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Research article

Application on third-order sandwich results for analytic functions defined by using a linear operator

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Abstract: The study examines third-order sandwich-type outcomes for analytic and univalent functions that are characterized by means of special functions like the Hurwitz–Lerch functions related to the Srivastava–Attiya linear operator. By employing techniques of differential subordination and superordination, we establish several new inclusion relationships and admissibility conditions that extend and unify earlier results in the literature. Further, optimal dominant and subordinant functions are identified, yielding sharp outcomes within the framework of third-order differential inequalities. The derived outcomes not only generalize a number of previously known results but also provide applications to special subclasses of analytic functions associated with the convolution operator N_{α}^{β}.

Keywords: analytic function; Srivastava–Attiya operator; function of Hurwitz–Lerch; differential subordination; superordination; sandwich theorem
Mathematics Subject Classification: 30C45, 30C50, 30C80

1. Introduction

Antonino and Miller [1] enlarged the context of second-order differential subordination, at the outset established by Miller and Mocanu [12], to include third-order differential subordinations. The methodologies proposed by Antonino and Miller provided the chance to get intriguing new outcomes, prompting several writers to pursue this area of inquiry [8]. The notion of broadening the duality theory of differential superordination, which includes third-order differential superordination, was executed in 2014, yielding intriguing results shortly thereafter [18, 24]. The subsequent notations and terminology furnish the foundational context for the investigation.

The class of analytic functions is denoted by $D(\Xi)$, where $\Xi = \{\xi \in \mathbb{C} : |\xi| < 1\}$, with $\overline{\Xi} = \{\xi \in \mathbb{C} : |\xi| \leq 1\}$ and $\partial\Xi = \{\xi \in \mathbb{C} : |\xi| = 1\}$. Let n be a positive integer and let $a \in \mathbb{C}$ be a complex number; the subsequent significant subclasses of $D(\Xi)$ are recognized:

$$D[a, n] = \{f \in D(\Xi) : f(\xi) = a + a_n\xi^n + a_{n+1}\xi^{n+1} + \cdots,\ \xi \in \Xi\},$$

denoting $D_0 = D[0, 1]$ and $D_1 = D[1, 1]$.

Let $K \subset D(\Xi)$ denote the class of functions that are analytic in Ξ and possess the standardized Taylor–Maclaurin series of the specified form:

$$f(\xi) = \xi + \sum_{n=2}^{\infty} a_n\xi^n, \quad (\xi \in \Xi). \tag{1.1}$$

Assume that the functions f and g are in $D(\Xi)$. We say that f is subordinate to g, or equivalently, that g is superordinate to f, expressed as $f \prec g$ in Ξ, or $f(\xi) \prec g(\xi)$, $(\xi \in \Xi)$, if a Schwarz function $\omega \in D(\Xi)$ exists that is analytic in Ξ, satisfies $\omega(0) = 0$ and $|\omega(\xi)| < 1$ $(\xi \in \Xi)$, and $f(\xi) = g(\omega(\xi))$, $(\xi \in \Xi)$. If the function g is univalent in Ξ, we then possess the ensuing equivalence relationship [11].

$$f(\xi) \prec g(\xi) \Leftrightarrow f(0) = g(0) \text{ and } f(\Xi) \subset g(\Xi).$$

We now recall the generalized Hurwitz–Lerch zeta function, defined by the series [9, 20]:

$$F(\xi, \alpha, \beta) = \sum_{n=0}^{\infty} \frac{\xi^n}{(\beta + n)^\alpha}, \tag{1.2}$$

where $\beta \in \mathbb{C} \setminus \xi_0^-$, $\alpha \in \mathbb{C}$, $\xi \in \Xi$, and $\xi \in \partial\Xi$.

Some special cases of the function $F(\xi, \alpha, \beta)$ are of interest; for additional information, refer to [19]. Srivastava and Attiya examined the subsequent normalized function:

$$T_\alpha^\beta(\xi) = (\alpha + n)^\beta \left[F(\xi, \alpha, \beta) - \beta^{-\alpha}\right] = \xi + \sum_{n=2}^{\infty} \left(\frac{\beta + 1}{\beta + n}\right)^\alpha \xi^n, \quad \xi \in \Xi. \tag{1.3}$$

By using $T_\alpha^\beta(\xi)$ we obtain the linear operator $N_\alpha^\beta : A \to A$ defined by the convolution

$$N_\alpha^\beta f(\xi) = T_\alpha^\beta(\xi) * f(\xi) = \xi + \sum_{n=2}^{\infty} \left(\frac{\beta + 1}{\beta + n}\right)^\alpha a_n\xi^n, \quad \xi \in \Xi. \tag{1.4}$$

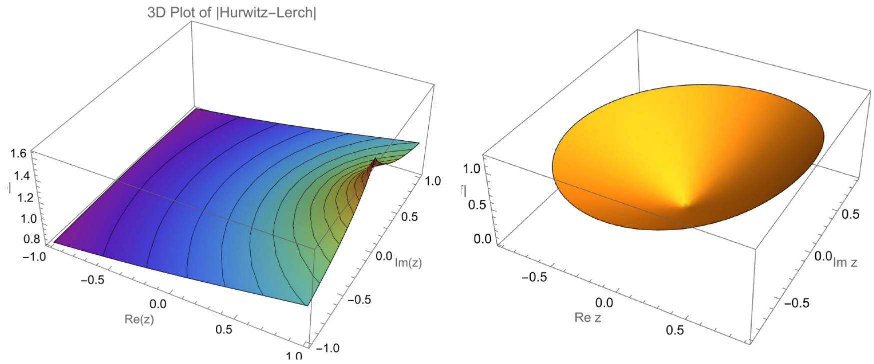

Figure 1. 3D plot of the Hurwitz–Lerch function $F(\xi, 2, 1)$ and the linear operator $N_1^{0.5} f(\xi)$.

There are many various applications for the $N_\alpha^\beta f(\xi)$ operator; see [6, 7]. From (1.4), it is clear that

$$\xi \left(N_\alpha^\beta f(\xi)\right)' = (1+\beta) N_{\alpha-1}^\beta f(\xi) - \beta N_\alpha^\beta f(\xi). \tag{1.5}$$

The notion of third-order differential subordination was examined in the work of Ponnusamy and Juneja [16]. Current research has been conducted by many authors (see, for instance, [23]). Second- and third-order differential subordination has received considerable interest from academics in this domain; for example, refer to [5, 21, 25].

Here, we investigate and identify an appropriate class of admissible functions related to the differential operator and establish sufficient criteria for the normalized analytic function according to the sandwich condition. For a visual illustration of the functions involved, Figure 1 present the three-dimension plots of the Hurwitz-Lerch function $F(\xi, 2, 1)$ and the corresponding linear operator $N_1^{0.5} f(\xi)$. The left panel illustrates the behavior of the Hurwitz-Lerch function, while the right panel shows the corresponding three-dimentional representation of the linear operator.

2. Preliminary considerations

We recall the ensuing definitions and lemmas needed to substantiate the findings.

Definition 2.1. [1] Assume $\psi : \mathbb{C}^4 \times \Xi \to \mathbb{C}$ and $\Omega(\xi)$ is univalent in Ξ. If $\gamma(\xi)$ is analytic in Ξ and fulfills the third-order differential subordination

$$\psi(\gamma(\xi), \xi\gamma'(\xi), \xi^2\gamma''(\xi), \xi^3\gamma'''(\xi); \xi) \prec \Omega(\xi), \tag{2.1}$$

then the function $\gamma(\xi)$ is referred to as a solution of the differential subordination (2.1). Furthermore, the univalent function $\delta(\xi)$ is designated as a dominant of the solution of (2.1), or more succinctly, a dominant, if $\gamma(\xi) \prec \delta(\xi)$ for all $\delta(\xi)$ satisfying (2.1). A dominant $\widetilde{\delta}(\xi)$ that fulfills $\widetilde{\delta}(\xi) \prec \delta(\xi)$ for all dominants $\delta(\xi)$ of (2.1) is regarded as the optimal (best) dominant.

