover, Bordbar et al. [16] studied N-subalgebras generated by hyperfuzzy structures, further extending the applications of hyperfuzzy sets in algebraic systems. Sun et al. [17] developed fuzzy hypergroups based on fuzzy relations and investigated their fundamental properties. Super hypergroups further extend hypergroups through operations on iterated power sets, while n-refined neutrosophic modules provide a refined representation of indeterminacy in module structures [18, 19].

The characteristics of anti fuzzy submodule was suggested by Sharma [20]. The intersection and sum were demonstrated by employing this phenomena. Also, discussed the basic definitions related to anti fuzzy relations. Furthermore, proved that the product of two anti fuzzy submodules was a submodule and vice versa. Hussein and Shukor [21] proposed the notion of fuzzy semi-essential submodule and also presented the criteria that are both adequate and essential for a fuzzy submodule of a fuzzy module to be a fuzzy semi-essential submodule. Davvaz and Firouzkouhi [22] defined the fuzzy hypermodules and various important facts related to this phenomena were analyzed.

Atanassov [23] presented the first ideology about fuzzy sets that are extended from intuitionistic fuzzy ($\mathbb{IF}$) sets that gives a more flexible and expressive representation of uncertainty in data. The scientific community confirms this theory to be more successful mainly due to its investigation in [0, 1] using varying association of membership grading ($\mathbb{MG}$) and non-membership grading ($\mathbb{NMG}$). Jin et al. [24] developed an enhanced min/max interaction-based interval-valued intuitionistic fuzzy distance measure. Ishtiaq et al. [25] proposed an intuitionistic pentagonal fuzzy model to evaluate psychological-disorder risks. Basuri et al. [26] applied generalized pentagonal intuitionistic fuzzy numbers to VIKOR for ranking the higher education institutions under uncertainty.

Biswas [27] contributed greatly in algebra and defined IF subgroups, and studied its properties. Davvaz et al. [28] initiated the foundation of $\mathbb{IF}$ submodules and discuss its various properties. Muslikh [29] discussed the submodules and their important basic results along with homomorphism by employing $\mathbb{IF}$ environments. Bhunia et al. [30] discussed the subgroups and its vital characteristic using Pythagorean fuzzy environments. Razaq and Alhamzi [31] investigated the ideals under the influenced of Pythagorean fuzzy environment. Sharma [32] defined direct sum of $\mathbb{IF}$ submodules of module over group and studied its properties. Further, proved the support of $\mathbb{IF}$ submodules. The notions of decomposability and indecomposability of $\mathbb{IF}$ $\mathbb{G}$-modules were provided. Artificial Intelligence has dealt with uncertainties arising from large amounts of industrial data in robotic engineering, and an $\mathbb{IF}$ set framework was proposed by Firouzkouhi et al. [33] as a flexible and efficient tool, generalized notions of $\mathbb{IF}$ Γ-subrings. A new notion of an $\mathbb{IF}$ Γ-submodule introduced as a generalization of fuzzy Γ-modules by Firouzkouhi et al. [34]. The relationship between α-level cuts, pre-image and image structures were also discussed and the notion is applied to spread of coronavirus and the aerial transmission of viral diseases. A theoretical setting of $\mathbb{IF}$ Γ-submodule was utilized to model the uncertainty and incompleteness inherent in multi-aspect systems and to formulate a theory of Global e-Commerce (GeC), the (α, β)-cuts, cartesian product and t-IFΓM were defined by [35].

Ramot et al. [36, 37] explored a complex fuzzy ($\mathbb{CF}$) set with a wider vary for the purpose of ($\mathbb{MG}$) beyond real to complex conversions using unit disc which is described as $\mu : A \rightarrow U = \{a, \nu_A(a)e^{2\pi\iota\phi_A(a)}\}$, $\iota = \sqrt{-1}$. Both ν_A and ϕ_A are real valued functions that lie between the interval [0,1] where ν_A represents the magnitude and ϕ_A denotes the direction of the $\mathbb{MG}$. Al-Husban et al. [38] defined the $\mathbb{CF}$ subgroup. Later a number of researchers explored the advanced structure of group theory employing the influence of $\mathbb{CF}$ framework [39–41]. Ali et al. [42, 43] applied (ϵ, δ)-complex anti fuzzy set to subgroups and isomorphic theorem were investigated using complex anti fuzzy set.

The notion of $\mathbb{CF}$ sets was extended to the complex intuitionistic fuzzy ($\mathbb{CIF}$) sets by Alkouri and Salleh [44] using the level of complexity of the $\mathbb{NMG}$ and the study of the $\mathbb{CIF}$ sets is particularly important since it has provided the two-dimensional information and a flexible way of representing the fuzziness and vagueness of facts. Razzaque et al. [45] analyzed the climate change utilizing the knowledge of $\mathbb{CIF}$ framework. Applications of $\mathbb{CIF}$ sets and interval-valued $\mathbb{CIF}$ sets to agricultural decision-making and renewable energy source selection, respectively, have also been investigated [46, 47]. Gulzar et al. [48] introduced $\mathbb{CIF}$ subgroups and also examined several of their basic algebraic properties. Group homomorphism was studied with the homomorphic pre-image and homomorphic image of a $\mathbb{CIF}$ subgroup. It was also demonstrated that a $\mathbb{CIF}$ level subset and product of $\mathbb{CIF}$ subgroups is also a $\mathbb{CIF}$ subgroup. Some important attributes of subgroups employing $\mathbb{CIF}$ sets were explored [49,50]. The abstract expression of the features of $\mathbb{CF}$ sets by the means of the (α, β)-$\mathbb{CF}$ sets was initiated by Gulzar et al. [51] and then (α, β)-$\mathbb{CF}$ subgroups based on a specific subset of the elements were generated using the pair of letters. Moreover, investigated the problem of (α, β)-$\mathbb{CF}$ cosets and some of their algebraic properties. The classical quotient group fuzzy subgroup was defined as (α, β)-$\mathbb{CF}$ subgroup.

Jawad et al. [52–54] discussed the fundamental theorem of isomorphism, Sylow Theorems, *BCK BCI*-algebra and several important characteristics using $\mathbb{CIF}$ environment. Gulzar et al. [55] explore the notion of direct product of CIF subrings and demonstrate that a CIF subring can also be the direct product of two other $\mathbb{CIF}$ subrings, Also described its many algebraic properties. Alnefaie et al. [56]

introduced the complex bipolar fuzzy subring and its application in global crypto investment company. Alsuraiheed et al. An interdisciplinary framework was proposed to generalize classical probability theory, fuzzy sets and fuzzy logic into neutrosophic probability, neutrosophic sets and neutrosophic logic examined by Smarandache [57]. Hummdi and Elrawy [58] investigated neutrosophic ideals in rings, where the concept of ring regularity was characterized through neutrosophic ideals, while neutrosophic prime ideals and all such ideals over the set of integers **Z** were identified. The relationship between algebraic structures and geometric symmetries was investigated through semimodules over semirings using neutrosophic subsemimodules and ideals were introduced by Muhiuddin et al. [60] Ali and Smarandache [61] introduced complex neutrosophic sets to overcome the limitations of complex fuzzy and intuitionistic fuzzy sets in representing imprecise periodic information, where complex-valued truth and falsehood membership functions with real-valued amplitude and phase terms and their operational properties, distance measures, applications, and comparative advantages were systematically investigated. Complex neutrosophic subsemigroups were investigated by examining their Cartesian products, characteristic functions and associated properties [62]. Alsuraiheed et al. [63] analyzed complex bipolar fuzzy submodule and various important aspects. Khan et al. [64] explored the idea of $\mathbb{CIF}$ set to ideals in subrings. Khamis and Ahmad [65] examined the notion of a $\mathbb{CIF}$ subfield and the direct product of a $\mathbb{CIF}$ subfield, which is derived form the idea of a $\mathbb{CF}$ subfield through integrating the idea of $\mathbb{IF}$ into a $\mathbb{CF}$ subfield. Mateen et al. [66] discovered numerous helpful algebraic elements of this idea for a complex Pythagorean fuzzy subfield and classified a special framework of complex Pythagorean fuzzy set in field theory.

$\mathbb{CIF}$ sets extend traditional $\mathbb{IF}$ sets by incorporating complex-valued membership and non-membership degrees, enabling the modeling of uncertainty together with periodicity and phase information. This makes them particularly powerful in applications where two-dimensional uncertainty, such as amplitude and phase, must be considered simultaneously. $\mathbb{CIF}$ sets have shown significant importance in decision-making, pattern recognition, and signal processing due to their enhanced capability to represent ambiguous and oscillatory data. In algebra, they also provide a generalized framework for studying fuzzy algebraic structures such as groups, rings, and modules under complex uncertainty. Moreover, their flexible mathematical structure offers a robust tool for handling complex uncertainties in modern scientific and engineering problems. Figure 1 and Table 1 illustrated the comparison between proposed model and existing models. This manuscript investigates the theory of complex intuitionistic fuzzy submodules of $\mathbb{R}$-modules. The main objective of this work is to examine the fundamental algebraic properties and structural characteristics of $\mathbb{CIF}$ submodules within the framework of module theory.

The manuscript is organized as: Section 2 recalls the preliminary concepts and basic definitions required for the development of the proposed theory. In Section 3, we introduce the formal definition of $\mathbb{CIF}$ submodules of $\mathbb{R}$-modules with example and establish several foundational results. Furthermore, important results of $\mathbb{CIF}$ quotient submodules are studied to understand their structural behavior. In Section 4, we analyze the level subsets and support sets of $\mathbb{CIF}$ submodules and present important theoretical results related to these concepts. Section 5 investigates the behavior of $\mathbb{CIF}$ submodules of $\mathbb{R}$-modules under homomorphisms and examines their structural preservation properties. Finally, Section 6 presents the conclusions and outlines directions for future research.

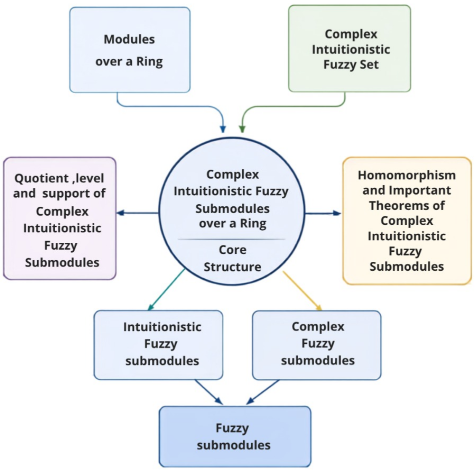

Figure 1. Relationship between the proposed model and existing models.

Table 1. Comparison of the proposed complex intuitionistic fuzzy submodule ($\mathbb{CIFSM}$) with existing module structures.

Module structure	**Information representation**	**Phase information**	**Purpose and capability**
Classical submodule [1]	$\chi_{\mathfrak{P}}(\mathfrak{l}) \in \{0, 1\}$	Not included	Represents the exact membership of $\mathfrak{l} \in \hat{\mathbb{M}}$ and preserves the algebraic operations of the underlying module.
Fuzzy submodule [11]	$\gamma_{\mathfrak{P}}(\mathfrak{l}) \in [0, 1]$	Not included	Represents the $\mathbb{MG}$ of an element $\mathfrak{l} \in \hat{\mathbb{M}}$ by a single real-valued grade.
Intuitionistic fuzzy submodule [28]	$\gamma_{\mathfrak{P}}(\mathfrak{l})$ and $\hat{\gamma}_{\mathfrak{P}}(\mathfrak{l})$, where $0 \leq \gamma_{\mathfrak{P}}(\mathfrak{l}) + \hat{\gamma}_{\mathfrak{P}}(\mathfrak{l}) \leq 1$	Not included	Represents the $\mathbb{MG}$ and non-$\mathbb{MG}$ of $\mathfrak{l}$ through real-valued grades.
Neutrosophic submodule [59]	$\gamma_{\mathfrak{P}}(\mathfrak{l})$, $\eta_{\mathfrak{P}}(\mathfrak{l})$, and $\hat{\gamma}_{\mathfrak{P}}(\mathfrak{l})$, where $0 \leq \gamma_{\mathfrak{P}}(\mathfrak{l}) + \eta_{\mathfrak{P}}(\mathfrak{l}) + \hat{\gamma}_{\mathfrak{P}}(\mathfrak{l}) \leq 3$	Not included	Represents the truth-$\mathbb{MG}$, indeterminacy-$\mathbb{MG}$, and falsity-$\mathbb{MG}$ of $\mathfrak{l}$ as independent real-valued grades.
Proposed $\mathbb{CIFSM}$	$\gamma_{\mathfrak{P}}(\mathfrak{l})e^{i\varpi_{\mathfrak{P}}(\mathfrak{l})}$ and $\hat{\gamma}_{\mathfrak{P}}(\mathfrak{l})e^{i\hat{\varpi}_{\mathfrak{P}}(\mathfrak{l})}$	Included in both the $\mathbb{MG}$ and non-$\mathbb{MG}$	Simultaneously represents the $\mathbb{MG}$, non-$\mathbb{MG}$, and phase-dependent information while preserving the algebraic structure of $\hat{\mathbb{M}}$ over $\mathbb{R}$. Also discuss quotient, level subsets, support sets and homomorphisms under the influence of $\mathbb{CIF}$ submodules in details.

2. Preliminaries

This section examines the basics of the $\mathbb{CIF}$ sets and ($\mathbb{CIFSM}$), both of which are essential to learning. We remind the basics of $\mathbb{FS}$, $\mathbb{IF}$ set, $\mathbb{IF}$ set, $\mathbb{CIF}$ set, $\mathbb{R}$-module, fuzzy submodule, and $\mathbb{IF}$ submodule, along with their fundamental properties.

Definition 2.1. *[6] Let $\mathbb{V}$ be a crisp set and $\mathfrak{a}$ is an object of $\mathbb{V}$. Then fuzzy set $ß$ is referred as $ß = \{(\mathfrak{a},\ \gamma_{ß}(\mathfrak{a}), \mathfrak{a} \in \mathbb{V})\}$, where $\gamma_{ß}(\mathfrak{a})$ is the $\mathbb{MG}$ of $\mathfrak{a}$ and $\gamma_{ß} \in [0, 1]$.*

Definition 2.2. *[23] Assuming that $ß$ is an $\mathbb{IFS}$ on a crisp set $\mathbb{V}$, then $ß = \{\mathfrak{a} : \mathfrak{a} \in \mathbb{V},\ \gamma(\mathfrak{a}),\ \hat{\gamma}(\mathfrak{a})\}$, where $\gamma(\mathfrak{a})$, $\hat{\gamma}(\mathfrak{a})$ represent the $\mathbb{MG}$ and non-$\mathbb{MG}$, respectively such that $\forall\ \mathfrak{a} \in \mathbb{V}$ and met the following criterion $0 \leq \gamma(\mathfrak{a}) + \hat{\gamma}(\mathfrak{a}) \leq 1$.*

Definition 2.3. *[36] Suppose $ß$ is a $\mathbb{CF}$ set of universal set $\mathbb{V}$. Then, complex $\mathbb{MG}$ $£_{ß}(\mathfrak{a}) = \gamma_{ß}(\mathfrak{a})e^{\phi i \varpi_{ß}(\mathfrak{a})}$, $i = \sqrt{-1}$ have the amplitude and phase term associated with an element $\mathfrak{a}$ in $ß$ are denoted as $\gamma_{ß}(\mathfrak{a})$ and $\varpi_{ß}(\mathfrak{a})$, with $\gamma_{ß} \in [0, 1]$ and $\varpi_{ß} \in [0, 2\pi]$, respectively.*

Definition 2.4. *[44] A $\mathbb{CIFS}$ $ß$ of $\mathbb{V} \neq \phi$ is defined as $ß = \{< \mathfrak{a}, \gamma_{ß}(\mathfrak{a})e^{i\varpi_{ß}(\mathfrak{a})}, \hat{\gamma}_{ß}(\mathfrak{a})e^{i\hat{\varpi}_{ß}(\mathfrak{a})} >: \mathfrak{a} \in \mathbb{V}\}$, where $\gamma_{ß} \in [0, 1]$, $\varpi_{ß} \in [0, 2\pi]$, $\hat{\gamma}_{ß} \in [0, 1]$ and $\hat{\varpi}_{ß} \in [0, 2\pi]$ define the $\mathbb{MG}$ and non-$\mathbb{MG}$ of the element $\mathfrak{a} \in \mathbb{V}$ accordingly and for arbitrary $\mathfrak{a} \in \mathbb{V}$, satisfied $0 \leq \gamma_{ß}(\mathfrak{a}) + \hat{\gamma}_{ß}(\mathfrak{a}) \leq 1$ and $0 \leq \varpi_{ß}(\mathfrak{a}) + \hat{\varpi}_{ß}(\mathfrak{a}) \leq 2\pi$.*

Definition 2.5. *A $\mathbb{FS}$ $ß$ of module $\hat{\mathbb{M}}$ over ring $\mathbb{R}$ is known as fuzzy submodule of $\hat{\mathbb{M}}$ over ring if it satisfied the conditions:*

1. $\gamma_{ß}(\mathfrak{k} - \mathfrak{b}) \geq \{\gamma_{ß}(\mathfrak{k}) \wedge \gamma_{ß}(\mathfrak{b})\},\ \forall \mathfrak{k}, \mathfrak{b} \in \hat{\mathbb{M}}$,
2. $\gamma_{ß}(r\mathfrak{k}) \geq \gamma_{ß}(\mathfrak{k}), r \in \mathbb{R}$, *and for all* $\mathfrak{k} \in \hat{\mathbb{M}}$,
3. $\gamma_{ß}(0) = 1$.

Definition 2.6. *[28] Assuming that $\hat{\mathbb{M}}$ be a module of the ring $\mathbb{R}$. An $\mathbb{IF}$ set $ß$ of a module $\hat{\mathbb{M}}$ over the ring is known as $\mathbb{IF}$ (left) submodule if:*

1. $\gamma_{ß}(0) = 1, \hat{\gamma}_{ß}(0) = 0$,
2. $\gamma_{ß}(\mathfrak{a} + \mathfrak{b}) \geq \{\gamma_{ß}(\mathfrak{a}) \wedge \gamma_{ß}(\mathfrak{b})\}$ *and* $\hat{\gamma}_{ß}(\mathfrak{a} + \mathfrak{b}) \leq \{\hat{\gamma}_{ß}(\mathfrak{a}) \vee \hat{\gamma}_{ß}(\mathfrak{b})\}, \forall \mathfrak{a}, \mathfrak{b} \in \hat{\mathbb{M}}$,
3. $\gamma_{ß}(r\mathfrak{a}) \geq \gamma_{ß}(\mathfrak{a})$ *and* $\hat{\gamma}_{ß}(r\mathfrak{a}) \leq \hat{\gamma}_{ß}(\mathfrak{a}), r \in \mathbb{R}$ *and for all a belongs to* $\hat{\mathbb{M}}$.

If we change the third axiom with $\gamma_{ß}(\mathfrak{a}r) \geq \gamma_{ß}(\mathfrak{a})$ and $\hat{\gamma}_{ß}(\mathfrak{a}r) \leq \hat{\gamma}_{ß}(\mathfrak{a})$, $r \in \mathbb{R}$ and for all a belongs to $\hat{\mathbb{M}}$, it is known as $\mathbb{IF}$ (right) module. Both of these modules coincide when $\mathbb{R}$ is a commutative ring. Then, $\mathbb{R}$ will be a commutative ring with unity.

