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Research article

Hermite-Hadamard-Mercer-type inequalities via proportional Caputo-hybrid-conformable operator

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Abstract: In this paper, we introduce a new proportional Caputo-hybrid-conformable (PCHC) fractional integral operator that combines proportional Caputo-hybrid and conformable fractional structures within a unified framework. By employing this operator, a fundamental integral identity is established, and several Hermite-Hadamard-Mercer-type inequalities for convex functions are derived. The proposed framework establishes a boundary limiting connection with the proportional Caputo-hybrid operator. In particular, as $\beta \to 1$ and $\eta \to 1^-$, and after setting $m = a$ and $n = b$, Theorem 2.1 reduces to Theorem 1.7 of [M. Z. Sarikaya, On Hermite-Hadamard type inequalities for proportional Caputo-hybrid operator, *Konuralp J. Math.*, **11** (2023), 31–39]. Several limiting cases and numerical examples are provided to illustrate the effectiveness and applicability of the proposed results. These findings contribute to the study of fractional integral inequalities and convex analysis in generalized fractional settings.

Keywords: Hermite-Hadamard-Mercer-type inequalities; proportional Caputo-hybrid operator; conformable operator; integral inequalities; fractional calculus; convex functions
Mathematics Subject Classification: 26D10, 26D15

1. Introduction

Fractional calculus has attracted considerable attention in recent years due to its ability to characterize memory and hereditary properties arising in numerous scientific and engineering problems. Compared with classical integer-order operators, fractional operators provide additional flexibility for describing nonlocal phenomena and complex dynamical behaviors. Consequently, a wide variety of fractional operators have been introduced and successfully applied in differential equations, mathematical physics, optimization, and integral inequalities.

The study of fractional integral inequalities originated from the classical Hermite-Hadamard inequality. A fundamental breakthrough was achieved by Sarikaya et al. [1], who established Hermite-

Hadamard-type inequalities via the Riemann-Liouville fractional integral operator. Their work initiated extensive research on fractional extensions of classical integral inequalities and demonstrated the effectiveness of fractional operators in convex analysis.

Since then, many researchers have developed new Hermite-Hadamard, Mercer, Ostrowski, Newton, and Simpson-type inequalities by employing different fractional operators. For example, Farid et al. [2] derived Hadamard inequalities for strongly m-convex functions using the Riemann-Liouville fractional integral framework. Kashuri [3] established Hermite-Hadamard inequalities involving Atangana-Baleanu-Katugampola (ABK)-fractional integrals, while Karim et al. [4] investigated Ostrowski-type inequalities through the Atangana-Baleanu fractional operator. These studies illustrate the importance of fractional operators in obtaining generalized bounds for convex functions.

Another active research direction concerns fractional operators with non-singular kernels. Atangana and Baleanu [5] introduced a new class of fractional derivatives possessing nonlocal and non-singular kernels, which significantly enriched the theory of fractional calculus. Subsequently, Abdeljawad and Baleanu [6] developed integration-by-parts formulas for these operators, providing useful analytical tools for the derivation of fractional inequalities. Based on this framework, Set et al. [7] established new Hermite-Hadamard inequalities, while Set et al. [8] further derived integral inequalities for differentiable convex functions via Atangana-Baleanu fractional integrals.

The Atangana-Baleanu setting has also been employed to study various Mercer-type and Simpson-type inequalities. Butt et al. [9] obtained Hadamard-Mercer inequalities associated with the Atangana-Baleanu operator in Katugampola sense. Jiu et al. [10] investigated Jensen-Mercer variants of Hermite-Hadamard inequalities, whereas Tariq et al. [11] established Simpson-Mercer-type inequalities involving Atangana-Baleanu fractional operators. More recently, Long et al. [12] derived Simpson-like inequalities for functions whose third derivatives satisfy s-convexity assumptions.

In parallel, several generalized Atangana-Baleanu fractional operators have been proposed. Butt et al. [13] introduced generalized integral inequalities for ABK-fractional integral operators. Kermausuor and Nwaeze [14] developed new inequalities involving k-Atangana-Baleanu fractional integrals, while Kermausuor et al. [15] extended these investigations to ψ-Hilfer-Atangana-Baleanu fractional integral operators. These developments reveal the continuing effort to construct more flexible fractional frameworks capable of producing stronger integral estimates.

Another important branch of fractional calculus is associated with Caputo-type operators. The classical Caputo derivative was introduced by Caputo [16] and has become one of the most widely used tools in fractional differential equations. Later, Caputo and Fabrizio [17] proposed a non-singular fractional derivative with an exponential kernel, which stimulated extensive research on new fractional models and applications.

Recently, proportional fractional operators have attracted growing interest because the proportional parameter provides additional freedom in modeling memory effects. Sarikaya [18] established Hermite-Hadamard-type inequalities involving the proportional Caputo-hybrid operator and demonstrated its effectiveness in convex analysis. Building upon this framework, Demir [19] derived Milne-type inequalities via the proportional Caputo-hybrid operator, while Mahajan and Nagar [20] obtained fractional Newton-type inequalities for the Caputo fractional operator. These results indicate that proportional fractional structures can produce refined integral estimates compared with several classical settings.

On the other hand, conformable fractional operators have become increasingly popular because of their relatively simple kernel structures and computational convenience. Jarad et al. [21] introduced a new class of fractional operators that unifies several existing fractional formulations. More recently, Ying et al. [22] established conformable fractional Milne-type inequalities and demonstrated the applicability of conformable operators to integral inequalities. Related developments include the Jensen-type inequality for harmonically convex functions established by Baloch et al. [23], the strengthened fractional Hermite-Hadamard inequalities based on the Jensen-Mercer inequality derived by Shi et al. [24], and the Hermite-Hadamard-type estimates for special functions presented by Ahmad et al. [25]. Collectively, these studies demonstrate that the choice of fractional kernel and convexity structure significantly affects the resulting integral estimates. However, none of them addresses the combined proportional Caputo-hybrid-conformable (PCHC) framework investigated in the present paper.

Despite the substantial progress described above, most existing studies investigate proportional Caputo-hybrid operators and conformable fractional operators separately. To the best of our knowledge, there are very few studies that combine these two fractional structures within a unified framework and apply such a framework to Hermite-Hadamard-Mercer type inequalities. Therefore, there remains a significant gap in the literature regarding the development of a fractional operator that simultaneously incorporates proportional effects, hybrid fractional behavior, and conformable kernels.

Motivated by this observation, we introduce a new fractional integral operator. The proposed PCHC fractional integral operator combines the proportional Caputo-hybrid structure with conformable fractional kernels. By establishing a fundamental fractional integral identity, we derive new Hermite-Hadamard-Mercer-type inequalities for convex functions. The proposed framework establishes a boundary limiting connection with the proportional Caputo-hybrid operator. In particular, as $\beta \to 1$ and $\eta \to 1^-$, and after setting $m = a$ and $n = b$, Theorem 2.1 reduces to Theorem 1.7 of [18].

Background on classical integral inequalities:

Definition 1.1. *[1] Let $f \in L_1[a, b]$. The left- and right-sided Riemann-Liouville fractional integrals $J^{\alpha}_{a^+} f$ and $J^{\alpha}_{b^-} f$ of order $\alpha > 0$ are defined by*

$$J^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \quad x > a$$

and

$$J^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \quad x < b,$$

respectively, where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1}\, du.$$

For $\alpha = 0$, we use the convention

$$J^0_{a^+} f(x) = J^0_{b^-} f(x) = f(x).$$

In [1], Sarikaya et al. established the Hermite-Hadamard inequality for fractional integrals as follows:

Theorem 1.1. *[1] Let $f : [a, b] \to \mathbb{R}$ be a function with $a < b$ and $f \in L_1([a, b])$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J^{\alpha}_{a^+} f(b) + J^{\alpha}_{b^-} f(a)\right] \le \frac{f(a)+f(b)}{2}$$

with $\alpha > 0$.

Remark. In general, the two fractional integral terms $J^{\alpha}_{a^+}f(b)$ and $J^{\alpha}_{b^-}f(a)$ are not equal. Indeed, their integral kernels are $(b-t)^{\alpha-1}$ and $(t-a)^{\alpha-1}$, respectively. Therefore, the factor 2 in the denominator of Theorem 1.1 cannot be cancelled unless an additional symmetry condition is imposed on f. The Caputo derivative operator is a fractional derivative operator that is widely used in the field of fractional calculus. It is defined as the fractional derivative of a function with respect to time, where the order of the derivative is a non-integer value. The Riemann-Liouville integral operator, on the other hand, is a fractional integral operator that is also commonly used in fractional calculus.

One of the significant definitions in fractional analysis is the following:

Definition 1.2. *[16] Let* $\alpha > 0$ *and* $\alpha \notin \{1, 2, \ldots\}$, $n = [\alpha] + 1$, *and* $f \in AC^n[a, b]$, *the space of functions having n-th derivatives absolutely continuous. The left-sided and right-sided Caputo fractional derivatives of order* α *are defined as follows:*

$$^{C}D^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\,dt, \quad x > a$$

and

$$^{C}D^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b (t-x)^{n-\alpha-1} f^{(n)}(t)\,dt, \quad x < b.$$

If $\alpha = n \in \{1, 2, \ldots\}$ *and usual derivative* $f^{(n)}(x)$ *of order n exists, then Caputo fractional derivative* $^{C}D^{\alpha}_{a^+}f(x)$ *coincides with* $f^{(n)}(x)$, *whereas* $^{C}D^{\alpha}_{b^-}f(x)$ *with exactness to a constant multiplier* $(-1)^n$ *. In particular, we have* $^{C}D^{0}_{a^+}f(x) = ^{C}D^{0}_{b^-}f(x) = f(x)$, *where* $n = 1$ *and* $\alpha = 0$.

Mahajan and Nagar [20] established the following fractional Newton-type inequality involving the Caputo fractional operator.