Definition 2.2. [11] Denote by O the collection of all functions f that are analytic and injective on $\overline{\Xi} \setminus E(f)$, where $\overline{\Xi} = \Xi \cup \{\xi \in \partial\Xi\}$, and

$$E(f) = \{\zeta \in \partial\Xi : f(\zeta) = \infty\}, \tag{2.2}$$

and are characterized by $f'(\zeta) \neq 0$ for $\zeta \in \partial\Xi \setminus E(f)$. Further, let the subclass of O for which $f(0) = a$ be denoted by $O(a)$, so that $O(0) = O_0$, $O(1) = O_1 = \{f \in O : f(0) = 1\}$.

Definition 2.3. [1] Assume Ω is a set in $\mathbb{C}$, $\delta \in O(a)$, and $n \in \mathbb{N} \setminus \{1\}$. The class of admissible functions $\Psi_n[\Omega, \delta]$ comprises those functions $\psi : \mathbb{C}^4 \times \Xi \to \mathbb{C}$ that fulfill the subsequent admission criteria:

$$\psi(r, s, t, u; \xi) \notin \Omega,$$

at any time

$$r = \delta(\zeta), \quad s = \varrho\zeta\delta'(\zeta), \quad \mathrm{Re}\left(\frac{t}{s} + 1\right) \geq \varrho\,\mathrm{Re}\left(1 + \frac{\zeta\delta''(\zeta)}{\delta'(\zeta)}\right),$$

$$\mathrm{Re}\left(\frac{u}{s}\right) \geq \varrho^2 \mathrm{Re}\left(\frac{\zeta^2\delta'''(\zeta)}{\delta'(\zeta)}\right),$$

where $\xi \in \Xi$, $\zeta \in \partial\Xi \setminus E(\delta)$, and $\varrho \geq n$.

Lemma 2.4. *[1] Consider $\gamma \in D[a, n]$ with $n \geq 2$. Also, consider $\delta \in O(a)$ satisfying the subsequent criteria:*

$$\mathrm{Re}\left(\frac{\zeta\delta''(\zeta)}{\delta'(\zeta)}\right) \geq 0, \qquad \left|\frac{\xi\gamma'(\xi)}{\delta'(\zeta)}\right| \leq \varrho,$$

where $\xi \in \Xi$, $\zeta \in \partial\Xi \setminus E(\delta)$, and $\varrho \geq n$. If Ω is a set in $\mathbb{C}$, $\psi \in \Psi_n[\Omega, \delta]$, and

$$\psi(\gamma(\xi), \xi\gamma'(\xi), \xi^2\gamma''(\xi), \xi^3\gamma'''(\xi); \xi) \in \Omega,$$

then $\gamma(\xi) \prec \delta(\xi)$, $(\xi \in \Xi)$.

Definition 2.5. [12] Consider $\psi : \mathbb{C}^4 \times \Xi \to \mathbb{C}$ and $\Omega(\xi)$ is analytic in Ξ. If the functions $\gamma(\xi)$ and $\psi(\gamma(\xi), \xi\gamma'(\xi), \xi^2\gamma''(\xi), \xi^3\gamma'''(\xi); \xi)$ are univalent in Ξ and comply with the third-order differential superordination

$$\Omega(\xi) \prec \psi(\gamma(\xi), \xi\gamma'(\xi), \xi^2\gamma''(\xi), \xi^3\gamma'''(\xi); \xi), \tag{2.3}$$

then the function $\gamma(\xi)$ is referred to as a solution of the differential superordination (2.3). An analytic function $\delta(\xi)$ is referred to as a subordinant of the solution of (2.3), or more succinctly, a subordinant, if $\delta(\xi) \prec \gamma(\xi)$ for all $\gamma(\xi)$ satisfying (2.3). A univalent subordinant $\widetilde{\delta}(\xi)$ that satisfies $\delta(\xi) \prec \widetilde{\delta}(\xi)$ for all subordinants $\delta(\xi)$ of (2.3) is regarded as the optimal (best) subordinant. We observe that both the optimal dominant and the optimal subordinant are unique modulo rotation of Ξ.

Definition 2.6. [12] Consider Ω a set in $\mathbb{C}$, $\delta \in D[a, n]$ with $\delta'(\xi) \neq 0$, and $n \in \mathbb{N} \setminus \{1\}$. The class of admissible functions $\Psi'_n[\Omega, \delta]$ comprises those functions $\psi : \mathbb{C}^4 \times \overline{\Xi} \to \mathbb{C}$ that fulfill the subsequent admission criteria:

$$\psi(r, s, t, u; \xi) \in \Omega,$$

at any time

$$r = \delta(\xi), \quad s = \frac{\xi\delta'(\xi)}{m}, \quad \mathrm{Re}\left(\frac{t}{s} + 1\right) \leq \frac{1}{m}\mathrm{Re}\left(1 + \frac{\xi\delta''(\xi)}{\gamma'(\xi)}\right),$$

$$\mathrm{Re}\left(\frac{u}{s}\right) \leq \frac{1}{m^2}\mathrm{Re}\left(\frac{\xi^2\delta'''(\xi)}{\delta'(\xi)}\right),$$

where $\xi \in \Xi$, $\zeta \in \partial\Xi \setminus E(\delta)$, and $m \geq n \geq 2$.

Lemma 2.7. *[12] Consider* $\psi \in \Psi'_n[\Omega, \delta]$. *If*

$$\psi(\gamma(\xi), \xi\gamma'(\xi), \xi^2\gamma''(\xi), \xi^3\gamma'''(\xi); \xi)$$

is univalent in Ξ, $\gamma \in O(a)$, *and* $\delta \in D[a, n]$ *fulfills the subsequent criteria:*

$$\operatorname{Re}\left(\frac{\zeta\delta''(\zeta)}{\delta'(\zeta)}\right) \geq 0, \qquad \left|\frac{\xi\gamma'(\xi)}{\delta'(\zeta)}\right| \leq m, \qquad \xi \in \Xi,\ \zeta \in \partial\Xi,\ m \geq n \geq 2,$$

then

$$\Omega \subset \left\{\psi(\gamma(\xi), \xi\gamma'(\xi), \xi^2\gamma''(\xi), \xi^3\gamma'''(\xi); \xi)\right\}$$

implies

$$\delta(\xi) \prec \gamma(\xi), \quad (\xi \in \Xi).$$

A foundational method in the examination of third-order differential superordination is to employ the core notion of admissible functions as delineated in [3]. This methodology has yielded significant outcomes for various authors examining appropriate categories of admissible functions, including generalized Bessel functions, fractional operators [26], linear operators [2, 14], meromorphic functions [4], and Mittag–Leffler functions [17]. The dual theories of third-order differential subordination and superordination are progressing well; current findings derived from this methodology are documented in publications such as [13]. Recent research has adopted a novel approach to third-order differential subordination, focusing on an additional fundamental term within the framework of the optimal dominant of differential subordination; in [15], an approach for identifying the principal dominant of a third-order differential subordination is also presented.

3. Outcomes on third-order differential subordination

We present certain outcomes regarding differential subordination utilizing the differential operator N_α^β.

Definition 3.1. Consider Ω is a set in $\mathbb{C}$ and $\delta \in O_0 \cap D_0$. The class of admissible functions $M_j[\Omega, \delta]$ comprises the functions $\psi : \mathbb{C}^4 \times \Xi \to \mathbb{C}$ that fulfill the subsequent criteria for admission:

$$\psi(e_1, e_2, e_3, e_4; \xi) \notin \Omega,$$

whenever

$$e_1 = \delta(\zeta), \qquad e_2 = \frac{\zeta\varrho\delta'(\zeta) + (1+\beta)\delta(\zeta)}{\beta},$$

$$\operatorname{Re}\left(\frac{((1+\beta)^2 - 2(2+\beta)(1+\beta))e_1 - (3+\beta)e_2 - e_3\beta(\beta+1)}{(1+\beta)e_1 - \beta e_2}\right) \geq \varrho \operatorname{Re}\left(\frac{\zeta\delta''(\zeta)}{\delta'(\zeta)} + 1\right),$$

and

$$\operatorname{Re}\left(\frac{e_4\beta(\beta+1)(\beta+2) - \beta(1+\beta)^3 e_1 + (3+2\beta)e_2 - (1+\beta)e_3}{(1+\beta)e_1 - \beta e_2}\right) \geq \varrho^2 \operatorname{Re}\left(\frac{\zeta^2\delta'''(\zeta)}{\delta'(\zeta)}\right),$$

where $\xi \in \Xi$, $\zeta \in \partial\Xi \setminus E(\delta)$, and $\varrho \geq 2$.