Definition 2.7. *[67] A $\mathbb{CIFS}$ $ß = \left\{\left\langle \mathfrak{a}, \gamma_{ß}(\mathfrak{a})e^{i\varpi_{ß}(\mathfrak{a})}, \hat{\gamma}_{ß}(\mathfrak{a})e^{i\hat{\varpi}_{ß}(\mathfrak{a})}\right\rangle : \mathfrak{a} \in \mathbb{V}\right\}$ is said to be homogeneous if, for all $\mathfrak{a}, \mathfrak{b} \in \mathbb{V}$, $\gamma_{ß}(\mathfrak{a}) \leq \gamma_{ß}(\mathfrak{b}) \iff \varpi_{ß}(\mathfrak{a}) \leq \varpi_{ß}(\mathfrak{b})$, and $\hat{\gamma}_{ß}(\mathfrak{a}) \leq \hat{\gamma}_{ß}(\mathfrak{b}) \iff \hat{\varpi}_{ß}(\mathfrak{a}) \leq \hat{\varpi}_{ß}(\mathfrak{b})$.*

Definition 2.8. *[67] Let $ß = \left\{\left\langle \mathfrak{a}, \gamma_{ß}(\mathfrak{a})e^{i\varpi_{ß}(\mathfrak{a})}, \hat{\gamma}_{ß}(\mathfrak{a})e^{i\hat{\varpi}_{ß}(\mathfrak{a})}\right\rangle : \mathfrak{a} \in \mathbb{V}\right\}$ and $\mathfrak{U} = \left\{\left\langle \mathfrak{a}, \gamma_{\mathfrak{U}}(\mathfrak{a})e^{i\varpi_{\mathfrak{U}}(\mathfrak{a})}, \hat{\gamma}_{\mathfrak{U}}(\mathfrak{a})e^{i\hat{\varpi}_{\mathfrak{U}}(\mathfrak{a})}\right\rangle : \mathfrak{a} \in \mathbb{V}\right\}$ be two $\mathbb{CIFS}$s on $\mathbb{V}$. Then, $ß$ is said to be homogeneous with respect to $\mathfrak{U}$ if, for every $\mathfrak{a} \in \mathbb{V}$, $\gamma_{ß}(\mathfrak{a}) \leq \gamma_{\mathfrak{U}}(\mathfrak{a}) \iff \varpi_{ß}(\mathfrak{a}) \leq \varpi_{\mathfrak{U}}(\mathfrak{a})$, and $\hat{\gamma}_{ß}(\mathfrak{a}) \leq \hat{\gamma}_{\mathfrak{U}}(\mathfrak{a}) \iff \hat{\varpi}_{ß}(\mathfrak{a}) \leq \hat{\varpi}_{\mathfrak{U}}(\mathfrak{a})$.*

In this case, $ß$ and $\mathfrak{U}$ are said to be mutually homogeneous $\mathbb{CIFS}$s. Throughout this paper, all CIFSs are assumed to be homogeneous. For the complex-valued membership function, $\gamma_{ß}(\mathfrak{l})e^{i\varpi_{ß}(\mathfrak{l})} \geq$

$\gamma_{\mathfrak{U}}(\mathfrak{l})e^{i\varpi_{\mathfrak{U}}(\mathfrak{l})}$. Here, $\geq$ denotes the componentwise order. It means that the magnitude term on the left-hand side is greater than or equal to the magnitude term on the right-hand side and, simultaneously, the phase term on the left-hand side is greater than or equal to the phase term on the right-hand side. Explicitly, $\gamma_{ß}(\mathfrak{l}) \geq \gamma_{\mathfrak{U}}(\mathfrak{l})$, and $\varpi_{ß}(\mathfrak{l}) \geq \varpi_{\mathfrak{U}}(\mathfrak{l})$. Similarly, $\gamma_{ß}(\mathfrak{l})e^{i\varpi_{ß}(\mathfrak{l})} \leq \gamma_{\mathfrak{U}}(\mathfrak{l})e^{i\varpi_{\mathfrak{U}}(\mathfrak{l})}$, where $\leq$ denotes the componentwise order. $\gamma_{ß}(\mathfrak{l}) \leq \gamma_{\mathfrak{U}}(\mathfrak{l})$, and $\varpi_{ß}(\mathfrak{l}) \leq \varpi_{\mathfrak{U}}(\mathfrak{l})$.

3. Quotient module of complex intuitionistic fuzzy submodule of $\mathbb{R}$-module

In this section, we introduced $\mathbb{CIF}$ submodule of a $\mathbb{R}$-module and some basic algebraic properties of $\mathbb{CIFSM}$ will be studied. Furthermore, $\mathbb{CIFSM}$ of a quotient module will be discussed. Some important algebraic results are also investigated.

Definition 3.1. *A $\mathbb{CIFS}$ ß of $\mathbb{R}$-module $\hat{\mathbb{M}}$ i.e., $ß = \{(\mathfrak{l}, £_{ß}(\mathfrak{l}), \hat{£}_{ß}(\mathfrak{l})) : \mathfrak{l} \in \hat{\mathbb{M}}\}$ with the $\mathbb{MF}$ $£_{ß}(\mathfrak{l}) = \gamma_{ß}(\mathfrak{l})e^{i\varpi_{ß}(\mathfrak{l})}$ and non-$\mathbb{MF}$ $\hat{£}_{ß}(\mathfrak{l}) = \hat{\gamma}_{ß}(\mathfrak{l})e^{i\hat{\varpi}_{ß}(\mathfrak{l})}$ is called $\mathbb{CIF}$ submodule if:*

1. $\gamma_{ß}(0) = 1$ *and* $\varpi_{ß}(0) = 2\pi$,
2. $\hat{\gamma}_{ß}(0) = 0$ *and* $\hat{\varpi}_{ß}(0) = 0$,
3. $\gamma_{ß}(\mathfrak{l} + \mathfrak{p})e^{i\varpi_{ß}(\mathfrak{l}+\mathfrak{p})} \geq \{\gamma_{ß}(\mathfrak{l}) \wedge \gamma_{ß}(\mathfrak{p})\}e^{i\{\varpi_{ß}(\mathfrak{l})\wedge\varpi_{ß}(\mathfrak{p})\}}$, $\forall \mathfrak{l}, \mathfrak{p} \in \hat{\mathbb{M}}$,
4. $\hat{\gamma}_{ß}(\mathfrak{l} + \mathfrak{p})e^{i\hat{\varpi}_{ß}(\mathfrak{l}+\mathfrak{p})} \leq \{\hat{\gamma}_{ß}(\mathfrak{l}) \vee \hat{\gamma}_{ß}(\mathfrak{p})\}e^{i\{\hat{\varpi}_{ß}(\mathfrak{l})\vee\hat{\varpi}_{ß}(\mathfrak{p})\}}$,
5. $\gamma_{ß}(r\mathfrak{l})e^{i\varpi_{ß}(r\mathfrak{l})} \geq \gamma_{ß}(\mathfrak{l})e^{i\varpi_{ß}(\mathfrak{l})}$ $\forall \mathfrak{l} \in \hat{\mathbb{M}}, r \in \mathbb{R}$,
6. $\hat{\gamma}_{ß}(r\mathfrak{l})e^{i\hat{\varpi}_{ß}(r\mathfrak{l})} \leq \hat{\gamma}_{ß}(\mathfrak{l})e^{i\hat{\varpi}_{ß}(\mathfrak{l})}$ $\forall \mathfrak{l} \in \hat{\mathbb{M}}, r \in \mathbb{R}$.

The $\mathbb{MF}$ $£_{ß}(\mathfrak{l})$ is defined as $£_{ß} : \hat{\mathbb{M}} \to \{z \in \mathfrak{C} : |z| \leq 1\}$ and the non-$\mathbb{MF}$ $\hat{£}_{ß} : \hat{\mathbb{M}} \to \{z \in \mathfrak{C} : |z| \leq 1\}$, as the set of complex numbers is denoted by $\mathfrak{C}$.

The following example illustrates the $\mathbb{CIFSM}$ over ring.

Example 3.2. *Let $\mathbb{R} = \mathbb{Z}$ and $\hat{\mathbb{M}} = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ be a $\mathbb{Z}$-module. Define a $\mathbb{CIFS}$ ß of $\hat{\mathbb{M}}$ by $ß = \left\{\left(\mathfrak{l}, £_{ß}(\mathfrak{l}), \hat{£}_{ß}(\mathfrak{l})\right) : \mathfrak{l} \in \hat{\mathbb{M}}\right\}$, where $£_{ß}(\mathfrak{l}) = \gamma_{ß}(\mathfrak{l})e^{i\varpi_{ß}(\mathfrak{l})}$ and $\hat{£}_{ß}(\mathfrak{l}) = \hat{\gamma}_{ß}(\mathfrak{l})e^{i\hat{\varpi}_{ß}(\mathfrak{l})}$. In particular, $ß = \{(0, 1e^{i2\pi}, 0e^{i0}), (1, 0.5e^{i0.2\pi}, 0.4e^{i0.4\pi}), (2, 0.5e^{i0.2\pi}, 0.4e^{i0.4\pi}), (3, 0.9e^{i0.4\pi}, 0.1e^{i0.2\pi}), (4, 0.5e^{i0.2\pi}, 0.4e^{i0.4\pi}), (5, 0.5e^{i0.2\pi}, 0.4e^{i0.4\pi})\}$. We verify the conditions of CIFSM.*

1. *At the zero element, $\gamma_{ß}(0) = 1$ and $\varpi_{ß}(0) = 2\pi$. Hence, $£_{ß}(0) = 1e^{i2\pi}$. Similarly, $\hat{\gamma}_{ß}(0) = 0$ and $\hat{\varpi}_{ß}(0) = 0$. Hence, $\hat{£}_{ß}(0) = 0e^{i0}$.*
2. *Let $\mathfrak{l}, \mathfrak{p} \in \hat{\mathbb{M}}$. The sets $\{0\} \subseteq \{0, 3\} \subseteq \mathbb{Z}_6$ are submodules of $\mathbb{Z}_6$. Since the membership amplitude and phase values decrease along this chain, we have $\gamma_{ß}(\mathfrak{l} + \mathfrak{p})e^{i\varpi_{ß}(\mathfrak{l}+\mathfrak{p})} \geq \{\gamma_{ß}(\mathfrak{l}) \wedge \gamma_{ß}(\mathfrak{p})\}\, e^{i\{\varpi_{ß}(\mathfrak{l})\wedge\varpi_{ß}(\mathfrak{p})\}}$. For example, take $\mathfrak{l} = 1$ and $\mathfrak{p} = 2$. Since $1+2 \equiv 3 \pmod 6$, we obtain $£_{ß}(1 + 2) = £_{ß}(3) = 0.9e^{i0.4\pi} \geq (0.5 \wedge 0.5)e^{i(0.2\pi\wedge0.2\pi)} = 0.5e^{i0.2\pi}$.*
3. *The non-$\mathbb{MF}$ amplitude and phase values increase along the above chain. Therefore, $\hat{\gamma}_{ß}(\mathfrak{l} + \mathfrak{p})e^{i\hat{\varpi}_{ß}(\mathfrak{l}+\mathfrak{p})} \leq \{\hat{\gamma}_{ß}(\mathfrak{l}) \vee \hat{\gamma}_{ß}(\mathfrak{p})\}\, e^{i\{\hat{\varpi}_{ß}(\mathfrak{l})\vee\hat{\varpi}_{ß}(\mathfrak{p})\}}$. For $\mathfrak{l} = 1$ and $\mathfrak{p} = 2$, we have $\hat{£}_{ß}(1 + 2) = \hat{£}_{ß}(3) = 0.1e^{i0.2\pi} \leq (0.4 \vee 0.4)e^{i(0.4\pi\vee0.4\pi)} = 0.4e^{i0.4\pi}$.*
4. *Let $r \in \mathbb{R} = \mathbb{Z}$ and $\mathfrak{l} \in \hat{\mathbb{M}}$. Scalar multiplication either preserves the membership level of $\mathfrak{l}$ or moves it to a higher membership level. Hence, $\gamma_{ß}(r\mathfrak{l})e^{i\varpi_{ß}(r\mathfrak{l})} \geq \gamma_{ß}(\mathfrak{l})e^{i\varpi_{ß}(\mathfrak{l})}$. For example, take $r = 3$ and $\mathfrak{l} = 2$. Since $3 \cdot 2 \equiv 0 \pmod 6$, we obtain $£_{ß}(3 \cdot 2) = £_{ß}(0) = 1e^{i2\pi} \geq 0.5e^{i0.2\pi} = £_{ß}(2)$.*
5. *Similarly, scalar multiplication either preserves the non-membership level or moves it to a lower non-membership level. Thus,*
$$\hat{\gamma}_{ß}(r\mathfrak{l})e^{i\hat{\varpi}_{ß}(r\mathfrak{l})} \leq \hat{\gamma}_{ß}(\mathfrak{l})e^{i\hat{\varpi}_{ß}(\mathfrak{l})}.$$

For $r = 3$ *and* $\text{l} = 2$, *we have* $\hat{\pounds}_{\text{ß}}(3 \cdot 2) = \hat{\pounds}_{\text{ß}}(0) = 0e^{i0} \leq 0.4e^{i0.4\pi} = \hat{\pounds}_{\text{ß}}(2)$.

Hence, all the conditions of Definition 3.1 are satisfied. Therefore, ß *is a* $\mathbb{CIF}$ *submodule of the* $\mathbb{Z}$-*module* $\hat{\mathbb{M}} = \mathbb{Z}_6$.

Definition 3.3. *Let* ß *and* $\mathfrak{U}$ *be two* $\mathbb{CIFS}$*s of module* $\hat{\mathbb{M}}$ *over a ring. Then, their sum* ß + $\mathfrak{U}$ *as the* $\mathbb{CIFS}$ ß + $\mathfrak{U}$ *is given as:*

$$\text{ß} + \mathfrak{U} = \{(\text{l}, \pounds_{\text{ß}+\mathfrak{U}}(\text{l}), \hat{\pounds}_{\text{ß}+\mathfrak{U}}(\text{l}), \text{l} \in M)\}.$$

$$\begin{aligned} Where, \pounds_{\text{ß}+\mathfrak{U}}(\text{l}) &= \vee\{\gamma_{\mathfrak{J}}(\mathfrak{w}) \wedge \gamma_{\text{ß}}(z) : \mathfrak{w}, z \in M, \text{l} = \mathfrak{w} + z\}e^{i\vee\{\varpi_{\text{ß}}(\mathfrak{w}) \wedge \varpi_{\text{ß}}(z) : \mathfrak{w}, z \in M, \text{l} = \mathfrak{w}+z\}} \\ and \qquad \hat{\pounds}_{\text{ß}+\mathfrak{U}}(\text{l}) &= \wedge\{\hat{\gamma}_{\text{ß}}(\mathfrak{w}) \vee \hat{\gamma}_{\text{ß}}(z) : \mathfrak{w}, z \in M, \text{l} = \mathfrak{w} + z\}e^{i\wedge\{\hat{\varpi}_{\text{ß}}(\mathfrak{w}) \vee \hat{\varpi}_{\text{ß}}(z) : \mathfrak{w}, z \in M, \text{l} = \mathfrak{w}+z\}}. \end{aligned}$$

Definition 3.4. *Suppose* $\mathfrak{J}$ *be a* $\mathbb{CIFS}$ *in a module* $\hat{\mathbb{M}}$ *over a ring. Then, for any* $r \in \mathbb{R}$, *we define the* $\mathbb{CIFS}$ $r\mathfrak{J}$ *as*

$$\begin{aligned} r\mathfrak{J} &= \{(\text{l}, \pounds_{r\mathfrak{J}}(\text{l}), \hat{\pounds}_{r\mathfrak{J}}(\text{l})) : \forall \text{l} \in \hat{\mathbb{M}}\} \\ \pounds_{r\mathfrak{J}}(\text{l}) &= \vee\{\gamma_{\mathfrak{J}}(\mathfrak{w}) : \mathfrak{w} \in \hat{\mathbb{M}}, \text{l} = r\mathfrak{w}\}e^{i\vee\{\varpi_{\mathfrak{J}}(\mathfrak{w}) : \mathfrak{w} \in \hat{\mathbb{M}}, \text{l} = r\mathfrak{w}\}} \\ and \qquad \hat{\pounds}_{r\mathfrak{J}}(\text{l}) &= \wedge\{\hat{\gamma}_{\mathfrak{J}}(\mathfrak{w}) : \mathfrak{w} \in \hat{\mathbb{M}}, \text{l} = r\mathfrak{w}\}e^{i\wedge\{\hat{\varpi}_{\mathfrak{J}}(\mathfrak{w}) : \mathfrak{w} \in \hat{\mathbb{M}}, \text{l} = r\mathfrak{w}\}} \end{aligned}$$

Definition 3.5. *Let* $\mathfrak{J} \in \mathbb{R}(\hat{\mathbb{M}})$ *and assume* $\hat{\mathbb{M}}$ *be a* $\mathbb{R}$-*submodule of* $\hat{\mathbb{M}}$. *Define* $\mathfrak{J}|_{\hat{\mathbb{N}}} \in \mathbb{R}^{\hat{\mathbb{N}}}$ *(where* $\mathbb{R}^{\hat{\mathbb{N}}}$ *is the* $\mathbb{CIF}$ *power set of* $\mathbb{R}$-*module* $\hat{\mathbb{N}}$ *as follows:*

$$\begin{aligned} \gamma_{\mathfrak{J}|_{\hat{\mathbb{N}}}}(\mathfrak{k})e^{i\varpi_{\mathfrak{J}}(\mathfrak{k})} &= \gamma_{\mathfrak{J}}(\mathfrak{k})e^{i\varpi_{\mathfrak{J}}(\mathfrak{k})} \ and \\ \hat{\gamma}_{\mathfrak{J}|_{\hat{\mathbb{N}}}}(\mathfrak{k})e^{i\hat{\varpi}_{\mathfrak{J}}(\mathfrak{k})} &= \hat{\gamma}_{\mathfrak{J}}(\mathfrak{k})e^{i\hat{\varpi}_{\mathfrak{J}}(\mathfrak{k})}, \ \ \forall\, \mathfrak{k} \in \hat{\mathbb{N}}. \end{aligned}$$

Then, $\mathfrak{J}|_{\hat{\mathbb{N}}} \in \mathbb{R}(\hat{\mathbb{N}})$.

If $\hat{\mathbb{N}}$ is a submodule of a module $\hat{\mathbb{M}}$ over a ring, then $\hat{\mathbb{M}}/\hat{\mathbb{N}}$ the set of all cosets of $\hat{\mathbb{N}}$ in $\hat{\mathbb{M}}$, is said to be the quotient module of $\hat{\mathbb{M}}$ in regard to $\hat{\mathbb{N}}$ and $[\text{l}]$ introduce the cosets $\text{l} + \hat{\mathbb{N}}$. Here, we discuss two ways of establishing $\mathbb{CIFSM}$ of quotient module $\hat{\mathbb{M}}/\hat{\mathbb{N}}$.

Theorem 3.6. *Let* $\text{ß} = (\gamma_{\text{ß}}, \hat{\gamma_{\text{ß}}})$ *be a* $\mathbb{CIFSM}$ *of* $\hat{\mathbb{M}}$ *and* $\hat{\mathbb{N}}$ *be a submodule over a ring. Define* $Q = (\gamma_Q, \hat{\gamma_Q})$ *as a complex intutionistic fuzzy subset in* $\hat{\mathbb{M}}/\hat{\mathbb{N}}$ *as follows.*

$$\begin{aligned} For \ \ \text{l} \in \hat{\mathbb{M}}, \quad \gamma_Q([\text{l}])e^{i\varpi_Q([\text{l}])} &= \vee\{\gamma_{\text{ß}}(\xi) : \xi \in [\text{l}]\}e^{i\vee\{\varpi_{\text{ß}}(\xi) : \xi \in [\text{l}]\}} \ \ and \\ \hat{\gamma}_Q([\text{l}])e^{i\hat{\varpi}_Q([\text{l}])} &= \wedge\{\hat{\gamma}_{\text{ß}}(\xi) : \xi \in [\text{l}]\}e^{i\wedge\{\varpi_{\text{ß}}(\xi) : \xi \in [\text{l}]\}}. \end{aligned}$$

Then, $Q = (\gamma_Q, \hat{\gamma_Q})$ *is a* $\mathbb{CIFSM}$ *of* $\hat{\mathbb{M}}/\hat{\mathbb{N}}$.