Theorem 1.2. *[20]* $f : [a, b] \to \mathbb{R}$ *is an absolutely continuous function on* (a, b) *such that* $f' \in L_1[a, b]$ *. If* $f \in C^{n+1}[a, b]$ *and* $\left|f^{n+1}\right|$ *is convex on* $[a, b]$, *then*

$$\begin{aligned} &\left| \frac{1}{8} \left[f^n(a) + 3f^n\left(\frac{2a+b}{3}\right) + 3f^n\left(\frac{a+2b}{3}\right) + f^n(b) \right] \right. \\ &\left. -\frac{2^{n-\alpha}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}} \left[C^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f^n(b) + (-1)^n C^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f^n(a) \right] \right| \\ \leq\ & \frac{b-a}{4} \left[\psi_1(\alpha) + \psi_2(\alpha)\right] \left[\left|f^{n+1}(a)\right| + \left|f^{n+1}(b)\right| \right], \end{aligned}$$

where $\lambda \in [0, 1]$ *denotes the integration variable,*

$$\psi_1(\alpha) = \int_0^{2/3} \left| \lambda^{n-\alpha} - \frac{1}{4} \right| d\lambda \quad \textit{and} \quad \psi_2(\alpha) = \int_{2/3}^1 \left(1 - \lambda^{n-\alpha}\right) d\lambda.$$

The proportional Caputo-hybrid operator is a mathematical operation that has been proposed as a non-local and singular operator, incorporating both a derivative and integral operator component in its definition. It can be expressed as a straightforward linear combination of a Riemann-Liouville integral and Caputo derivative operators.

Definition 1.3. *Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I^o and $f, f' \in L^1(I)$. Then the proportional Caputo-hybrid operator may be defined as*

$$ {}_0^{PC}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \left[K_1(\alpha,\tau) f(\tau) + K_0(\alpha,\tau) f'(\tau) \right] (t-\tau)^{-\alpha} \, d\tau, $$

where $\alpha \in [0,1]$, and K_0 and K_1 are functions satisfying

$$ \lim_{\alpha\to 0^+} K_0(\alpha,\tau) = 0; \ \lim_{\alpha\to 1} K_0(\alpha,\tau) = 1; \quad K_0(\alpha,\tau) \neq 0, \ \alpha \in (0,1]; $$
$$ \lim_{\alpha\to 0} K_1(\alpha,\tau) = 0; \ \lim_{\alpha\to 1^-} K_1(\alpha,\tau) = 1; \quad K_1(\alpha,\tau) \neq 0, \ \alpha \in [0,1). $$

Afterwards, Sarikaya presented a novel definition by employing distinct K_1 and K_0 functions based as following definition:

Definition 1.4. *[18] Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I^o and $f, f' \in L^1(I)$. The left-sided and right-sided proportional Caputo-hybrid operators of order α are defined as follows:*

$$ {}_{a^+}^{PC}D_b^{\alpha} f(b) = \frac{1}{\Gamma(1-\alpha)} \int_a^b \left[K_1(\alpha, b-\tau) f(\tau) + K_0(\alpha, b-\tau) f'(\tau) \right] (b-\tau)^{-\alpha} \, d\tau $$

and

$$ {}_{b^-}^{PC}D_a^{\alpha} f(a) = \frac{1}{\Gamma(1-\alpha)} \int_a^b \left[K_1(\alpha, \tau-a) f(\tau) + K_0(\alpha, \tau-a) f'(\tau) \right] (\tau-a)^{-\alpha} \, d\tau, $$

where $\alpha \in [0,1]$, $K_0(\alpha,t) = (1-\alpha)^2 t^{1-\alpha}$, and $K_1(\alpha,t) = \alpha^2 t^{\alpha}$.

Sarikaya [18] established Hermite-Hadamard inequalities for convex functions via the proportional Caputo-hybrid operator.

Theorem 1.3. *[18] Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I^o, the interior of the interval I, where $a, b \in I^o$ with $a < b$, and f, f' be convex functions on I. For $\alpha \in [0,1]$, the following inequalities hold:*

$$ \begin{aligned} & \alpha^2 (b-a)^{\alpha} f\left(\frac{a+b}{2}\right) + \frac{1}{2}(1-\alpha)(b-a)^{1-\alpha} f'\left(\frac{a+b}{2}\right) \\ \leq \ & \frac{\Gamma(1-\alpha)}{2(b-a)^{1-\alpha}} \left[{}_{a^+}^{PC}D_b^{\alpha} f(b) + {}_{b^-}^{PC}D_a^{\alpha} f(a) \right] \\ \leq \ & \alpha^2 (b-a)^{\alpha} \left[\frac{f(a)+f(b)}{2} \right] + (1-\alpha)(b-a)^{1-\alpha} \left[\frac{f'(a)+f'(b)}{4} \right]. \end{aligned} $$

Theorem 1.4. *[18] Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on (a,b). Assume that $|f'|$ and $\left|f''\right|$ are convex functions on $[a,b]$. For $\alpha \in [0,1]$, the following inequalities hold:*

$$ \begin{aligned} & \left| \alpha^2 (b-a)^{\alpha} \left(\frac{f(a)+f(b)}{2} \right) + (1-\alpha)(b-a)^{1-\alpha} \left(\frac{f'(a)+f'(b)}{4} \right) \right. \\ & \left. - \frac{\Gamma(1-\alpha)}{2(b-a)^{1-\alpha}} \left[{}_{a^+}^{PC}D_b^{\alpha} f(b) + {}_{b^-}^{PC}D_a^{\alpha} f(a) \right] \right| \\ \leq \ & \frac{\alpha^2 (b-a)^{1+\alpha}}{2} \left(\frac{|f'(a)+f'(b)|}{4} \right) + \frac{(1-\alpha)(b-a)}{(3-2\alpha)^2} \left[1 - \frac{1}{2^{3-2\alpha}} \right] \left(\frac{|f''(a)+f''(b)|}{2} \right). \end{aligned} $$

Recent developments in fractional calculus have shown that different fractional operators possess distinct advantages in describing memory effects, nonlocal behaviors, and anomalous dynamics. Among them, proportional Caputo-hybrid operators provide additional flexibility through proportional parameters, while conformable fractional operators preserve several important properties of classical calculus. Nevertheless, most existing Hermite-Hadamard and Mercer-type inequalities have been established separately for these operators.

This observation naturally raises the following question: Can one construct a unified fractional framework that simultaneously incorporates proportional and conformable structures and derive corresponding integral inequalities in a more general setting? Motivated by this problem, we introduce a new fractional integral operator and investigate Hermite-Hadamard-Mercer type inequalities associated with the proposed PCHC fractional integral operator. The proposed approach not only unifies several existing fractional models but also provides a broader framework for studying convexity-related inequalities.

In [21], Jarad et al. established a new fractional operator called conformable fractional integrals.

Definition 1.5. *The fractional conformable integral operator $I_{a^+}^{\beta} f(x)$ and $I_{b^-}^{\beta} f(x)$ of order $\beta \in \mathbb{C}$, $\mathrm{Re}(\beta) > 0$, and $\alpha \in (0,1]$ are given by*

$$^{Conf}I_{a^+}^{\beta,\alpha} f(t) = \frac{1}{\Gamma(\beta)} \int_a^t \frac{1}{(s-a)^{1-\alpha}} \left[\frac{(t-a)^{\alpha} - (s-a)^{\alpha}}{\alpha}\right]^{\beta-1} f(s)\, ds,$$

$$^{Conf}I_{b^-}^{\beta,\alpha} f(t) = \frac{1}{\Gamma(\beta)} \int_t^b \frac{1}{(b-s)^{1-\alpha}} \left[\frac{(b-t)^{\alpha} - (b-s)^{\alpha}}{\alpha}\right]^{\beta-1} f(s)\, ds.$$

In [22], Ying et al. gave the fractional Milne-type inequailties via conformable fractional operator.

Theorem 1.5. *Let $f : [a,b] \to \mathbb{R}$ be differentiable. Assume that $f'^1([a,b])$ and $|f'|$ is convex on $[a,b]$. For $\beta > 0$ and $\alpha \in (0,1]$, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{\alpha^{\beta} 2^{\alpha\beta-1}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}} W(\alpha,\beta,f)\right| \\
\leq\ & \frac{b-a}{4}\left[\frac{2\alpha - 3B\left(\frac{2}{\alpha},\beta+1\right)}{3\alpha}|f'(a)| + \frac{2\alpha - 3B\left(\frac{2}{\alpha},\beta+1\right)}{3\alpha}|f'(b)|\right. \\
&\left.+\frac{4\alpha - 6B\left(\frac{1}{\alpha},\beta+1\right) + 6B\left(\frac{2}{\alpha},\beta+1\right)}{3}\left|f'\left(\frac{a+b}{2}\right)\right|\right],
\end{aligned}$$

where

$$W(\alpha,\beta,f) = {}^{Conf}I_{\left(\frac{a+b}{2}\right)^+}^{\beta,\alpha} f(b) + {}^{Conf}I_{\left(\frac{a+b}{2}\right)^-}^{\beta,\alpha} f(a)$$

and $B(\cdot,\cdot)$ is the beta function.

Fractional integral inequalities have attracted considerable attention in recent years due to their significant roles in convex analysis, optimization theory, and the qualitative study of fractional differential equations. Classical Hermite-Hadamard and Hermite-Hadamard-Mercer inequalities have been generalized in various directions through different fractional operators, including the Riemann-Liouville fractional integral, Caputo fractional derivative, proportional fractional operators, Atangana-Baleanu operators, and conformable fractional integrals.

Among these developments, the proportional Caputo-hybrid operator provides a flexible framework by incorporating both derivative and integral information through proportional kernel functions. On the other hand, conformable fractional integrals preserve several important properties of classical calculus while introducing a fractional measure that depends on the conformable transformation. Although both approaches have been successfully employed in the study of integral inequalities, they have largely been investigated independently.

Motivated by this observation, we seek to establish a unified framework that simultaneously incorporates proportional and conformable fractional structures. The purpose is not merely to combine two existing operators formally, but rather to construct a new fractional operator whose kernel structure reflects the interaction between proportional weighting mechanisms and conformable fractional measures. Such a framework allows the resulting inequalities to depend on multiple fractional parameters, thereby providing additional flexibility in controlling the associated bounds.

More precisely, the proposed PCHC fractional integral operator introduces a conformable deformation into the proportional Caputo-hybrid kernel. Consequently, the resulting operator possesses a richer kernel structure characterized by the parameters α, β, and η. This additional degree of generality enables the derivation of new Hermite-Hadamard-Mercer-type inequalities that cannot be obtained directly from existing proportional or conformable frameworks.

The main contributions of this paper may be summarized as follows:

(1) A new PCHC fractional integral operator is introduced by embedding proportional Caputo-hybrid kernels into a conformable fractional measure.

(2) A fundamental integral identity associated with the proposed operator is established, which serves as the basis for deriving various Hermite-Hadamard-Mercer-type inequalities.