Theorem 3.2. *Consider $\psi \in M_j[\Omega,\delta]$. If the functions $f \in K$ and $\delta \in O_0 \cap D_0$ fulfill the subsequent criteria:*

$$\operatorname{Re}\left(\frac{\zeta\delta''(\zeta)}{\delta'(\zeta)}\right) \geq 0, \qquad \left|\frac{N^{\beta}_{\alpha-1}f(\xi)}{\delta'(\zeta)}\right| \leq \varrho, \tag{3.1}$$

and

$$\left\{\psi(N^{\beta}_{\alpha}f(\xi), N^{\beta}_{\alpha-1}f(\xi), N^{\beta}_{\alpha-2}f(\xi), N^{\beta}_{\alpha-3}f(\xi);\xi) : \xi \in \Xi\right\} \subset \Omega,$$

then

$$N^{\beta}_{\alpha}f(\xi) \prec \delta(\xi), \quad (\xi \in \Xi). \tag{3.2}$$

Proof. Define the analytic function $\gamma(\xi)$ in Ξ by

$$\gamma(\xi) = N^{\beta}_{\alpha}f(\xi). \tag{3.3}$$

From Eqs (1.5) and (3.3), we derive

$$N^{\beta}_{\alpha-1}f(\xi) = \frac{\beta\gamma(\xi) + \xi\gamma'(\xi)}{1+\beta}. \tag{3.4}$$

By a similar argument, we obtain

$$N^{\beta}_{\alpha-2}f(\xi) = \frac{(1+\beta)^2\gamma(\xi) + 2(1+\beta)\xi\gamma'(\xi) + \xi^2\gamma''(\xi)}{(1+\beta)^2}, \tag{3.5}$$

and

$$N^{\beta}_{\alpha-3}f(\xi) = \frac{(1+\beta)^3\gamma(\xi) + 3\xi\gamma'(\xi)(1+\beta)^2 + 3(1+\beta)\xi^2\gamma''(\xi) + \xi^3\gamma'''(\xi)}{(1+\beta)^3}. \tag{3.6}$$

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ as follows:

$$e_1(r,s,t,u) = r, \qquad e_2 = \frac{(1+\beta)r + s}{1+\beta}, \tag{3.7}$$

$$e_3(r,s,t,u) = \frac{(1+\beta)^2 r + 2(1+\beta)s + t}{(1+\beta)^2}, \tag{3.8}$$

and

$$e_4(r,s,t,u) = \frac{(1+\beta)^3 r + 3(1+\beta)^2 s + 3(1+\beta)t + u}{(1+\beta)^3}. \tag{3.9}$$

Let

$$\begin{aligned}\varphi(r,s,t,u) &= \psi(e_1,e_2,e_3,e_4;\xi)\\ &= \psi\left(r, \frac{(1+\beta)r+s}{1+\beta}, \frac{(1+\beta)^2r+2(1+\beta)s+t}{(1+\beta)^2}, \frac{(1+\beta)^3r+3(1+\beta)^2s+3(1+\beta)t+u}{(1+\beta)^3}\right).\end{aligned} \tag{3.10}$$

The proof employs Lemma 2.4. Utilizing Eq (3.3) through (3.6), along with (3.10), we obtain

$$\varphi(\gamma(\xi), \xi\gamma'(\xi), \xi^2\gamma''(\xi), \xi^3\gamma'''(\xi);\xi) = \psi(N^{\beta}_{\alpha}f(\xi), N^{\beta}_{\alpha-1}f(\xi), N^{\beta}_{\alpha-2}f(\xi), N^{\beta}_{\alpha-3}f(\xi);\xi). \tag{3.11}$$

Therefore, (3.2) leads to

$$\varphi(\gamma(\xi), \xi\gamma'(\xi), \xi^2\gamma''(\xi), \xi^3\gamma'''(\xi); \xi) \in \Omega.$$

We note that

$$\frac{t}{s} + 1 = \frac{(1+\beta)(e_3 - 2e_2 + e_1)}{(e_2 - e_1)} + 1,$$

and

$$\frac{u}{s} = \frac{(1+\beta)^2(e_4 - 3e_3 + 3e_2 - e_1)}{(e_2 - e_1)}.$$

Consequently, the criterion for admissibility for $\psi \in M_j[\Omega, \delta]$ in Definition 3.1 is identical to the admissibility criterion for $\varphi \in \Psi_2[\Omega, \delta]$ as specified in Definition 2.3 with $n = 2$. Consequently, utilizing (3.1) and Lemma 2.4, we derive

$$N_\alpha^\beta f(\xi) \prec \delta(\xi).$$

This completes the proof of Theorem 3.2. □

The subsequent result constitutes an extension of Theorem 3.2, applicable to the scenario when the behavior of $\delta(\xi)$ on $\partial\Xi$ remains unidentified.

Corollary 3.3. *Consider $\Omega \subset \mathbb{C}$ and the function δ is univalent in Ξ with $\delta(0) = 1$. Consider $\psi \in M_j[\Omega, \delta_\rho]$; for certain $\rho \in (0, 1)$, let $\delta_\rho(\xi) = \delta(\xi\rho)$. If the functions $f \in K$ and δ_ρ fulfill the following terms:*

$$\operatorname{Re}\left(\frac{\zeta\delta_\rho''(\zeta)}{\delta_\rho'(\zeta)}\right) \geq 0, \qquad \left|\frac{N_{\alpha-1}^\beta f(\xi)}{\delta_\rho'(\zeta)}\right| \leq \varrho, \qquad (\xi \in \Xi,\ \zeta \in \partial\Xi \setminus E(\delta_\rho),\ \varrho \geq 2),$$

and

$$\psi(N_\alpha^\beta f(\xi), N_{\alpha-1}^\beta f(\xi), N_{\alpha-2}^\beta f(\xi), N_{\alpha-3}^\beta f(\xi); \xi) \in \Omega,$$

then

$$N_\alpha^\beta f(\xi) \prec \delta_\rho(\xi), \qquad (\xi \in \Xi).$$

Proof. Utilizing Theorem 3.2, we obtain

$$N_\alpha^\beta f(\xi) \prec \delta_\rho(\xi), \qquad (\xi \in \Xi).$$

The result posited by Corollary 3.3 is now derived from the subsequent subordination attribute:

$$\delta_\rho(\xi) \prec \delta(\xi), \qquad (\xi \in \Xi).$$

This concludes the proof of Corollary 3.3. □

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = \Omega(\Xi)$ for some conformal mapping $\Omega(\xi)$ of Ξ onto Ω. In this case, the class $M_j[\Omega(\Xi), \delta]$ is written as $M_j[\Omega, \delta]$. The following is an immediate consequence of Theorem 3.2.

Theorem 3.4. *Consider $\psi \in M_j[\Omega, \delta]$. If the functions $f \in K$ and $\delta \in O_0 \cap D_0$ fulfill the subsequent criteria:*

$$\mathrm{Re}\left(\frac{\zeta\delta''(\zeta)}{\delta'(\zeta)}\right) \geq 0, \qquad \left|\frac{N^{\beta}_{\alpha-1} f(\xi)}{\delta'(\zeta)}\right| \leq \varrho, \tag{3.12}$$

and

$$\psi(N^{\beta}_{\alpha} f(\xi), N^{\beta}_{\alpha-1} f(\xi), N^{\beta}_{\alpha-2} f(\xi), N^{\beta}_{\alpha-3} f(\xi); \xi) \prec \Omega(\xi), \tag{3.13}$$

then

$$N^{\beta}_{\alpha} f(\xi) \prec \delta(\xi), \qquad (\xi \in \Xi).$$

The next outcome is a straightforward consequence of Corollary 3.3.