Proof.

$$\begin{aligned} \text{As,} \ \ \gamma_Q([0])e^{i\varpi_Q([0])} &= \vee\{\gamma_{\text{ß}}(\xi) : \xi \in [0]\}e^{i\vee\{\varpi_{\text{ß}}(\xi) : \xi \in [0]\}} = \gamma_{\text{ß}}(0)e^{i\varpi_{\text{ß}}(0)} = 1 \ \text{ and} \\ \hat{\gamma}_Q([0])e^{i\hat{\varpi}_Q([0])} &= \wedge\{\hat{\gamma}_{\text{ß}}(\xi) : \xi \in [0]\}e^{i\wedge\{\hat{\varpi}_{\text{ß}}(\xi) : \xi \in [0]\}} = \hat{\gamma}_{\text{ß}}(0)e^{i\hat{\varpi}_{\text{ß}}(0)} = 0. \end{aligned}$$

$$For \ \ \text{l} \in \hat{\mathbb{M}}, \quad \gamma_Q([\text{l}])e^{i\varpi_Q([\text{l}])} = \vee\{\gamma_{\text{ß}}(\xi) : \xi \in [\text{l}]\}e^{i\text{ß}\{\varpi_{\text{ß}}(\xi \in [\text{l}])\}} \quad \text{and}$$

$$\hat{\gamma}_Q([\mathfrak{l}])e^{i\hat{\varpi}_Q([\mathfrak{l}])} = \wedge\{\hat{\gamma}_{\text{ß}}(\xi) : \xi \in [\mathfrak{l}]\}e^{i\varpi_{\text{ß}}(\xi\in[\mathfrak{l}])}.$$

Now for $\mathfrak{l}, \mathfrak{p} \in \hat{\mathbb{M}}$.

$$\begin{aligned}
\gamma_Q([\mathfrak{l}]+[\mathfrak{p}])e^{i\varpi_Q([\mathfrak{l}]+[\mathfrak{p}])} &= \vee\{\gamma_{\text{ß}}(\xi) : \xi \in [\mathfrak{l}]+[\mathfrak{p}]\}e^{i\vee\{\varpi_{\text{ß}}(\xi):\xi\in[\mathfrak{l}]+[\mathfrak{p}]\}} \\
&= \vee\{\gamma_{\text{ß}}(\xi_1+\xi_2) : \xi_1+\xi_2 \in [\mathfrak{l}]+[\mathfrak{p}]\}e^{i\vee\{\varpi_{\text{ß}}(\xi_1+\xi_2):\xi_1+\xi_2\in[\mathfrak{l}]+[\mathfrak{p}]\}} \\
&\quad \text{Since ß} = (\gamma_{\text{ß}}, \hat{\gamma}_{\text{ß}}) \text{ is a } \mathbb{CIFSM} \text{ and using Definition 3.1} \\
&\geq \vee\{\gamma_{\text{ß}}(\xi_1+\xi_2) : \xi_1 \in [\mathfrak{l}], \xi_2 \in [\mathfrak{p}]\}e^{i\vee\{\varpi_{\text{ß}}(\xi_1+\xi_2):\xi_1\in[\mathfrak{l}],\xi_2\in[\mathfrak{p}]\}} \\
&\geq \vee\{\gamma_{\text{ß}}(\xi_1) \wedge \gamma_{\text{ß}}(\xi_2) : \xi_1 \in [\mathfrak{l}], \xi_2 \in [\mathfrak{p}]\}e^{i\vee\{\varpi_{\text{ß}}(\xi_1)\wedge\varpi_{\text{ß}}(\xi_2):\xi_1\in[\mathfrak{l}],\xi_2\in[\mathfrak{p}]\}} \\
&= (\vee\{\gamma_{\text{ß}}(\xi_1) : \xi_1 \in [\mathfrak{l}]\})e^{i(\text{ß}\{\varpi_{\text{ß}}(\xi_1):\xi_1\in[\mathfrak{l}]\})} \\
&\quad \wedge(\vee\{\gamma_{\text{ß}}(\xi_2) : \xi_2 \in [\mathfrak{p}]\})e^{i(\vee\{\varpi_{\text{ß}}(\xi_2):\xi_2\in[\mathfrak{p}]\})} \\
&= \gamma_Q([\mathfrak{l}])e^{i\varpi_Q([\mathfrak{l}])} \wedge \gamma_Q([\mathfrak{p}])e^{i\varpi_Q([\mathfrak{p}])}. \\
\text{This implies } \gamma_Q([\mathfrak{l}]+[\mathfrak{p}])e^{i\varpi_Q([\mathfrak{l}]+[\mathfrak{p}])} &\geq \gamma_Q([\mathfrak{l}])e^{i\varpi_Q([\mathfrak{l}])} \wedge \gamma_Q([\mathfrak{p}])e^{i\varpi_Q([\mathfrak{p}])},
\end{aligned}$$

$$\begin{aligned}
\text{and } \hat{\gamma}_Q([\mathfrak{l}]+[\mathfrak{p}])e^{i\varpi_Q([\mathfrak{l}]+[\mathfrak{p}])} &= \wedge\{\hat{\gamma}_{\text{ß}}(\xi) : \xi \in [\mathfrak{l}]+[\mathfrak{p}]\}e^{i\wedge\{\hat{\varpi}_{\text{ß}}(\xi):\xi\in[\mathfrak{l}]+[\mathfrak{p}]\}} \\
&= \wedge\{\hat{\gamma}_{\text{ß}}(\xi_1+\xi_2) : \xi_1+\xi_2 \in [\mathfrak{l}]+[\mathfrak{p}]\}e^{i\wedge\{\hat{\varpi}_{\text{ß}}(\xi_1+\xi_2):\xi_1+\xi_2\in[\mathfrak{l}]+[\mathfrak{p}]\}} \\
&\quad \text{As ß} = (\gamma_{\text{ß}}, \hat{\gamma}_{\text{ß}}) \text{ is a } \mathbb{CIFSM} \text{ and by employing Definition 3.1} \\
&\leq \wedge\{\hat{\gamma}_{\text{ß}}(\xi_1+\xi_2) : \xi_1 \in [\mathfrak{l}], \xi_2 \in [\mathfrak{p}]\}e^{i\wedge\{\hat{\varpi}_{\text{ß}}(\xi_1+\xi_2):\xi_1\in[\mathfrak{l}],\xi_2\in[\mathfrak{p}]\}} \\
&\leq \wedge\{\hat{\gamma}_{\text{ß}}(\xi_1) \vee \hat{\gamma}_{\text{ß}}(\xi_2) : \xi_1 \in [\mathfrak{l}], \xi_2 \in [\mathfrak{p}]\}e^{i\wedge\{\varpi_{\text{ß}}(\xi_1)\vee\hat{\varpi}_{\text{ß}}(\xi_2):\xi_1\in[\mathfrak{l}],\xi_2\in[\mathfrak{p}]\}} \\
&= (\wedge\{\hat{\gamma}_{\text{ß}}(\xi_1) : \xi_1 \in [\mathfrak{l}]\})e^{i(\text{ß}\{\hat{\varpi}_{\text{ß}}(\xi_1):\xi_1\in[\mathfrak{l}]\})} \\
&\quad \vee(\wedge\{\hat{\gamma}_{\text{ß}}(\xi_2) : \xi_2 \in [\mathfrak{p}]\})e^{i(\wedge\{\hat{\varpi}_{\text{ß}}(\xi_2):\xi_2\in[\mathfrak{p}]\})} \\
&= \hat{\gamma}_Q([\mathfrak{l}])e^{i\hat{\varpi}_Q([\mathfrak{l}])} \vee \hat{\gamma}_Q([\mathfrak{p}])e^{i\hat{\varpi}]_Q([\mathfrak{p}])}. \\
\text{This implies } \hat{\gamma}_Q([\mathfrak{l}]+[\mathfrak{p}])e^{i\hat{\varpi}_Q([\mathfrak{l}]+[\mathfrak{p}])} &\leq \hat{\gamma}_Q([\mathfrak{l}])e^{i\hat{\varpi}_Q([\mathfrak{l}])} \wedge \hat{\gamma}_Q([\mathfrak{p}])e^{i\hat{\varpi}_Q([\mathfrak{p}])}, \ \forall\ \mathfrak{l}, \mathfrak{p} \in \hat{\mathbb{M}}.
\end{aligned}$$

Now $\forall$, $r \in \mathbb{R}, \mathfrak{l} \in \hat{\mathbb{M}}$, $\text{ß} = (\gamma_{\text{ß}}, \hat{\gamma}_{\text{ß}})$ is a $\mathbb{CIFSM}$ and using Definition 3.1, we have

$$\begin{aligned}
\gamma_Q(r[\mathfrak{l}])e^{i\varpi_Q(r[\mathfrak{l}])} &= \gamma_Q([r\mathfrak{l}])e^{i\varpi_Q([r\mathfrak{l}])} = \vee\{\gamma_{\text{ß}}(\xi) : \xi \in [r\mathfrak{l}]\}e^{i\vee\{\varpi_{\text{ß}}(\xi):\xi\in[r\mathfrak{l}]\}} \\
&= \vee\{\gamma_{\text{ß}}(rx+\psi) : \psi \in \hat{\mathbb{N}}\}e^{i\vee\{\varpi_{\text{ß}}(rx+\psi):\psi\in\hat{\mathbb{N}}\}} \\
&\geq \vee\{\gamma_{\text{ß}}(rx+r\psi_1) : \psi_1 \in \hat{\mathbb{N}}\}e^{i\vee\{\varpi_{\text{ß}}(rx+r\psi_1):\psi_1\in\hat{\mathbb{N}}\}} \\
&= \vee\{\gamma_{\text{ß}}(r(\mathfrak{a}+\psi_1)) : \psi_1 \in \hat{\mathbb{N}}\}e^{i\vee\{\varpi_{\text{ß}}(r(\mathfrak{l}+\psi_1)):\psi_1\in\hat{\mathbb{N}}\}} \\
&\geq \vee\{\gamma_{\text{ß}}(\mathfrak{l}+\psi_1) : \psi_1 \in \hat{\mathbb{N}}\}e^{i\vee\{\varpi_{\text{ß}}(\mathfrak{l}+\psi_1):\psi_1\in\hat{\mathbb{N}}\}} \\
&= \vee\{\gamma_{\text{ß}}(\xi_1) : \xi_1 \in [\mathfrak{l}]\}e^{i\vee\{\varpi_{\text{ß}}(\xi_1):\xi_1\in[\mathfrak{l}]\}} \\
&= \gamma_Q([\mathfrak{l}])e^{i\varpi_Q([\mathfrak{l}])}. \\
\text{This implies that } \gamma_Q(r[\mathfrak{l}])e^{i\varpi_Q(r[\mathfrak{l}])} &\geq \gamma_Q([\mathfrak{l}])e^{i\varpi_Q([\mathfrak{l}])}. \\
\text{And } \hat{\gamma}_Q(r[\mathfrak{l}])e^{i\hat{\varpi}_Q(r[\mathfrak{l}])} &= \hat{\gamma}_Q([r\mathfrak{l}])e^{i\hat{\varpi}_Q([r\mathfrak{l}])} = \wedge\{\hat{\gamma}_{\text{ß}}(\xi) : \xi \in [r\mathfrak{l}]\}e^{i\wedge\{\hat{\varpi}_{\text{ß}}(\xi):\xi\in[r\mathfrak{l}]\}} \\
&= \wedge\{\hat{\gamma}_{\text{ß}}(rx+\psi) : \psi \in \hat{\mathbb{N}}\}e^{i\wedge\{\hat{\varpi}_{\text{ß}}(rx+\psi):\psi\in\hat{\mathbb{N}}\}} \\
&\leq \wedge\{\hat{\gamma}_{\text{ß}}(rx+r\psi_1) : \psi_1 \in \hat{\mathbb{N}}\}e^{i\wedge\{\hat{\varpi}_{\text{ß}}(rx+r\psi_1):\psi_1\in\hat{\mathbb{N}}\}} \\
&= \wedge\{\hat{\gamma}_{\text{ß}}(r(\mathfrak{a}+\psi_1)) : \psi_1 \in \hat{\mathbb{N}}\}e^{i\wedge\{\hat{\varpi}_{\text{ß}}(r(\mathfrak{l}+\psi_1)):\psi_1\in\hat{\mathbb{N}}\}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \wedge\{\hat{\gamma}_{ß}(\mathfrak{l}+\psi_1) : \psi_1 \in \hat{\mathbb{N}}\} e^{i\wedge\{\hat{\varpi}_{ß}(\mathfrak{l}+\psi_1):\psi_1\in\hat{\mathbb{N}}\}} \\
&= \wedge\{\hat{\gamma}_{ß}(\xi_1) : \xi_1 \in [\mathfrak{l}]\} e^{i\wedge\{\hat{\varpi}_{ß}(\xi_1):\xi_1\in[\mathfrak{l}]\}} \\
&= \hat{\gamma}_Q([\mathfrak{l}]) e^{i\hat{\varpi}_Q([\mathfrak{l}])}.
\end{aligned}
$$

This implies that $\hat{\gamma}_Q(r[\mathfrak{l}]) e^{i\hat{\varpi}_Q(r[\mathfrak{l}])} \leq \hat{\gamma}_Q([\mathfrak{l}]) e^{i\hat{\varpi}_Q([\mathfrak{l}])} \ \forall\ r \in \mathbb{R}, \mathfrak{l} \in \hat{\mathbb{M}}$.

Thus, $Q = (\gamma_Q, \hat{\gamma_Q})$ is a $\mathbb{CIFSM}$ of $\hat{\mathbb{M}}/\hat{\mathbb{N}}$. □

Remark 3.7. *Let* $ß = (\gamma_{ß}, \hat{\gamma_{ß}})$ *and* $\mathfrak{U} = (\gamma_{\mathfrak{U}}, \hat{\gamma_{\mathfrak{U}}})$ *be* $\mathbb{CIFSM}$ *of* $\hat{\mathbb{M}}$ *such that* $ß \subseteq \mathfrak{U}$. *Then, we have* $\hat{ß}$ *and* $\hat{\mathfrak{U}}$ *are submodules of* $\hat{\mathbb{M}}$ *such that* $\hat{ß} \subseteq \hat{\mathfrak{U}}$. *Therefore,* $\hat{ß}$ *is a submodule of* $\hat{\mathfrak{U}}$, *and clearly* $\mathfrak{U}/\hat{\mathfrak{U}}$ *is a* $\mathbb{CIFSM}$ *of* $\hat{\mathfrak{U}}$. *Now, define* $Q = (\gamma_Q, \hat{\gamma_Q})$ *i.e., a* $\mathbb{CIFS}$ *in* $\hat{\mathfrak{U}}/\hat{ß}$, *where for* $\mathfrak{l} \in \hat{\mathfrak{U}}$.

$$
\begin{aligned}
\gamma_Q([\mathfrak{l}]) e^{i\varpi_Q([\mathfrak{l}])} &= \vee\{\gamma_{\mathfrak{U}}(\xi) : \xi \in [\mathfrak{l}]\} e^{i\vee\{\varpi_{\mathfrak{U}}(\xi):\xi\in[\mathfrak{l}]\}}, \\
\hat{\gamma}_Q([\mathfrak{l}]) e^{i\hat{\varpi}_Q([\mathfrak{l}])} &= \wedge\{\hat{\gamma}_{\mathfrak{U}}(\xi) : \upsilon \in [\mathfrak{l}]\} e^{i\wedge\{\hat{\varpi}_{\mathfrak{U}}(\xi):\upsilon\in[\mathfrak{l}]\}},
\end{aligned}
$$

$[\mathfrak{l}]$ *illustrate the coset* $\mathfrak{l} + \hat{ß}$. *Therefore, using the Theorem 3.6,* $Q = (\gamma_Q, \hat{\gamma_Q})$ *is a* $\mathbb{CIFSM}$ *of* $\hat{\mathfrak{U}}/\hat{ß}$ *and is known as the quotient of* $\mathfrak{U}$ *with respect to* $ß$, *expressed as* $\mathfrak{U}/ß$.

Theorem 3.8. *Assume that* $\hat{\mathbb{N}}$ *is a prime submodule of divisible module* $\hat{\mathbb{M}}$ *over the integral domain R. Let* $ß = (\gamma_{ß}, \hat{\gamma_{ß}})$ *be a* $\mathbb{CIF}$ *submodule of* $\hat{\mathbb{M}}$. *Define Q as a* $\mathbb{CIFS}$ *in* $\hat{\mathbb{M}}/\hat{\mathbb{N}}$ *as follows. For* $\mathfrak{l} \in \hat{\mathbb{M}}$,

$$
\begin{aligned}
\gamma_Q([\mathfrak{l}]) &= \begin{cases} 1 & \textit{if } [\mathfrak{l}] = \hat{\mathbb{N}} \\ \wedge\{\gamma_{ß}(\upsilon) : \upsilon \in [\mathfrak{l}]\}, & \text{otherwise} \end{cases} \\
\varpi_Q([\mathfrak{l}]) &= \begin{cases} 2\pi & \textit{if } [\mathfrak{l}] = \hat{\mathbb{N}} \\ \wedge\{\varpi_{ß}(\upsilon) : \upsilon \in [\mathfrak{l}]\}, & \text{otherwise} \end{cases} \\
&\textit{and} \\
\hat{\gamma}_Q([\mathfrak{l}]) &= \begin{cases} 0 & \textit{if } [\mathfrak{l}] = \hat{\mathbb{N}} \\ \vee\{\hat{\gamma}_{ß}(\upsilon) : \upsilon \in [\mathfrak{l}]\}, & \text{otherwise} \end{cases} \\
\hat{\varpi}_Q([\mathfrak{l}]) &= \begin{cases} 0 & \textit{if } [\mathfrak{l}] = \hat{\mathbb{N}} \\ \vee\{\hat{\varpi}_{ß}([\upsilon]) : \upsilon \in [\mathfrak{l}]\}, & \text{otherwise.} \end{cases}
\end{aligned}
$$

Then, Q is a $\mathbb{CIFSM}$ *of* $\hat{\mathbb{M}}/\hat{\mathbb{N}}$.

