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Research article

Lie group classification and invariant solutions of a class of quasilinear evolution equations with nonlinear gradient-power flux and state-dependent diffusivity

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Abstract: In this paper, we perform a Lie group classification of the quasilinear evolution equation $u_t = (\Phi(u)(u_x)^n)_x + F(u)$, $n \neq -1, 0, 1$, where $\Phi(u)$ is a state-dependent diffusivity and $F(u)$ is an arbitrary reaction term. The kernel Lie algebra admitted for arbitrary nonconstant $\Phi(u)$ and $F(u)$ is identified. All symmetry extensions beyond the kernel algebra are then classified with respect to the arbitrary elements, and the corresponding infinitesimal generators are explicitly constructed. For each classified symmetry algebra, one-dimensional optimal systems of subalgebras are obtained and used to derive similarity reductions of the governing equation. Several reduced ordinary differential equations are solved explicitly, leading to exact invariant solutions for selected nonlinear models in the classification. In addition, conservation laws are constructed by the multiplier method for the admissible reaction case, yielding explicit conserved vectors.

Keywords: Lie symmetry analysis; group classification; quasilinear diffusion equations; invariant solutions; conservation laws
Mathematics Subject Classification: 35K55, 35A30, 35C06, 58J70

1. Introduction and motivation

The classical heat equation

$$u_t = u_{xx}, \tag{1.1}$$

has long served as a prototype in the theory of parabolic partial differential equations (PDEs) and in the development of symmetry analysis. Its linear structure gives rise to a rich invariance algebra involving translations, scalings, Galilean transformations, and an infinite-dimensional Abelian ideal associated with the superposition principle. Detailed descriptions of the point-symmetry pseudogroup of the heat equation, including discrete and pseudo-discrete transformations, were obtained in [1]. The heat equation also motivated major developments in nonclassical and conditional symmetry analysis, beginning with the work of Bluman and Cole [2] and later extended by Fushchych et al. [3], and Popovych [4]. Its conservation laws and adjoint structures have likewise been extensively investigated; see, for example, Popovych et al. [5], and the direct multiplier method of Anco and Bluman [6], and Ibragimov's theory of nonlinear self-adjointness [7]. These developments establish the heat equation as a central reference point for symmetry methods applied to evolution equations.

Nonlinear diffusion equations, however, exhibit substantially richer and more delicate structures. Nonlinear constitutive laws usually destroy the infinite-dimensional symmetry algebra associated with linear superposition, leaving finite-dimensional symmetry algebras whose form depends strongly on the nonlinear diffusion and reaction terms. Such equations arise naturally in nonlinear heat transfer, filtration in porous media, population dynamics, diffusion in heterogeneous materials, nonlinear transport, non-Newtonian fluid mechanics, and anomalous diffusion. In many of these models, the diffusivity depends on the state variable, while the flux may also depend nonlinearly on the spatial gradient.

Motivated by these considerations, we study the nonlinear diffusion-reaction equation

$$u_t = \left(\Phi(u)(u_x)^n\right)_x + F(u), \qquad n \neq -1,\ 0,\ 1, \tag{1.2}$$

or, equivalently

$$u_t = n\,\Phi(u)\,(u_x)^{n-1}\,u_{xx} + \Phi'(u)\,(u_x)^{n+1} + F(u). \tag{1.3}$$

Here, $\Phi(u) \in C^2$ is a state-dependent diffusivity, $(u_x)^n$ represents a nonlinear gradient-power flux, and $F(u) \in C^1$ is a reaction term modeling internal sources, sinks, growth, decay, or interaction effects. Thus, Eq (1.2) combines three nonlinear mechanisms: The dependence of diffusivity on the state variable, nonlinear gradient-driven transport, and reaction dynamics.

The integer exponent n controls the strength and type of the gradient-dependent diffusion. Depending on its value, the model may describe degenerate diffusion, fast diffusion, nonlinear flux amplification, or strongly nonlinear transport effects. Such gradient-power structures occur in non-Newtonian transport theory, turbulent diffusion, nonlinear conductivity, generalized filtration, and interface propagation. The additional dependence of Φ on u allows the medium to modify its transport properties dynamically as the solution evolves. Therefore, Eq (1.2) extends the standard heat equation, porous medium-type models, and several gradient-dependent diffusion equations. At the same time, the simultaneous presence of an arbitrary $\Phi(u)$, nonlinear flux power $(u_x)^n$, and an arbitrary reaction $F(u)$ places the model beyond the standard porous medium and p-Laplacian frameworks. Throughout this paper, we assume that the nonlinear gradient term $(u_x)^n$ is well defined and no singular powers of u_x arise.

From the viewpoint of Lie symmetry analysis, Eq (1.2) leads to a genuine group classification problem. For generic Φ and F, the Lie symmetry algebra is expected to be small, usually consisting only of translations in t and x. Symmetry extensions occur only for special forms of the arbitrary elements or for special parameter regimes. The restriction $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$ is imposed to exclude, respectively, the classical linear diffusion regime, first-order degeneration, and a singular case in which the determining equations change structure.

The Lie symmetry method provides a systematic way to analyze such equations. One begins with a general infinitesimal generator, applies its prolongation to the differential equation, and splits the resulting invariance condition with respect to independent derivatives. This yields an overdetermined system for the infinitesimal coefficients and the arbitrary elements. In equations containing arbitrary functions, equivalence transformations are essential: They identify transformations preserving the class of equations and allow redundant subclasses to be normalized.

Analytical symmetry classification serves a different purpose from numerical and data-driven approaches. Lie group methods identify admissible point transformations, symmetry extensions, and exact similarity reductions at the level of the differential equation itself, whereas numerical schemes and artificial intelligence (AI)-assisted methods are primarily designed for approximation, data-driven discovery, or symbolic assistance. In the present work, we focus on the complete Lie point-symmetry classification of the local quasilinear diffusion-reaction class (1.2).