Corollary 3.5. *Consider $\Omega \subset \mathbb{C}$ and the function δ is univalent in Ξ with $\delta(0) = 1$. Consider $\psi \in M_j[\Omega, \delta_\rho]$ for certain $\rho \in (0, 1)$, where $\delta_\rho(\xi) = \delta(\xi\rho)$. If the functions $f \in K$ and δ_ρ fulfill the subsequent criteria:*

$$\mathrm{Re}\left(\frac{\zeta\delta''_\rho(\zeta)}{\delta'_\rho(\zeta)}\right) \geq 0, \qquad \left|\frac{N^{\beta}_{\alpha-1} f(\xi)}{\delta'_\rho(\zeta)}\right| \leq \varrho, \qquad (\xi \in \Xi,\ \zeta \in \partial\Xi \setminus E(\delta_\rho),\ \varrho \geq 2),$$

and

$$\psi(N^{\beta}_{\alpha} f(\xi), N^{\beta}_{\alpha-1} f(\xi), N^{\beta}_{\alpha-2} f(\xi), N^{\beta}_{\alpha-3} f(\xi); \xi) \prec \Omega(\xi),$$

then

$$N^{\beta}_{\alpha} f(\xi) \prec \delta_\rho(\xi), \qquad (\xi \in \Xi).$$

The ensuing result yields the ideal (best) dominant in differential subordination (3.13).

Theorem 3.6. *Consider Ω is a univalent function in Ξ, and let $\varphi : \mathbb{C}^4 \times \Xi \to \mathbb{C}$ be defined by* (3.10). *Consider the subsequent differential equation:*

$$\varphi(\gamma(\xi), \xi\gamma'(\xi), \xi^2\gamma''(\xi), \xi^3\gamma'''(\xi); \xi) = \Omega(\xi), \tag{3.14}$$

which possesses a solution $\delta(\xi)$ with $\delta(0) = 1$ that fulfills the criterion (3.1). *If $f \in K$ fulfills the criterion* (3.13) *and*

$$\psi(N^{\beta}_{\alpha} f(\xi), N^{\beta}_{\alpha-1} f(\xi), N^{\beta}_{\alpha-2} f(\xi), N^{\beta}_{\alpha-3} f(\xi); \xi)$$

is analytic in Ξ, then

$$N^{\beta}_{\alpha} f(\xi) \prec \delta(\xi), \qquad (\xi \in \Xi),$$

and $\delta(\xi)$ is the optimal (best) dominant.

Proof. It is evident in accordance with Theorem 3.2 that δ functions as a dominant of (3.13). Since δ is also a solution of (3.13), because it fulfills (3.14), all dominants shall precede δ. Therefore, the optimal dominant is δ. This concludes the proof of Theorem 3.6. □

Considering Definition 3.1, particularly in the specific instance when $\delta(\xi) = T\xi$, $T > 0$, the category of admissible functions $M_j[\Omega, \delta]$, designated by $M_j[\Omega, T]$, is articulated as stated below.

Definition 3.7. Consider Ω is a set in $\mathbb{C}$ and $T > 0$. The class $M_j[\Omega, T]$ of acceptable functions comprises those functions $\psi : \mathbb{C}^4 \times \Xi \to \mathbb{C}$ such that

$$\psi\left(Te^{i\vartheta}, \frac{(\varrho-(1+\beta))Te^{i\vartheta}}{\beta}, \frac{L-[(3+2\beta)\varrho+(1+\beta)^2]Te^{i\vartheta}}{\beta(\beta+1)}, \frac{N-(3+2\beta)\varrho+(1+\beta)^2L+[(3+2\beta)\varrho+(1+\beta)^2]\varrho-\beta(1+\beta)^3Te^{i\vartheta}}{\beta(\beta+1)(\beta+2)};\xi\right) \notin \Omega, \tag{3.15}$$

whenever $\xi \in \Xi$,

$$\mathrm{Re}(Le^{-i\vartheta}) \geq (\varrho-1)T\varrho, \qquad \mathrm{Re}(Ne^{-i\vartheta}) \geq 0, \qquad \forall\vartheta \in \mathbb{R},\ \varrho \geq 2.$$

Corollary 3.8. *Consider $\psi \in M_j[\Omega, T]$. If the function $f \in K$ fulfills the subsequent criteria:*

$$\left|N^{\beta}_{\alpha-1}f(\xi)\right| \leq T\varrho, \qquad (\xi \in \Xi;\ \varrho \geq 2;\ T > 0),$$

and

$$\psi(N^{\beta}_{\alpha}f(\xi), N^{\beta}_{\alpha-1}f(\xi), N^{\beta}_{\alpha-2}f(\xi), N^{\beta}_{\alpha-3}f(\xi);\xi) \in \Omega,$$

then

$$\left|N^{\beta}_{\alpha}f(\xi)\right| < T.$$

In this particular instance, $\Omega = \Omega(\Xi) = \{\omega : |\omega| < T\}$, and the class $M_j[\Omega, T]$ is simply designated by $M_j[\Omega, T]$. Corollary 3.8 is now able to be reformulated in the subsequent manner.

Corollary 3.9. *Assume $\psi \in M_j[\Omega, T]$. If the function $f \in K$ fulfills the subsequent criteria:*

$$\left|N^{\beta}_{\alpha-1}f(\xi)\right| \leq T\varrho, \qquad (\xi \in \Xi;\ \varrho \geq 2;\ T > 0),$$

and

$$\left|\psi(N^{\beta}_{\alpha}f(\xi), N^{\beta}_{\alpha-1}f(\xi), N^{\beta}_{\alpha-2}f(\xi), N^{\beta}_{\alpha-3}f(\xi);\xi)\right| < T,$$

then

$$\left|N^{\beta}_{\alpha}f(\xi)\right| < T.$$

Corollary 3.10. *Consider $\varrho \geq 2$ and $T > 0$. If the function $f \in K$ fulfills the subsequent criteria:*

$$\left|N^{\beta}_{\alpha-1}f(\xi)\right| \leq T\varrho,$$

and

$$\left|N^{\beta}_{\alpha-1}f(\xi) - N^{\beta}_{\alpha}f(\xi)\right| \leq \frac{(\varrho-1)T}{\beta},$$

then

$$\left|N^{\beta}_{\alpha}f(\xi)\right| \leq T.$$

Proof. Let $\psi(e_1, e_2, e_3, e_4;\xi) = e_2 - e_1$ and $\Omega = \Omega(\Xi)$, where $\Omega(\xi) = \left\{w \in \mathbb{C} : |w| < \frac{(\varrho-1)T}{\beta}\right\}$, $\varrho \geq 2$ and $T > 0$. Utilizing Corollary 3.8, we must demonstrate that $\psi \in M_j[\Omega, T]$, i.e., the admissibility criterion (3.15) is fulfilled. This is evident, as it is observed that

$$|\psi(e_1, e_2, e_3, e_4;\xi)| = |e_2 - e_1| = \left|\frac{(\varrho-1-2\beta)T}{\beta}e^{i\vartheta}\right|.$$

Since $\varrho \geq 2$, we get

$$|\psi(e_1, e_2, e_3, e_4; \xi)| \geq \frac{(\varrho - 1)T}{\beta}.$$

Thus,

$$\psi(e_1, e_2, e_3, e_4; \xi) \notin \Omega,$$

and consequently the admissibility condition (3.15) holds. Therefore, $\psi \in M_j[\Omega, T]$. Applying Corollary 3.8, we conclude that $\left|N_\alpha^\beta f(\xi)\right| \leq T$. □

Definition 3.11. Consider Ω is a set in $\mathbb{C}$ and $\delta \in O_1 \cap D_1$. The class of admissible functions $M_{j,1}[\Omega, \delta]$ comprises those functions $\psi : \mathbb{C}^4 \times \Xi \to \mathbb{C}$ that fulfill the subsequent admission standards:

$$\psi(e_1, e_2, e_3, e_4; \xi) \notin \Omega,$$

whenever

$$e_1 = \delta(\zeta), \qquad e_2 = \frac{\zeta\varrho\delta'(\zeta) + \beta\delta(\zeta)}{\beta},$$

$$\operatorname{Re}\left(\frac{\beta(\beta+1)(e_3+e_1) + 2e_2 + \beta(e_2 - e_1)}{\beta(e_2 - e_1)}\right) \geq \varrho \operatorname{Re}\left(\frac{\zeta\delta''(\zeta)}{\delta'(\zeta)} + 1\right),$$

and

$$\operatorname{Re}\left(\frac{e_4(\beta(\beta+1)(\beta+2)) - (\beta+1)e_3 + \beta(e_3+e_2) - \beta(\beta+1)^2 e_1}{\beta(e_2 - e_1)}\right) \geq \varrho^2 \operatorname{Re}\left(\frac{\zeta^2\delta'''(\zeta)}{\delta'(\zeta)}\right),$$

where $\xi \in \Xi$, $\zeta \in \partial\Xi \setminus E(\delta)$, and $\varrho \geq 2$.