Proof.

$$
\begin{aligned}
Since\ 0 \in \hat{\mathbb{N}}, \gamma_Q([0]) &= 1, \quad \varpi_Q([0]) = 2\pi \quad \text{and} \\
\hat{\gamma}_Q([0]) &= 0, \quad \hat{\varpi}_Q([0]) = 0.
\end{aligned}
$$

For some $r \in \mathbb{R}, \mathfrak{l} \in \hat{\mathbb{M}}$. Consider $\gamma_Q(r[\mathfrak{l}]) e^{i\varpi_Q(r[\mathfrak{l}])}$ and $\hat{\gamma}_Q(r[\mathfrak{l}]) e^{i\hat{\varpi}_Q(r[\mathfrak{l}])}$. If $r[\mathfrak{l}] = \hat{\mathbb{N}}$, then $\gamma_Q(r[\mathfrak{l}]) = 1 \geq \gamma_Q([\mathfrak{l}])$, $\varpi_Q(r[\mathfrak{l}]) = 2\pi \geq \varpi_Q([\mathfrak{l}])$ and $\hat{\gamma}_Q(r[\mathfrak{l}]) \leq \hat{\gamma}_Q([\mathfrak{l}])$, $\hat{\varpi}_Q(r[\mathfrak{l}]) \leq \hat{\varpi}_Q([\mathfrak{l}])$. Now $r[\mathfrak{l}] \neq \hat{\mathbb{N}} \Rightarrow rx \notin \hat{\mathbb{N}} \Rightarrow r \neq 0, \mathfrak{l} \notin \hat{\mathbb{N}}$, and in this case

$$
\begin{aligned}
\gamma_Q(r[\mathfrak{l}]) e^{i\varpi_Q(r[\mathfrak{l}])} &= \gamma_Q([r\mathfrak{l}]) e^{i\varpi_Q([r\mathfrak{l}])}, \\
&= \wedge\{\gamma_{ß}(\xi) : \xi \in [r\mathfrak{l}]\} e^{i\wedge\{\varpi_{ß}(\xi):\xi\in[r\mathfrak{l}]\}},
\end{aligned}
$$

$$\begin{aligned} &= \wedge\{\gamma_{\text{ß}}(r\mathfrak{l}+\psi) : \psi \in \hat{\mathbb{N}}\}e^{i\wedge\{\varpi_{\text{ß}}(r\mathfrak{l}+\psi):\psi\in\hat{\mathbb{N}}\}},\\ &= \wedge\{\gamma_{\text{ß}}(r\mathfrak{l}+r\mathfrak{z}) : \mathfrak{z} \in \hat{\mathbb{N}}\}e^{i\wedge\{\varpi_{\text{ß}}(r\mathfrak{l}+r\mathfrak{z}):\mathfrak{z}\in\hat{\mathbb{N}}\}}.\end{aligned}$$

So, $\hat{\mathbb{N}}$ is a prime-submodule of divisible module $\hat{\mathbb{M}}$, this implies $N = \{rz : z \in \hat{\mathbb{N}}\}$

$$\begin{aligned} &= \wedge\{\gamma_{\text{ß}}(r(\mathfrak{l}+z)) : z \in \hat{\mathbb{N}}\}e^{i\wedge\{\varpi_{\text{ß}}(r(\mathfrak{l}+z)):z\in\hat{\mathbb{N}}\}}\\ &\geq \wedge\{\gamma_{\text{ß}}(\mathfrak{l}+z) : z \in \hat{\mathbb{N}}\}e^{i\wedge\{\varpi_{\text{ß}}(\mathfrak{l}+z):z\in\hat{\mathbb{N}}\}}\\ &= \wedge\{\gamma_{\text{ß}}(\omega) : \omega \in [\mathfrak{l}]\}e^{i\wedge\{\varpi_{\text{ß}}(\omega):\omega\in[\mathfrak{l}]\}}\\ &= \gamma_Q([\mathfrak{l}])e^{i\varpi_Q([\mathfrak{l}])}.\\ \text{This implies that } \gamma_Q(r[\mathfrak{l}])e^{i\varpi_Q(r[\mathfrak{l}])} &\geq \gamma_Q([\mathfrak{l}])e^{i\varpi_Q([\mathfrak{l}])}.\\ \text{and } \gamma_Q(r[\mathfrak{l}])e^{i\varpi_Q(r[\mathfrak{l}])} &= \gamma_Q([r\mathfrak{l}])e^{i\hat{\varpi}_Q([r\mathfrak{l}])},\\ &= \vee\{\gamma_{\text{ß}}(\xi) : \xi \in [r\mathfrak{l}]\}e^{i\vee\{\hat{\varpi}_{\text{ß}}(\xi):\xi\in[r\mathfrak{l}]\}},\\ &= \vee\{\gamma_{\text{ß}}(r\mathfrak{l}+\psi) : \psi \in \hat{\mathbb{N}}\}e^{i\vee\{\hat{\varpi}_{\text{ß}}(r\mathfrak{l}+\psi):\psi\in\hat{\mathbb{N}}\}},\\ &= \vee\{\gamma_{\text{ß}}(r\mathfrak{l}+r\mathfrak{z}) : \mathfrak{z} \in \hat{\mathbb{N}}\}e^{i\vee\{\hat{\varpi}_{\text{ß}}(r\mathfrak{l}+r\mathfrak{z}):\mathfrak{z}\in\hat{\mathbb{N}}\}}.\end{aligned}$$

(So, $\hat{\mathbb{N}}$ is a prime-submodule of divisible module $\hat{\mathbb{M}}$, this implies $N = \{rz : z \in \hat{\mathbb{N}}\}$)

$$\begin{aligned} &= \vee\{\hat{\gamma}_{\text{ß}}(r(\mathfrak{l}+z)) : z \in \hat{\mathbb{N}}\}e^{i\vee\{\hat{\varpi}_{\text{ß}}(r(\mathfrak{l}+z)):z\in\hat{\mathbb{N}}\}}\\ &\leq \vee\{\hat{\gamma}_{\text{ß}}(\mathfrak{l}+z) : z \in \hat{\mathbb{N}}\}e^{i\vee\{\hat{\varpi}_{\text{ß}}(\mathfrak{l}+z):z\in\hat{\mathbb{N}}\}}\\ &= \vee\{\hat{\gamma}_{\text{ß}}(\omega) : \omega \in [\mathfrak{l}]\}e^{i\vee\{\hat{\varpi}_{\text{ß}}(\omega):\omega\in[\mathfrak{l}]\}}\\ &= \hat{\gamma}_Q([\mathfrak{l}])e^{i\hat{\varpi}_Q([\mathfrak{l}])}.\end{aligned}$$

Thus, $\hat{\gamma}_Q(r[\mathfrak{l}])e^{i\hat{\varpi}_Q(r[\mathfrak{l}])} \leq \hat{\gamma}_Q([\mathfrak{l}])e^{i\hat{\varpi}_Q([\mathfrak{l}])}$.

Now for $\mathfrak{l}, \mathfrak{p} \in \hat{\mathbb{M}}$, consider $\gamma_Q([\mathfrak{l}]+[\mathfrak{p}])e^{i\varpi_Q([\mathfrak{l}]+[\mathfrak{p}])}$ and $\hat{\gamma}_Q([\mathfrak{l}]+[\mathfrak{p}])e^{i\hat{\varpi}_Q([\mathfrak{l}]+[\mathfrak{p}])}$. When $[\mathfrak{l}] = [\mathfrak{p}] = N$ and $[\mathfrak{p}] \neq \hat{\mathbb{N}}$

$$\begin{aligned} \gamma_Q([\mathfrak{l}]+[\mathfrak{p}])e^{i\varpi_Q([\mathfrak{l}]+[\mathfrak{p}])} &= \wedge\{\gamma_{\text{ß}}(\xi) : \xi \in [\mathfrak{l}]+[\mathfrak{p}]\}e^{i\varpi_{\text{ß}}(\xi):\xi\in[\mathfrak{l}]+[\mathfrak{p}]\}},\\ &= \wedge\{\gamma_{\text{ß}}((\mathfrak{l}+\xi_1)+(\mathfrak{p}+\xi_2)) : \xi_1,\xi_2 \in \hat{\mathbb{N}}\}e^{i\varpi_{\text{ß}}(((\mathfrak{l}+\xi_1)+(\mathfrak{p}+\xi_2)):\xi_1,\xi_2\in\hat{\mathbb{N}}\}},\\ &\geq \wedge\{\gamma_{\text{ß}}(\mathfrak{p}+\psi) : \psi \in \hat{\mathbb{N}}\}e^{i\wedge\{\varpi_{\text{ß}}(\mathfrak{p}+\psi):\psi\in\hat{\mathbb{N}}\}}(\text{Since } \mathfrak{l},\xi_1,\xi_2 \in \hat{\mathbb{N}}),\\ &= \gamma_Q([\mathfrak{p}])e^{i\varpi_Q([\mathfrak{p}])},\\ &= \gamma_Q([\mathfrak{l}])e^{i\varpi_Q([\mathfrak{l}])} \wedge \gamma_Q([\mathfrak{p}])e^{i\varpi_Q([\mathfrak{p}])}. \ (\text{Since } \gamma_Q([\mathfrak{l}])e^{i\varpi_Q([\mathfrak{l}])} = 1).\\ \text{So, we have } \gamma_Q([\mathfrak{l}]+[\mathfrak{p}])e^{i\varpi_Q([\mathfrak{l}]+[\mathfrak{p}])} &\geq \gamma_Q([\mathfrak{l}])e^{i\varpi_Q([\mathfrak{l}])} \wedge \gamma_Q([\mathfrak{p}])e^{i\varpi_Q([\mathfrak{p}])}.\end{aligned}$$

Similarly the case when $[\mathfrak{l}] \neq \hat{\mathbb{N}}$ and $[\mathfrak{p}] = \hat{\mathbb{N}}$. Now suppose that $[\mathfrak{l}] \neq \hat{\mathbb{N}}$ and $[\mathfrak{p}] \neq \hat{\mathbb{N}}$. In this case, if $[\mathfrak{l}]+[\mathfrak{p}] = \hat{\mathbb{N}}$, clearly $\gamma_Q([\mathfrak{l}]+[\mathfrak{p}])e^{i\varpi_Q([\mathfrak{l}]+[\mathfrak{p}])} \geq \gamma_Q([\mathfrak{l}])e^{i\varpi_Q([\mathfrak{l}])} \wedge \gamma_Q([\mathfrak{p}])e^{i\varpi_Q([\mathfrak{p}])}$.
If $[\mathfrak{l}]+[\mathfrak{p}] \neq \hat{\mathbb{N}}$, then

$$\begin{aligned} \gamma_Q([\mathfrak{l}]+[\mathfrak{p}])e^{i\varpi_Q([\mathfrak{l}]+[\mathfrak{p}])} &= \wedge\{\gamma_{\text{ß}}(\xi) : \xi \in [\mathfrak{l}]+[\mathfrak{p}]\}e^{i\wedge\{\varpi_{\text{ß}}(\xi):\xi\in[\mathfrak{l}]+[\mathfrak{p}]\}},\\ &= \wedge\{\gamma_{\text{ß}}((\mathfrak{l}+\xi_1)+(\mathfrak{p}+\xi_2)) : \xi_1,\xi_2 \in \hat{\mathbb{N}}\}e^{i\wedge\{\varpi_{\text{ß}}((\mathfrak{l}+\xi_1)+(\mathfrak{p}+\xi_2)):\xi_1,\xi_2\in\hat{\mathbb{N}}\}},\\ &\geq \wedge\{\gamma_{\text{ß}}(\mathfrak{l}+\xi_1) \wedge \gamma_{\text{ß}}(\mathfrak{p}+\xi_2) : \xi_1,\xi_2 \in \hat{\mathbb{N}}\}e^{i\wedge\{\varpi_{\text{ß}}((\mathfrak{l}+\xi_1))\wedge\varpi_{\text{ß}}(\mathfrak{p}+\xi_2):\xi_1,\xi_2\in\hat{\mathbb{N}}\}},\\ &\geq (\wedge\{\gamma_{\text{ß}}(\mathfrak{l}+\xi_1) : \xi_1 \in \hat{\mathbb{N}}\})e^{i(\wedge\{\varpi_{\text{ß}}(\mathfrak{l}+\xi_1):\xi_1\in\hat{\mathbb{N}}\})}\end{aligned}$$

$$
\begin{aligned}
& && \wedge(\wedge\{\gamma_{ß}(\mathfrak{p}+\xi_2) : \xi_2 \in \hat{\mathbb{N}}\})e^{i(\wedge\{\varpi_{ß}(\mathfrak{p}+\xi_2):\xi_2\in\hat{\mathbb{N}}\})},\\
& = && (\wedge\{\gamma_{ß}(\xi_1) : \xi \in [\mathfrak{l}]\}) \wedge (\wedge\{\gamma_{ß}(\xi_2) : \xi_2 \in [\mathfrak{p}]\}),\\
& = && \gamma_Q([\mathfrak{l}]) \wedge \gamma_Q([\mathfrak{p}]).\\
\text{This implies } \gamma_Q([\mathfrak{l}] + [\mathfrak{p}])e^{i\varpi_Q([\mathfrak{l}]+[\mathfrak{p}])} & \geq && \gamma_Q([\mathfrak{l}]) \wedge \gamma_Q([\mathfrak{p}]).
\end{aligned}
$$

Thus, in all cases we get $\gamma_Q([\mathfrak{l}] + [\mathfrak{p}]) \geq \gamma_Q([\mathfrak{l}]) \wedge \gamma_Q([\mathfrak{p}])$. In same way we get $\nu_Q([\mathfrak{l}] + [\mathfrak{p}]) \leq \nu_Q([\mathfrak{l}]) \vee \nu_Q([\mathfrak{p}]) \forall \mathfrak{l},\ \mathfrak{p} \in \hat{\mathbb{M}}$. Hence, Q is a $\mathbb{CIFSM}$ of $\hat{\mathbb{M}}/\hat{\mathbb{N}}$. Thus, the theorem is proved. □

We are now made to see that the integral domain $\mathbb{R}$ is to be a ring. In such a case, the space of the $\hat{\mathbb{M}}$ will become a vector space in the $\mathbb{R}$ and every non-zero element in the $\mathbb{R}$ have a multiplication inverse in the $\mathbb{R}$. Thus, it is straightforward to demonstrate that every submodule of $\hat{\mathbb{M}}$ is a prime submodule and that $\hat{\mathbb{M}}$ is a divisible module. Thus, the Theorem 3.8 has the following conclusion.

Corollary 3.9. *Suppose that* $ß = (\gamma_{ß}, \hat{\gamma_{ß}})$ *is a* $\mathbb{CIFSM}$ *of* $\hat{\mathbb{M}}$, *where* $\hat{\mathbb{N}}$ *is a submodule of a module* $\hat{\mathbb{M}}$ *over a field* $\mathbb{R}$. *Then, the* $\mathbb{CIFS}$ $Q = (\gamma_Q, \hat{\gamma_Q})$, *defined as in Theorem 3.8, is a complex intuitionistic submodule of the quotient module* $\hat{\mathbb{M}}/\hat{\mathbb{N}}$.

4. Attributes of $\mathbb{CIFSM}$ over R using support and level sets

In this section, some important results related to $\mathbb{CIFSM}$ are analyzed using the idea of support and level sets.

Definition 4.1. *Let* $\mathbb{U}$ *be a non-empty set and let* $\mathbb{E}$ *be a* $\mathbb{CIF}$ *set in* $\mathbb{U}$. *Then the support of* $\mathbb{E}$, *denoted by* $\mathbb{E}^*$ *and expressed as* $\mathbb{E}^* = \{l \in \mathbb{U} : \gamma_{\mathbb{E}}(l) > 0,\ \varpi_{\mathbb{E}}(l) > 0$ *and* $\hat{\gamma}_{\mathbb{E}}(l) < 1,\ \hat{\varpi}_{\mathbb{E}}(l) < 2\pi\}$.

The following theorem illustrates that if $\mathbb{K}$ $\mathbb{CIF}$ set is contains $\mathbb{CIF}$ set $\mathbb{E}$ in $\mathbb{R}$-module then support set also follows the same pattern.

Theorem 4.2. *Suppose* $\mathbb{E}$ *and* $\mathbb{K}$ *are* $\mathbb{CIF}$ *sets in a module* $\mathbb{M}$ *over* $\mathbb{R}$. *If* $\mathbb{E} \subseteq \mathbb{K}$. *Then,* $\mathbb{E}^* \subseteq \mathbb{K}^*$.

Proof. Since $\mathbb{E} \subseteq \mathbb{K}$, this implies $\gamma_{\mathbb{E}}(l) \leq \gamma_{\mathbb{K}}(l)$, $\varpi_{\mathbb{E}}(l) \leq \varpi_{\mathbb{K}}(l)$ and $\hat{\alpha}_{\mathbb{E}}(l) \geq \hat{\alpha}_{\mathbb{K}}(l)$ for all l belongs to $\mathbb{M}$. Suppose $l \in \mathbb{E}^*$, then $\gamma_{\mathbb{E}}(l) > 0,\ \varpi_{\mathbb{E}}(l) > 0$ *and* $\hat{\gamma}_{\mathbb{E}}(l) < 1,\ \hat{\varpi}_{\mathbb{E}}(l) < 2\pi$. So we get $\gamma_{\mathbb{K}}(l) \geq \gamma_{\mathbb{E}}(l) > 0,\ \varpi_{\mathbb{K}}(l) \geq \varpi_{\mathbb{E}}(l) > 0$ *and* $\hat{\gamma}_{\mathbb{K}}(l) \leq \hat{\gamma}_{\mathbb{E}}(l) < 1,\ \hat{\varpi}_{\mathbb{K}}(l) \leq \varpi_{\mathbb{E}}(l) < 2\pi$, and hence $l \in \mathbb{K}^*$. Therefore, $\mathbb{E}^* \subseteq \mathbb{K}^*$. □

The upcoming results shows that the support set of $\mathbb{CIFSM}$ is also $\mathbb{CIFSM}$.

Theorem 4.3. *Assume that* $\mathbb{E}$ *is a* $\mathbb{CIFSM}$ *of a module* $\mathbb{M}$ *over* $\mathbb{R}$. *Then,* $\mathbb{E}^*$ *is a submodule of* $\mathbb{M}$.

Proof. Suppose $l, w \in \mathbb{E}^*$. Then, $\gamma_{\mathbb{E}}(l) > 0,\ \varpi_{\mathbb{E}}(l) > 0,\ \gamma_{\mathbb{E}}(w) > 0,\ \varpi_{\mathbb{E}}(w) > 0$ *and* $\hat{\gamma}_{\mathbb{E}}(l) < 1,\ \hat{\gamma}_{\mathbb{E}}(w) < 1,\ \hat{\varpi}_{\mathbb{E}}(l) < 2\pi,\ \hat{\varpi}_{\mathbb{E}}(w) < 2\pi$. Now $\gamma_{\mathbb{E}}(l-w) \geq \gamma_{\mathbb{E}}(l) \wedge \gamma_{\mathbb{E}}(w) > 0,\ \varpi_{\mathbb{E}}(l-w) \geq \varpi_{\mathbb{E}}(l) \wedge \varpi_{\mathbb{E}}(w) > 0$ *and* $\hat{\gamma}_{\mathbb{E}}(l-w) \leq \hat{\gamma}_{\mathbb{E}}(l) \vee \hat{\gamma}_{\mathbb{E}}(w) < 1,\ \hat{\varpi}_{\mathbb{E}}(l-w) \leq \hat{\varpi}_{\mathbb{E}}(l) \vee \varpi_{\mathbb{E}}(w) < 2\pi$. So, $l - w \in \mathbb{E}^*$. Now for $l \in \mathbb{E}^*$ *and* $r \in \mathbb{R}, \gamma_{\mathbb{E}}(rl) \leq \gamma_{\mathbb{E}}(l)$ as $\gamma_{\mathbb{E}}(l) > 0$ this implies that $\gamma_{\mathbb{E}}(rl) > 0,\ \varpi_{\mathbb{E}}(rl) \leq \varpi_{\mathbb{E}}(l)$, because $\varpi_{\mathbb{E}}(l) > 0$, thus $\varpi_{\mathbb{E}}(rl) > 0$ and $\hat{\gamma}_{\mathbb{E}}(rl) \leq \hat{\gamma}_{\mathbb{E}}(l) < 1,\ \hat{\varpi}_{\mathbb{E}}(rl) \leq \hat{\hat{\varpi}}_{\mathbb{E}}(l) < 2\pi$. So, we get $rl \in \mathbb{E}^*$. Hence, by Definition 3.1, $\mathbb{E}^*$ is a submodule of $\mathbb{M}$. □

Lemma 4.4. *Assume that* $\mathfrak{J}$ *and* $\mathfrak{U}$ *are two* $\mathbb{CIFSM}$*s of the module* $\hat{\mathbb{M}}$ *over the ring* $\mathbb{R}$. *Then,* $\mathfrak{J} + \mathfrak{U}$ *is also a* $\mathbb{CIFSM}$ *of* $\hat{\mathbb{M}}$.