(3) Several new inequalities are obtained under different convexity assumptions involving first and second derivatives.

(4) The proposed framework establishes a boundary limiting connection with the proportional Caputo-hybrid operator. More precisely, as $\beta \to 1$ and $\eta \to 1^-$, and after setting $m = a$ and $n = b$, Theorem 2.1 reduces to Theorem 1.7 of [18].

(5) Numerical examples and parameter sensitivity analyses are provided to illustrate the validity of the theoretical results and to demonstrate the influence of the fractional parameters on the resulting bounds.

To avoid ambiguity, throughout this paper, we use the unified notation

$$ {}^{\mathrm{PCHC}}_{a+}I^{\alpha,\beta,\eta} \quad \text{and} \quad {}^{\mathrm{PCHC}}_{b-}I^{\alpha,\beta,\eta} $$

for the left- and right-sided PCHC integral operators, respectively. This notation is used consistently in all definitions, lemmas, theorems, proofs, corollaries, numerical examples, and limiting cases.

Definition 1.6. *Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I^o and $f, f' \in L^1(I)$. The left- and right-sided PCHC fractional integral operators with parameters $\alpha \in (0, 1]$, $\beta \in (0, 1]$, $\eta > 0$, and*

$\alpha\eta \in (0,1)$ *are defined as follows:*

$$
\begin{aligned}
{}^{\mathrm{PCHC}}_{a^+}I^{\alpha,\beta,\eta} f(t) &= \frac{1}{\Gamma(1-\alpha\eta)} \int_a^t \left[K_1(\alpha,\eta,t,\tau) f(\tau) + K_0(\alpha,\eta,t,\tau) f'(\tau)\right] \\
&\quad \times \frac{1}{(\tau-a)^{1-\beta}} \left[\frac{(t-a)^\beta - (\tau-a)^\beta}{\beta}\right]^{-\alpha\eta} d\tau,
\end{aligned}
$$

where

$$
K_0(\alpha,\eta,t,\tau) = (1-\alpha\eta)^2 \left[\frac{(t-a)^\beta - (\tau-a)^\beta}{\beta}\right]^{1-\alpha\eta}
$$

and

$$
K_1(\alpha,\eta,t,\tau) = (\alpha\eta)^2 \left[\frac{(t-a)^\beta - (\tau-a)^\beta}{\beta}\right]^{\alpha\eta},
$$

and

$$
\begin{aligned}
{}^{\mathrm{PCHC}}_{b^-}I^{\alpha,\beta,\eta} f(t) &= \frac{1}{\Gamma(1-\alpha\eta)} \int_t^b \left[L_1(\alpha,\eta,t,\tau) f(\tau) + L_0(\alpha,\eta,t,\tau) f'(\tau)\right] \\
&\quad \times \frac{1}{(b-\tau)^{1-\beta}} \left[\frac{(b-t)^\beta - (b-\tau)^\beta}{\beta}\right]^{-\alpha\eta} d\tau,
\end{aligned}
$$

where

$$
L_0(\alpha,\eta,t,\tau) = (1-\alpha\eta)^2 \left[\frac{(b-t)^\beta - (b-\tau)^\beta}{\beta}\right]^{1-\alpha\eta}
$$

and

$$
L_1(\alpha,\eta,t,\tau) = (\alpha\eta)^2 \left[\frac{(b-t)^\beta - (b-\tau)^\beta}{\beta}\right]^{\alpha\eta}.
$$

This parameter convention is used throughout all definitions, theorems, proofs, corollaries, remarks, and numerical examples.

The remainder of this paper is organized as follows. Section 2 presents the main identities and Hermite-Hadamard-Mercer-type inequalities associated with the proposed PCHC integral operators. Section 3 provides numerical examples and comparisons of the obtained upper bounds. Finally, Section 4 contains the conclusions and possible directions for future research.

2. Main results

Although many Hermite-Hadamard and Hermite-Hadamard-Mercer-type inequalities have been studied under various fractional operators, most of the existing results are derived separately for different operators. There are still very few studies that combine proportional fractional operators with conformable fractional integrals into a unified framework. Therefore, it is meaningful to construct a more general operator that integrates these two fractional structures and to investigate integral inequalities under this generalized operator.

The motivation of this work is to construct a more general fractional integral operator that combines proportional fractional operators and conformable fractional integrals and to investigate Hermite-Hadamard-Mercer-type inequalities in this unified fractional framework. The results obtained in this paper extend and generalize many existing inequalities in fractional calculus and convex analysis.

We first establish a Hermite-Hadamard-type inequality associated with the proposed PCHC fractional integral operator.

Theorem 2.1. *Let $I \subset \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be a differentiable function such that $f, f' \in L^1([a,b])$. Assume that both f and f' are convex on $[a,b]$. Let $a, b \in I^\circ$ with $a \leq m < n \leq b$. If $\alpha \in (0,1]$, $\beta \in (0,1]$, $\eta > 0$, and $\alpha\eta \in (0,1)$, then the following inequalities hold:*

$$\begin{aligned}
&(\alpha\eta)^2 \beta^{1-2\alpha\eta} (n-m)^{\alpha\beta\eta+\beta-1} f\left(a+b-\frac{m+n}{2}\right) + \frac{(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta-1}}{2} f'\left(a+b-\frac{m+n}{2}\right) \\
\leq\ & \frac{\Gamma(1-\alpha\eta)\beta^{2-2\alpha\eta}}{2(n-m)^{1-\alpha\beta\eta}} \left[{}^{\text{PCHC}}_{a+b-m^-}I^{\alpha,\beta,\eta}\{f(a+b-n)\} + {}^{\text{PCHC}}_{a+b-n^+}I^{\alpha,\beta,\eta}\{f(a+b-m)\}\right] \\
\leq\ & \frac{(\alpha\eta)^2 \beta^{1-2\alpha\eta}(n-m)^{\alpha\beta\eta+\beta-1}}{2}[2f(a)+2f(b)-f(m)-f(n)] \qquad (2.1)\\
&+\frac{(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta-1}}{4}[2f'(a)+2f'(b)-f'(m)-f'(n)].
\end{aligned}$$

Proof. Since $m < n$, we have $n - m > 0$. Therefore, all denominators involving $n - m$ are nonzero. The degenerate case $m = n$ is excluded by the assumptions of Theorem 2.1. By the convexity of f and the Jensen-Mercer inequality, for every $t \in [0,1]$, we have

$$f\left(a+b-\frac{m+n}{2}\right) \leq \frac{f(a+b-[tm+(1-t)n]) + f(a+b-[(1-t)m+tn])}{2}.$$

Similarly, since f' is convex on $[a,b]$,

$$f'\left(a+b-\frac{m+n}{2}\right) \leq \frac{f'(a+b-[tm+(1-t)n]) + f'(a+b-[(1-t)m+tn])}{2}.$$

Step 1. Application of the Jensen-Mercer inequality. Since both f and f' are convex functions on $[a,b]$, we obtain

$$2f\left(a+b-\frac{x+y}{2}\right) \leq f(a+b-x) + f(a+b-y)$$

and

$$2f'\left(a+b-\frac{x+y}{2}\right) \leq f'(a+b-x) + f'(a+b-y)$$

for all $x, y \in [a,b]$.

Replacing $x = tm + (1-t)n$ and $y = (1-t)m + tn$, for all $m, n \in [a,b]$ and $t \in [0,1]$, we have

$$f\left(a+b-\frac{m+n}{2}\right) \leq \frac{1}{2}\{f(a+b-[tm+(1-t)n]) + f(a+b-[(1-t)m+tn])\}$$

and

$$f'\left(a+b-\frac{m+n}{2}\right) \leq \frac{1}{2}\{f'(a+b-[tm+(1-t)n]) + f'(a+b-[(1-t)m+tn])\}.$$

Step 2. Multiplication by the PCHC kernel weights and integration. By multiplying the results by $(\alpha\eta)^2\beta^{2-2\alpha\eta}(n-m)^{\alpha\beta\eta+\beta-1}\ t^{\beta-1}$ and $(1-\alpha\eta)^2\beta(n-m)^{2\beta-\alpha\beta\eta-1}\left(1-t^{\beta}\right)^{1-2\alpha\eta}t^{\beta-1}$ on both sides of two inequalities, then integrating both inequalities with respect to t over $[0,1]$, we obtain

$$\begin{aligned}&(\alpha\eta)^2\beta^{2-2\alpha\eta}(n-m)^{\alpha\beta\eta+\beta-1}f\left(a+b-\frac{m+n}{2}\right)\int_0^1 t^{\beta-1}dt\\ \leq\ &\frac{(\alpha\eta)^2\beta^{2-2\alpha\eta}(n-m)^{\alpha\beta\eta+\beta-1}}{2}\int_0^1 t^{\beta-1}\left\{f\left(a+b-[tm+(1-t)n]\right)+f\left(a+b-[(1-t)m+tn]\right)\right\}dt\end{aligned}$$

and

$$\begin{aligned}&(1-\alpha\eta)^2\beta(n-m)^{2\beta-\alpha\beta\eta-1}f'\left(a+b-\frac{m+n}{2}\right)\times\int_0^1\left(1-t^{\beta}\right)^{1-2\alpha\eta}t^{\beta-1}dt\\ \leq\ &\frac{(1-\alpha\eta)^2\beta(n-m)^{2\beta-\alpha\beta\eta-1}}{2}\int_0^1\left(1-t^{\beta}\right)^{1-2\alpha\eta}t^{\beta-1}\left\{f'\left(a+b-[tm+(1-t)n]\right)\right.\\ &\left.+f'\left(a+b-[(1-t)m+tn]\right)\right\}dt.\end{aligned}$$