Theorem 3.12. *Consider $\psi \in M_{j,1}[\Omega, \delta]$. If the functions $f \in K$ and $\delta \in O_1 \cap D_1$ fulfill the subsequent criteria:*

$$\operatorname{Re}\left(\frac{\zeta\delta''(\zeta)}{\delta'(\zeta)}\right) \geq 0, \qquad \left|\frac{N_{\alpha-1}^\beta f(\xi)}{\xi\,\delta'(\zeta)}\right| \leq \varrho, \tag{3.16}$$

and

$$\left\{\psi\left(\frac{N_\alpha^\beta f(\xi)}{\xi}, \frac{N_{\alpha-1}^\beta f(\xi)}{\xi}, \frac{N_{\alpha-2}^\beta f(\xi)}{\xi}, \frac{N_{\alpha-3}^\beta f(\xi)}{\xi}; \xi\right) : \xi \in \Xi\right\} \subset \Omega,$$

then

$$\frac{N_\alpha^\beta f(\xi)}{\xi} \prec \delta(\xi), \qquad (\xi \in \Xi). \tag{3.17}$$

Proof. Define the analytic function $\gamma(\xi)$ in Ξ by

$$\gamma(\xi) = \frac{N_\alpha^\beta f(\xi)}{\xi}. \tag{3.18}$$

From Eqs (1.5) and (3.18), we derive

$$\frac{N_{\alpha-1}^\beta f(\xi)}{\xi} = \frac{\beta\gamma(\xi) - \xi\gamma'(\xi)}{\beta}. \tag{3.19}$$

By an analogous argument, we derive

$$\frac{N^{\beta}_{\alpha-2}f(\xi)}{\xi} = \frac{\beta(\beta+1)\gamma(\xi) - 2\xi\gamma'(\xi) + \xi^2\gamma''(\xi)}{\beta(\beta+1)}, \tag{3.20}$$

and

$$\frac{N^{\beta}_{\alpha-3}f(\xi)}{\xi} = \frac{\beta(\beta+1)^3\gamma(\xi) - \beta(3+\beta)\xi\gamma'(\xi) + (\beta+1)\xi^2\gamma''(\xi) - \xi^3\gamma'''(\xi)}{\beta(\beta+1)(\beta+2)}. \tag{3.21}$$

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ via

$$e_1(r,s,t,u) = r, \qquad e_2 = \frac{\beta r - s}{\beta}, \tag{3.22}$$

$$e_3(r,s,t,u) = \frac{\beta(\beta+1)r - 2s + t}{\beta(\beta+1)}, \tag{3.23}$$

and

$$e_4(r,s,t,u) = \frac{\beta(\beta+1)^3 r - \beta(3+\beta)s + (\beta+1)t - u}{\beta(\beta+1)(\beta+2)}. \tag{3.24}$$

Let

$$\varphi(r,s,t,u) = \psi(e_1,e_2,e_3,e_4;\xi) = \psi\left(r, \frac{\beta r - s}{\beta}, \frac{\beta(\beta+1)r - 2s + t}{\beta(\beta+1)}, \frac{\beta(\beta+1)^3 r - \beta(3+\beta)s + (\beta+1)t - u}{\beta(\beta+1)(\beta+2)}\right). \tag{3.25}$$

The proof employs Lemma 2.4. Utilizing Eq (3.18) to (3.21), along with (3.25), we obtain

$$\varphi(\gamma(\xi), \xi\gamma'(\xi), \xi^2\gamma''(\xi), \xi^3\gamma'''(\xi); \xi) = \psi\left(\frac{N^{\beta}_{\alpha}f(\xi)}{\xi}, \frac{N^{\beta}_{\alpha-1}f(\xi)}{\xi}, \frac{N^{\beta}_{\alpha-2}f(\xi)}{\xi}, \frac{N^{\beta}_{\alpha-3}f(\xi)}{\xi}; \xi\right). \tag{3.26}$$

Hence, clearly, (3.17) becomes

$$\varphi(\gamma(\xi), \xi\gamma'(\xi), \xi^2\gamma''(\xi), \xi^3\gamma'''(\xi); \xi) \in \Omega,$$

and we note that

$$\frac{t}{s} + 1 = \frac{\beta(\beta+1)(e_3+e_1) + 2e_2 + \beta(e_2 - e_1)}{\beta(e_2 - e_1)},$$

and

$$\frac{u}{s} = \frac{e_4(\beta(\beta+1)(\beta+2)) - (\beta+1)e_3 + \beta(e_3+e_2) - \beta(\beta+1)^2 e_1}{\beta(e_2 - e_1)}.$$

Therefore, the criterion for admission $\psi \in M_{j,1}[\Omega, \delta]$ in Definition 3.11 is identical to the admissibility criterion for $\varphi \in \Psi_2[\Omega, \delta]$ as specified in Definition 2.3 with $n = 2$. Consequently, via employing (3.16) and Lemma 2.4, we acquire

$$\frac{N^{\beta}_{\alpha}f(\xi)}{\xi} \prec \delta(\xi).$$

This concludes the proof of Theorem 3.12. □

If $\Omega \notin \mathbb{C}$ is just a simply connected region, and $\Omega = \Omega(\Xi)$ for certain conformal mappings $\Omega(\xi)$ of Ξ onto Ω, then in this instance, the class $M_{j,1}[\Omega(\Xi), \delta]$ is rewritten as $M_{j,1}[\Omega, \delta]$. This is a direct implication of Theorem 3.12, stated beneath.

Theorem 3.13. *Assume $\psi \in M_{j,1}[\Omega, \delta]$. If the functions $f \in K$ and $\delta \in O_1$ fulfill the subsequent criteria:*

$$\mathrm{Re}\left(\frac{\zeta\delta''(\zeta)}{\delta'(\zeta)}\right) \geq 0, \qquad \left|\frac{N_{\alpha-1}^{\beta} f(\xi)}{\xi\,\delta'(\zeta)}\right| \leq \varrho, \tag{3.27}$$

and

$$\left\{\psi\left(\frac{N_{\alpha}^{\beta} f(\xi)}{\xi}, \frac{N_{\alpha-1}^{\beta} f(\xi)}{\xi}, \frac{N_{\alpha-2}^{\beta} f(\xi)}{\xi}, \frac{N_{\alpha-3}^{\beta} f(\xi)}{\xi}; \xi\right) : \xi \in \Xi\right\} \prec \Omega(\xi),$$

then

$$\frac{N_{\alpha}^{\beta} f(\xi)}{\xi} \prec \delta(\xi), \qquad (\xi \in \Xi). \tag{3.28}$$

Considering Definition 3.11 and the specific instance $\delta(\xi) = T\xi$, $T > 0$, the class of admissible functions $M_{j,1}[\Omega, \delta]$, designated by $M_{j,1}[\Omega, T]$, is articulated as follows.