Proof. For each $\mathfrak{l} \in \hat{\mathbb{M}}$, define

$$\mathfrak{I} + \mathfrak{U} = \left\{ \left(\mathfrak{l}, \pounds_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l}), \hat{\pounds}_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l}) \right) : \mathfrak{l} \in \hat{\mathbb{M}} \right\},$$

where

$$\pounds_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l}) = \bigvee_{\mathfrak{l}=\mathfrak{w}+z} \{\gamma_{\mathfrak{I}}(\mathfrak{w}) \wedge \gamma_{\mathfrak{U}}(z)\}\, e^{i \bigvee_{\mathfrak{l}=\mathfrak{w}+z}\{\varpi_{\mathfrak{I}}(\mathfrak{w}) \wedge \varpi_{\mathfrak{U}}(z)\}}$$

and

$$\hat{\pounds}_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l}) = \bigwedge_{\mathfrak{l}=\mathfrak{w}+z} \{\hat{\gamma}_{\mathfrak{I}}(\mathfrak{w}) \vee \hat{\gamma}_{\mathfrak{U}}(z)\}\, e^{i \bigwedge_{\mathfrak{l}=\mathfrak{w}+z}\{\hat{\varpi}_{\mathfrak{I}}(\mathfrak{w}) \vee \hat{\varpi}_{\mathfrak{U}}(z)\}}.$$

For convenience, write $\pounds_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l}) = \gamma_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l})e^{i\varpi_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l})}$ and $\hat{\pounds}_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l}) = \hat{\gamma}_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l})e^{i\hat{\varpi}_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l})}$. Since $0 = 0 + 0$ and both $\mathfrak{I}$ and $\mathfrak{U}$ are $\mathbb{CIFSM}$s, we obtain $\gamma_{\mathfrak{I}+\mathfrak{U}}(0) \geq \gamma_{\mathfrak{I}}(0) \wedge \gamma_{\mathfrak{U}}(0) = 1$, and $\hat{\gamma}_{\mathfrak{I}+\mathfrak{U}}(0) \leq \hat{\gamma}_{\mathfrak{I}}(0) \vee \hat{\gamma}_{\mathfrak{U}}(0) = 0$. Therefore, $\gamma_{\mathfrak{I}+\mathfrak{U}}(0) = 1 \quad \text{and} \quad \hat{\gamma}_{\mathfrak{I}+\mathfrak{U}}(0) = 0$. Similarly,

$$\varpi_{\mathfrak{I}+\mathfrak{U}}(0) = 2\pi \quad \text{and} \quad \hat{\varpi}_{\mathfrak{I}+\mathfrak{U}}(0) = 0.$$

Now, let $\mathfrak{l}, \mathfrak{p} \in \hat{\mathbb{M}}$. Suppose

$$\mathfrak{l} = \mathfrak{w}_1 + z_1 \quad \text{and} \quad \mathfrak{p} = \mathfrak{w}_2 + z_2.$$

Then,

$$\mathfrak{l} + \mathfrak{p} = (\mathfrak{w}_1 + \mathfrak{w}_2) + (z_1 + z_2).$$

Because $\mathfrak{I}$ and $\mathfrak{U}$ are $\mathbb{CIFSM}$s, $\gamma_{\mathfrak{I}}(\mathfrak{w}_1 + \mathfrak{w}_2) \geq \gamma_{\mathfrak{I}}(\mathfrak{w}_1) \wedge \gamma_{\mathfrak{I}}(\mathfrak{w}_2)$ and $\gamma_{\mathfrak{U}}(z_1 + z_2) \geq \gamma_{\mathfrak{U}}(z_1) \wedge \gamma_{\mathfrak{U}}(z_2)$. Consequently,

$$\gamma_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l} + \mathfrak{p}) \geq (\gamma_{\mathfrak{I}}(\mathfrak{w}_1) \wedge \gamma_{\mathfrak{U}}(z_1)) \quad \wedge (\gamma_{\mathfrak{I}}(\mathfrak{w}_2) \wedge \gamma_{\mathfrak{U}}(z_2)).$$

Taking the supremum over all representations of $\mathfrak{l}$ and $\mathfrak{p}$ gives

$$\gamma_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l} + \mathfrak{p}) \geq \gamma_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l}) \wedge \gamma_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{p}).$$

The same argument for the membership phases gives $\varpi_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l} + \mathfrak{p}) \geq \varpi_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l}) \wedge \varpi_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{p})$. Therefore, $\pounds_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l} + \mathfrak{p}) \geq \pounds_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l}) \wedge \pounds_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{p})$. Similarly, using

$$\hat{\gamma}_{\mathfrak{I}}(\mathfrak{w}_1 + \mathfrak{w}_2) \leq \hat{\gamma}_{\mathfrak{I}}(\mathfrak{w}_1) \vee \hat{\gamma}_{\mathfrak{I}}(\mathfrak{w}_2)$$

and

$$\hat{\gamma}_{\mathfrak{U}}(z_1 + z_2) \leq \hat{\gamma}_{\mathfrak{U}}(z_1) \vee \hat{\gamma}_{\mathfrak{U}}(z_2),$$

and then taking the infimum over all representations, we obtain $\hat{\gamma}_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l} + \mathfrak{p}) \leq \hat{\gamma}_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l}) \vee \hat{\gamma}_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{p})$, and $\hat{\varpi}_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l} + \mathfrak{p}) \leq \hat{\varpi}_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l}) \vee \hat{\varpi}_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{p})$. Thus, $\hat{\pounds}_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l} + \mathfrak{p}) \leq \hat{\pounds}_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l}) \vee \hat{\pounds}_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{p})$. Finally, let $r \in \mathbb{R}$. If $\mathfrak{l} = \mathfrak{w} + z$, then

$$r\mathfrak{l} = r\mathfrak{w} + rz.$$

Since $\mathfrak{I}$ and $\mathfrak{U}$ are $\mathbb{CIFSM}$s,

$$\gamma_{\mathfrak{I}}(r\mathfrak{w}) \geq \gamma_{\mathfrak{I}}(\mathfrak{w}) \quad \text{and} \quad \gamma_{\mathfrak{U}}(rz) \geq \gamma_{\mathfrak{U}}(z).$$

Therefore, $\gamma_{\mathfrak{I}+\mathfrak{U}}(r\mathfrak{l}) \geq \gamma_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l})$. Similarly, $\varpi_{\mathfrak{I}+\mathfrak{U}}(r\mathfrak{l}) \geq \varpi_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l})$, $\hat{\gamma}_{\mathfrak{I}+\mathfrak{U}}(r\mathfrak{l}) \leq \hat{\gamma}_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l})$, and $\hat{\varpi}_{\mathfrak{I}+\mathfrak{U}}(r\mathfrak{l}) \leq \hat{\varpi}_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l})$. Hence, $\pounds_{\mathfrak{I}+\mathfrak{U}}(r\mathfrak{l}) \geq \pounds_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l})$ and $\hat{\pounds}_{\mathfrak{I}+\mathfrak{U}}(r\mathfrak{l}) \leq \hat{\pounds}_{\mathfrak{I}+\mathfrak{U}}(\mathfrak{l})$. Therefore, $\mathfrak{I} + \mathfrak{U}$ satisfies all the defining conditions of a complex intuitionistic fuzzy submodule. Hence, $\mathfrak{I} + \mathfrak{U}$ is a $\mathbb{CIFSM}$ of $\hat{\mathbb{M}}$. □

Theorem 4.5. *Suppose that* $\mathfrak{B}, \mathfrak{U} \in \mathbb{R}(\hat{\mathbb{M}})$ *be such that* $\mathfrak{B} \subseteq \mathfrak{U}$. *Then,* $(\mathfrak{U}|_{\mathfrak{U}^*})_{\mathfrak{B}^*} = \mathfrak{U}/\mathfrak{B}^*$.

Proof. Let $\mathfrak{B}$ and $\mathfrak{U}$ are $\mathbb{CIFSM}$s of $\mathbb{R}-$module $\hat{\mathbb{M}}$ such that $\mathfrak{B} \subseteq \mathfrak{U}$. Therefore, $\mathfrak{B}^*$ and $\mathfrak{U}^*$ are $\mathbb{R}-$submodules of $\hat{\mathbb{M}}$ such that $\mathfrak{B}* \subseteq \mathfrak{U}*$. Clearly, both $(\mathfrak{U}|_{\mathfrak{U}^*})_{\mathfrak{B}^*}$ and $\mathfrak{U}/\mathfrak{B}$ are $\mathbb{CIFSM}$ of $\mathbb{R}-$module $\mathfrak{U}^*/\mathfrak{B}^*$. Let $l+\mathfrak{B}^* \in \mathfrak{U}^*/\mathfrak{B}^*$ random element, where $\mathfrak{a} \in \mathfrak{U}^*$. So, we get $\gamma_{(\mathfrak{U}|\mathfrak{U}^*)_{\mathfrak{B}^*}}(\mathfrak{a}+\mathfrak{B}^*)e^{i\varpi_{(\mathfrak{U}|\mathfrak{U}^*)_{\mathfrak{B}^*}}(\mathfrak{a}+\mathfrak{B}^*)} = \vee\{\gamma_{(\mathfrak{U}|\mathfrak{U}^*)_{\mathfrak{B}^*}}(\mathfrak{a}+\mathfrak{p}) : \mathfrak{p} \in \mathfrak{B}^*\}e^{i\vee\{\varpi_{(\mathfrak{U}|\mathfrak{U}^*)_{\mathfrak{B}^*}}(\mathfrak{a}+\mathfrak{p}):\mathfrak{p}\in\mathfrak{B}^*\}} = \vee\{\gamma_{\mathfrak{U}}(\mathfrak{a}+\mathfrak{p}) : \mathfrak{p} \in \mathfrak{B}^*\}e^{i\vee\{\varpi_{\mathfrak{U}}(\mathfrak{a}+\mathfrak{p}):\mathfrak{p}\in\mathfrak{B}^*\}} = \gamma_{\mathfrak{U}/\mathfrak{B}}(\mathfrak{a}+\mathfrak{B}^*)e^{i\varpi_{\mathfrak{U}/\mathfrak{B}}(\mathfrak{a}+\mathfrak{B}^*)}$. Similarly, we can show that: $\hat{\gamma}_{(\mathfrak{U}|\mathfrak{U}^*)_{\mathfrak{B}^*}}(\mathfrak{a}+\mathfrak{B}^*)e^{i\hat{\varpi}_{(\mathfrak{U}|\mathfrak{U}^*)_{\mathfrak{B}^*}}(\mathfrak{a}+\mathfrak{B}^*)} = \hat{\gamma}_{\mathfrak{U}/\mathfrak{B}}(\mathfrak{a}+\mathfrak{B}^*)e^{i\hat{\varpi}_{\mathfrak{U}/\mathfrak{B}}(\mathfrak{a}+\mathfrak{B}^*)}$. Hence, $(\mathfrak{U}|_{\mathfrak{U}^*})_{\mathfrak{B}^*} = \mathfrak{U}/\mathfrak{B}^*$. □

Definition 4.6. *Let* $\mathbb{E}$ *be a* $\mathbb{CIF}$ *set in* $\mathbb{U} \neq \phi$, $\alpha, \widehat{\alpha} \in [0,1]$, *and* $\beta, \widehat{\beta} \in [0, 2\pi]$. *Then, the* $\mathbb{E}^{(\widehat{\alpha},\widehat{\beta})}_{(\alpha,\beta)}$*- level set of* $\mathbb{E}$ *is expressed as* $\mathbb{E}^{(\widehat{\alpha},\widehat{\beta})}_{(\alpha,\beta)} = \{l \in \mathbb{U} : \gamma_{\mathbb{E}}(l) \geq \alpha, \varpi_{\mathbb{E}}(l) \geq \beta, \hat{\gamma}_{\mathbb{E}}(l) \leq \widehat{\alpha}, \hat{\varpi}_{\mathbb{E}}(l) \leq \widehat{\beta}\}$.

Lemma 4.7. *Let* $\mathbb{E}$ *and* $\mathbb{K}$ *be* $\mathbb{CIF}$ *sets in a nonempty set* $\mathbb{U}$, *we have the following:*

1. $\mathbb{E} \subseteq \mathbb{K}, \alpha, \widehat{\alpha} \in [0,1],\ \beta, \widehat{\beta} \in [0, 2\pi]$ *with* $\alpha + \widehat{\alpha} \leq 1,\ \beta + \widehat{\beta} \leq 2\pi \Rightarrow \mathbb{E}^{(\widehat{\alpha},\widehat{\beta})}_{(\alpha,\beta)} \subseteq \mathbb{K}^{(\widehat{\alpha},\widehat{\beta})}_{(\alpha,\beta)}$.
2. $\alpha_1, \widehat{\alpha}_1, \alpha_2, \widehat{\alpha}_2 \in [0,1],\ \beta_1, \widehat{\beta}_1, \beta_2, \widehat{\beta}_2 \in [0, 2\pi]$ *with* $\alpha_1 + \widehat{\alpha}_1 \leq 1,\ \alpha_2 + \widehat{\alpha}_2 \leq 1,\ \alpha_1 \leq \alpha_2,\ \beta_1 + \widehat{\beta}_1 \leq 2\pi,\ \beta_2 + \widehat{\beta}_2 \leq 2\pi,\ \beta_1 \leq \beta_2$ *and* $\widehat{\alpha}_1 \geq \widehat{\alpha}_2,\ \widehat{\beta}_1 \geq \widehat{\beta}_2 \Rightarrow \mathbb{E}^{(\widehat{\alpha}_2,\widehat{\beta}_2)}_{(\alpha_2,\beta_2)} \subseteq \mathbb{K}^{(\widehat{\alpha}_1,\widehat{\beta}_1)}_{(\alpha_1,\beta_1)}$.
3. $\mathbb{E} = \mathbb{K} \Leftrightarrow \mathbb{E}^{(\widehat{\alpha},\widehat{\beta})}_{(\alpha,\beta)} = \mathbb{K}^{(\widehat{\alpha},\widehat{\beta})}_{(\alpha,\beta)} =\ \forall\, \alpha, \widehat{\alpha} \in [0,1],\ \beta, \widehat{\beta} \in [0, 2\pi]$ *with* $\alpha + \widehat{\alpha} \leq 1,\ \beta + \widehat{\beta} \leq 2\pi$.

The following theorem shows that the union of the support sets of a family of $\mathbb{CIFS}$s is contained in the support set of their union, whereas the intersection of their support sets is equal to the support set of their intersection.

Theorem 4.8. *Let* $\mathbb{E}_t;\ t \in T$. $|T| > 1$, *be a family of complex intuitionistic fuzzy sets in* $\mathbb{U}$. *Then, for any* $\alpha, \widehat{\alpha} \in [0,1],\ \beta, \widehat{\beta} \in [0, 2\pi]$ *with* $\alpha + \widehat{\alpha} \leq 1,\ \beta + \widehat{\beta} \leq 2\pi$ *the following hold:*

1. $\bigcup_{t\in T}(\mathbb{E}_t)^{(\widehat{\alpha},\widehat{\beta})}_{(\alpha,\beta)} \subseteq (\bigcup_{t\in T}\mathbb{E}_t)^{(\widehat{\alpha},\widehat{\beta})}_{(\alpha,\beta)}$
2. $\bigcap_{t\in T}(\mathbb{E}_t)^{(\widehat{\alpha},\widehat{\beta})}_{(\alpha,\beta)} = (\bigcap_{t\in T}\mathbb{E}_t)^{(\widehat{\alpha},\widehat{\beta})}_{(\alpha,\beta)}$

Proof.

$$
\begin{aligned}
(i) \text{ We have, } l \in \bigcup_{t\in T}(\mathbb{E}_t)^{(\widehat{\alpha},\widehat{\beta})}_{(\alpha,\beta)} \quad &\Rightarrow \quad l \in \mathbb{E}^{(\widehat{\alpha},\widehat{\beta})}_{j(\alpha,\beta)} \text{ for some } j \in T,\\
&\Rightarrow \quad \gamma_{\mathbb{E}_j}(l) \geq \alpha, \varpi_{\mathbb{E}_j}(l) \geq \beta, \hat{\gamma}_{\mathbb{E}_j}(l) \leq \widehat{\alpha}, \hat{\varpi}_{\mathbb{E}_j}(l) \leq \widehat{\beta}\\
&\Rightarrow \quad \vee_{t\in T}\ \gamma_{\mathbb{E}_r}(l) \geq \alpha,\ \ \vee_{t\in T}\ \ \varpi_{\mathbb{E}_t}(l) \geq \beta,\ \ \wedge_{t\in T}\ \ \hat{\gamma}_{\mathbb{E}_t}(l) \leq \widehat{\alpha}\ \ \wedge_{t\in T}\ \ \hat{\varpi}_{\mathbb{E}_t}(l) \leq \widehat{\beta}\\
&\Rightarrow \quad \gamma \cup_{t\in T} \mathbb{E}_t(l) \geq \alpha,\ \varpi \cup_{t\in T} \mathbb{E}_t(l) \geq \beta,\ \hat{\gamma} \cup_{t\in T} \mathbb{E}_t(l) \leq \widehat{\alpha},\ \hat{\varpi} \cup_{t\in T} \mathbb{E}_t(l) \leq \widehat{\beta}\\
&\Rightarrow \quad l \in (\bigcup_{t\in T}\mathbb{E}_t)^{(\widehat{\alpha},\widehat{\beta})}_{(\alpha,\beta)}.
\end{aligned}
$$

$$
\begin{aligned}
(ii) \text{ Also, } l \in \bigcap_{t\in T}(\mathbb{E}_t)^{(\widehat{\alpha},\widehat{\beta})}_{(\alpha,\beta)} \quad &\Leftrightarrow \quad l \in \mathbb{E}^{(\widehat{\alpha},\widehat{\beta})}_{t(\alpha,\beta)}\ \ \forall t \in T,\\
&\Leftrightarrow \quad \gamma_{\mathbb{E}_j}(l) \geq \alpha, \varpi_{\mathbb{E}_j}(l) \geq \beta, \hat{\gamma}_{\mathbb{E}_j}(l) \leq \widehat{\alpha}, \hat{\varpi}_{\mathbb{E}_j}(l) \leq \widehat{\beta},\ \forall t \in T,\\
&\Leftrightarrow \quad \wedge_{t\in T}\ \gamma_{\mathbb{E}_t}(l) \geq \alpha,\ \ \wedge_{t\in T}\ \ \varpi_{\mathbb{E}_t}(l) \geq \beta,\ \ \vee_{t\in T}\ \ \hat{\gamma}_{\mathbb{E}_t}(l) \leq \widehat{\alpha},\ \ \vee_{t\in T}\ \ \hat{\varpi}_{\mathbb{E}_t}(l) \leq \widehat{\beta}\\
&\Leftrightarrow \quad \gamma \cap_{t\in T} \mathbb{E}_t(l) \geq \alpha,\ \varpi \cap_{t\in T} \mathbb{E}_t(l) \geq \beta,\ \hat{\gamma} \cap_{t\in T} \mathbb{E}_t(l) \leq \widehat{\alpha},\ \hat{\varpi} \cap_{t\in T} \mathbb{E}_t(l) \leq \widehat{\beta}\\
&\Leftrightarrow \quad l \in (\bigcap_{t\in T}\mathbb{E}^{(\widehat{\alpha},\widehat{\beta})}_{t(\alpha,\beta)}.
\end{aligned}
$$

□

The following theorem establishes how varying level parameters determine the union and intersection of the level subsets of a $\mathbb{CIFS}$.

Theorem 4.9. *Let $\mathbb{E}$ be a $\mathbb{CIFS}$ on a nonempty set $\mathbb{U}$, and let $J \neq \emptyset$ be an index set. For each $i \in J$, let $\alpha_i, \widehat{\alpha_i} \in [0,1]$ and $\beta_i, \widehat{\beta_i} \in [0, 2\pi]$ satisfy $\alpha_i + \widehat{\alpha_i} \leq 1$ and $\beta_i + \widehat{\beta_i} \leq 2\pi$. Define $\lambda_1 = \wedge_{i\in J}\alpha_i$, $\delta_1 = \wedge_{i\in J}\beta_i$, $\widehat{\lambda_1} = \vee_{i\in J}\widehat{\alpha_i}$, $\widehat{\delta_1} = \vee_{i\in J}\widehat{\beta_i}$, and $\lambda_2 = \vee_{i\in J}\alpha_i$, $\delta_2 = \vee_{i\in J}\beta_i$, $\widehat{\lambda_2} = \wedge_{i\in J}\widehat{\alpha_i}$, $\widehat{\delta_2} = \wedge_{i\in J}\widehat{\beta_i}$. Then, the following relations hold:*

1. $\bigcup_{i\in J} \mathbb{E}^{(\widehat{\alpha_i},\widehat{\beta_i})}_{(\alpha_i,\beta_i)} \subseteq \mathbb{E}^{(\widehat{\lambda_1},\widehat{\delta_1})}_{(\lambda_1,\delta_1)}$.
2. $\bigcap_{i\in J} \mathbb{E}^{(\widehat{\alpha_i},\widehat{\beta_i})}_{(\alpha_i,\beta_i)} = \mathbb{E}^{(\widehat{\lambda_2},\widehat{\delta_2})}_{(\lambda_2,\delta_2)}$.