Step 3. Change of variables. Adding the preceding two inequalities and applying the changes of variables

$$x=a+b-[tm+(1-t)n]\quad\text{and}\quad y=a+b-[(1-t)m+tn],$$

we obtain

$$\begin{aligned}&(\alpha\eta)^2\beta^{1-2\alpha\eta}(n-m)^{\alpha\beta\eta+\beta-1}f\left(a+b-\frac{m+n}{2}\right)+\frac{(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta-1}}{2}f'\left(a+b-\frac{m+n}{2}\right)\\ \leq\ &\frac{(\alpha\eta)^2\beta^{2-2\alpha\eta}(n-m)^{\alpha\beta\eta+\beta-1}}{2}\left[\int_{a+b-n}^{a+b-m}\left[\frac{x-(a+b-n)}{n-m}\right]^{\beta-1}f(x)\frac{dx}{n-m}\right.\\ &\left.+\int_{a+b-m}^{a+b-n}\left[\frac{(a+b-m)-y}{n-m}\right]^{\beta-1}f(y)\frac{(-1)\,dy}{n-m}\right]\\ &+\frac{(1-\alpha\eta)^2\beta(n-m)^{2\beta-\alpha\beta\eta-1}}{2}\left[\int_{a+b-n}^{a+b-m}\left[1-\left(\frac{x-(a+b-n)}{n-m}\right)^{\beta}\right]^{1-2\alpha\eta}\right.\\ &\times\left[\frac{x-(a+b-n)}{n-m}\right]^{\beta-1}f'(x)\frac{dx}{n-m}\\ &\left.+\int_{a+b-m}^{a+b-n}\left[1-\left(\frac{(a+b-m)-y}{n-m}\right)^{\beta}\right]^{1-2\alpha\eta}\times\left[\frac{(a+b-m)-y}{n-m}\right]^{\beta-1}f'(y)\frac{(-1)\,dy}{n-m}\right].\end{aligned}$$

Step 4. Identification of the PCHC fractional integral terms. By Definition 1.6 and the preceding changes of variables, the two integrals can be written in terms of the left- and right-sided PCHC integral operators ${}^{\mathrm{PCHC}}_{a^+}I^{\alpha,\beta,\eta}f(t)$ and ${}^{\mathrm{PCHC}}_{b^-}I^{\alpha,\beta,\eta}f(t)$, and then renaming the integration variable y as x, we obtain

$$\begin{aligned}&(\alpha\eta)^2\beta^{1-2\alpha\eta}(n-m)^{\alpha\beta\eta+\beta-1}f\left(a+b-\frac{m+n}{2}\right)+\frac{(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta-1}}{2}f'\left(a+b-\frac{m+n}{2}\right)\\ \leq\ &\frac{\beta^{2-2\alpha\eta}}{2(n-m)^{1-\alpha\beta\eta}}\left\{\int_{a+b-n}^{a+b-m}\frac{1}{[x-(a+b-n)]^{1-\beta}}f(x)\,dx+\int_{a+b-n}^{a+b-m}\frac{1}{[(a+b-m)-x]^{1-\beta}}f(x)\,dx\right.\end{aligned}$$

$$+\int_{a+b-n}^{a+b-m}\left[\frac{[(a+b-m)-(a+b-n)]^{\beta}-[x-(a+b-n)]^{\beta}}{\beta}\right]^{1-2\alpha\eta}\times\frac{1}{[x-(a+b-n)]^{1-\beta}}f'(x)\,dx$$
$$+\int_{a+b-n}^{a+b-m}\left[\frac{[(a+b-m)-(a+b-n)]^{\beta}-[x-(a+b-n)]^{\beta}}{\beta}\right]^{1-2\alpha\eta}$$
$$\left.\times\frac{1}{[(a+b-m)-x]^{1-\beta}}f'(x)\,dx\right\}$$
$$=\frac{\Gamma(1-\alpha\eta)\beta^{2-2\alpha\eta}}{2(n-m)^{1-\alpha\beta\eta}}\left[{}^{\text{PCHC}}_{a+b-m^{-}}I^{\alpha,\beta,\eta}\{f(a+b-n)\}+{}^{\text{PCHC}}_{a+b-n^{+}}I^{\alpha,\beta,\eta}\{f(a+b-m)\}\right].$$

Step 5. Upper bound. We now prove the upper-bound inequality in (2.1). Since both f and f' are convex on $[a,b]$, the Jensen-Mercer inequality gives, for every $t\in[0,1]$,

$$f(a+b-[tm+(1-t)n])+f(a+b-[(1-t)m+tn])\le 2f(a)+2f(b)-f(m)-f(n)$$

and

$$f'(a+b-[tm+(1-t)n])+f'(a+b-[(1-t)m+tn])\le 2f'(a)+2f'(b)-f'(m)-f'(n).$$

By multiplying the results by

$$\frac{(\alpha\eta)^2\beta^{2-2\alpha\eta}(n-m)^{\alpha\beta\eta+\beta-1}t^{\beta-1}}{2}\quad\text{and}\quad\frac{(1-\alpha\eta)^2\beta(n-m)^{2\beta-\alpha\beta\eta-1}\left(1-t^{\beta}\right)^{1-2\alpha\eta}t^{\beta-1}}{2}$$

on both sides of two inequalities and integrating the resulting inequalities with respect to t over $[0,1]$, we obtain

$$\frac{(\alpha\eta)^2\beta^{2-2\alpha\eta}(n-m)^{\alpha\beta\eta+\beta-1}}{2}\int_0^1 t^{\beta-1}\left[f(a+b-[tm+(1-t)n])+f(a+b-[(1-t)m+tn])\right]dt$$
$$\le\frac{(\alpha\eta)^2\beta^{2-2\alpha\eta}(n-m)^{\alpha\beta\eta+\beta-1}}{2}\int_0^1 t^{\beta-1}\left[2f(a)+2f(b)-f(m)-f(n)\right]dt$$

and

$$\frac{(1-\alpha\eta)^2\beta(n-m)^{2\beta-\alpha\beta\eta-1}}{2}\int_0^1\left(1-t^{\beta}\right)^{1-2\alpha\eta}t^{\beta-1}\left[f'(a+b-[tm+(1-t)n])\right.$$
$$\left.+f'(a+b-[(1-t)m+tn])\right]dt$$
$$\le\frac{(1-\alpha\eta)^2\beta(n-m)^{2\beta-\alpha\beta\eta-1}}{2}\int_0^1\left(1-t^{\beta}\right)^{1-2\alpha\eta}t^{\beta-1}\left[2f'(a)+2f'(b)-f'(m)-f'(n)\right]dt.$$

Adding the preceding two inequalities and applying the same changes of variables, we obtain

$$\frac{(\alpha\eta)^2\beta^{2-2\alpha\eta}(n-m)^{\alpha\beta\eta+\beta-1}}{2}\left[\int_{a+b-n}^{a+b-m}\left[\frac{x-(a+b-n)}{n-m}\right]^{\beta-1}f(x)\frac{dx}{n-m}\right.$$
$$\left.+\int_{a+b-m}^{a+b-n}\left[\frac{(a+b-m)-y}{n-m}\right]^{\beta-1}f(y)\frac{(-1)\,dy}{n-m}\right]$$

$$+\frac{(1-\alpha\eta)^{2}\beta(n-m)^{2\beta-\alpha\beta\eta-1}}{2}\left[\int_{a+b-n}^{a+b-m}\left[1-\left(\frac{x-(a+b-n)}{n-m}\right)^{\beta}\right]^{1-2\alpha\eta}\right.$$
$$\times\left[\frac{x-(a+b-n)}{n-m}\right]^{\beta-1}f'(x)\frac{dx}{n-m}$$
$$\left.+\int_{a+b-m}^{a+b-n}\left[1-\left(\frac{(a+b-m)-y}{n-m}\right)^{\beta}\right]^{1-2\alpha\eta}\times\left[\frac{(a+b-m)-y}{n-m}\right]^{\beta-1}f'(y)\frac{(-1)\,dy}{n-m}\right]$$
$$\leq\frac{(\alpha\eta)^{2}\beta^{1-2\alpha\eta}(n-m)^{\alpha\beta\eta+\beta-1}}{2}\left[2f(a)+2f(b)-f(m)-f(n)\right]$$
$$+\frac{(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta-1}}{4}\left[2f'(a)+2f'(b)-f'(m)-f'(n)\right].$$

By the definition of ${}_{a^{+}}^{\mathrm{PCHC}}I^{\alpha,\beta,\eta}f(t)$ and ${}_{b^{-}}^{\mathrm{PCHC}}I^{\alpha,\beta,\eta}f(t)$, and then renaming the integration variable y as x, we obtain

$$\frac{\Gamma(1-\alpha\eta)\beta^{2-2\alpha\eta}}{2(n-m)^{1-\alpha\beta\eta}}\left[{}_{a+b-m^{-}}^{\mathrm{PCHC}}I^{\alpha,\beta,\eta}\{f(a+b-n)\}+{}_{a+b-n^{+}}^{\mathrm{PCHC}}I^{\alpha,\beta,\eta}\{f(a+b-m)\}\right]$$
$$\leq\frac{(\alpha\eta)^{2}\beta^{1-2\alpha\eta}(n-m)^{\alpha\beta\eta+\beta-1}}{2}\left[2f(a)+2f(b)-f(m)-f(n)\right]$$
$$+\frac{(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta-1}}{4}\left[2f'(a)+2f'(b)-f'(m)-f'(n)\right].$$

This completes the proof of the stated inequalities.

Remark 2.1. *Taking the boundary limits $\beta\to1$ and $\eta\to1^{-}$ in Theorem 2.1, the inequalities for proportional Caputo-hybrid fractional integral operators:*

$$\alpha^{2}(n-m)^{\alpha}f\left(a+b-\frac{m+n}{2}\right)+\frac{(1-\alpha)(n-m)^{1-\alpha}}{2}f'\left(a+b-\frac{m+n}{2}\right)$$
$$\leq\frac{\Gamma(1-\alpha)}{2(n-m)^{1-\alpha}}\left[{}^{\mathrm{PCH}}{}_{a+b-m^{-}}I^{\alpha}\{f(a+b-n)\}+{}^{\mathrm{PCH}}{}_{a+b-n^{+}}I^{\alpha}\{f(a+b-m)\}\right]$$
$$\leq\frac{\alpha^{2}(n-m)^{\alpha}}{2}\left[2f(a)+2f(b)-f(m)-f(n)\right]$$
$$+\frac{(1-\alpha)(n-m)^{1-\alpha}}{4}\left[2f'(a)+2f'(b)-f'(m)-f'(n)\right].$$

Remark 2.2. *Taking the boundary limits $\beta\to1^{-}$ and $\eta\to1^{-}$ in Theorem 2.1 and setting $m=a$ and $n=b$, we recover the Hermite-Hadamard inequality for the proportional Caputo-hybrid operator established in Theorem 1.7 of [18].*