Group classification of nonlinear diffusion equations has been widely studied. Cherniha et al. [8] classified equations of the form

$$u_t = D(u_x)u_{xx} + Q(u),$$

including cases admitting infinite-dimensional symmetry algebras and linearization by hodograph transformations. Opanasenko et al. [9] refined such classifications using furcate splitting and generalized equivalence transformations. Connections between symmetry algebras and exact invariant solutions for nonlinear diffusion equations were also developed by Galaktionov and Svirshchevskii [10]. These works provide an important foundation for the present analysis.

Despite this progress, diffusion-reaction equations involving both state-dependent diffusivity and nonlinear gradient-power flux have received comparatively limited attention from the viewpoint of complete Lie group classification. In particular, a systematic classification of Eq (1.2) for the arbitrary smooth functions $\Phi(u)$ and $F(u)$ appears to be absent from the literature. The coexistence of the functions Φ and F, together with the exponent n, leads to a determining system that is more involved than those arising in many standard nonlinear diffusion models.

The aim of this paper is to fill this gap. We derive the determining equations for Eq (1.2), identify the kernel Lie algebra, construct the equivalence transformations of the class, and perform a complete group classification with respect to the arbitrary elements $\Phi(u)$ and $F(u)$. We then classify relevant low-dimensional subalgebras, derive invariant reductions and exact solutions, and investigate conservation laws using the multiplier method. In this way, the symmetry framework developed for classical heat-type equations is extended to a broader class of doubly nonlinear diffusion-reaction models.

Fractional differential models provide a complementary framework for describing transport processes involving nonlocal interactions, long-range dependence, or memory effects. In this direction, Min and Guo [11] investigated a fractional advection-dispersion equation involving both instantaneous and non-instantaneous impulses. Their model is based on the symmetric fractional flux

$$Q_\gamma[u](s) = \frac{1}{2}\,{}_0D_s^{-\gamma}(u'(s)) + \frac{1}{2}\,{}_sD_T^{-\gamma}(u'(s)), \qquad 0 \le \gamma < 1,$$

and the equation

$$-\frac{d}{ds}Q_\gamma[u](s) + h(s)u(s) = \lambda f_j(s, u(s)), \qquad s \in (t_j, s_{j+1}].$$

The instantaneous effects are described by jump conditions of the form

$$\Delta Q_\gamma[u](s_j) = I_j(u(s_j)),$$

whereas the noninstantaneous effects act over prescribed subintervals. Using a variational formulation and critical-point methods, the authors established, under suitable growth assumptions, the conditions for the existence and nonexistence of nontrivial solutions. This work illustrates the broader role of fractional and impulsive mechanisms in nonlinear transport models.

2. Determining equations and kernel Lie algebra

In this section, we derive the determining equations associated with the Lie point symmetries of the class

$$u_t = \left(\Phi(u)(u_x)^n\right)_x + F(u), \qquad n \neq -1, 0, 1.$$

Starting from a general infinitesimal generator, we compute the relevant first-and second-order prolongation coefficients and impose the standard infinitesimal invariance criterion. The resulting identity is then restricted to the solution manifold and split with respect to the independent derivative monomials. This procedure yields the full determining system for the infinitesimals. We then reduce this system to its essential classifying equations and identify the kernel Lie algebra admitted by every equation in the class.

We consider the general Lie point symmetry generator

$$X = \xi(x,t,u)\,\partial_x + \tau(x,t,u)\,\partial_t + \eta(x,t,u)\,\partial_u$$

with the characteristic

$$Q = \eta - \xi u_x - \tau u_t.$$

The first prolongation coefficients are

$$\eta^x = D_x(Q) + \xi u_{xx} + \tau u_{xt}, \qquad \eta^t = D_t(Q) + \xi u_{xt} + \tau u_{tt},$$

while the second prolongation coefficient is

$$\eta^{xx} = D_x(\eta^x) - u_{xx} D_x(\xi) - u_{xt} D_x(\tau).$$

Introduce

$$\Delta = u_t - n\Phi(u)(u_x)^{n-1} u_{xx} - \Phi'(u)(u_x)^{n+1} - F(u).$$

The invariance criterion is

$$\mathrm{pr}^{(2)}\, X(\Delta)\Big|_{\Delta=0} = 0.$$

Substituting the restricted prolongation coefficients into the invariance condition and splitting with respect to the independent derivative monomials, we obtain the determining equations

$$
\begin{aligned}
e_1 &: \tau_x = 0,\\
e_2 &: \tau_u = 0,\\
e_3 &: \eta_x = 0,\\
e_4 &: \xi_u = 0,\\
e_5 &: \xi_t = 0,
\end{aligned}
$$

$$
\begin{aligned}
&e_6 : \xi_{xx} = 0,\\
&e_7 : \Phi\tau_t + \Phi'(u)\eta + (n-1)\Phi\eta_u - (n+1)\Phi\xi_x = 0,\\
&e_8 : ((n+1)\xi_x - \tau_t - n\eta_u)\Phi'(u) - \eta\Phi''(u) - n\Phi(u)\eta_{uu} = 0,\\
&e_9 : \eta_t + F(u)(\eta_u - \tau_t) - \eta F'(u) = 0.
\end{aligned}
$$

Equivalently, we have

$$\tau = \tau(t), \qquad \xi = \xi(x), \qquad \eta = \eta(t,u),$$

and from e_6, we have

$$\xi = c_1 x + c_0.$$

Hence, the infinitesimal generator takes the form

$$X = \tau(t)\partial_t + (c_1 x + c_0)\partial_x + \eta(t,u)\partial_u. \tag{2.1}$$

Substituting $\xi_x = c_1$ into e_7 and e_8, the reduced classifying system becomes

$$\Phi\tau_t + \Phi'(u)\eta + (n-1)\Phi\eta_u - (n+1)c_1\Phi = 0, \tag{2.2}$$

$$\eta_t + F(u)(\eta_u - \tau_t) - \eta F'(u) = 0, \tag{2.3}$$

$$((n+1)c_1 - \tau_t - n\eta_u)\Phi'(u) - \eta\Phi''(u) - n\Phi(u)\eta_{uu} = 0. \tag{2.4}$$

These are the determining equations for the Lie point symmetries of (1.2).