Definition 3.14. Assume Ω is a set in $\mathbb{C}$ and $T > 0$. The class of admissible functions $M_{j,1}[\Omega, T]$ consists of those functions $\psi : \mathbb{C}^4 \times \Xi \to \mathbb{C}$ such that

$$\psi\left(Te^{i\vartheta}, \frac{(\varrho+\beta)Te^{i\vartheta}}{\beta}, \frac{L + [\beta(\beta+1)\varrho + \beta(\beta+1)^2]Te^{i\vartheta}}{\beta(\beta+1)}, \frac{N + (\beta+1)L + [\beta(3+\beta)\varrho + \beta(\beta+1)^3]Te^{i\vartheta}}{\beta(\beta+1)(\beta+2)}; \xi\right) \notin \Omega, \tag{3.29}$$

whenever

$$\mathrm{Re}(Le^{-i\vartheta}) \geq (\varrho - 1)T\varrho, \qquad \xi \in \Xi,$$

and

$$\mathrm{Re}(Ne^{-i\vartheta}) \geq 0, \qquad \forall \vartheta \in \mathbb{R},\ \varrho \geq 2.$$

Corollary 3.15. *Consider $\psi \in M_{j,1}[\Omega, T]$. If the function $f \in K$ satisfies the subsequent circumstances:*

$$\left|\frac{N_{\alpha-1}^{\beta} f(\xi)}{\xi}\right| \leq T\varrho, \qquad (\xi \in \Xi;\ \varrho \geq 2;\ T > 0),$$

and

$$\psi\left(\frac{N_{\alpha}^{\beta} f(\xi)}{\xi}, \frac{N_{\alpha-1}^{\beta} f(\xi)}{\xi}, \frac{N_{\alpha-2}^{\beta} f(\xi)}{\xi}, \frac{N_{\alpha-3}^{\beta} f(\xi)}{\xi}; \xi\right) \in \Omega,$$

then

$$\left|\frac{N_{\alpha}^{\beta} f(\xi)}{\xi}\right| < T.$$

In this particular instance, $\Omega = \Omega(\Xi) = \{\omega : |\omega| < T\}$, and the class $M_{j,1}[\Omega, T]$ is simply designated by $M_{j,1}[\Omega, T]$. Corollary 3.15 may currently be reformulated as follows.

Corollary 3.16. *Consider $\psi \in M_{j,1}[\Omega, T]$. If the function $f \in K$ meets the subsequent criteria:*

$$\left|N^{\beta}_{\alpha-1} f(\xi)\right| \leq T\varrho, \qquad (\xi \in \Xi;\ \varrho \geq 2;\ T > 0),$$

and

$$\left|\psi(N^{\beta}_{\alpha} f(\xi), N^{\beta}_{\alpha-1} f(\xi), N^{\beta}_{\alpha-2} f(\xi), N^{\beta}_{\alpha-3})\right| < T,$$

then

$$\left|\frac{N^{\beta}_{\alpha} f(\xi)}{\xi}\right| < T.$$

4. Outcomes of third-order differential superordination

This section presents results on third-order differential superordination. To this end, the class of admissible functions is delineated as follows:

Definition 4.1. Consider Ω is a set in $\mathbb{C}$ and $\delta \in O_0 \cap D_0$ with $\delta'(\xi) \neq 0$. The category of admissible functions $M'_j[\Omega, \delta]$ includes those functions $\psi : \mathbb{C}^4 \times \overline{\Xi} \to \mathbb{C}$ that meet the subsequent admission standards:

$$\psi(e_1, e_2, e_3, e_4; \zeta) \in \Omega,$$

at any time

$$e_1 = \delta(\xi), \qquad e_2 = \frac{\xi\delta'(\xi) + (1+\beta)m\delta(\xi)}{\beta m},$$

$$\mathrm{Re}\left(\frac{((1+\beta)^2 - 2(2+\beta)(1+\beta))r - (3+\beta)e_2 - e_3\beta(\beta+1)}{(1+\beta)r - \beta e_2}\right) \leq \frac{1}{m}\mathrm{Re}\left(\frac{\xi\delta''(\xi)}{\delta'(\xi)} + 1\right),$$

and

$$\mathrm{Re}\left(\frac{e_4\beta(\beta+1)(\beta+2) - \beta(1+\beta)^3 r + (3+2\beta)s - (1+\beta)t}{(1+\beta)r - \beta e_2}\right) \leq \frac{1}{m^2}\mathrm{Re}\left(\frac{\xi^2\delta'''(\xi)}{\delta'(\xi)}\right),$$

where $\xi \in \Xi$, $\zeta \in \partial\Xi \setminus E(\delta)$, and $m \geq 2$.

Theorem 4.2. *Consider $\psi \in M'_j[\Omega, \delta]$. If the functions $f \in K$, $N^{\beta}_{\alpha} f(\xi) \in O_0$, and $\delta \in D_0$ with $\delta'(\xi) \neq 0$ fulfill the subsequent criteria:*

$$\mathrm{Re}\left(\frac{\zeta\delta''(\zeta)}{\delta'(\zeta)}\right) \geq 0, \qquad \left|\frac{N^{\beta}_{\alpha-1} f(\xi)}{\delta'(\zeta)}\right| \leq m, \tag{4.1}$$

and

$$\psi(N^{\beta}_{\alpha} f(\xi), N^{\beta}_{\alpha-1} f(\xi), N^{\beta}_{\alpha-2} f(\xi), N^{\beta}_{\alpha-3} f(\xi); \xi)$$

is a univalent function in Ξ, then

$$\Omega \subset \left\{\psi(N^{\beta}_{\alpha} f(\xi), N^{\beta}_{\alpha-1} f(\xi), N^{\beta}_{\alpha-2} f(\xi), N^{\beta}_{\alpha-3} f(\xi); \xi) : \xi \in \Xi\right\} \tag{4.2}$$

implies

$$\delta(\xi) \prec N^{\beta}_{\alpha} f(\xi), \qquad (\xi \in \Xi).$$

Proof. Consider the function $\gamma(\xi)$ characterized by (3.3) and φ characterized by (3.10). Since $\psi \in M'_j[\Omega, \delta]$, from Eqs (3.11) and (4.2), we derive

$$\Omega \subset \left\{\varphi(\gamma(\xi), \xi\gamma'(\xi), \xi^2\gamma''(\xi), \xi^3\gamma'''(\xi); \xi) : \xi \in \Xi\right\}.$$

From Eqs (3.8) and (3.9), we note that the admissibility criterion for $\psi \in M'_j[\Omega, \delta]$ in Definition 4.1 is synonymous with the admissibility criterion for $\varphi \in \Psi'_n[\Omega, \delta]$ as stated in Definition 2.6 with $n = 2$. Therefore, $\varphi \in \Psi'_2[\Omega, \delta]$. Utilizing (4.2) and Lemma 2.7, we derive

$$\delta(\xi) \prec N^{\beta}_{\alpha} f(\xi), \qquad (\xi \in \Xi).$$

This concludes the proof of Theorem 4.2. □

If $\Omega \neq \mathbb{C}$ constitutes a simply connected domain and $\Omega = \Omega(\Xi)$ for certain conformal mappings $\Omega(\xi)$ of Ξ onto Ω, then in this instance, the class $M'_j[\Omega(\Xi), \delta]$ is rewritten as $M'_j[\Omega, \delta]$. This is a direct consequence of Theorem 4.2.

Theorem 4.3. *Consider $\psi \in M'_j[\Omega, \delta]$ and Ω is analytic in Ξ. If the functions $f \in K$, $N^{\beta}_{\alpha} f(\xi) \in O_0$, and $\delta \in D_0$ with $\delta'(\xi) \neq 0$ satisfy the requirements of* (4.1) *and*

$$\psi(N^{\beta}_{\alpha} f(\xi), N^{\beta}_{\alpha-1} f(\xi), N^{\beta}_{\alpha-2} f(\xi), N^{\beta}_{\alpha-3} f(\xi); \xi)$$

is a univalent function in Ξ, then

$$\Omega(\xi) \prec \psi(N^{\beta}_{\alpha} f(\xi), N^{\beta}_{\alpha-1} f(\xi), N^{\beta}_{\alpha-2} f(\xi), N^{\beta}_{\alpha-3} f(\xi); \xi), \tag{4.3}$$

implies

$$\delta(\xi) \prec N^{\beta}_{\alpha} f(\xi), \qquad (\xi \in \Xi).$$

Theorems 4.2 and 4.3 are applicable only for deriving subordination in the context of third-order differential superordination as delineated in (4.2) and (4.3). The subsequent theorem establishes the existence of the optimal subordinant of (4.3) under appropriate conditions on ψ.