Lemma 2.1. *Let $f:[a,b]\to\mathbb{R}$ be twice differentiable on (a,b), with $f',f''\in L^{1}([a,b])$. Let $a\leq m<n\leq b$, $\alpha\in(0,1]$, $\beta\in(0,1]$, $\eta>0$, and $0<\alpha\eta<1$. Then the following identity holds:*

$$\frac{(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta}}{4}$$
$$\times\left[\int_{0}^{1}\left(1-(1-t)^{\beta}\right)^{2-2\alpha\eta}f''(a+b-[tm+(1-t)n])\,dt\right.$$

$$
\begin{aligned}
&-\int_0^1 \left(1-t^\beta\right)^{2-2\alpha\eta} f''(a+b-[tm+(1-t)n])\,dt\Bigg] \\
&-\frac{(\alpha\eta)^2 \beta^{1-2\alpha\eta}(n-m)^{\beta+\alpha\beta\eta}}{2} \\
&\times\left[\int_0^1 (1-t)^\beta f'(a+b-[tm+(1-t)n])\,dt-\int_0^1 t^\beta f'(a+b-[tm+(1-t)n])\,dt\right] \\
= \; & (\alpha\eta)^2 \beta^{1-2\alpha\eta}(n-m)^{\alpha\beta\eta+\beta-1}\left[\frac{f(a+b-n)+f(a+b-m)}{2}\right] \\
&+(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta-1}\left[\frac{f'(a+b-n)+f'(a+b-m)}{4}\right] \\
&-\frac{\Gamma(1-\alpha\eta)\beta^{2-2\alpha\eta}}{2(n-m)^{1-\alpha\beta\eta}}\left[{}^{\mathrm{PCHC}}_{a+b-m^-}I^{\alpha,\beta,\eta}\{f(a+b-n)\}+{}^{\mathrm{PCHC}}_{a+b-n^+}I^{\alpha,\beta,\eta}\{f(a+b-m)\}\right].
\end{aligned}
$$

Proof. For convenience, we set

$$
A=\frac{(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta}}{4} \quad \text{and} \quad B=\frac{(\alpha\eta)^2\beta^{1-2\alpha\eta}(n-m)^{\beta+\alpha\beta\eta}}{2}.
$$

Step 1. Integration by parts for the left kernel terms. By integration by parts, we have

$$
\begin{aligned}
I_1 &= \int_0^1 \left(1-(1-t)^\beta\right)^{2-2\alpha\eta} f''(a+b-[tm+(1-t)n])\,dt \\
&= \frac{f'(a+b-m)}{n-m}-\frac{\beta(2-2\alpha\eta)}{n-m}\int_0^1 f'(a+b-[tm+(1-t)n])\left(1-(1-t)^\beta\right)^{1-2\alpha\eta}(1-t)^{\beta-1}\,dt.
\end{aligned}
$$

Step 2. Using the change of the variable by $x=a+b-[tm+(1-t)n]$, we have

$$
\begin{aligned}
I_1 &= \int_0^1 \left(1-(1-t)^\beta\right)^{2-2\alpha\eta} f''(a+b-[tm+(1-t)n])\,dt \\
&= \frac{f'(a+b-m)}{n-m}-\frac{\beta(2-2\alpha\eta)}{n-m}\int_0^1 f'(a+b-[tm+(1-t)n])\left(1-(1-t)^\beta\right)^{1-2\alpha\eta}(1-t)^{\beta-1}\,dt \\
&= \frac{f'(a+b-m)}{n-m}-\frac{\beta^{2-2\alpha\eta}(2-2\alpha\eta)}{(n-m)^{2\beta-2\alpha\beta\eta+1}}\int_{a+b-n}^{a+b-m} f'(x)\frac{1}{[(a+b-m)-x]^{1-\beta}} \\
&\quad\times\left\{\frac{[(a+b-m)-(a+b-n)]^\beta-[(a+b-m)-x]^\beta}{\beta}\right\}^{1-2\alpha\eta}dx
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \int_0^1 (1-t)^\beta f'(a+b-[tm+(1-t)n])\,dt \\
&= \frac{-f(a+b-n)}{n-m}+\frac{\beta}{(n-m)^2}\int_0^1 f(a+b-[tm+(1-t)n])(1-t)^{\beta-1}\,dt \\
&= \frac{-f(a+b-n)}{n-m}+\frac{\beta}{(n-m)^{1+\beta}}\int_{a+b-n}^{a+b-m} f(x)\frac{1}{[(a+b-m)-x]^{1-\beta}}dx.
\end{aligned}
$$

Step 3. Multiplying I_1 by A and multiplying I_2 by B, we have

$$\begin{aligned}
J_1 &= AI_1 - BI_2 \\
&= \frac{(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta}}{4}\frac{f'(a+b-m)}{n-m} \\
&\quad -\frac{(1-\alpha\eta)^2\beta^{2-2\alpha\eta}(n-m)^{\alpha\beta\eta-1}}{2}\int_{a+b-n}^{a+b-m} f'(x)\frac{1}{[(a+b-m)-x]^{1-\beta}} \\
&\quad \times\left\{\frac{[(a+b-m)-(a+b-n)]^{\beta}-[(a+b-n)-x]^{\beta}}{\beta}\right\}^{1-2\alpha\eta}dx \\
&\quad +\frac{(\alpha\eta)^2\beta^{1-2\alpha\eta}(n-m)^{\beta+\alpha\beta\eta}}{2}\frac{f(a+b-n)}{n-m} \\
&\quad -\frac{(\alpha\eta)^2\beta^{2-2\alpha\eta}(n-m)^{\alpha\beta\eta-1}}{2}\int_{a+b-n}^{a+b-m} f(x)\frac{1}{[(a+b-m)-x]^{1-\beta}}dx.
\end{aligned}$$

Step 4. By the definition of ${}^{\mathrm{PCHC}}_{b^-}I^{\alpha,\beta,\eta}f(t)$ in Definition 1.6, we get

$$\begin{aligned}
J_1 &= \frac{(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta}}{4}\frac{f'(a+b-m)}{n-m}+\frac{(\alpha\eta)^2\beta^{1-2\alpha\eta}(n-m)^{\beta+\alpha\beta\eta}}{2}\frac{f(a+b-n)}{n-m} \\
&\quad -\frac{\beta^{2-2\alpha\eta}(n-m)^{\alpha\beta\eta-1}\Gamma(1-\alpha\eta)}{2}\,{}^{\mathrm{PCHC}}_{a+b-m^-}I^{\alpha,\beta,\eta}f(a+b-n).
\end{aligned}$$

Next, we show the right kernel terms.

Step 5. Integration by parts for the right kernel terms. Using the similar way, we have

$$\begin{aligned}
I_3 &= \int_0^1\left(1-t^{\beta}\right)^{2-2\alpha\eta} f''(a+b-[tm+(1-t)n])\,dt \\
&= \frac{-f'(a+b-n)}{n-m}+\frac{\beta(2-2\alpha\eta)}{n-m}\int_0^1 f'(a+b-[tm+(1-t)n])\left(1-t^{\beta}\right)^{1-2\alpha\eta}t^{\beta-1}dt
\end{aligned}$$

and

$$\begin{aligned}
I_4 &= \int_0^1 t^{\beta}f'(a+b-[tm+(1-t)n])\,dt \\
&= \frac{f(a+b-m)}{n-m}-\frac{\beta}{(n-m)^{1+\beta}}\int_0^1 f(a+b-[tm+(1-t)n])\,t^{\beta-1}dt.
\end{aligned}$$

Step 6. Using the change of the variable by $x=a+b-[tm+(1-t)n]$, we have

$$\begin{aligned}
I_3 &= \int_0^1\left(1-t^{\beta}\right)^{2-2\alpha\eta} f''(a+b-[tm+(1-t)n])\,dt \\
&= \frac{-f'(a+b-n)}{n-m}+\frac{\beta^{2-2\alpha\eta}(2-2\alpha\eta)}{(n-m)^{2\beta-2\alpha\beta\eta+1}}\int_{a+b-n}^{a+b-m} f'(x)\frac{1}{[x-(a+b-n)]^{1-\beta}} \\
&\quad \times\left\{\frac{[(a+b-m)-(a+b-n)]^{\beta}-[x-(a+b-n)]^{\beta}}{\beta}\right\}^{1-2\alpha\eta}dx
\end{aligned}$$

and

$$
\begin{aligned}
I_4 &= \int_0^1 t^\beta f'(a+b-[tm+(1-t)n])\,dt \\
&= \frac{f(a+b-m)}{n-m} - \frac{\beta}{(n-m)^{1+\beta}} \int_0^1 f(a+b-[tm+(1-t)n])\, t^{\beta-1} dt \\
&= \frac{f(a+b-m)}{n-m} - \frac{\beta}{(n-m)^{1+\beta}} \int_{a+b-n}^{a+b-m} f(x) \frac{1}{[x-(a+b-n)]^{1-\beta}} dx.
\end{aligned}
$$

Step 7. Multiplying I_3 by A and multiplying I_4 by B, we have

$$
\begin{aligned}
J_2 &= AI_3 - BI_4 \\
&= \frac{-(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta}}{4} \frac{f'(a+b-n)}{n-m} \\
&\quad + \frac{(1-\alpha\eta)^2 \beta^{2-2\alpha\eta} (n-m)^{\alpha\beta\eta-1}}{2} \int_{a+b-n}^{a+b-m} f'(x) \frac{1}{[x-(a+b-n)]^{1-\beta}} \\
&\quad \times \left\{ \frac{[(a+b-m)-(a+b-n)]^\beta - [x-(a+b-n)]^\beta}{\beta} \right\}^{1-2\alpha\eta} dx \\
&\quad - \frac{(\alpha\eta)^2 \beta^{1-2\alpha\eta} (n-m)^{\beta+\alpha\beta\eta}}{2} \frac{f(a+b-m)}{n-m} \\
&\quad + \frac{(\alpha\eta)^2 \beta^{2-2\alpha\eta} (n-m)^{\alpha\beta\eta-1}}{2} \int_{a+b-n}^{a+b-m} f(x) \frac{1}{[x-(a+b-n)]^{1-\beta}} dx.
\end{aligned}
$$

Step 8. By Definition 1.6 and the changes of variables established above, the remaining integral terms can be expressed in terms of the left- and right-sided PCHC integral operators. Hence, we obtain

$$
\begin{aligned}
J_2 &= \frac{-(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta}}{4} \frac{f'(a+b-n)}{n-m} \\
&\quad - \frac{(\alpha\eta)^2 \beta^{1-2\alpha\eta} (n-m)^{\beta+\alpha\beta\eta}}{2} \frac{f(a+b-m)}{n-m} \\
&\quad + \frac{(1-\alpha\eta)^2 \beta^{2-2\alpha\eta} (n-m)^{\alpha\beta\eta-1}\Gamma(1-\alpha\eta)}{2} \, {}^{\mathrm{PCHC}}_{a+b-n^+} I^{\alpha,\beta,\eta} f(a+b-m).
\end{aligned}
$$

Collecting the boundary terms obtained from J_1 and J_2, retaining their signs before combining the two identities, we obtain

$$
\begin{aligned}
J_1 - J_2 &= (\alpha\eta)^2 \beta^{1-2\alpha\eta} (n-m)^{\alpha\beta\eta+\beta-1} \left[\frac{f(a+b-n)+f(a+b-m)}{2} \right] \\
&\quad + (1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta-1} \left[\frac{f'(a+b-n)+f'(a+b-m)}{4} \right] \\
&\quad - \frac{\Gamma(1-\alpha\eta)\beta^{2-2\alpha\eta}}{2(n-m)^{1-\alpha\beta\eta}} \left[{}^{\mathrm{PCHC}}_{a+b-m^-} I^{\alpha,\beta,\eta} \{f(a+b-n)\} + {}^{\mathrm{PCHC}}_{a+b-n^+} I^{\alpha,\beta,\eta} \{f(a+b-m)\} \right].
\end{aligned}
$$

This completes the proof.