Differentiating (2.2) with respect to u gives

$$\Phi'\tau_t + \Phi''\eta + \Phi'\eta_u + (n-1)\Phi'\eta_u + (n-1)\Phi\eta_{uu} - (n+1)c_1\Phi' = 0.$$

Hence

$$\eta\Phi'' - ((n+1)c_1 - \tau_t - n\eta_u)\Phi' + (n-1)\Phi\eta_{uu} = 0. \tag{2.5}$$

On the other hand, (2.4) is equivalent to

$$\eta\Phi'' - ((n+1)c_1 - \tau_t - n\eta_u)\Phi' + n\Phi\eta_{uu} = 0. \tag{2.6}$$

Subtracting (2.5) from (2.6), we obtain

$$\Phi\eta_{uu} = 0.$$

Assuming $\Phi(u) \neq 0$, this gives

$$\eta_{uu} = 0. \tag{2.7}$$

Therefore

$$\eta(t,u) = a(t)u + b(t). \tag{2.8}$$

So, any admissible infinitesimal generator must have the reduced form

$$X = \tau(t)\partial_t + (c_1 x + c_0)\partial_x + (a(t)u + b(t))\partial_u.$$

Remark 1. *The splitting is justified by factoring the common nonzero term $n\Phi(u)(u_x)^{n-2}$ from the u_{xx} terms. Since $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$, $\Phi(u) \neq 0$, and the calculation is carried out on a regular branch where $u_x \neq 0$, no additional exceptional cases arise. The remaining powers $1, u_x, (u_x)^n$, and $(u_x)^{n+1}$ are distinct and linearly independent as functions of u_x. Hence, the special values $n = -2, 2, 3$ do not yield extra splittings, while $n = -1, 0, 1$ are genuinely degenerate and are excluded from the present classification.*

Proposition 1 (Kernel Lie algebra)**.** *Consider the class*

$$u_t = \left(\Phi(u)(u_x)^n\right)_x + F(u), \qquad n \neq -1, 0, 1,$$

where $\Phi(u) \neq 0$, $\Phi'(u) \not\equiv 0$, and $F'(u) \not\equiv 0$. Then the kernel Lie symmetry algebra of the class is

$$\mathfrak{g}^{\text{ker}} = \langle \partial_t, \partial_x \rangle.$$

Proof. Substituting

$$\tau = \tau(t), \qquad \xi = c_1 x + c_0, \qquad \eta = a(t)u + b(t),$$

into the determining equations gives the classifying equations

$$\left(a(t)u + b(t)\right)\Phi'(u) + \left((n-1)a(t) + \tau_t - (n+1)c_1\right)\Phi(u) = 0,$$

and

$$a'(t)\,u + b'(t) + \left(a(t) - \tau_t\right)F(u) - \left(a(t)u + b(t)\right)F'(u) = 0.$$

For a symmetry to belong to the kernel algebra, these identities must hold for every admissible pair (Φ, F). Since Φ and F are arbitrary nonconstant functions, this forces

$$a(t) = 0, \qquad b(t) = 0, \qquad c_1 = 0, \qquad \tau_t = 0.$$

Hence

$$\tau = c_2, \qquad \xi = c_0, \qquad \eta = 0.$$

Therefore, the kernel algebra is

$$\mathfrak{g}^{\text{ker}} = \langle \partial_t, \partial_x \rangle.$$

□

3. Group classification

In this section, we perform the group classification of the class

$$u_t = \left(\Phi(u)(u_x)^n\right)_x + F(u), \qquad n \neq -1, 0, 1,$$

with respect to the arbitrary elements $\Phi(u)$ and $F(u)$. Since the kernel algebra is

$$\mathfrak{g}^{\text{ker}} = \langle \partial_t, \partial_x \rangle,$$

the classification problem consists of identifying all nonconstant pairs (Φ, F) for which the admitted Lie algebra is strictly larger than the kernel algebra. We therefore solve the classifying equations for Φ and F, separating the analysis according to the possible forms of the infinitesimal coefficient η. This leads to two principal branches, namely power-type diffusivities and exponential-type diffusivities, together with several resonant and affine subcases.

Theorem 1 (Group classification)**.** *Consider the class of nonlinear evolution equations*

$$u_t = \left(\Phi(u)(u_x)^n\right)_x + F(u), \qquad n \neq -1, 0, 1,$$

where $\Phi'(u) \not\equiv 0$, and $F'(u) \not\equiv 0$. Then, up to the equivalence group $G^{\sim}$, the equation admits Lie point symmetries beyond the kernel algebra only in the cases listed below. Here, each ε_i denotes a normalized nonzero sign, $\varepsilon_i = \pm 1$ that may be further reduced under equivalence transformations.
Case I: Power diffusivity.

$$\Phi(u) = \varepsilon_1 u^m, \qquad m \neq 0.$$

Case I-(1) *If*

$$F(u) = \varepsilon_2 u^r, \qquad r \neq 0, 1,$$

then

$$X_3 = (1-r)(n+1)t\partial_t + (m+n-r)x\partial_x + (n+1)u\partial_u.$$

Case I-(2) *If*

$$F(u) = \varepsilon_2\, u^{m+n} - \frac{\varepsilon_3}{m+n-1}\, u, \qquad m+n \neq 1,$$

then

$$X_3 = e^{\varepsilon_3 t}\,\partial_t - \frac{\varepsilon_3}{m+n-1}\, e^{\varepsilon_3 t}\, u\,\partial_u.$$

Case I-(3) *If*

$$m+n = 1, \qquad F(u) = \varepsilon_2 u \ln|u|,$$

then

$$X_3 = e^{\varepsilon_2 t} u\partial_u.$$

Case I-(4) *If*

$$F(u) = \varepsilon_2 u, \qquad m+n \neq 1,$$

then the equation admits

$$X_3 = \frac{m+n-1}{n+1} x\partial_x + u\partial_u,$$

and

$$X_4 = e^{\varepsilon_2(1-m-n)t}\,\partial_t + \varepsilon_2\, e^{\varepsilon_2(1-m-n)t}\, u\,\partial_u.$$