Proof.

$$\begin{aligned}
\text{We have, } l \in \bigcup_{i\in J} \mathbb{E}^{(\widehat{\alpha_i},\widehat{\beta_i})}_{(\alpha_i,\beta_i)} \quad &\Rightarrow \quad l \in \mathbb{E}^{(\widehat{\alpha_i},\widehat{\beta_i})}_{(\alpha_i,\beta_i)}, \text{ for some } j \in J,\\
&\Rightarrow \quad \gamma_{\mathbb{E}}(l) \geq \alpha_j,\ \varpi_{\mathbb{E}}(l) \geq \beta_j,\ \hat{\gamma}_{\mathbb{E}}(l) \leq \widehat{\alpha}_j, \hat{\varpi}_{\mathbb{E}}(l) \leq \widehat{\beta}_j,\\
&\Rightarrow \quad \gamma_{\mathbb{E}}(l) \geq \alpha_j \geq \wedge_{i\in J}\, \alpha_i = \lambda_1, \varpi_A\mathbb{E}(l) \geq \beta_j \geq \wedge_{i\in J}\, \beta_i = \delta_1,\\
&\qquad\ \ \hat{\gamma}_{\mathbb{E}}(l) \leq \widehat{\alpha}_j \leq \vee_{i\in J}\, \widehat{\alpha}_i = \widehat{\lambda_1},\ \hat{\varpi}_{\mathbb{E}}(l) \leq \widehat{\beta}_j \leq \vee_{i\in J}\, \widehat{\beta}_i = \widehat{\delta_1},\\
&\Rightarrow \quad l \in \mathbb{E}^{(\widehat{\gamma_1},\widehat{\delta_1})}_{(\lambda_1,\delta_1)}.
\end{aligned}$$

$$\begin{aligned}
\text{Also, } l \in \bigcap_{i\in J} \mathbb{E}^{(\widehat{\alpha_i},\widehat{\beta_i})}_{(\alpha_i,\beta_i)} \quad &\Rightarrow \quad l \in \mathbb{E}^{(\widehat{\alpha_i},\widehat{\beta_i})}_{(\alpha_i,\beta_i)},\ \forall i \in J,\\
&\Rightarrow \quad \gamma_{\mathbb{E}}(l) \geq \alpha_i,\ \hat{\gamma}_{\mathbb{E}}(l) \leq \widehat{\alpha_i},\ \varpi_{\mathbb{E}}(l) \geq \beta_i,\ \hat{\varpi}_{\mathbb{E}}(l) \leq \widehat{\beta_i}\\
&\Rightarrow \quad \gamma_{\mathbb{E}}(l) \geq \alpha_i \geq \vee_{i\in J}\, \alpha_i = \lambda_1, \varpi_{\mathbb{E}}(l) \geq \beta_i \geq \vee_{i\in J}\, \beta_i = \delta_1,\\
&\qquad\ \ \hat{\gamma}_{\mathbb{E}}(l) \leq \widehat{\alpha}_j \leq \wedge_{i\in J}\, \widehat{\alpha}_i = \widehat{\lambda_1}, \hat{\varpi}_{\mathbb{E}}(l) \leq \widehat{\beta_i} \leq \wedge_{i\in J}\, \widehat{\beta_i} = \widehat{\delta_1},\\
&\Rightarrow \quad l \in \mathbb{E}^{(\widehat{\gamma_2},\widehat{\delta_2})}_{(\lambda_2,\delta_2)}.
\end{aligned}$$

□

This theorem provides a level-subset characterization of complex intuitionistic fuzzy submodules.

Theorem 4.10. *Let $\mathbb{E}$ be a $\mathbb{CIF}$ set in a module $\mathbb{M}$ over $\mathbb{R}$. Then, $\mathbb{E}$ is a $\mathbb{CIFSM}$ of $\mathbb{M}$ $\Leftrightarrow$ for every $\alpha, \widehat{\alpha} \in [0,1]$, $\beta, \widehat{\beta} \in [0, 2\pi]$ with $\alpha + \widehat{\alpha} \leq 1$, $\beta + \widehat{\beta} \leq 2\pi$, $\mathbb{E}^{(\widehat{\alpha},\widehat{\beta})}_{(\alpha,\beta)}$ is a submodule of $\mathbb{M}$.*

Proof. Assume that $\mathbb{E}$ is a $\mathbb{CIFSM}$ of a module $\mathbb{M}$ over $\mathbb{R}$. Let $\alpha, \widehat{\alpha} \in [0,1]$, $\beta, \widehat{\beta} \in [0, 2\pi]$ implies that $\alpha + \widehat{\alpha} \leq 1$, $\beta + \widehat{\beta} \leq 2\pi$. We have

$$\begin{aligned}
l, w \in \mathbb{E}^{(\widehat{\alpha},\widehat{\beta})}_{(\alpha,\beta)} \quad &\Rightarrow \quad \gamma_{\mathbb{E}}(l) \geq \alpha,\ \gamma_{\mathbb{E}}(w) \geq \alpha,\ \varpi_{\mathbb{E}}(l) \geq \beta,\ \varpi_{\mathbb{E}}(w) \geq \beta,\\
\text{and} \quad &\ \ \hat{\gamma}_{\mathbb{E}}(l) \leq \widehat{\alpha},\ \hat{\gamma}_{\mathbb{E}}(w) \leq \widehat{\alpha},\ \hat{\varpi}_{\mathbb{E}}(l) \leq \widehat{\beta},\ \hat{\varpi}_{\mathbb{E}}(w) \leq \widehat{\beta},\\
&\Rightarrow \quad \gamma_{\mathbb{E}}(l-w) \geq \gamma_{\mathbb{E}}(l) \wedge \gamma_{\mathbb{E}}(w) \geq \alpha,\ \varpi_{\mathbb{E}}(l-w) \geq \varpi_{\mathbb{E}}(l) \wedge \varpi_{\mathbb{E}}(w) \geq \beta,\\
\text{and} \quad &\ \ \hat{\gamma}_{\mathbb{E}}(l-w) \leq \hat{\gamma}_{\mathbb{E}}(l) \vee \hat{\gamma}_{\mathbb{E}}(w) \leq \widehat{\alpha},\ \hat{\varpi}_{\mathbb{E}}(l-w) \leq \hat{\varpi}_{\mathbb{E}}(l) \vee \hat{\varpi}_{\mathbb{E}}(w) \leq \widehat{\beta},\\
&\Rightarrow \quad l - w \in \mathbb{E}^{(\widehat{\alpha},\widehat{\beta})}_{(\alpha,\beta)}.
\end{aligned}$$

$$
\begin{aligned}
\text{Also, } r \in \mathbb{R}, l \in \mathbb{E}_{(\alpha,\beta)}^{(\widehat{\alpha},\widehat{\beta})} \quad &\Rightarrow \quad \gamma_{\mathbb{E}}(l) \geq \alpha,\ \varpi_{\mathbb{E}}(l) \geq \beta,\\
&\textit{and} \quad \hat{\gamma}_{\mathbb{E}}(l) \leq \widehat{\alpha},\ \hat{\varpi}_{\mathbb{E}}(l) \leq \widehat{\beta},\\
&\Rightarrow \quad \gamma_{\mathbb{E}}(rl) \geq \gamma_{\mathbb{E}}(l) \geq \alpha,\ \varpi_{\mathbb{E}}(rl) \geq \varpi_{\mathbb{E}}(l) \geq \beta,\\
&\textit{and} \quad \hat{\gamma}_{\mathbb{E}}(rl) \leq \hat{\gamma}_{\mathbb{E}}(l) \leq \widehat{\alpha},\ \hat{\varpi}_{\mathbb{E}}(rl) \leq \hat{\varpi}_{\mathbb{E}}(l) \leq \widehat{\beta},\\
&\Rightarrow \quad rl \in \mathbb{E}_{(\alpha,\beta)}^{(\widehat{\alpha},\widehat{\beta})}.
\end{aligned}
$$

Hence, $\mathbb{E}_{(\alpha,\beta)}^{(\widehat{\alpha},\widehat{\beta})}$ is a submodule of $\mathbb{M}$. Conversely, Suppose $\mathbb{E}_{(\alpha,\beta)}^{(\widehat{\alpha},\widehat{\beta})}$ is a submodule of $\mathbb{M}$ for every $\alpha, \widehat{\alpha} \in [0,1]$, $\beta, \widehat{\beta} \in [0, 2\pi]$ with $\alpha+\widehat{\alpha} \leq 1$, $\beta+\widehat{\beta} \leq 2\pi$. Now let $l, w \in \mathbb{M}$ be arbitrarily fixed. Let $\gamma_{\mathbb{E}}(l) \wedge \gamma_{\mathbb{E}}(w) = \alpha$, $\gamma_{\mathbb{E}}(l) \wedge \gamma_{\mathbb{E}}(w) = \beta$ *and* $\hat{\varpi}_{\mathbb{E}}(l) \vee \hat{\varpi}_{\mathbb{E}}(w) = \widehat{\alpha}$, $\hat{\varpi}_{\mathbb{E}}(l) \vee \hat{\varpi}_{\mathbb{E}}(w) = \widehat{\beta}$. Then, obviously $\gamma_{\mathbb{E}}(l) \geq \alpha, \hat{\gamma}_{\mathbb{E}}(l) \leq \widehat{\alpha}$, $\varpi_{\mathbb{E}}(l) \geq \beta, \hat{\varpi}_{\mathbb{E}}(l) \leq \widehat{\beta}$. This implies that $l \in \mathbb{E}_{(\alpha,\beta)}^{(\widehat{\alpha},\widehat{\beta})}$. Similarly, we have $w \in \mathbb{E}_{(\alpha,\beta)}^{(\widehat{\alpha},\widehat{\beta})}$. Hence $l + w \in \mathbb{E}_{(\alpha,\beta)}^{(\widehat{\alpha},\widehat{\beta})}$, since $\mathbb{E}_{(\alpha,\beta)}^{(\widehat{\alpha},\widehat{\beta})}$ is a submodule of $\mathbb{M}$. From this, we get

$$
\begin{aligned}
\gamma_{\mathbb{E}}(l+w) &\geq \alpha = \gamma_{\mathbb{E}}(l) \wedge \gamma_{\mathbb{E}}(w),\\
\varpi_{\mathbb{E}}(l+w) &\geq \beta = \varpi_{\mathbb{E}}(l) \wedge \varpi_{\mathbb{E}}(w) \text{ and}\\
\hat{\gamma}_{\mathbb{E}}(l+w) &\leq \widehat{\alpha} = \gamma_{\mathbb{E}}(l) \wedge \hat{\gamma}_{\mathbb{E}}(w)\ \forall\, l, w \in \mathbb{M},\\
\hat{\varpi}_{\mathbb{E}}(l+w) &\leq \widehat{\beta} = \hat{\varpi}_{\mathbb{E}}(l) \wedge \hat{\varpi}_{\mathbb{E}}(w)\ \forall\, l, w \in \mathbb{M}.
\end{aligned}
$$

Also, we have $l \in \mathbb{E}_{(\alpha_1,\beta_1)}^{(\widehat{\alpha_1},\widehat{\beta_1})}$, where $\alpha_1 = \gamma_{\mathbb{E}}(l)$, $\beta_1 = \varpi_{\mathbb{E}}(l)$ and $\widehat{\alpha_1} = \hat{\gamma}_{\mathbb{E}}(l)$, $\widehat{\beta_1} = \hat{\varpi}_{\mathbb{E}}(l)$. Therefore, for any $r \in \mathbb{R}$, we get $rl \in \mathbb{E}_{(\alpha_1,\beta_1)}^{(\widehat{\alpha_1},\widehat{\beta_1})}$. From this we get $\gamma_{\mathbb{E}}(rl) \geq \alpha_1 = \gamma_{\mathbb{E}}(l)$, $\varpi_{\mathbb{E}}(rl) \geq \beta_1 = \varpi_{\mathbb{E}}(l)$ and $\hat{\gamma}_{\mathbb{E}}(rl) \leq \widehat{\alpha_1} = \hat{\gamma}_{\mathbb{E}}(l)$, $\hat{\varpi}_{\mathbb{E}}(rl) \leq \widehat{\beta_1} = \hat{\varpi}_{\mathbb{E}}(l)$. That is $\gamma_{\mathbb{E}}(rl) \geq \gamma_{\mathbb{E}}(l)$, $\varpi_{\mathbb{E}}(rl) \geq \varpi_{\mathbb{E}}(l)$ and $\hat{\gamma}_{\mathbb{E}}(rl) \leq \hat{\gamma}_{\mathbb{E}}(l)$, $\hat{\varpi}_{\mathbb{E}}(rl) \leq \hat{\varpi}_{\mathbb{E}}(l)\ \forall r \in \mathbb{R}$, and $\forall\, l \in \mathbb{M}$. Thus, $\mathbb{E}$ is a $\mathbb{CIFSM}$ of $\mathbb{M}$. □

5. Complex intuitionistic fuzzy submodules homomorphisms

The homomorphism theorems for $\mathbb{CIFSM}$ over $\mathbb{R}$-modules are initiated.

Definition 5.1. *Let $\mathcal{P} \neq \phi$ and $\mathcal{U} \neq \phi$ be two sets and $\mathfrak{I} : \mathcal{P} \longrightarrow \mathcal{U}$ be a function. Also let us assume* ß $= (\gamma_{ß}, \hat{\gamma_{ß}})$ *and* $\mathfrak{U} = (\gamma_{\mathfrak{U}}, \hat{\gamma_{\mathfrak{U}}})$ *be $\mathbb{CIFS}$s in $\mathcal{P}$ and $\mathcal{U}$, accordingly. Then, image of* ß *under $\mathfrak{I}$ represented by $\mathfrak{I}$(ß) is a $\mathbb{CIFS}$ in $\mathcal{U}$ given by:*

$$
\begin{aligned}
\mathfrak{I}(ß) &= \{(\mathfrak{k}, £_{\mathfrak{I}(ß)}(\mathfrak{k}), \hat{£}_{\mathfrak{I}(ß)}(\mathfrak{k})) : \mathfrak{k} \in \mathcal{U}\},\\
\textit{where} \quad £_{(\mathfrak{I}(ß))}(\mathfrak{k}) &= \begin{cases} \vee\{\gamma_{ß}(\mathfrak{l}) : \mathfrak{l} \in \mathcal{P}, \mathfrak{I}(\mathfrak{l}) = \mathfrak{k}\} e^{i\vee\{\varpi_{ß}(\mathfrak{l}) : \mathfrak{l}\in\mathcal{P}, \mathfrak{I}(\mathfrak{l})=\mathfrak{k}\}}, & \text{if } \mathfrak{I}^{-1}(\mathfrak{k}) \neq \phi\\ \quad 0 \quad , & \text{if } \mathfrak{I}^{-1}(\mathfrak{k}) = \phi \end{cases}\\
\textit{and} \quad &\\
\hat{£}_{(\mathfrak{I}(ß))}(\mathfrak{k}) &= \begin{cases} \wedge\{\hat{\gamma}_{ß}(\mathfrak{l}) : \mathfrak{l} \in \mathcal{P}, \mathfrak{I}(\mathfrak{l}) = \mathfrak{k}\} e^{i\wedge\{\hat{\varpi}_{ß}(\mathfrak{l}) : \mathfrak{l}\in\mathcal{P}, \mathfrak{I}(\mathfrak{l})=\mathfrak{k}\}}, & \text{if } \mathfrak{I}^{-1}(\mathfrak{k}) \neq \phi\\ 1, & \text{if } \mathfrak{I}^{-1}(\mathfrak{k}) = \phi. \end{cases}
\end{aligned}
$$

Definition 5.2. *The inverse image of $\mathfrak{U}$ under $\mathfrak{I}$, represented by $\mathfrak{I}^{-1}(\mathfrak{U})$ is a $\mathbb{CIFS}$ in $\mathcal{P}$ is expressed as $\mathfrak{I}^{-1}(\mathfrak{U}) = \{(\mathfrak{k}, £_{\mathfrak{I}^{-1}_{(\mathfrak{U})}}(\mathfrak{k}), \hat{£}_{\mathfrak{I}^{-1}(\mathfrak{U})}(\mathfrak{k})) : \mathfrak{k} \in \mathcal{P}\}$. Where $\gamma_{\mathfrak{I}^{-1}(\mathfrak{U})}(\mathfrak{k}) e^{i\varpi_{\mathfrak{I}^{-1}(\mathfrak{U})}(\mathfrak{k})} = \gamma_{\mathfrak{U}}(\mathfrak{I}(\mathfrak{k})) e^{i\varpi_{\mathfrak{U}}(\mathfrak{I}(\mathfrak{k}))}$ and $\hat{\gamma}_{\mathfrak{I}^{-1}(\mathfrak{U})}(\mathfrak{k}) e^{i\hat{\varpi}_{\mathfrak{I}^{-1}(\mathfrak{U})}(\mathfrak{k})} = \hat{\gamma}_{\mathfrak{U}}(\mathfrak{I}(\mathfrak{k})) e^{i\hat{\varpi}_{\mathfrak{U}}(\mathfrak{I}(\mathfrak{k}))}$.*

Here, we examine the characteristics of the image and inverse-image of a $\mathbb{CIFSM}$ under an $\mathbb{R}-$module homomorphism. For this, we review the following definition.

Definition 5.3. *Assume that $\hat{\mathbb{M}}$ and $\hat{\mathbb{M}}'$ are two modules over $\mathbb{R}$. A mapping $\mathfrak{I} : \hat{\mathbb{M}} \to \hat{\mathbb{M}}'$ is a $\mathbb{R}-$module homomorphism if:*

1. $\mathfrak{I}(k_1\mathfrak{m}'_1 + k_2\mathfrak{m}'_2) = k_1\mathfrak{I}(\mathfrak{m}'_1) + k_2\mathfrak{I}(\mathfrak{m}'_2)$,
2. $\mathfrak{I}(r\mathfrak{m}') = r\mathfrak{I}(\mathfrak{m}'), \forall k_1, k_2 \in K; \mathfrak{m}', \mathfrak{m}'_1, \mathfrak{m}'_2 \in \hat{\mathbb{M}}$, *and* $r \in \mathbb{R}$.

Theorem 5.4. *Consider that $\hat{\mathbb{M}}$ and $\hat{\mathbb{M}}'$ are two modules over $\mathbb{R}$. A mapping $\mathfrak{I} : \hat{\mathbb{M}} \to \hat{\mathbb{M}}'$ is a homomorphism. Assume that $r, \mathfrak{s} \in \mathbb{R}$ and ß and $\mathfrak{U}$ be two $\mathbb{CIFS}$s in $\hat{\mathbb{M}}$. Then,*

1. $\mathfrak{I}(ß + \mathfrak{U}) = \mathfrak{I}(ß) + \mathfrak{I}(\mathfrak{U})$,
2. $\mathfrak{I}(rß) = r\mathfrak{I}(ß)$,
3. $\mathfrak{I}(rß + \mathfrak{s}\mathfrak{U}) = r\mathfrak{I}(ß) + \mathfrak{s}\mathfrak{I}(\mathfrak{U})$.