Corollary 2.1. *Taking the boundary limits $\beta \to 1$ and $\eta \to 1^-$ in Lemma 2.1 and setting $m = a$ and $n = b$, we obtain the following identity for the proportional Caputo-hybrid integral operators:*

$$\begin{aligned}
&\frac{(1-\alpha)(b-a)^{2-\alpha}}{4}\times\left[\int_0^1 t^{2-2\alpha} f''((1-t)a+tb)\,dt - \int_0^1 (1-t)^{2-2\alpha} f''((1-t)a+tb)\,dt\right]\\
&-\frac{\alpha^2(b-a)^{1+\alpha}}{2}\times\left[\int_0^1 (1-t) f'((1-t)a+tb)\,dt - \int_0^1 t f'((1-t)a+tb)\,dt\right]\\
= \;& \alpha^2 (b-a)^{\alpha}\left[\frac{f(a)+f(b)}{2}\right] + (1-\alpha)(b-a)^{1-\alpha}\left[\frac{f'(a)+f'(b)}{4}\right]\\
&-\frac{\Gamma(1-\alpha)}{2(b-a)^{1-\alpha}}\left[{}^{\mathrm{PCH}}_{b^-}I^{\alpha}\{f(a)\} + {}^{\mathrm{PCH}}_{a^+}I^{\alpha}\{f(b)\}\right].
\end{aligned}$$

Theorem 2.2. *Let $f : [a, b] \to \mathbb{R}$ be twice differentiable on (a, b), with $f', f'' \in L^1([a, b])$. Assume that $|f'|$ and $|f''|$ are convex on $[a, b]$. Let $a \le m < n \le b$, $\alpha \in (0, 1]$, $\beta \in (0, 1]$, $\eta > 0$, and $\alpha\eta \in (0, 1)$. Then the following inequality holds:*

$$\begin{aligned}
&\Bigg|(\alpha\eta)^2 \beta^{1-2\alpha\eta}(n-m)^{\alpha\beta\eta+\beta-1}\left[\frac{f(a+b-n)+f(a+b-m)}{2}\right]\\
&+(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta-1}\left[\frac{f'(a+b-n)+f'(a+b-m)}{4}\right]\\
&-\frac{\Gamma(1-\alpha\eta)\beta^{2-2\alpha\eta}}{2(n-m)^{1-\alpha\beta\eta}}\left[{}^{\mathrm{PCHC}}_{a+b-m^-}I^{\alpha,\beta,\eta}\{f(a+b-n)\} + {}^{\mathrm{PCHC}}_{a+b-n^+}I^{\alpha,\beta,\eta}\{f(a+b-m)\}\right]\Bigg|\\
\le\;& \frac{(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta}}{4}A_1\left\{[|f''(a)|+|f''(b)|] - \frac{|f''(m)|+|f''(n)|}{2}\right\}\\
&+\frac{(\alpha\eta)^2\beta^{1-2\alpha\eta}(n-m)^{\beta+\alpha\beta\eta}}{2}\left\{\frac{2[|f'(a)|+|f'(b)|]-[|f'(m)|+|f'(n)|]}{\beta+1}\right\},
\end{aligned}$$

where

$$A_1 = 2\int_0^1 \left(1-t^{\beta}\right)^{2-2\alpha\eta} dt = \frac{2}{\beta}B\left(\frac{1}{\beta}, 3-2\alpha\eta\right)$$

and $B(\cdot,\cdot)$ denotes the beta function.

Proof. By applying Lemma 2.1 and the Jensen-Mercer inequality, we obtain

$$\begin{aligned}
&\Bigg|(\alpha\eta)^2 \beta^{1-2\alpha\eta}(n-m)^{\alpha\beta\eta+\beta-1}\left[\frac{f(a+b-n)+f(a+b-m)}{2}\right]\\
&+(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta-1}\left[\frac{f'(a+b-n)+f'(a+b-m)}{4}\right]\\
&-\frac{\Gamma(1-\alpha\eta)\beta^{2-2\alpha\eta}}{2(n-m)^{1-\alpha\beta\eta}}\left[{}^{\mathrm{PCHC}}_{a+b-m^-}I^{\alpha,\beta,\eta}\{f(a+b-n)\} + {}^{\mathrm{PCHC}}_{a+b-n^+}I^{\alpha,\beta,\eta}\{f(a+b-m)\}\right]\Bigg|\\
\le\;& \frac{(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta}}{4}\times\left[\int_0^1 \left|\left(1-(1-t)^{\beta}\right)^{2-2\alpha\eta}\right| |f''(a+b-[tm+(1-t)n])|\,dt\right.\\
&\left.+\int_0^1 \left|\left(1-t^{\beta}\right)^{2-2\alpha\eta}\right| |f''(a+b-[tm+(1-t)n])|\,dt\right] + \frac{(\alpha\eta)^2\beta^{1-2\alpha\eta}(n-m)^{\beta+\alpha\beta\eta}}{2}
\end{aligned}$$

$$
\begin{aligned}
&\times\left[\int_0^1 \left|(1-t)^{\beta}\right| \left|f'\left(a+b-[tm+(1-t)n]\right)\right| dt + \int_0^1 \left|t^{\beta}\right| \left|f'\left(a+b-[tm+(1-t)n]\right)\right| dt\right] \\
\leq\ & \frac{(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta}}{4}\left\{\int_0^1 \left|\left(1-(1-t)^{\beta}\right)^{2-2\alpha\eta}\right| \right. \\
&\times\left[\left|f''(a)\right|+\left|f''(b)\right|-t\left|f''(m)\right|-(1-t)\left|f''(n)\right|\right] dt \\
&\left. +\int_0^1 \left|\left(1-t^{\beta}\right)^{2-2\alpha\eta}\right|\left[\left|f''(a)\right|+\left|f''(b)\right|-t\left|f''(m)\right|-(1-t)\left|f''(n)\right|\right] dt\right\} \\
&+\frac{(\alpha\eta)^2\beta^{1-2\alpha\eta}(n-m)^{\beta+\alpha\beta\eta}}{2}\left\{\int_0^1 \left|(1-t)^{\beta}\right| \times \left[\left|f'(a)\right|+\left|f'(b)\right|-t\left|f'(m)\right|-(1-t)\left|f'(n)\right|\right] dt\right. \\
&\left. +\int_0^1 \left|t^{\beta}\right|\left[\left|f'(a)\right|+\left|f'(b)\right|-t\left|f'(m)\right|-(1-t)\left|f'(n)\right|\right] dt\right\} \\
=\ & \frac{(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta}}{4}A_1\left\{\left[\left|f''(a)\right|+\left|f''(b)\right|\right]-\frac{\left|f''(m)\right|+\left|f''(n)\right|}{2}\right\} \\
&+\frac{(\alpha\eta)^2\beta^{1-2\alpha\eta}(n-m)^{\beta+\alpha\beta\eta}}{2}\left\{\left[\left|f'(a)\right|+\left|f'(b)\right|\right]B_1-\left|f'(m)\right|B_2-\left|f'(n)\right|B_3\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
B_1 &= \int_0^1 \left|(1-t)^{\beta}\right|+\left|t^{\beta}\right| dt = \frac{2}{\beta+1}, \\
B_2 &= \int_0^1 t\left[\left|(1-t)^{\beta}\right|+\left|t^{\beta}\right|\right] dt = \frac{1}{\beta+1}, \\
B_3 &= \int_0^1 (1-t)\left[\left|(1-t)^{\beta}\right|+\left|t^{\beta}\right|\right] dt = \frac{1}{\beta+1},
\end{aligned}
$$

so that

$$
\begin{aligned}
&\left|(\alpha\eta)^2\beta^{1-2\alpha\eta}(n-m)^{\alpha\beta\eta+\beta-1}\left[\frac{f(a+b-n)+f(a+b-m)}{2}\right]\right. \\
&+(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta-1}\left[\frac{f'(a+b-n)+f'(a+b-m)}{4}\right] \\
&\left.-\frac{\Gamma(1-\alpha\eta)\beta^{2-2\alpha\eta}}{2(n-m)^{1-\alpha\beta\eta}}\left[{}^{\mathrm{PCHC}}_{a+b-m^-}I^{\alpha,\beta,\eta}\left\{f(a+b-n)\right\} + {}^{\mathrm{PCHC}}_{a+b-n^+}I^{\alpha,\beta,\eta}\left\{f(a+b-m)\right\}\right]\right| \\
\leq\ & \frac{(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta}}{4}A_1\left\{\left[\left|f''(a)\right|+\left|f''(b)\right|\right]-\frac{\left|f''(m)\right|+\left|f''(n)\right|}{2}\right\} \\
&+\frac{(\alpha\eta)^2\beta^{1-2\alpha\eta}(n-m)^{\beta+\alpha\beta\eta}}{2}\left\{\frac{2\left[\left|f'(a)\right|+\left|f'(b)\right|\right]-\left[\left|f'(m)\right|+\left|f'(n)\right|\right]}{\beta+1}\right\}.
\end{aligned}
$$

This completes the proof.