Case I-(5) *If*

$$F(u) = \varepsilon_2 u, \qquad m+n = 1,$$

then the equation admits

$$X_3 = u\partial_u,$$

and

$$X_4 = (n+1)t\partial_t + x\partial_x + \varepsilon_2(n+1)tu\partial_u.$$

Case II: Exponential diffusivity

$$\Phi(u) = \varepsilon_1 e^u.$$

Case II-(1) *If*

$$F(u) = \varepsilon_2 e^{qu}, \qquad q \neq 0,$$

then

$$X_3 = -qt\partial_t + \frac{1-q}{n+1}x\partial_x + \partial_u.$$

Case II-(2) *If*

$$F(u) = \varepsilon_2 e^u - \varepsilon_3,$$

then

$$X_3 = e^{\varepsilon_3 t}\partial_t - \varepsilon_3 e^{\varepsilon_3 t}\partial_u.$$

For all other nonconstant choices of $\Phi(u)$ *and* $F(u)$*, the maximal Lie invariance algebra coincides with the kernel Lie symmetry algebra.*

Proof. Let

$$X = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$$

be a Lie point symmetry generator. From the determining equations obtained in the preceding section, we have

$$\tau = \tau(t), \qquad \xi = c_1 x + c_0, \qquad \eta = a(t)u + b(t). \tag{3.1}$$

Substituting (3.1) into the reduced determining equations yields the two classifying equations

$$(a(t)u + b(t))\Phi'(u) + ((n-1)a(t) + \tau_t - (n+1)c_1)\Phi(u) = 0, \tag{3.2}$$

and

$$a'(t)u + b'(t) + (a(t) - \tau_t)F(u) - (a(t)u + b(t))F'(u) = 0. \tag{3.3}$$

We determine all nontrivial solutions of (3.2) and (3.3) and split according to whether $a(t)$ vanishes identically.

Case I: $a(t) \not\equiv 0$**.**

Assume $a(t) \neq 0$ on an interval. Dividing (3.2) by $a(t)$, we get

$$\left(u + \frac{b(t)}{a(t)}\right)\Phi'(u) + \left(n - 1 + \frac{\tau_t - (n+1)c_1}{a(t)}\right)\Phi(u) = 0.$$

Since $\Phi = \Phi(u)$, the coefficients must be independent of t. Hence, there exist constants λ and m such that

$$\frac{b(t)}{a(t)} = \lambda, \qquad n - 1 + \frac{\tau_t - (n+1)c_1}{a(t)} = -m.$$

Thus

$$b(t) = \lambda a(t), \qquad \tau_t = (n+1)c_1 - (m+n-1)a(t),$$

and

$$(u+\lambda)\Phi'(u) - m\Phi(u) = 0.$$

Therefore

$$\Phi(u) = \Phi_0(u+\lambda)^m, \qquad \Phi_0 \neq 0.$$

Substituting

$$b(t) = \lambda a(t), \qquad \tau_t = (n+1)c_1 - (m+n-1)a(t)$$

into (3.3), and using

$$\eta = a(t)(u+\lambda),$$

we obtain

$$a'(t)(u+\lambda) + \big((m+n)a(t) - (n+1)c_1\big)F(u) - a(t)(u+\lambda)F'(u) = 0.$$

Dividing by $a(t) \neq 0$, this becomes

$$(u+\lambda)F'(u) - \left(m+n-\frac{(n+1)c_1}{a(t)}\right)F(u) = \frac{a'(t)}{a(t)}(u+\lambda). \tag{3.4}$$

Set

$$\sigma(t) = \frac{(n+1)c_1}{a(t)}, \qquad \rho(t) = \frac{a'(t)}{a(t)}.$$

Then

$$(u+\lambda)F'(u) - (m+n-\sigma(t))F(u) = \rho(t)(u+\lambda).$$

Differentiating with respect to t gives

$$\sigma'(t)F(u) - \rho'(t)(u+\lambda) = 0.$$

Case I-(1): $F(u)$ is not linearly proportional to $u+\lambda$.

Then $F(u)$ and $u+\lambda$ are linearly independent, and therefore

$$\sigma'(t) = 0, \qquad \rho'(t) = 0.$$

Hence

$$\sigma(t) = \sigma_0, \qquad \rho(t) = k.$$

Case I-(1a): $c_1 \neq 0$.

Since

$$\sigma(t) = \frac{(n+1)c_1}{a(t)} = \sigma_0$$

and $c_1 \neq 0$, we get

$$a(t) = a_0, \qquad \rho(t) = 0.$$

Equation (3.4) reduces to

$$(u+\lambda)F'(u) - rF(u) = 0, \qquad r = m+n-\sigma_0.$$

Therefore

$$F(u) = F_0(u+\lambda)^r.$$

The assumption that F is not linearly proportional to $u+\lambda$ gives $r \neq 1$. Moreover

$$c_1 = \frac{a_0(m+n-r)}{n+1}, \qquad \tau_t = a_0(1-r).$$

After the normalization $a_0 = 1$, the corresponding additional generator is

$$X_3 = (1 - r)t\partial_t + \frac{m + n - r}{n + 1} x\partial_x + (u + \lambda)\partial_u.$$

This gives Case I-(1) of the theorem.

Case I-(1b): $c_1 = 0$.