Proof. 1. We have

$$\begin{aligned} \mathfrak{I}(ß + \mathfrak{U}) &= \{(\mathfrak{p}, \gamma_{\mathfrak{I}(ß+\mathfrak{U})}(\mathfrak{p})e^{i\varpi_{\mathfrak{I}(ß+\mathfrak{U})}(\mathfrak{p})}, \hat{\gamma}_{\mathfrak{I}(ß+\mathfrak{U})}(\mathfrak{p})e^{i\hat{\varpi}_{\mathfrak{I}(ß+\mathfrak{U})}(\mathfrak{p})}) : \mathfrak{p} \in \hat{\mathbb{N}}\} \\ \text{and} \quad \mathfrak{I}(ß) + \mathfrak{I}(\mathfrak{U}) &= \{(\mathfrak{p}, \gamma_{\mathfrak{I}(ß)+\mathfrak{I}(\mathfrak{U})}(\mathfrak{p})e^{i\varpi_{\mathfrak{I}(ß)+\mathfrak{I}(\mathfrak{U})}(\mathfrak{p})}, \hat{\gamma}_{\mathfrak{I}(ß)+\mathfrak{I}(\mathfrak{U})}(\mathfrak{p})e^{i\hat{\varpi}_{\mathfrak{I}(ß)+\mathfrak{I}(\mathfrak{U})}(\mathfrak{p})} : \mathfrak{p} \in \hat{\mathbb{N}})\}. \end{aligned}$$

Assume $\mathfrak{p} \in \hat{\mathbb{N}}$. If $\mathfrak{I}^{-1}(\mathfrak{p}) = \phi$ then $\gamma_{\mathfrak{I}(ß+\mathfrak{U})}(\mathfrak{p})e^{i\varpi_{\mathfrak{I}(ß+\mathfrak{U})}(\mathfrak{p})} = 0$.

$$\begin{aligned} \text{Also,} \quad \gamma_{\mathfrak{I}(ß)+\mathfrak{I}(\mathfrak{U})}(\mathfrak{p})e^{i\varpi_{\mathfrak{I}(ß)+\mathfrak{I}(\mathfrak{U})}(\mathfrak{p})} &= \vee\{\gamma_{\mathfrak{I}(ß)}(\mathfrak{p}_1) \wedge \gamma_{\mathfrak{I}(\mathfrak{U})}(\mathfrak{p}_2) : \\ &\quad \mathfrak{p}_1, \mathfrak{p}_2 \in \hat{\mathbb{N}}, \mathfrak{p} = \mathfrak{p}_1 + \mathfrak{p}_2\}e^{i\vee\{\varpi_{\mathfrak{I}(ß)}(\mathfrak{p}_1) \wedge i\varpi_{\mathfrak{I}(\mathfrak{U})}(\mathfrak{p}_2):\mathfrak{p}_1,\mathfrak{p}_2\in\hat{\mathbb{N}},\mathfrak{p}=\mathfrak{p}_1+\mathfrak{p}_2\}}. \\ &= 0, \quad \text{Since } \mathfrak{I}^{-1}(\mathfrak{p}_1) = \phi \text{ or } \mathfrak{I}^{-1}(\mathfrak{p}_2) = \phi \text{ as} \mathfrak{I}^{-1}(\mathfrak{p}) = \phi. \\ \text{Thus,} \quad \gamma_{\mathfrak{I}(ß+\mathfrak{U})}(\mathfrak{p})e^{i\varpi_{\mathfrak{I}(ß+\mathfrak{U})}(\mathfrak{p})} &= \gamma_{\mathfrak{I}(ß)+\mathfrak{I}(\mathfrak{U})}(\mathfrak{p})e^{i\varpi_{\mathfrak{I}(ß)+\mathfrak{I}(\mathfrak{U})}(\mathfrak{p})} \qquad if\ \mathfrak{I}^{-1}(\mathfrak{p}) = \phi. \end{aligned}$$

If $\mathfrak{I}^{-1}(\mathfrak{p}) \neq \phi$, then also we have,
$\gamma_{\mathfrak{I}(ß+\mathfrak{U})}(\mathfrak{p})e^{i\varpi_{\mathfrak{I}(ß+\mathfrak{U})}(\mathfrak{p})} = \vee\{\gamma_{ß+\mathfrak{U}}(\mathfrak{l}) : \mathfrak{l} \in \hat{\mathbb{M}}, \mathfrak{p} = \mathfrak{I}(\mathfrak{l})\}e^{i\vee\{\varpi_{ß+\mathfrak{U}}(\mathfrak{l}):\mathfrak{l}\in\hat{\mathbb{M}},\mathfrak{p}=\mathfrak{I}(\mathfrak{l})\}}$

$$\begin{aligned} &= \vee\{\vee\{\gamma_ß(\mathfrak{l}_1) \wedge \gamma_\mathfrak{U}(\mathfrak{l}_2) : \mathfrak{l}_1, \mathfrak{l}_2 \in \hat{\mathbb{M}}, \mathfrak{l} = \mathfrak{l}_1 + \mathfrak{l}_2\} : \mathfrak{l} \in \hat{\mathbb{M}}, \mathfrak{p} = \mathfrak{I}(\mathfrak{l})\} \\ &\quad e^{i\vee\{\vee\{\varpi_ß(\mathfrak{l}_1)\wedge\varpi_\mathfrak{U}(\mathfrak{l}_2):\mathfrak{l}_1,\mathfrak{l}_2\in\hat{\mathbb{M}},\mathfrak{l}=\mathfrak{l}_1+\mathfrak{l}_2\}:\mathfrak{l}\in\hat{\mathbb{M}},\mathfrak{p}=\mathfrak{I}(\mathfrak{l})\}} \\ &= \vee\{\vee\{\gamma_ß(\mathfrak{l}_1) \wedge \gamma_\mathfrak{U}(\mathfrak{l}_2) : \mathfrak{l}_1, \mathfrak{l}_2 \in \hat{\mathbb{M}}, \mathfrak{z}_1 = \mathfrak{I}(\mathfrak{l}_1), \mathfrak{z}_2 = \mathfrak{I}(\mathfrak{l}_2)\} : \\ &\quad \mathfrak{z}_1, \mathfrak{z}_2 \in \hat{\mathbb{N}}, \mathfrak{p} = \mathfrak{z}_1 + \mathfrak{z}_2\}e^{i\vee\{\vee\{\varpi_ß(\mathfrak{l}_1)\wedge\varpi_\mathfrak{U}(\mathfrak{l}_2):\mathfrak{l}_1,\mathfrak{l}_2\in\hat{\mathbb{M}},\mathfrak{z}_1=\mathfrak{I}(\mathfrak{l}_1),\mathfrak{z}_2=\mathfrak{I}(\mathfrak{l}_2)\}:\mathfrak{z}_1,\mathfrak{z}_2\in\hat{\mathbb{N}},\mathfrak{p}=\mathfrak{z}_1+\mathfrak{z}_2\}} \\ &= \vee\{(\vee\{\gamma_ß(\mathfrak{l}_1) : \mathfrak{l}_1 \in \hat{\mathbb{M}}, \mathfrak{z}_1 = \mathfrak{I}(\mathfrak{l}_1)\})e^{i\vee\{(\vee\{\varpi_ß(\mathfrak{l}_1):\mathfrak{l}_1\in\hat{\mathbb{M}},\mathfrak{z}_1=\mathfrak{I}(\mathfrak{l}_1)\})} \wedge (\vee\{\gamma_\mathfrak{U}(\mathfrak{l}_2) : \mathfrak{l}_2 \in \hat{\mathbb{M}}, \mathfrak{z}_2 = \mathfrak{I}(\mathfrak{l}_2)\}) \\ &\quad : \mathfrak{z}_1, \mathfrak{z}_2 \in \hat{\mathbb{N}}, \mathfrak{p} = \mathfrak{z}_1 + \mathfrak{z}_2\}e^{i(\vee\{\varpi_\mathfrak{U}(\mathfrak{l}_2):\mathfrak{z}_1,\mathfrak{z}_2\in\hat{\mathbb{N}},\mathfrak{p}=\mathfrak{z}_1+\mathfrak{z}_2\})} \\ &= \vee\{\gamma_{\mathfrak{I}(ß)}(\mathfrak{z}_1) \wedge \gamma_{\mathfrak{I}(\mathfrak{U})}(z_2) : \mathfrak{z}_1, \mathfrak{z}_2 \in \hat{\mathbb{N}}, \mathfrak{p} = \mathfrak{z}_1 + \mathfrak{z}_2\}e^{i\vee\{\varpi_\mathfrak{I}(ß)(\mathfrak{z}_1)\wedge\varpi_\mathfrak{I}(\mathfrak{U})(\mathfrak{z}_2):\mathfrak{z}_1,\mathfrak{z}_2\in\hat{\mathbb{N}},\mathfrak{p}=\mathfrak{z}_1+\mathfrak{z}_2\}} \\ &= \gamma_{\mathfrak{I}(ß)+\mathfrak{I}(\mathfrak{U})}(\mathfrak{p})e^{i\varpi_{\mathfrak{I}(ß)+\mathfrak{I}(\mathfrak{U})}(\mathfrak{p})}. \end{aligned}$$

Thus, in case we get $\gamma_{\mathfrak{I}(ß+\mathfrak{U})}(\mathfrak{p})e^{i\varpi_{\mathfrak{I}(ß+\mathfrak{U})}(\mathfrak{p})} = \gamma_{\mathfrak{I}(ß)+\mathfrak{I}(\mathfrak{U})}(\mathfrak{p})e^{i\varpi_{\mathfrak{I}(ß)+\mathfrak{I}(\mathfrak{U})}(\mathfrak{p})}$.
Similarly, we can obtain $\hat{\gamma}_{\mathfrak{I}(ß+\mathfrak{U})}(\mathfrak{p})e^{i\hat{\varpi}_{\mathfrak{I}(ß+\mathfrak{U})}(\mathfrak{p})} = \hat{\gamma}_{\mathfrak{I}(ß)+\mathfrak{I}(\mathfrak{U})}(\mathfrak{p})e^{i\hat{\varpi}_{\mathfrak{I}(ß)+\mathfrak{I}(\mathfrak{U})}(\mathfrak{p})}$
$\forall\ \mathfrak{p} \in \hat{\mathbb{N}}$. Hence, we have $\mathfrak{I}(ß + \mathfrak{U}) = \mathfrak{I}(ß) + \mathfrak{I}(\mathfrak{U})$.
2. Now,

$$\begin{aligned} \mathfrak{I}(rß) &= \{(\mathfrak{p}, \gamma_{\mathfrak{I}(rß)}(\mathfrak{p})e^{i\varpi_{\mathfrak{I}(rß)}(\mathfrak{p})}, \hat{\gamma}_{\mathfrak{I}(rß)}(\mathfrak{p})e^{i\hat{\varpi}_{\mathfrak{I}(rß)}(\mathfrak{p})} : \mathfrak{p} \in \hat{\mathbb{N}}\} \\ \text{and} \quad r\mathfrak{I}(ß) &= \{(\mathfrak{p}, \gamma_{r\mathfrak{I}(ß)}(\mathfrak{p})e^{i\varpi_{r\mathfrak{I}(ß)}(\mathfrak{p})}, \hat{\gamma}_{r\mathfrak{I}(ß)}(\mathfrak{p})e^{i\hat{\varpi}_{r\mathfrak{I}(ß)}(\mathfrak{p})} : \mathfrak{p} \in \hat{\mathbb{N}}\}. \end{aligned}$$

Assume that $\mathfrak{p} \in \hat{\mathbb{N}}$. If $\mathfrak{I}^{-1}(\mathfrak{p}) = \phi$, then, naturally $\gamma_{\mathfrak{I}(r\text{ß})}(\mathfrak{p})e^{i\varpi_{\mathfrak{I}(r\text{ß})}(\mathfrak{p})} = 0$.
Also, $\gamma_{r\mathfrak{I}(\text{ß})}(\mathfrak{p})e^{i\varpi_{r\mathfrak{I}(\text{ß})}(\mathfrak{p})} = \vee\{\gamma_{\mathfrak{I}(\text{ß})}(\xi) : \xi \in \hat{\mathbb{N}}, \mathfrak{p} = r\xi\}e^{i\vee\{\varpi_{\mathfrak{I}(\text{ß})}(\xi):\xi\in\hat{\mathbb{N}},\mathfrak{p}=r\xi\}} = 0$. Therefore, $\mathfrak{I}^{-1}(\xi) = \phi$ as $\mathfrak{I}^{-1}(r\xi) = \mathfrak{I}^{-1}(\mathfrak{p}) = \phi$. Thus, $\gamma_{\mathfrak{I}(r\text{ß})}(\mathfrak{p})e^{i\varpi_{\mathfrak{I}(r\text{ß})}(\mathfrak{p})} = \gamma_{r\mathfrak{I}(\text{ß})}(\mathfrak{p})e^{i\varpi_{r\mathfrak{I}(\text{ß})}(\mathfrak{p})}$ if $\mathfrak{I}^{-1}(\mathfrak{p}) = \phi$. Now, if $\mathfrak{I}^{-1}(\mathfrak{p}) \neq \phi$, then,

$$\begin{aligned}
\gamma_{\mathfrak{I}(r\text{ß})}(\mathfrak{p})e^{i\varpi_{\mathfrak{I}(r\text{ß})}(\mathfrak{p})} &= \vee\{\gamma_{r\text{ß}}(\mathfrak{a}) : \mathfrak{a} \in \hat{\mathbb{M}}, \mathfrak{p} = \mathfrak{I}(\mathfrak{a})\}e^{i\vee\{\varpi_{r\text{ß}}(\mathfrak{a}):\mathfrak{a}\in\hat{\mathbb{M}},\mathfrak{p}=\mathfrak{I}(\mathfrak{a})\}} \\
&= \vee\{\vee\{\gamma_{\text{ß}}(\psi) : \psi \in \hat{\mathbb{M}}, \mathfrak{a} = r\psi\} : \mathfrak{a} \in \hat{\mathbb{M}}, \mathfrak{p} = \mathfrak{I}(\mathfrak{a})\}e^{i\vee\{\vee\{\varpi_{\text{ß}}(\psi):\mathfrak{a}\in\hat{\mathbb{M}},\mathfrak{p}=\mathfrak{I}(\mathfrak{a})\}} \\
&= \vee\{\vee\{\gamma_{\text{ß}}(\psi) : \psi \in \hat{\mathbb{M}}, \xi = \mathfrak{I}(\psi)\} : \xi \in \hat{\mathbb{N}}, \mathfrak{p} = r\xi\}e^{i\vee\{\vee\{\varpi_{\text{ß}}(\psi):\xi\in\hat{\mathbb{N}},\mathfrak{p}=r\xi\}} \\
&= \vee\{\gamma_{\mathfrak{I}(\text{ß})}(\xi) : \xi \in \hat{\mathbb{N}}, \mathfrak{p} = r\xi\}e^{i\vee\{\varpi_{\mathfrak{I}(\text{ß})}(\xi):\xi\in\hat{\mathbb{N}},\mathfrak{p}=r\xi\}} \\
&= \gamma_{r\mathfrak{I}(\text{ß})}(\mathfrak{p})e^{i\varpi_{r\mathfrak{I}(\text{ß})}(\mathfrak{p})}.
\end{aligned}$$

Thus, we get $\gamma_{\mathfrak{I}(r\text{ß})}(\mathfrak{p})e^{i\varpi_{\mathfrak{I}(r\text{ß})}(\mathfrak{p})} = \gamma_{r\mathfrak{I}(\text{ß})}(\mathfrak{p})e^{i\varpi_{r\mathfrak{I}(\text{ß})}(\mathfrak{p})}$ $\forall$ $\mathfrak{y} \in \hat{\mathbb{N}}$. Similarly we can obtain $\hat{\gamma}_{\mathfrak{I}(r\text{ß})}(\mathfrak{p})e^{i\hat{\varpi}_{\mathfrak{I}(r\text{ß})}(\mathfrak{p})} = \hat{\gamma}_{r\mathfrak{I}(\text{ß})}(\mathfrak{p})e^{i\hat{\varpi}_{r\mathfrak{I}(\text{ß})}(\mathfrak{p})}$ $\forall$ $y \in \hat{\mathbb{N}}$. Hence, we get $\mathfrak{I}(r\text{ß}) = r\mathfrak{I}(\text{ß})$.

3. This is a repercussion of (1) and (2). □

The following result shows that homomorphic image of $\mathbb{CIFSM}$ is $\mathbb{CIFSM}$.

Theorem 5.5. *Suppose that* $\hat{\mathbb{M}}$ *and* $\hat{\mathbb{N}}$ *are modules over* $\mathbb{R}$ *and* $\mathfrak{I}$ *is a homomorphism of* $\hat{\mathbb{M}}$ *into* $\hat{\mathbb{N}}$*. If* ß *is a* $\mathbb{CIFSM}$ *of* $\hat{\mathbb{M}}$*, then* $\mathfrak{I}(\text{ß})$ *is* $\mathbb{CIFSM}$ *of* $\hat{\mathbb{N}}$*.*

Proof. We have $\mathfrak{I}(\text{ß}) = \{(\mathfrak{p}, \gamma_{\mathfrak{I}(\text{ß})}(\mathfrak{p})e^{i\varpi_{\mathfrak{I}(\text{ß})}(\mathfrak{p})}, \hat{\gamma}_{\mathfrak{I}(\text{ß})}(\mathfrak{p})e^{i\hat{\varpi}_{\mathfrak{I}(\text{ß})}(\mathfrak{p})} : \mathfrak{y} \in \hat{\mathbb{N}}\}$. First of all we note that

$$\begin{aligned}
\gamma_{\mathfrak{I}(\text{ß})}(0)e^{i\varpi_{\mathfrak{I}(\text{ß})}(0)} &= \vee\{\gamma_{\text{ß}}(\mathfrak{a}) : \mathfrak{a} \in \hat{\mathbb{M}}, \mathfrak{I}(\mathfrak{a}) = 0\}e^{i\vee\{\varpi_{\text{ß}}(\mathfrak{a}):\mathfrak{a}\in\hat{\mathbb{M}},\mathfrak{I}(\mathfrak{a})=0\}} = \gamma_{\text{ß}}(0)e^{i\varpi_{\text{ß}}(0)} = 1 \\
\text{and} \quad \hat{\gamma}_{\mathfrak{I}(\text{ß})}(0)e^{i\hat{\varpi}_{\mathfrak{I}(\text{ß})}(0)} &= \wedge\{\hat{\gamma}_{\text{ß}}(\mathfrak{a}) : \mathfrak{a} \in \hat{\mathbb{M}}, \mathfrak{I}(\mathfrak{a}) = 0\}e^{i\wedge\{\hat{\varpi}_{\text{ß}}(\mathfrak{a}):\mathfrak{a}\in\hat{\mathbb{M}},\mathfrak{I}(\mathfrak{a})=0\}} = \hat{\gamma}_{\text{ß}}(0)e^{i\hat{\varpi}_{\text{ß}}(0)} = 0.
\end{aligned}$$

To show the second axioms of a $\mathbb{CIFSM}$, now take $\mathfrak{p}_1, \mathfrak{p}_2 \in \hat{\mathbb{N}}$. If $\mathfrak{I}^{-1}(\mathfrak{p}_1) = \phi$ or $\mathfrak{I}^{-1}(\mathfrak{p}_2) = \phi$, then correspondingly $\gamma_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_1)e^{i\varpi_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_1)} = 0$ or $\gamma_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_2)e^{i\varpi_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_2)} = 0$. So in this case,

$$\begin{aligned}
\gamma_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_1)e^{i\varpi_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_1)} &\wedge \gamma_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_2)e^{i\varpi_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_2)} = 0 \quad \text{and} \\
\gamma_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_1 + \mathfrak{p}_2)e^{i\varpi_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_1+\mathfrak{p}_2)} &\geq \gamma_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_1)e^{i\varpi_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_1)} \wedge \gamma_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_2)e^{i\varpi_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_2)}.
\end{aligned}$$