Theorem 4.4. *Consider the function Ω is univalent in Ξ, and let $\psi : \mathbb{C}^4 \times \overline{\Xi} \to \mathbb{C}$ and φ be defined as in* (3.10)*. Consider the subsequent differential equation:*

$$\varphi(\delta(\xi), \xi\delta'(\xi), \xi^2\delta''(\xi), \xi^3\delta'''(\xi); \xi) = \Omega(\xi), \tag{4.4}$$

which has a solution $\delta(\xi) \in Q_0$. If the functions $f \in K$, $N^{\beta}_{\alpha} f(\xi) \in O_0$, and $\delta \in D_0$ with $\delta'(\xi) \neq 0$ satisfy the conditions of (4.1) *and the function*

$$\psi(N^{\beta}_{\alpha} f(\xi), N^{\beta}_{\alpha-1} f(\xi), N^{\beta}_{\alpha-2} f(\xi), N^{\beta}_{\alpha-3} f(\xi); \xi)$$

is univalent in Ξ, then

$$\Omega(\xi) \prec \psi(N^{\beta}_{\alpha} f(\xi), N^{\beta}_{\alpha-1} f(\xi), N^{\beta}_{\alpha-2} f(\xi), N^{\beta}_{\alpha-3} f(\xi); \xi)$$

implies that

$$\delta(\xi) \prec N^{\beta}_{\alpha} f(\xi), \qquad (\xi \in \Xi),$$

and $\delta(\xi)$ is the most exemplary (optimal) subordinant.

Proof. Utilizing Theorems 4.2 and 4.3, we infer that δ is a subordinant of (4.3). Since δ satisfies (4.4), it is therefore a solution of (4.3); consequently, δ will be subordinate to all subordinants. Hence, δ is the ideal subordinant. This completes the proof of Theorem 4.4. □

Definition 4.5. Consider Ω is a set in $\mathbb{C}$ and $\delta \in D_1$ with $\delta'(\xi) \neq 0$. The class of admissible functions $M'_{j,1}[\Omega, \delta]$ comprises those functions $\psi : \mathbb{C}^4 \times \overline{\Xi} \to \mathbb{C}$ that fulfill the subsequent admission criteria:

$$\psi(e_1, e_2, e_3, e_4; \zeta) \in \Omega,$$

at any time

$$e_1 = \delta(\xi), \qquad e_2 = \frac{\xi\delta'(\xi) + \beta m\delta(\xi)}{\beta m},$$

$$\mathrm{Re}\left(\frac{\beta(\beta+1)(e_3+r) + 2s + \beta(e_2 - r)}{\beta(e_2 - r)}\right) \leq \frac{1}{m}\mathrm{Re}\left(\frac{\xi\delta''(\xi)}{\delta'(\xi)} + 1\right),$$

and

$$\mathrm{Re}\left(\frac{e_4\big(\beta(\beta+1)(\beta+2)\big) - (\beta+1)t + \beta(\beta+3)s - \beta(\beta+1)^2 r}{\beta(e_2 - r)}\right) \leq \frac{1}{m^2}\mathrm{Re}\left(\frac{\xi^2\delta'''(\xi)}{\delta'(\xi)}\right),$$

where $\xi \in \Xi$, $\zeta \in \partial\Xi$, and $m \geq 2$.

Theorem 4.6. *Consider $\psi \in M'_{j,1}[\Omega, \delta]$. If the functions $f \in K$, $\dfrac{N^\beta_\alpha f(\xi)}{\xi} \in O_1$, and $\delta \in D_1$ with $\delta'(\xi) \neq 0$ fulfill the subsequent criteria:*

$$\mathrm{Re}\left(\frac{\zeta\delta''(\zeta)}{\delta'(\zeta)}\right) \geq 0, \qquad \left|\frac{N^\beta_{\alpha-1} f(\xi)}{\xi\,\delta'(\zeta)}\right| \leq m, \tag{4.5}$$

and

$$\psi\left(\frac{N^\beta_\alpha f(\xi)}{\xi}, \frac{N^\beta_{\alpha-1} f(\xi)}{\xi}, \frac{N^\beta_{\alpha-2} f(\xi)}{\xi}, \frac{N^\beta_{\alpha-3} f(\xi)}{\xi}; \xi\right)$$

is univalent in Ξ, then

$$\Omega \subset \left\{\psi\left(\frac{N^\beta_\alpha f(\xi)}{\xi}, \frac{N^\beta_{\alpha-1} f(\xi)}{\xi}, \frac{N^\beta_{\alpha-2} f(\xi)}{\xi}, \frac{N^\beta_{\alpha-3} f(\xi)}{\xi}; \xi\right) : \xi \in \Xi\right\} \tag{4.6}$$

implies that

$$\delta(\xi) \prec \frac{N^\beta_\alpha f(\xi)}{\xi}, \qquad (\xi \in \Xi).$$

Proof. Assume the function $\gamma(\xi)$ is characterized by (3.18) and φ is delineated via (3.25). Given $\psi \in M'_{j,1}[\Omega, \delta]$, from Eqs (3.26) and (4.6), we obtain

$$\Omega \subset \left\{\varphi(\gamma(\xi), \xi\gamma'(\xi), \xi^2\gamma''(\xi), \xi^3\gamma'''(\xi); \xi) : \xi \in \Xi\right\}.$$

Equations (3.23) and (3.24) indicate that the admissibility criterion for $\psi \in M'_{j,1}[\Omega, \delta]$ in Definition 4.5 is synonymous with the admission criterion for φ as provided in Definition 2.6 with $n = 2$. Therefore, $\varphi \in \Psi'_2[\Omega, \delta]$. Utilizing (4.6) and Lemma 2.7, we obtain

$$\delta(\xi) \prec \frac{N^\beta_\alpha f(\xi)}{\xi}, \qquad (\xi \in \Xi).$$

This concludes the proof of Theorem 4.6. □

If $\Omega \notin \mathbb{C}$ is a simply connected region and $\Omega = \Omega(\Xi)$ for certain conformal mappings $\Omega(\xi)$ of Ξ onto Ω, then in this instance, the class $M'_{j,1}[\Omega(\Xi), \delta]$ is written as $M'_{j,1}[\Omega, \delta]$. The immediate implication of Theorem 4.6 is articulated below.

Theorem 4.7. *Consider $\psi \in M'_{j,1}[\Omega, \delta]$ and Ω is analytic in Ξ. If the functions $f \in K$ and $\delta \in D_1$ with $\delta'(\xi) \neq 0$ fulfill the conditions of* (4.5) *and the function*

$$\psi\left(\frac{N_\alpha^\beta f(\xi)}{\xi}, \frac{N_{\alpha-1}^\beta f(\xi)}{\xi}, \frac{N_{\alpha-2}^\beta f(\xi)}{\xi}, \frac{N_{\alpha-3}^\beta f(\xi)}{\xi}; \xi\right)$$

is univalent in Ξ, then

$$\Omega(\xi) \prec \psi\left(\frac{N_\alpha^\beta f(\xi)}{\xi}, \frac{N_{\alpha-1}^\beta f(\xi)}{\xi}, \frac{N_{\alpha-2}^\beta f(\xi)}{\xi}, \frac{N_{\alpha-3}^\beta f(\xi)}{\xi}; \xi\right)$$

implies that

$$\delta(\xi) \prec \frac{N_\alpha^\beta f(\xi)}{\xi}, \qquad (\xi \in \Xi).$$

5. Sandwich outcomes

The subsequent sandwich theorem is derived from the amalgamation of Theorems 3.4 and 4.3.

Theorem 5.1. *Consider Ω_1 and δ_1 are analytic functions in Ξ. Also assume Ω_2 is a univalent function in Ξ and $\delta_2 \in O_0$, with $\delta_1(0) = \delta_2(0) = 1$, and $\psi \in M_j[\Omega_2, \delta_2] \cap M'_j[\Omega_1, \delta_1]$. If the function $f \in K$, satisfies $N_\alpha^\beta f(\xi) \in O_0 \cap D_0$, and the function*

$$\psi(N_\alpha^\beta f(\xi), N_{\alpha-1}^\beta f(\xi), N_{\alpha-2}^\beta f(\xi), N_{\alpha-3}^\beta f(\xi); \xi)$$

is univalent in Ξ, and if conditions (3.1) *and* (4.1) *are met, then*

$$\Omega_1(\xi) \prec \psi(N_\alpha^\beta f(\xi), N_{\alpha-1}^\beta f(\xi), N_{\alpha-2}^\beta f(\xi), N_{\alpha-3}^\beta f(\xi); \xi) \prec \Omega_2(\xi)$$

implies that

$$\delta_1(\xi) \prec N_\alpha^\beta f(\xi) \prec \delta_2(\xi), \qquad (\xi \in \Xi). \tag{5.1}$$

By integrating Theorems 3.13 and 4.7, we derive the subsequent sandwich theorem.