Then $\sigma(t) = 0$, and

$$(u + \lambda)F'(u) - (m + n)F(u) = \rho(t)(u + \lambda).$$

Since $\rho(t) = k$, we have

$$a(t) = a_0 e^{kt}.$$

Thus F satisfies

$$(u + \lambda)F'(u) - (m + n)F(u) = k(u + \lambda).$$

If

$$m + n \neq 1,$$

then

$$F(u) = F_0(u + \lambda)^{m+n} - \frac{k}{m + n - 1}(u + \lambda),$$

and

$$\tau_t = -(m + n - 1)a_0 e^{kt}.$$

After normalization, the additional generator is

$$X_3 = e^{kt}\partial_t - \frac{k}{m + n - 1} e^{kt}(u + \lambda)\partial_u.$$

This gives Case I-(2) of the theorem.

Now, if

$$m + n = 1,$$

then the equation for F becomes

$$(u + \lambda)F'(u) - F(u) = k(u + \lambda),$$

whose solution is

$$F(u) = (u + \lambda)(F_0 + k \ln |u + \lambda|).$$

In this case $\tau_t = 0$, and the corresponding additional generator is

$$X_3 = e^{kt}(u + \lambda)\partial_u.$$

This gives Case I-(3) of the theorem.

Case I-(2): $F(u) = K(u + \lambda),\quad K \neq 0$.

This affine case is excluded from Case I-(1) and must be treated separately. Substituting

$$F(u) = K(u + \lambda), \qquad F'(u) = K \neq 0$$

into (3.4) gives

$$\rho(t) = K(1 - m - n + \sigma(t)),$$

or equivalently

$$a'(t) = K(1 - m - n)a(t) + K(n + 1)c_1.$$

If

$$m + n \neq 1,$$

then two independent choices of the constants give two independent additional generators. The first is obtained from the formal $r = 1$ power branch as follows:

$$X_3 = \frac{m + n - 1}{n + 1} x\partial_x + (u + \lambda)\partial_u.$$

The second is obtained from the solution of the linear equation above for $a(t)$, or equivalently, from

$$k = K(1 - m - n),$$

and is

$$X_4 = e^{K(1-m-n)t}\partial_t + Ke^{K(1-m-n)t}(u + \lambda)\partial_u.$$

This gives Case I-(4) of the theorem.

Next, if

$$m + n = 1,$$

then

$$a'(t) = K(n + 1)c_1.$$

Together with

$$\tau_t = (n + 1)c_1,$$

this yields the two independent additional generators

$$X_3 = (u + \lambda)\partial_u,$$

and

$$X_4 = (n + 1)t\partial_t + x\partial_x + K(n + 1)t(u + \lambda)\partial_u.$$

This gives Case I-(5) of the theorem.

Case II: $a(t) \equiv 0$**.**

Now

$$\eta = b(t).$$

Equation (3.2) becomes

$$b(t)\Phi'(u) + (\tau_t - (n + 1)c_1)\Phi(u) = 0.$$

The subcase $b(t) \equiv 0$ gives only the kernel algebra or constant reaction branches, which are excluded by the hypotheses. Hence, we assume

$$b(t) \not\equiv 0.$$

Dividing by $b(t)$, we get

$$\Phi'(u) + \frac{\tau_t - (n+1)c_1}{b(t)}\Phi(u) = 0.$$

Since $\Phi = \Phi(u)$, the ratio must be constant. Let

$$\frac{\tau_t - (n+1)c_1}{b(t)} = -\alpha.$$

Then

$$\tau_t = (n+1)c_1 - \alpha b(t),$$

and

$$\Phi'(u) - \alpha\Phi(u) = 0.$$

Thus

$$\Phi(u) = \Phi_0 e^{\alpha u}, \qquad \alpha \neq 0.$$

The F-equation becomes

$$b'(t) + \alpha b(t)F(u) - (n+1)c_1 F(u) - b(t)F'(u) = 0,$$

or, equivalently

$$b(t)F'(u) - \big(\alpha b(t) - (n+1)c_1\big)F(u) = b'(t). \tag{3.5}$$

Case II-(1): $c_1 \neq 0$**.**

Dividing (3.5) by $b(t)$, set

$$\sigma(t) = \frac{(n+1)c_1}{b(t)}, \qquad \rho(t) = \frac{b'(t)}{b(t)}.$$

Then

$$F'(u) - (\alpha - \sigma(t))F(u) = \rho(t).$$

Differentiating with respect to t gives

$$\sigma'(t)F(u) - \rho'(t) = 0.$$

Since F is nonconstant,

$$\sigma'(t) = 0, \qquad \rho'(t) = 0.$$

Because $c_1 \neq 0$, this implies

$$b(t) = b_0, \qquad \rho = 0.$$

Therefore

$$F'(u) - \beta F(u) = 0, \qquad \beta = \alpha - \frac{(n+1)c_1}{b_0}.$$

Hence

$$F(u) = F_0 e^{\beta u}, \qquad \beta \neq 0.$$

After normalizing $b_0 = 1$, we obtain

$$c_1 = \frac{\alpha - \beta}{n+1}, \qquad \tau_t = -\beta.$$

Thus, the additional generator is

$$X_3 = -\beta t\partial_t + \frac{\alpha-\beta}{n+1}x\partial_x + \partial_u.$$

This gives Case II-(1) of the theorem.

Case II-(2): $c_1 = 0$**.**

Equation (3.5) becomes

$$F'(u) - \alpha F(u) = \frac{b'(t)}{b(t)}.$$

Thus

$$\frac{b'(t)}{b(t)} = k, \qquad b(t) = b_0 e^{kt}.$$

The equation for F is

$$F'(u) - \alpha F(u) = k.$$

Since $\alpha \neq 0$, its solution is

$$F(u) = F_0 e^{\alpha u} - \frac{k}{\alpha}.$$

Furthermore,

$$\tau_t = -\alpha b_0 e^{kt}, \qquad \eta = b_0 e^{kt}.$$

After normalization, the additional generator is

$$X_3 = e^{kt}\partial_t - \frac{k}{\alpha}e^{kt}\partial_u.$$

This gives Case II-(2) of the theorem.