Now if $\mathfrak{I}^{-1}(\mathfrak{p}_1) \neq \phi \neq \mathfrak{I}^{-1}(\mathfrak{p}_2)$, then
$\gamma_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_1 + \mathfrak{p}_2)e^{i\varpi_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_1+\mathfrak{p}_2)} = \vee\{\gamma_{\text{ß}}(\mathfrak{a}) : \mathfrak{a} \in \hat{\mathbb{M}}, \mathfrak{p}_1 + \mathfrak{p}_2 = \mathfrak{I}(\mathfrak{a})\}e^{i\vee\{\varpi_{\text{ß}}(\mathfrak{a}):\mathfrak{a}\in\hat{\mathbb{M}},\mathfrak{p}_1+\mathfrak{p}_2=\mathfrak{I}(\mathfrak{a})\}}$

$$\begin{aligned}
&= \vee\{\gamma_{\text{ß}}(\mathfrak{a}_1 + \mathfrak{a}_2) : \mathfrak{a}_1 + \mathfrak{a}_2 \in \hat{\mathbb{M}}, \mathfrak{p}_1 + \mathfrak{p}_2 \in \mathfrak{I}(\mathfrak{p}_1 + \mathfrak{p}_2)\}e^{i\varpi_{\text{ß}}(\mathfrak{a}_1+\mathfrak{a}_2):\mathfrak{a}_1+\mathfrak{a}_2\in\hat{\mathbb{M}},\mathfrak{p}_1+\mathfrak{p}_2\in\mathfrak{I}(\mathfrak{p}_1+\mathfrak{p}_2)\}} \\
&\geq \vee\{\gamma_{\text{ß}}(\mathfrak{a}_1 + \mathfrak{a}_2) : \mathfrak{a}_1 \in \mathfrak{I}^{-1}(\mathfrak{p}_1), \mathfrak{a}_2 \in \mathfrak{I}^{-1}(\mathfrak{p}_2)\}e^{i\vee\{\varpi_{\text{ß}}(\mathfrak{a}_1+\mathfrak{a}_2):\mathfrak{a}_1\in\mathfrak{I}^{-1}(\mathfrak{p}_1),\mathfrak{a}_2\in\mathfrak{I}^{-1}(\mathfrak{p}_2)\}} \\
&= \vee\{\gamma_{\text{ß}}(\mathfrak{a}_1 + \mathfrak{a}_2) : \mathfrak{a}_1, \mathfrak{a}_2 \in \hat{\mathbb{M}}, \mathfrak{p}_1 = \mathfrak{I}(\mathfrak{a}_1), \mathfrak{p}_2 = \mathfrak{I}(\mathfrak{a}_2)\}e^{i\vee\{\varpi_{\text{ß}}(\mathfrak{a}_1+\mathfrak{a}_2):\mathfrak{a}_1,\mathfrak{a}_2\in\hat{\mathbb{M}},\mathfrak{p}_1=\mathfrak{I}(\mathfrak{a}_1),\mathfrak{p}_2=\mathfrak{I}(\mathfrak{a}_2)\}} \\
&\geq \vee\{\gamma_{\text{ß}}(\mathfrak{a}_1) \wedge \gamma_{\text{ß}}(\mathfrak{a}_2) : \mathfrak{a}_1, \mathfrak{a}_2 \in \hat{\mathbb{M}}, \mathfrak{p}_1 = \mathfrak{I}(\mathfrak{a}_1), \mathfrak{p}_2 = \mathfrak{I}(\mathfrak{a}_2)\}e^{i\vee\{\varpi_{\text{ß}}(\mathfrak{a}_1)\wedge\varpi_{\text{ß}}(\mathfrak{a}_2):\mathfrak{a}_1,\mathfrak{a}_2\in\hat{\mathbb{M}},\mathfrak{p}_1=\mathfrak{I}(\mathfrak{a}_1),\mathfrak{p}_2=\mathfrak{I}(\mathfrak{a}_2)\}} \\
&\geq (\vee\{\gamma_{\text{ß}}(\mathfrak{a}_1) : \mathfrak{a}_1 \in \hat{\mathbb{M}}, \mathfrak{p}_1 = \mathfrak{I}(\mathfrak{a}_1)\}e^{i\vee\{\varpi_{\text{ß}}(\mathfrak{a}_1):\mathfrak{a}_1\in\hat{\mathbb{M}},\mathfrak{p}_1=\mathfrak{I}(\mathfrak{a}_1)\}}) \wedge \\
&\quad (\vee\{\gamma_{\text{ß}}(\mathfrak{a}_2) : \mathfrak{a}_2 \in \hat{\mathbb{M}}, \mathfrak{p}_2 = \mathfrak{I}(\mathfrak{a}_2)\}e^{i\vee\{\varpi_{\text{ß}}(\mathfrak{a}_2):\mathfrak{a}_2\in\hat{\mathbb{M}},\mathfrak{p}_2=\mathfrak{I}(\mathfrak{a}_2)\}}) \\
&= \gamma_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_1)e^{i\varpi_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_1)} \wedge \gamma_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_2)e^{i\varpi_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_2)}.
\end{aligned}$$

Thus, $\gamma_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_1 + \mathfrak{p}_2)e^{i\varpi_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_1+\mathfrak{p}_2)} \geq \gamma_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_1)e^{i\varpi_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_1)} \wedge \gamma_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_2)e^{i\varpi_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_2)}$ $\forall \mathfrak{p}_1, \mathfrak{p}_2 \in \hat{\mathbb{N}}$. Similarly we can prove that $\hat{\gamma}_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_1 + \mathfrak{p}_2)e^{i\hat{\varpi}_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_1+\mathfrak{p}_2)} \leq \hat{\gamma}_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_1)e^{i\hat{\varpi}_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_1)} \vee \hat{\gamma}_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_2)e^{i\hat{\varpi}_{\mathfrak{I}(\text{ß})}(\mathfrak{p}_2)}$ $\forall \mathfrak{p}_1, \mathfrak{p}_2 \in \hat{\mathbb{N}}$.

Now, demonstrate the third premise of a $\mathbb{CIFSM}$, let $y \in \hat{\mathbb{N}}$. If $\mathfrak{I}^{-1}(\mathfrak{p}) = \phi$, then $\gamma_{\mathfrak{I}(\text{ß})}(\mathfrak{p})e^{i\varpi_{\mathfrak{I}(\text{ß})}(\mathfrak{p})} = 0$. So in the case $\gamma_{\mathfrak{I}(\text{ß})}(r\mathfrak{p})e^{i\varpi_{\mathfrak{I}(\text{ß})}(r\mathfrak{p})} \geq \gamma_{\mathfrak{I}(\text{ß})}(\mathfrak{p})e^{i\varpi_{\mathfrak{I}(\text{ß})}(\mathfrak{p})}$. Now, if $\mathfrak{I}^{-1} \neq \phi$, then

$$
\begin{aligned}
\gamma_{\mathfrak{I}(\text{ß})}(r\mathfrak{p})e^{i\varpi_{\mathfrak{I}(\text{ß})}(r\mathfrak{p})} &= \vee\{\gamma_{\text{ß}}(\mathfrak{a}): \ \mathfrak{a} \in \hat{\mathbb{M}}, r\mathfrak{p} = \mathfrak{I}(\mathfrak{a})\}e^{i\vee\{\varpi_{\text{ß}}(\mathfrak{a}): \ \mathfrak{a}\in\hat{\mathbb{M}}, r\mathfrak{p}=\mathfrak{I}(\mathfrak{a})\}} \\
&\geq \vee\{\gamma_{\text{ß}}(r\mathfrak{a}_1) : r\mathfrak{a}_1 \in \hat{\mathbb{M}}, r\mathfrak{p} = \mathfrak{I}(r\mathfrak{a}_1)\}e^{i\vee\{\varpi_{\text{ß}}(r\mathfrak{a}_1):r\mathfrak{a}_1\in\hat{\mathbb{M}}, r\mathfrak{p}=\mathfrak{I}(r\mathfrak{a}_1)\}} \\
&\geq \vee\{\gamma_{\text{ß}}(r\mathfrak{a}_1)e^{i\varpi_{\text{ß}}(r\mathfrak{a}_1)} : \mathfrak{a}_1 \in \hat{\mathbb{M}}, \mathfrak{a}_1 \in \mathfrak{I}^{-1}(\mathfrak{p})\}e^{i\vee\{\varpi_{\text{ß}}(r\mathfrak{a}_1):\mathfrak{a}_1\in\hat{\mathbb{M}}, \mathfrak{a}_1\in\mathfrak{I}^{-1}(\mathfrak{p})\}} \\
&= \vee\{\gamma_{\text{ß}}(r\mathfrak{a}_1) : \mathfrak{a}_1 \in \hat{\mathbb{M}}, \mathfrak{p} = \mathfrak{I}(\mathfrak{a}_1)\}e^{i\vee\{\varpi_{\text{ß}}(r\mathfrak{a}_1):\mathfrak{a}_1\in\hat{\mathbb{M}}, \mathfrak{p}=\mathfrak{I}(\mathfrak{a}_1)\}} \\
&\geq \vee\{\gamma_{\text{ß}}(\mathfrak{a}_1) : \mathfrak{a}_1 \in \hat{\mathbb{M}}, \mathfrak{p} = \mathfrak{I}(\mathfrak{a}_1)\}e^{i\vee\{\varpi_{\text{ß}}(\mathfrak{a}_1):\mathfrak{a}_1\in\hat{\mathbb{M}}, \mathfrak{p}=\mathfrak{I}(\mathfrak{a}_1)\}} \\
&= \gamma_{\mathfrak{I}(\text{ß})}(\mathfrak{p})e^{i\varpi_{\mathfrak{I}(\text{ß})}(\mathfrak{p})}.
\end{aligned}
$$

Similarly, we can show that $\hat{\gamma}_{\mathfrak{I}(\text{ß})}(r\mathfrak{p})e^{i\hat{\varpi}_{\mathfrak{I}(\text{ß})}(r\mathfrak{p})} \leq \hat{\gamma}_{\mathfrak{I}(\text{ß})}(\mathfrak{p})e^{i\hat{\varpi}_{\mathfrak{I}(\text{ß})}(\mathfrak{p})}$ $\forall \mathfrak{y} \in \hat{\mathbb{N}}$. Thus $\mathfrak{I}(\text{ß})$ is a $\mathbb{CIFSM}$ of $\hat{\mathbb{N}}$. □

Inverse image of $\mathbb{CIFSM}$ is also $\mathbb{CIFSM}$.

Theorem 5.6. *Assume that $\hat{\mathbb{M}}$ and $\hat{\mathbb{N}}$ are two modules over $\mathbb{R}$. A mapping $\mathfrak{I} : \hat{\mathbb{M}} \to \hat{\mathbb{N}}$ is a homomorphism. If $\mathfrak{U}$ is $\mathbb{CIFSM}$ of $\hat{\mathbb{N}}$, then $\mathfrak{I}^{-1}(\mathfrak{U})$is a $\mathbb{CIFSM}$ of $\hat{\mathbb{M}}$.*

Proof. As $\mathfrak{I}^{-1}(\mathfrak{U}) = \{(\mathfrak{a}, \gamma_{\mathfrak{I}^{-1}(\mathfrak{U})}(\mathfrak{a})e^{i\varpi_{\mathfrak{I}^{-1}(\mathfrak{U})}}(\mathfrak{a}), \hat{\gamma}_{\mathfrak{I}^{-1}(\mathfrak{U})}(\mathfrak{a})e^{i\hat{\varpi}_{\mathfrak{I}^{-1}(\mathfrak{U})}}(\mathfrak{a}) : \mathfrak{a} \in \hat{\mathbb{M}}\}$, where $\gamma_{\mathfrak{I}^{-1}(\mathfrak{U})}(\mathfrak{a})e^{i\varpi_{\mathfrak{I}^{-1}(\mathfrak{U})}}(\mathfrak{a}) = \gamma_{\mathfrak{U}}(\mathfrak{I}(\mathfrak{a}))e^{i\varpi_{\mathfrak{U}}(\mathfrak{I}(\mathfrak{a}))}$ and $\hat{\gamma}_{\mathfrak{I}^{-1}(\mathfrak{U})}(\mathfrak{a})e^{i\hat{\varpi}_{\mathfrak{I}^{-1}(\mathfrak{U})}}(\mathfrak{a}) = \hat{\gamma}_{\mathfrak{U}}(\mathfrak{I}(\mathfrak{a}))e^{i\hat{\varpi}_{\mathfrak{U}}(\mathfrak{I}(\mathfrak{a}))}$ $\forall \mathfrak{a} \in \hat{\mathbb{M}}$. So $\gamma_{\mathfrak{I}^{-1}(\mathfrak{U})}(0)e^{i\varpi_{\mathfrak{I}^{-1}(\mathfrak{U})}}(0) = \gamma_{\mathfrak{U}}(\mathfrak{I}(0))e^{i\varpi_{\mathfrak{U}}(\mathfrak{I}(0))} = \gamma_{\mathfrak{U}}(0)e^{i\varpi_{\mathfrak{U}}(0)} = 1$ and $\hat{\gamma}_{\mathfrak{I}^{-1}(\mathfrak{U})}(0)e^{i\hat{\varpi}_{\mathfrak{I}^{-1}(\mathfrak{U})}}(0) = \hat{\gamma}_{\mathfrak{U}}(\mathfrak{I}(0))e^{i\hat{\varpi}_{\mathfrak{U}}(\mathfrak{I}(0))} = \hat{\gamma}_{\mathfrak{U}}(0)e^{i\hat{\varpi}_{\mathfrak{U}}(0)} = 0$. Now, $\gamma_{\mathfrak{I}^{-1}(\mathfrak{U})}(\mathfrak{a} + \mathfrak{p})e^{i\varpi_{\mathfrak{I}^{-1}(\mathfrak{U})}}(\mathfrak{a} + \mathfrak{p}) = \gamma_{\mathfrak{U}}(\mathfrak{I}(\mathfrak{a} + \mathfrak{p}))e^{i\varpi_{\mathfrak{U}}(\mathfrak{I}(\mathfrak{a}+\mathfrak{p}))} = \gamma_{\mathfrak{U}}(\mathfrak{I}(\mathfrak{a}) + \mathfrak{I}(\mathfrak{p}))e^{i\varpi_{\mathfrak{U}}(\mathfrak{I}(\mathfrak{a})+\mathfrak{I}(\mathfrak{p}))} \geq \gamma_{\mathfrak{U}}(\mathfrak{I}(\mathfrak{a}) \wedge \mathfrak{I}(\mathfrak{p}))e^{i\varpi_{\mathfrak{U}}(\mathfrak{I}(\mathfrak{a})\wedge\mathfrak{I}(\mathfrak{p}))} = \gamma_{\mathfrak{I}^{-1}(\mathfrak{U})}(\mathfrak{a})e^{i\varpi_{\mathfrak{I}^{-1}(\mathfrak{U})}}(\mathfrak{a}) \wedge \gamma_{\mathfrak{I}^{-1}(\mathfrak{U})}(\mathfrak{p})e^{i\varpi_{\mathfrak{I}^{-1}(\mathfrak{U})}}(\mathfrak{p})$ $\forall \mathfrak{a}, \mathfrak{p} \in \hat{\mathbb{M}}$. This implies that $\gamma_{\mathfrak{I}^{-1}(\mathfrak{U})}(\mathfrak{a} + \mathfrak{p})e^{i\varpi_{\mathfrak{I}^{-1}(\mathfrak{U})}}(\mathfrak{a} + \mathfrak{p}) \geq \gamma_{\mathfrak{I}^{-1}(\mathfrak{U})}(\mathfrak{a})e^{i\varpi_{\mathfrak{I}^{-1}(\mathfrak{U})}}(\mathfrak{a}) \wedge \gamma_{\mathfrak{I}^{-1}(\mathfrak{U})}(\mathfrak{p})e^{i\varpi_{\mathfrak{I}^{-1}(\mathfrak{U})}}(\mathfrak{p})$. Similarly we can show that $\hat{\gamma}_{\mathfrak{I}^{-1}(\mathfrak{U})}(\mathfrak{a} + \mathfrak{p})e^{i\hat{\varpi}_{\mathfrak{I}^{-1}(\mathfrak{U})}}(\mathfrak{a} + \mathfrak{p}) \leq \hat{\gamma}_{\mathfrak{I}^{-1}(\mathfrak{U})}(\mathfrak{a})e^{i\hat{\varpi}_{\mathfrak{I}^{-1}(\mathfrak{U})}}(\mathfrak{a}) \vee \hat{\gamma}_{\mathfrak{I}^{-1}(\mathfrak{U})}(\mathfrak{p})e^{i\hat{\varpi}_{\mathfrak{I}^{-1}(\mathfrak{U})}}(\mathfrak{p})$ $\forall \mathfrak{a}, \mathfrak{p} \in \hat{\mathbb{M}}$. Also,

$$
\begin{aligned}
\gamma_{\mathfrak{I}^{-1}(\mathfrak{U})}(r\mathfrak{a})e^{i\varpi_{\mathfrak{I}^{-1}(\mathfrak{U})}}(r\mathfrak{a}) &= \gamma_{\mathfrak{U}}(r\mathfrak{I}(\mathfrak{a}))e^{i\varpi_{\mathfrak{U}}(r\mathfrak{I}(\mathfrak{a}))} \\
&\geq \gamma_{\mathfrak{U}}(\mathfrak{I}(\mathfrak{a}))e^{i\varpi_{\mathfrak{U}}(\mathfrak{I}(\mathfrak{a}))} \\
&= \gamma_{\mathfrak{I}^{-1}(\mathfrak{U})}(\mathfrak{a})e^{i\varpi_{\mathfrak{I}^{-1}(\mathfrak{U})}}(\mathfrak{a}). \\
\text{So, we have } \gamma_{\mathfrak{I}^{-1}(\mathfrak{U})}(r\mathfrak{a})e^{i\varpi_{\mathfrak{I}^{-1}(\mathfrak{U})}}(r\mathfrak{a}) &\geq \gamma_{\mathfrak{I}^{-1}(\mathfrak{U})}(\mathfrak{a})e^{i\varpi_{\mathfrak{I}^{-1}(\mathfrak{U})}}(\mathfrak{a}).
\end{aligned}
$$

Similarly we get $\hat{\gamma}_{\mathfrak{I}^{-1}(\mathfrak{U})}(r\mathfrak{a})e^{i\hat{\varpi}_{\mathfrak{I}^{-1}(\mathfrak{U})}}(r\mathfrak{a}) \leq \hat{\gamma}_{\mathfrak{I}^{-1}(\mathfrak{U})}(\mathfrak{a})e^{i\hat{\varpi}_{\mathfrak{I}^{-1}(\mathfrak{U})}}(\mathfrak{a})$ $\forall\ r \in V, \mathfrak{a} \in \mathbb{R}$. Thus, $\mathfrak{I}^{-1}(\mathfrak{U})$ is a $\mathbb{CIFSM}$ of $\hat{\mathbb{M}}$. □

6. Conclusions

In this paper, we introduced the concept of complex intuitionistic fuzzy submodules of $\mathbb{R}$-module and examined their fundamental structural properties. Important notions, such as complex intuitionistic fuzzy quotient submodules, level submodules, and supports, were defined and their algebraic behavior was investigated. Furthermore, the preservation of complex intuitionistic fuzzy submodules under module homomorphisms was studied, and the fundamental homomorphism theorems were extended to the complex intuitionistic fuzzy module setting. The limitation of the proposed framework is its inability to independently represent indeterminacy and contradictions among multiple attribute values. Future research may address these limitations by developing complex neutrosophic [61] and complex plithogenic [68] extensions of the submodule framework.

Author contributions

Sanaa Ahmed Bajri, Muhammad Haris Mateen, Sarka Hoskova-Mayerova: Conceptualization and supervision; Kholood Alnefaie, Muhammad Haris Mateen, Sanaa Ahmed Bajri: Methodology and Original draft preparation; Bijan Davvaz and Sarka Hoskova-Mayerova: Modified and verified the result. All authors have read and approved the finalised version of the article.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

There are not any conflicts of interest.

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