Theorem 2.3. *Let $f:[a,b]\to\mathbb{R}$ be twice differentiable on (a,b), with $f', f''^1([a,b])$. Assume that $|f'^q$ and $|f''^q$ are convex on $[a,b]$, where $q>1$ and $p>1$ satisfy*

$$\frac{1}{p}+\frac{1}{q}=1.$$

Let $\alpha \in (0,1]$, $\beta \in (0,1]$, $\eta > 0$, and $0 < \alpha\eta < 1$. Then, for $a \leq m < n \leq b$, the following inequality holds:

$$\begin{aligned}
&\Bigg|(\alpha\eta)^2 \beta^{1-2\alpha\eta} (n-m)^{\alpha\beta\eta+\beta-1} \left[\frac{f(a+b-n)+f(a+b-m)}{2}\right] \\
&+(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta-1} \left[\frac{f'(a+b-n)+f'(a+b-m)}{4}\right] \\
&-\frac{\Gamma(1-\alpha\eta)\beta^{2-2\alpha\eta}}{2(n-m)^{1-\alpha\beta\eta}} \left[{}^{\mathrm{PCHC}}_{a+b-m^-}I^{\alpha,\beta,\eta}\{f(a+b-n)\} + {}^{\mathrm{PCHC}}_{a+b-n^+}I^{\alpha,\beta,\eta}\{f(a+b-m)\}\right]\Bigg| \\
\leq\ & \frac{(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta}}{4} \times \left[\left(\int_0^1 \left|\left(1-(1-t)^\beta\right)\right|^{(2-2\alpha\eta)p} dt\right)^{\frac{1}{p}} + \left(\int_0^1 \left|\left(1-t^\beta\right)\right|^{(2-2\alpha\eta)p} dt\right)^{\frac{1}{p}}\right] \\
&\times \left[|f''(a)|^q + |f''(b)|^q - \frac{|f''(m)|^q + |f''(n)|^q}{2}\right]^{\frac{1}{q}} + \frac{(\alpha\eta)^2 \beta^{1-2\alpha\eta}(n-m)^{\beta+\alpha\beta\eta}}{2} \\
&\times \left[\left(\int_0^1 \left|(1-t)^\beta\right|^p dt\right)^{\frac{1}{p}} + \left(\int_0^1 \left|t^\beta\right|^p dt\right)^{\frac{1}{p}}\right] \times \left[|f'(a)|^q + |f'(b)|^q - \frac{|f'(m)|^q + |f'(n)|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

Proof. Applying Lemma 2.1, Hölder's inequality, and Jensen-Mercer inequality successively, together with the convexity of $|f'|^q$ and $|f''|^q$, we obtain

$$\begin{aligned}
&\Bigg|(\alpha\eta)^2 \beta^{1-2\alpha\eta} (n-m)^{\alpha\beta\eta+\beta-1} \left[\frac{f(a+b-n)+f(a+b-m)}{2}\right] \\
&+(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta-1} \left[\frac{f'(a+b-n)+f'(a+b-m)}{4}\right] \\
&-\frac{\Gamma(1-\alpha\eta)\beta^{2-2\alpha\eta}}{2(n-m)^{1-\alpha\beta\eta}} \left[{}^{\mathrm{PCHC}}_{a+b-m^-}I^{\alpha,\beta,\eta}\{f(a+b-n)\} + {}^{\mathrm{PCHC}}_{a+b-n^+}I^{\alpha,\beta,\eta}\{f(a+b-m)\}\right]\Bigg| \\
\leq\ & \frac{(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta}}{4} \times \left[\left(\int_0^1 \left|\left(1-(1-t)^\beta\right)\right|^{(2-2\alpha\eta)p} dt\right)^{\frac{1}{p}} \left(\int_0^1 |f''(a+b-[tm+(1-t)n])|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 \left|\left(1-t^\beta\right)\right|^{(2-2\alpha\eta)p} dt\right)^{\frac{1}{p}} \left(\int_0^1 |f''(a+b-[tm+(1-t)n])|^q dt\right)^{\frac{1}{q}}\right] \\
&+\frac{(\alpha\eta)^2 \beta^{1-2\alpha\eta}(n-m)^{\beta+\alpha\beta\eta}}{2} \times \left[\left(\int_0^1 \left|(1-t)^\beta\right|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 |f'(a+b-[tm+(1-t)n])|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 \left|t^\beta\right|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 |f'(a+b-[tm+(1-t)n])|^q dt\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta}}{4} \Bigg\{\left(\int_0^1 \left|\left(1-(1-t)^\beta\right)\right|^{(2-2\alpha\eta)p} dt\right)^{\frac{1}{p}} \\
&\times \left(\int_0^1 \left[|f''(a)|^q + |f''(b)|^q - t|f''(m)|^q - (1-t)|f''(n)|^q\right] dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_0^1 \left|\left(1-t^{\beta}\right)\right|^{(2-2\alpha\eta)p} dt\right)^{\frac{1}{p}} \left(\int_0^1 \left[\left|f''(a)\right|^q+\left|f''(b)\right|^q-t\left|f''(m)\right|^q-(1-t)\left|f''(n)\right|^q\right] dt\right)^{\frac{1}{q}}\Bigg\}\\
&+\frac{(\alpha\eta)^2\beta^{1-2\alpha\eta}(n-m)^{\beta+\alpha\beta\eta}}{2}\left\{\left(\int_0^1 \left|(1-t)^{\beta}\right|^p dt\right)^{\frac{1}{p}}\right.\\
&\times\left(\int_0^1 \left[\left|f'(a)\right|^q+\left|f'(b)\right|^q-t\left|f'(m)\right|^q-(1-t)\left|f'(n)\right|^q\right] dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 \left|t^{\beta}\right|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left[\left|f'(a)\right|^q+\left|f'(b)\right|^q-t\left|f'(m)\right|^q-(1-t)\left|f'(n)\right|^q\right] dt\right)^{\frac{1}{q}}\Bigg\}\\
=\ &\frac{(1-\alpha\eta)(n-m)^{2\beta-\alpha\beta\eta}}{4}\times\left[\left(\int_0^1 \left|\left(1-(1-t)^{\beta}\right)\right|^{(2-2\alpha\eta)p} dt\right)^{\frac{1}{p}}+\left(\int_0^1 \left|\left(1-t^{\beta}\right)\right|^{(2-2\alpha\eta)p} dt\right)^{\frac{1}{p}}\right]\\
&\times\left[\left|f''(a)\right|^q+\left|f''(b)\right|^q-\frac{\left|f''(m)\right|^q+\left|f''(n)\right|^q}{2}\right]^{\frac{1}{q}}\\
&+\frac{(\alpha\eta)^2\beta^{1-2\alpha\eta}(n-m)^{\beta+\alpha\beta\eta}}{2}\times\left[\left(\int_0^1 \left|(1-t)^{\beta}\right|^p dt\right)^{\frac{1}{p}}+\left(\int_0^1 \left|t^{\beta}\right|^p dt\right)^{\frac{1}{p}}\right]\\
&\times\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q-\frac{\left|f'(m)\right|^q+\left|f'(n)\right|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}
$$

This completes the proof.

Limiting relation. At the boundary limit $\beta \to 1$ and $\eta \to 1^-$,

$$\lim_{\substack{\beta\to 1\\ \eta\to 1^-}} {}^{\mathrm{PCHC}}I^{\alpha,\beta,\eta} = {}^{\mathrm{PCH}}I^{\alpha},$$

and the PCHC operator tends to the proportional Caputo-hybrid operator. If, in addition, $m = a$ and $n = b$, the lower and upper bounds in Theorem 2.1 tend to those in Theorem 1.7 of [18]. Since $\eta < 1$, this is a boundary limiting relation rather than an admissible parameter specialization.

3. Numerical examples and bound-range comparisons

To compare the upper bounds in Theorems 1.4 and 2.2, we take

$$[a,b]=[0,1], \qquad m=\frac{1}{3}, \qquad n=\frac{2}{3}$$

and use

$$\alpha,\eta \in \{0.2, 0.4, 0.6, 0.8\}, \qquad \beta \in \{0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}.$$

Thus, 96 admissible parameter triples are examined for each test function. Figures 1 and 2 illustrate the corresponding upper-bound comparisons for the two test functions. In Tables 1 and 2, $\alpha = \eta = 0.8$ is fixed in order to display clearly how the two estimates vary as β increases. The difference $U_{2.2} - U_{1.4}$ is reported, so a negative value means that Theorem 2.2 provides the sharper upper bound.

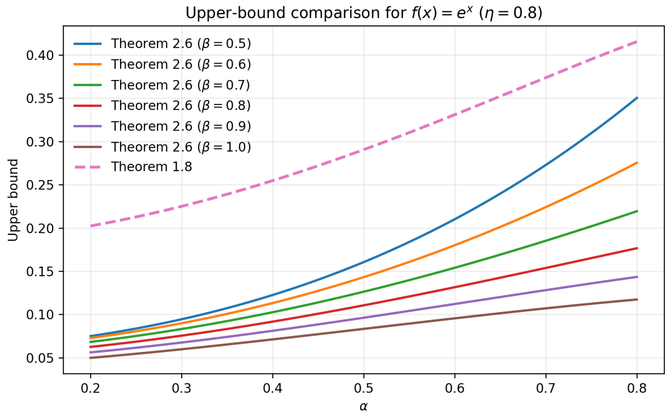

Figure 1. Upper-bound comparison for $f(x) = e^x$ with $\eta = 0.8$ and $\beta \in \{0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$.

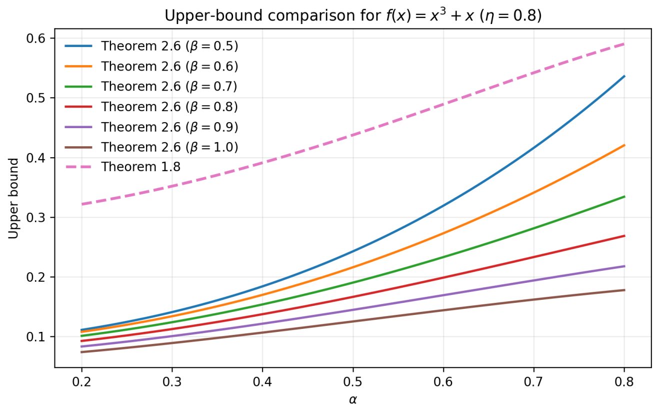

Figure 2. Upper-bound comparison for $f(x) = x^3 + x$ with $\eta = 0.8$ and $\beta \in \{0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$.