The preceding cases exhaust all possibilities because every nonkernel symmetry must satisfy either $a(t) \not\equiv 0$ or $a(t) \equiv 0$, and the corresponding classifying equations force respectively either the power diffusivity branch or the exponential diffusivity branch. Hence, outside the listed cases, the only admitted generators are ∂_t and ∂_x. The theorem follows. □

4. Equivalence transformations

In this section, we determine the equivalence group of the class

$$u_t = (\Phi(u)(u_x)^n)_x + F(u) = n\Phi(u)u_x^{n-1}u_{xx} + \Phi'(u)u_x^{n+1} + F(u), \qquad n \neq -1, 0, 1.$$

We also show that any point equivalence transformation preserving the class is necessarily of the form

$$\tilde{t} = \delta_1 t + \delta_0, \qquad \tilde{x} = \delta_2 x + \delta_3, \qquad \tilde{u} = \delta_4 u + \delta_5, \qquad \delta_1\delta_2\delta_4 \neq 0,$$

where δ_0–δ_5 are arbitrary constants and $\delta_1\delta_2\delta_4 \neq 0$. Consider the arbitrary point transformation in the (x,t)-space

$$\tilde{t} = f^1(x,t), \qquad \tilde{x} = f^2(x,t).$$

The chain rule for the total derivative in (x, t)-space is

$$D_t = (D_t f^1)\tilde{D}_{\tilde{t}} + (D_t f^2)\tilde{D}_{\tilde{x}},$$
$$D_x = (D_x f^1)\tilde{D}_{\tilde{t}} + (D_x f^2)\tilde{D}_{\tilde{x}}.$$

For

$$f^1(x,t) = \delta_1 t + \delta_0, \qquad f^2(x,t) = \delta_2 x + \delta_3,$$

we have

$$D_t f^1 = \delta_1, \qquad D_x f^1 = 0, \qquad D_t f^2 = 0, \qquad D_x f^2 = \delta_2.$$

Simplification leads to the following conditions:

$$\tilde{D}_{\tilde{t}} = \frac{1}{\delta_1} D_t, \qquad \tilde{D}_{\tilde{x}} = \frac{1}{\delta_2} D_x.$$

Thus, the continuous equivalence group is

$$G^{\sim} : (t, x, u, \Phi, F, n) \mapsto (\tilde{t}, \tilde{x}, \tilde{u}, \tilde{\Phi}, \tilde{F}, \tilde{n})$$

with the transformations presented above. In particular, translations and scalings of u remove the shift parameter λ in the power branch, while scalings of t, x, and u normalize the nonzero constants multiplying Φ and F. The total derivatives are transformed as

$$D_t = \delta_1 \tilde{D}_{\tilde{t}}, \qquad D_x = \delta_2 \tilde{D}_{\tilde{x}}.$$

Hence

$$u_t = \frac{\delta_1}{\delta_4}\tilde{u}_{\tilde{t}}, \qquad u_x = \frac{\delta_2}{\delta_4}\tilde{u}_{\tilde{x}}, \qquad u_{xx} = \frac{\delta_2^2}{\delta_4}\tilde{u}_{\tilde{x}\tilde{x}}.$$

Substituting these expressions into the equation gives

$$\tilde{u}_{\tilde{t}} = \left(\tilde{\Phi}(\tilde{u})(\tilde{u}_{\tilde{x}})^n\right)_{\tilde{x}} + \tilde{F}(\tilde{u}),$$

where

$$\tilde{n} = n, \quad \tilde{\Phi}(\tilde{u}) = \frac{\delta_2^{n+1}\delta_4^{1-n}}{\delta_1}\Phi\left(\frac{\tilde{u} - \delta_5}{\delta_4}\right), \quad \tilde{F}(\tilde{u}) = \frac{\delta_4}{\delta_1}F\left(\frac{\tilde{u} - \delta_5}{\delta_4}\right).$$

Thus, the continuous equivalence group of the class is generated by translations and scalings of t, x and u, together with the induced transformations of Φ and F shown above.

Cases I-(1)–Case I-(5) of Section 3: Power diffusivity

In the power case

$$\Phi(u) = \Phi_0(u + \lambda)^m,$$

the shift parameter λ is inessential. Taking

$$\tilde{u} = \delta_4(u + \lambda),$$

removes λ, and

$$\tilde{\Phi}(\tilde{u}) = \frac{\delta_2^{n+1}\delta_4^{1-n-m}}{\delta_1}\Phi_0\tilde{u}^m.$$

Hence, up to equivalence transformations, we get

$$\Phi(u) = \varepsilon_1 u^m, \qquad m \neq 0, \qquad \varepsilon_1 = \pm 1.$$

The corresponding normalized reaction terms are

$$F(u) = \varepsilon_2\, u^r, \qquad r \neq 0,\ 1,$$
$$F(u) = \varepsilon_2\, u^{m+n} - \frac{\varepsilon_3}{m+n-1}\, u, \qquad m+n \neq 1,$$
$$F(u) = \varepsilon_2\, u \ln|u|, \qquad m+n = 1,$$

and

$$F(u) = \varepsilon_2 u,$$

where each $\varepsilon_i = \pm 1$.

Cases II-(1), Case II-(2) of Section 3: Exponential diffusivity

Similarly, in the exponential case, we have

$$\Phi(u) = \Phi_0 e^{\alpha u}, \qquad \alpha \neq 0,$$

the transformation $\tilde{u} = \alpha u + \delta_5$ normalizes the exponent. Thus one may take

$$\Phi(u) = \varepsilon_1 e^u.$$

For

$$F(u) = F_0 e^{\beta u},$$

the essential exponent is

$$q = \frac{\beta}{\alpha},$$

so the normalized representative is

$$F(u) = \varepsilon_2 e^{qu}.$$

For the shifted exponential reaction, the normalized representative is

$$F(u) = \varepsilon_2 e^u - \varepsilon_3.$$

Thus, the parameters $\lambda, \Phi_0, F_0, K, k, \alpha$ appearing in the unnormalized classification are not essential. The essential continuous parameters are m, r, q, together with the fixed diffusion exponent n.