Theorem 5.2. *Consider Ω_1 and δ_1 are analytic functions in Ξ. Also assume Ω_2 is a univalent function in Ξ and $\delta_2 \in O_1$, with $\delta_1(0) = \delta_2(0) = 1$, and $\psi \in M_{j,1}[\Omega_2, \delta_2] \cap M'_{j,1}[\Omega_1, \delta_1]$. If the function $f \in K$ satisfies $\frac{N_\alpha^\beta f(\xi)}{\xi} \in O_1 \cap D_1$, the function*

$$\psi\left(\frac{N_\alpha^\beta f(\xi)}{\xi}, \frac{N_{\alpha-1}^\beta f(\xi)}{\xi}, \frac{N_{\alpha-2}^\beta f(\xi)}{\xi}, \frac{N_{\alpha-3}^\beta f(\xi)}{\xi}; \xi\right)$$

is univalent in Ξ*, and the requirements* (3.16) *and* (4.5) *are fulfilled, then*

$$\Omega_1(\xi) \prec \psi\left(\frac{N_\alpha^\beta f(\xi)}{\xi}, \frac{N_{\alpha-1}^\beta f(\xi)}{\xi}, \frac{N_{\alpha-2}^\beta f(\xi)}{\xi}, \frac{N_{\alpha-3}^\beta f(\xi)}{\xi}; \xi\right) \prec \Omega_2(\xi)$$

implies that

$$\delta_1(\xi) \prec \frac{N_\alpha^\beta f(\xi)}{\xi} \prec \delta_2(\xi), \qquad (\xi \in \Xi). \tag{5.2}$$

6. Conclusions

This paper presents substantial progress in the theory of differential subordination and superordination for univalent functions with an innovative Hadamard (convolution) product operator. The research establishes novel sandwich theorems that link dominant and subordinant functions under specified geometric restrictions, offering a cohesive framework for the analysis of these functions in the unit disk. The results improve the comprehension of convexity and starlikeness features in this setting, providing a basis for further applications in analytic function theory. Subsequent research may broaden these findings to encompass higher-order equations and investigate novel operators for additional theoretical and practical progress. The investigation successfully establishes third-order sandwich-type outcomes for analytic and univalent functions defined via the Srivastava–Attiya operator associated with Hurwitz–Lerch functions. Beyond proving several new inclusion relationships and admissibility conditions, the study highlights how these results unify earlier works and sharpen existing inequalities.

Author contributions

All authors contributed equally to this work and have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

References

1. J. A. Antonino, S. S. Miller, Third-order differential inequalities and subordination in the complex plane, *Complex Var. Elliptic*, **56** (2011), 439–454. https://doi.org/10.1080/17476931003728404
2. D. Breaz, S. Khan, F. M. O. Tawfiq, F. Tchier, Applications of fuzzy differential subordination to the subclass of analytic functions involving Riemann–Liouville fractional integral operator, *Mathematics*, **11** (2023), 4975. https://doi.org/10.3390/math11244975

3. T. Bulboacă, *Differential subordinations and superordinations: Recent results*, House of Scientific Book Publishing, Cluj-Napoca, Romania, 2005.

4. R. M. El-Ashwah, A. H. Hassan, Some third-order differential subordination and superordination results of some meromorphic functions using a Hurwitz–Lerch zeta type operator, *Ilir. J. Math.*, **4** (2015), 1–15.

5. H. A. Farzana, B. A. Stephen, M. P. Jeyaraman, Third-order differential subordination of analytic functions defined by functional derivative operator, *Ann. Alexandru Ioan Cuza Univ.-Math.*, **2014** (2014), 1–16. https://doi.org/10.2478/aicu-2014-0028

6. P. Gochhayat, Sandwich-type results for a class of functions defined by a generalized differential operator, *Mat. Vesn.*, **65** (2013), 178–186.

7. P. Gochhayat, Sandwich-type theorems of some subclasses of multivalent functions involving Dziok–Srivastava operator, *Acta Univ. Apulensis Math. Inform.*, **32** (2012), 31–47.

8. M. P. Jeyaraman, T. K. Suresh, Third-order differential subordination of analytic functions, *Acta Univ. Apulensis Math. Inform.*, **35** (2013), 187–202.

9. B. K. Mihsin, W. G. Atshan, On new sandwich results of univalent function defined by a linear operator, *Iraqi J. Sci.*, **63** (2022), 5467–5475. https://doi.org/10.24996/ijs.2022.63.12.32

10. S. S. Miller, P. T. Mocanu, Subordinants of differential superordinations, *Complex Var. Theory Appl.*, **48** (2003), 815–826. https://doi.org/10.1080/02781070310001599322

11. S. S. Miller, P. T. Mocanu, Differential subordinations and univalent functions, *Mich. Math. J.*, **28** (1981), 157–171. https://doi.org/10.1307/mmj/1029002507

12. S. S. Miller, P. T. Mocanu, *Differential subordinations: Theory and applications*, CRC Press, 2000.

13. A. K. Mishra, A. Prajapati, P. Gochhayat, Third-order differential subordination and superordination results for analytic functions involving the Hohlov operator, *Tbil. Math. J.*, **13** (2020), 95–109. https://doi.org/10.32513/tbilisi/1601344901

14. J. Morais, H. M. Zayed, Applications of differential subordination and superordination theorems to fluid mechanics involving a fractional higher-order integral operator, *Alex. Eng. J.*, **60** (2021), 3901–3914. https://doi.org/10.1016/j.aej.2021.02.037

15. G. I. Oros, G. Oros, L. F. Preluca, New applications of Gaussian hypergeometric function for developments on third-order differential subordinations, *Symmetry*, **15** (2023), 1306. https://doi.org/10.3390/sym15071306

16. S. Ponnusamy, O. P. Juneja, *Third-order differential inequalities in the complex plane*, In: Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, London, 1992, 274–290. https://doi.org/10.1142/9789814355896_0023

17. D. Răducanu, Third-order differential subordinations for analytic functions associated with generalized Mittag-Leffler functions, *Mediterr. J. Math.*, **14** (2017), 167. https://doi.org/10.1007/s00009-017-0969-8

18. Q. A. Shakir, F. M. Sakar, On third-order differential subordination and superordination properties of analytic functions defined by Tayyah–Atshan fractional integral operator, *Adv. Nonlinear Var. Inequal.*, **28** (2025), 176–200.

19. H. M. Srivastava, A. A. Attiya, An integral operator associated with the Hurwitz–Lerch zeta function and differential subordination, *Integr. Transf. Spec. F.*, **18** (2007), 207–216. https://doi.org/10.1080/10652460701208577

20. H. M. Srivastava, J. Choi, *Series associated with the Zeta and related functions*, Springer Science & Business Media, **530** (2001).

21. S. R. Swamy, Sandwich theorems for p-valent functions defined by certain integral operator, *Int. J. Math. Arch.*, **4** (2013), 101–107.

22. H. A. Janaby, F. Ghanim, M. Darus, On the third-order complex differential inequalities of ζ-generalized-Hurwitz-Lerch zeta functions, *Mathematics*, **8** (2020), 845. https://doi.org/10.3390/math8050845

23. H. Tang, H. M. Srivastava, S. Li, L. Ma, Third-order differential subordination and superordination results for meromorphically multivalent functions associated with the Liu–Srivastava operator, *Abstr. Appl. Anal.*, **2014** (2014), 792175. https://doi.org/10.1155/2014/792175

24. A. S. Tayyah, W. G. Atshan, G. I. Oros, Third-order differential subordination results for meromorphic functions associated with the inverse of the Legendre chi function via the Mittag-Leffler identity, *Mathematics*, **13** (2025), 2089. https://doi.org/10.3390/math13132089

25. A. S. Tayyah, W. G. Atshan, Differential subordination and superordination results for p-valent analytic functions associated with (r, k)-Srivastava fractional integral calculus, *MethodsX*, **13** (2024), 103079. https://doi.org/10.1016/j.mex.2024.103079

26. H. M. Zayed, T. Bulboacă, Applications of differential subordinations involving a generalized fractional differintegral operator, *J. Inequal. Appl.*, **2019** (2019), 242. https://doi.org/10.1186/s13660-019-2198-0