Table 1. Upper-bound comparison for $f(x) = e^x$ with $\alpha = \eta = 0.8$.

β	$U_{2.2}$	$U_{1.4}$	$U_{2.2} - U_{1.4}$	Sharper bound
0.5	0.35025656	0.41528482	−0.06502826	Theorem 2.2
0.6	0.27528961	0.41528482	−0.13999521	Theorem 2.2
0.7	0.21928596	0.41528482	−0.19599885	Theorem 2.2
0.8	0.17653377	0.41528482	−0.23875105	Theorem 2.2
0.9	0.14334701	0.41528482	−0.27193780	Theorem 2.2
1.0	0.11723712	0.41528482	−0.29804770	Theorem 2.2

Table 2. Upper-bound comparison for $f(x) = x^3 + x$ with $\alpha = \eta = 0.8$.

β	$U_{2.2}$	$U_{1.4}$	$U_{2.2} - U_{1.4}$	Sharper bound
0.5	0.53586737	0.59012373	−0.05425637	Theorem 2.2
0.6	0.42023052	0.59012373	−0.16989322	Theorem 2.2
0.7	0.33400985	0.59012373	−0.25611389	Theorem 2.2
0.8	0.26832206	0.59012373	−0.32180168	Theorem 2.2
0.9	0.21743574	0.59012373	−0.37268800	Theorem 2.2
1.0	0.17748329	0.59012373	−0.41264044	Theorem 2.2

Example 3.1. Let $f(x) = e^x$ on $[0, 1]$. Then $f'(x) = f''(x) = e^x$, so $|f'|$ and $|f''|$ are convex on $[0, 1]$.

For the representative setting $\alpha = \eta = 0.8$, Theorem 2.2 gives a smaller upper bound for every displayed value of β. Its advantage increases as β increases. On the complete 96-point parameter grid, Theorem 2.2 is sharper at all 96 points; no equality or parameter choice favoring Theorem 1.4 occurs.

Example 3.2. Let $f(x) = x^3 + x$ on $[0, 1]$. Then $f'(x) = 3x^2 + 1$ and $f''(x) = 6x$, and hence $|f'|$ and $|f''|$ are convex on $[0, 1]$.

The same pattern is observed for $f(x) = x^3 + x$. At $\alpha = \eta = 0.8$, Theorem 2.2 is sharper for all six displayed values of β. Across the complete 96-point parameter grid, Theorem 2.2 again gives the smaller upper bound at every point. Consequently, for both test functions and all parameter choices investigated here, the PCHC estimate from Theorem 2.2 is numerically sharper than the proportional Caputo-hybrid estimate from Theorem 1.4. This conclusion concerns the tested parameter grid and is not asserted as a universal dominance result.

4. Conclusions

In this paper, we introduced a new PCHC fractional integral operator that unifies proportional Caputo-hybrid and conformable fractional structures within a single framework. Based on this operator, we established a fundamental integral identity and derived several Hermite-Hadamard-Mercer-type inequalities under various convexity assumptions.

Compared with the existing approaches, the proposed framework involves three adjustable parameters, namely α, β, and η. These parameters provide additional flexibility in controlling the kernel structure and the corresponding inequality bounds.

Furthermore, the joint boundary limits $\beta \to 1$ and $\eta \to 1^-$ establish a precise connection between the proposed PCHC framework and the proportional Caputo-hybrid results available in the literature. Since the admissible parameter range requires $\eta < 1$, this connection must be understood as a limiting relation rather than as a direct admissible parameter specialization. In particular, after taking these boundary limits and setting $m = a$ and $n = b$, the corresponding result reduces to Theorem 1.7 of [18].

Numerical examples were also presented to compare the upper bounds in Theorems 1.4 and 2.2 and to examine the influence of the parameters on the resulting estimates. For both $f(x) = e^x$ and $f(x) = x^3 + x$, Theorem 2.2 provides the smaller upper bound at all 96 investigated parameter triples. The numerical advantage becomes more pronounced as β increases in the representative comparisons. These observations show that the additional PCHC parameters can produce sharper estimates under the selected settings, although the numerical evidence is not intended as a proof of universal dominance.

Future research may extend the proposed approach to other classes of generalized convex functions and related Mercer-, midpoint-, trapezoidal-, and other integral inequalities. Further applications to fractional differential and integral equations may also be investigated.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

References

1. M. Z. Sarikaya, E. Set, H. Yaldiz, N. Basak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.*, **57** (2013), 2403–2407. https://doi.org/10.1016/j.mcm.2011.12.048
2. G. Farid, S. B. Akbar, L. Rathour, L. N. Mishra, V. N. Mishra, Riemann-Liouville fractional versions of Hadamard inequality for strongly *m*-convex functions, *Int. J. Anal. Appl.*, **20** (2022), 5. https://doi.org/10.28924/2291-8639-20-2022-5
3. A. Kashuri, Hermite-Hadamard type inequalities for the ABK-fractional integrals, *J. Comput. Anal. Appl.*, **29** (2021), 309–326.
4. M. Karim, A. Fahmi, Z. Ullah, M. A. T. Bhatti, A. Qayyum, On certain Ostrowski type integral inequalities for convex function via AB-fractional operator, *AIMS Math.*, **8** (2023), 9166–9184. https://doi.org/10.3934/math.2023459
5. A. Atangana, D. Baleanu, New fractional derivatives with non-local and non-singular kernel: theory and applications to heat transfer model, *Thermal Sci.*, **20** (2016), 763–769.
6. T. Abdeljawad, D. Baleanu, Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel, *J. Nonlinear Sci. Appl.*, **10** (2017), 1098–1107. https://doi.org/10.22436/jnsa.010.03.20
7. E. Set, A. O. Akdemir, A. Karaoĝlan, T. Abdeljawad, W. Shatanawi, On new generalizations of Hermite-Hadamard type inequalities via Atangana-Baleanu fractional integral operator, *Axioms*, **10** (2021), 223. https://doi.org/10.3390/axioms10030223
8. E. Set, S. I. Butt, A. O. Akdemir, A. Karaoglan, T. Abdeljaward, New integral inequalities for differentiable convex functions via Atangana-Baleanu fractional integral operators, *Chaos Solitons Fract.*, **143** (2021), 110554. https://doi.org/10.1016/j.chaos.2020.110554
9. S. I. Butt, P. Agarwal, J. J. Neito, New Hadamard-Mercer inequalities pertaining Atangana-Baleanu operator in Katugampola sense with applications, *Mediterr. J. Math.*, **21** (2024), 9. https://doi.org/10.1007/s00009-023-02547-3
10. J. B. Jiu, S. I. Butt, J. Nasir, A. Aslam, A. Fahad, J. Soontharanon, Jensen-Mercer variant of Hermite-Hadamard type inequalities via Atangana-Baleanu fractional operator, *AIMS Math.*, **7** (2022), 2123–2141. https://doi.org/10.3934/math.2022121

11. M. Tariq, H. Ahmad, S. K. Sahoo, A. Kashuri, T. A. Nofal, C. H. Hsu, Inequalities of Simpson-Mercer-type including Atangana-Baleanu fractional operators and their applications, *AIMS Math.*, **7** (2022), 15159–15181. https://doi.org/10.3934/math.2022831
12. Y. Long, X. M. Yuan, T. S. Du, Simpson-like inequalities for functions whose third derivatives belong to s-convexity involving Atangana-Baleanu fractional integrals and their applications, *Filomat*, **38** (2024), 9373–9397.
13. S. I. Butt, E. Set, S. Yousaf, T. Abdeljawad, W. Shatanawi, Generalized integral inequalities for ABK-fractional integral operators, *AIMS Math.*, **6** (2021), 10164–10191. https://doi.org/10.3934/math.2021589
14. S. Kermausuor, E. R. Nwaeze, New fractional integral inequalities via k-Atangana-Baleanu fractional integral operators, *Fractal Fract.*, **7** (2023), 740. https://doi.org/10.3390/fractalfract7100740
15. S. Kermausuor, D. Denu, P. O. Olanipekun, ψ-Hilfer-Atangana-Baleanu fractional integral operators and some fractional integral inequalities, *Kragujevac J. Math.*, **50** (2026), 1585–1604.
16. M. Caputo, Linear model of dissipation whose Q is almost frequency independent–II, *Geophys. J. Int.*, **13** (1967), 529–539. https://doi.org/10.1111/j.1365-246X.1967.tb02303.x
17. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.*, **1** (2015), 73–85.
18. M. Z. Sarikaya, On Hermite-Hadamard type inequalities for proportional Caputo-hybrid operator, *Konuralp J. Math.*, **11** (2023), 31–39.
19. İ Demir, A novel method for Milne-type inequalities using proportional Caputo-hybrid operator, *Miskolc Math. Notes*, **26** (2025), 701–725. https://doi.org/10.18514/MMN.2025.4903
20. Y. Mahajan, H. Nagar, Fractional Newton-type integral inequalities for the Caputo fractional operator, *Math. Methods Appl. Sci.*, **48** (2025), 5244–5254. https://doi.org/10.1002/mma.10600
21. F. Jarad, E. Ugurlu, T. Abdeljawad, D. Baleanu, On a new class of fractional operators, *Adv. Differ. Equ.*, **2017** (2017), 247. https://doi.org/10.1186/s13662-017-1306-z
22. R. Ying, A. Lakhdari, H. Y. Xu, W. Saleh, B. Meftah, On conformable fractional Milne-type inequalities, *Symmetry*, **16** (2024), 196. https://doi.org/10.3390/sym16020196
23. I. A. Baloch, A. A. Mughal, Y. M. Chu, A. U. Haq, M. De La Sen, A variant of Jensen-type inequality and related results for harmonic convex functions, *AIMS Math.*, **5** (2020), 6404–6418. https://doi.org/10.3934/math.2020412
24. F. F. Shi, M. A. Ali, W. Liu, D. D. Gao, Strengthened version of fractional Hermite-Hadamard-Mercer inequalities for harmonically convex functions based on Jensen-Mercer inequality, *Fractals*, **34** (2026), 2550119. https://doi.org/10.1142/S0218348X25501191
25. H. Ahmad, M. Tariq, S. K. Sahoo, J. Baili, C. Cesarano, New estimations of Hermite-Hadamard type integral inequalities for special functions, *Fractal Fract.*, **5** (2021), 144. https://doi.org/10.3390/fractalfract5040144