Remark 2 (Boundary parameter cases)**.** *The boundary value $r = m + n$ is included in Case I-(1). Indeed, in this case, we have*

$$F(u) = F_0(u + \lambda)^{m+n},$$

and the corresponding additional generator becomes

$$X_3 = (1 - m - n)t\partial_t + (u + \lambda)\partial_u.$$

Similarly, the boundary value $\beta = \alpha$ is included in Case II-(1), for which

$$F(u) = F_0 e^{\alpha u}$$

and

$$X_3 = -\alpha t\partial_t + \partial_u.$$

The classification theorem above concerns the regular subclass $\Phi'(u) \not\equiv 0$ and $F'(u) \not\equiv 0$. The degenerate cases $\Phi' \equiv 0$, $F' \equiv 0$, and $F \equiv 0$ are not obtained as limiting cases of the theorem, because the determining equations and even the kernel algebra may change. Indeed, when $F \equiv 0$, the equation

$$u_t = \left(\Phi(u)(u_x)^n\right)_x$$

admits, for an arbitrary nonconstant Φ, the additional scaling symmetry

$$(n+1)t\partial_t + x\partial_x.$$

Thus, the kernel algebra in the subclass $F = 0$ is already larger than $\langle \partial_t, \partial_x \rangle$. Similarly, if $\Phi' \equiv 0$, then $\Phi = \Phi_0 \neq 0$ and the diffusion term becomes independent of u. In this constant-diffusivity subclass, the condition

$$\eta\Phi' + ((n-1)\eta_u + \tau_t - (n+1)\xi_x)\Phi = 0$$

reduces to

$$(n-1)\eta_u + \tau_t - (n+1)\xi_x = 0,$$

which is different from the regular case and may allow further branches, for example u-translations, when also $F \equiv 0$. If $F' \equiv 0$, then $F = F_0$ is constant. For $F_0 = 0$, one obtains the homogeneous subclass just mentioned. For $F_0 \neq 0$, the reaction equation reduces to

$$a'(t)u + b'(t) + (a(t) - \tau_t)F_0 = 0,$$

again giving a separate subclassification problem. Therefore, these degenerate cases are excluded from the present theorem.

5. Optimal systems of one-dimensional subalgebras

In this section we list optimal systems of one-dimensional subalgebras for the Lie algebras obtained in the classification theorem. The construction of one-dimensional optimal systems follows the standard adjoint-reduction method for Lie algebras of differential equations [12–14]. In this approach, one classifies one-dimensional subalgebras up to conjugacy under the adjoint action of the associated Lie group, so that the resulting list provides inequivalent representatives for similarity reductions [15, 16]. This procedure has also been used in recent symmetry-based studies of nonlinear diffusion, reaction–diffusion, and related evolution equations, where optimal systems are combined with reductions, exact solutions, and conservation laws [17, 18].

The purpose of an optimal system is to give one representative from each equivalence class of one-dimensional subalgebras under the adjoint action of the corresponding Lie group. We use the standard adjoint formula

$$\mathrm{Ad}\left(e^{\varepsilon X_i}\right) X_j = X_j - \varepsilon[X_i, X_j] + \frac{\varepsilon^2}{2}[X_i, [X_i, X_j]] - \cdots .$$

Since a one-dimensional subalgebra is unchanged by multiplication of its generator by a nonzero constant, each representative below is understood up to nonzero scalar multiplication.

Theorem 2 (One-dimensional optimal systems)**.** *For the classified Lie symmetry algebras of*

$$u_t = \left(\Phi(u)(u_x)^n\right)_x + F(u), \qquad n \neq -1, 0, 1,$$

the following lists give optimal systems of one-dimensional subalgebras.

(0) Principal algebra. *For*

$$\mathfrak{g}_0 = \langle X_1, X_2 \rangle, \qquad X_1 = \partial_t, \qquad X_2 = \partial_x,$$

where $[X_1, X_2] = 0$*, an optimal system is*

$$\langle X_1 \rangle, \quad \langle X_2 \rangle, \quad \langle X_1 + aX_2 \rangle, \quad a \neq 0.$$

(I-(1)) Power diffusivity with a power reaction. *For*

$$\Phi(u) = \varepsilon_1 u^m, \qquad F(u) = \varepsilon_2 u^r, \qquad m \neq 0, \quad r \neq 0, 1,$$

with

$$X_3 = (n+1)(1-r)t\partial_t + (m+n-r)x\partial_x + (n+1)u\partial_u,$$

put

$$A = (n+1)(1-r), \qquad B = m+n-r.$$

The nonzero brackets are

$$[X_1, X_3] = AX_1, \qquad [X_2, X_3] = BX_2.$$

Since $r \neq 1$*, we have* $A \neq 0$*. If* $B \neq 0$ *and* $A \neq B$*, an optimal system is*

$$\langle X_1 \rangle, \quad \langle X_2 \rangle, \quad \langle X_3 \rangle, \quad \langle X_1 + \varepsilon X_2 \rangle, \quad \varepsilon = \pm 1.$$

If $B \neq 0$ *and* $A = B$*, the last representative is replaced by*

$$\langle X_1 + aX_2 \rangle, \quad a \neq 0.$$

If $B = 0$*, or equivalently,* $r = m + n$*, then* $[X_2, X_3] = 0$*, and an optimal system is*

$$\langle X_1 \rangle, \quad \langle X_2 \rangle, \quad \langle X_1 + \varepsilon X_2 \rangle, \quad \langle X_3 + aX_2 \rangle, \quad \varepsilon = \pm 1, \quad a \in \mathbb{R}.$$

(I-(2)) Power diffusivity with resonant reaction. *For*

$$\Phi(u) = \varepsilon_1 u^m, \qquad F(u) = \varepsilon_2 u^{m+n} - \frac{\varepsilon_3}{m+n-1}u, \qquad m+n \neq 1,$$

with

$$X_3 = e^{\varepsilon_3 t}\partial_t - \frac{\varepsilon_3}{m+n-1}e^{\varepsilon_3 t}u\partial_u.$$

The only nonzero bracket is

$$[X_1, X_3] = \varepsilon_3 X_3.$$

Therefore, with respect to the adjoint group, an optimal system is

$$\langle X_1\rangle,\quad \langle X_2\rangle,\quad \langle X_3\rangle,\quad \langle X_1+aX_2\rangle,\quad a\neq 0,$$

and

$$\langle X_2+\delta X_3\rangle,\quad \delta=\pm 1.$$

Here, the constants λ, F_0, k *have been removed by the equivalence group, while the signs* ε_i *are retained as normalized representatives.*

(I-(3)) Power diffusivity with logarithmic reaction. *For*

$$\Phi(u)=\varepsilon_1 u^m,\qquad F(u)=\varepsilon_2 u\ln|u|,\qquad m+n=1,$$

with

$$X_3=e^{\varepsilon_2 t}u\partial_u.$$

The commutator structure is the same as in Case I-(2) as follows:

$$[X_1,X_3]=\varepsilon_2 X_3.$$

Hence, with respect to the adjoint group, an optimal system is

$$\langle X_1\rangle,\quad \langle X_2\rangle,\quad \langle X_3\rangle,\quad \langle X_1+aX_2\rangle,\quad a\neq 0,$$

and

$$\langle X_2+\delta X_3\rangle,\quad \delta=\pm 1.$$

Here, the shift λ*, the additive logarithmic constant* F_0*, and the nonzero coefficient* k *have been removed by the equivalence group.*

(I-(4)) Power diffusivity with linear reaction, $m+n\neq 1$***.*** *For*

$$\Phi(u)=\varepsilon_1 u^m,\qquad F(u)=\varepsilon_2 u,\qquad m+n\neq 1,$$

with

$$X_1=\partial_t,\qquad X_2=\partial_x,\qquad X_3=\frac{m+n-1}{n+1}x\partial_x+u\partial_u,$$

$$X_4=e^{\varepsilon_2(1-m-n)t}\partial_t+\varepsilon_2 e^{\varepsilon_2(1-m-n)t}u\partial_u.$$

Put

$$C=\frac{m+n-1}{n+1},\qquad D=\varepsilon_2(1-m-n).$$

Then $C\neq 0$*,* $D\neq 0$*, and the nonzero brackets are*

$$[X_1,X_4]=DX_4,\qquad [X_2,X_3]=CX_2.$$

Equivalently, we have

$$[X_1,X_4]=-\varepsilon_2(m+n-1)X_4.$$

An optimal system is represented by

$$\langle X_1 + aX_3\rangle,\ \langle X_1 + \delta X_2\rangle,\ \langle X_3 + \delta X_4\rangle,$$
$$\langle X_3\rangle,\ \langle X_2\rangle,\ \langle X_4\rangle,\ \langle X_2 + \delta X_4\rangle, \quad \delta = \pm 1, \quad a \in \mathbb{R}.$$

Here, a is an essential adjoint parameter.

(I-(5)) Power diffusivity with a linear reaction, $m + n = 1$. *For*

$$\Phi(u) = \varepsilon_1 u^m, \qquad F(u) = \varepsilon_2 u, \qquad m + n = 1,$$

with

$$X_3 = u\partial_u,$$
$$X_4 = (n+1)t\partial_t + x\partial_x + \varepsilon_2(n+1)tu\partial_u.$$

The nonzero brackets are

$$[X_1, X_4] = (n+1)(X_1 + \varepsilon_2 X_3), \quad [X_2, X_4] = X_2.$$

An optimal system is represented by

$$\langle X_4 + aX_3\rangle, \quad \langle X_1 + \varepsilon_2 X_3\rangle, \quad \langle X_2\rangle, \quad \langle X_1 + \varepsilon_2 X_3 + \delta X_2\rangle,$$

and

$$\langle X_3\rangle, \quad \langle X_3 + \delta(X_1 + \varepsilon_2 X_3)\rangle, \quad \langle X_3 + b(X_1 + \varepsilon_2 X_3) + \delta X_2\rangle,$$

where

$$\delta = \pm 1, \qquad a, b \in \mathbb{R}.$$

Here, the shift λ and the nonzero coefficient K have been removed by equivalence transformations, with only the sign ε_2 retained.

(II-(1)) Exponential diffusivity with exponential reaction. *For*

$$\Phi(u) = \varepsilon_1 e^u, \qquad F(u) = \varepsilon_2 e^{qu}, \qquad q \neq 0,$$

with

$$X_3 = -(n+1)qt\partial_t + (1-q)x\partial_x + (n+1)\partial_u,$$

put

$$A = -(n+1)q, \qquad B = 1 - q.$$

The nonzero brackets are

$$[X_1, X_3] = AX_1, \qquad [X_2, X_3] = BX_2.$$

If $B \neq 0$ and $A \neq B$, that is

$$q \neq 1, \qquad nq + 1 \neq 0,$$

an optimal system is

$$\langle X_1\rangle, \quad \langle X_2\rangle, \quad \langle X_3\rangle, \quad \langle X_1 + \delta X_2\rangle, \quad \delta = \pm 1.$$

If $B \neq 0$ and $A = B$ or, equivalently, $nq + 1 = 0$, the last representative is replaced by

$$\langle X_1 + aX_2 \rangle, \quad a \neq 0.$$

If $B = 0$, equivalently $q = 1$, then $[X_2, X_3] = 0$, and an optimal system is

$$\langle X_1 \rangle, \quad \langle X_2 \rangle, \quad \langle X_1 + \delta X_2 \rangle, \quad \langle X_3 + aX_2 \rangle, \quad \delta = \pm 1, \quad a \in \mathbb{R}.$$

(II-(2)) Exponential diffusivity with a shifted exponential reaction. *For*

$$\Phi(u) = \varepsilon_1 e^u, \qquad F(u) = \varepsilon_2 e^u - \varepsilon_3,$$

with

$$X_3 = e^{\varepsilon_3 t}\partial_t - \varepsilon_3 e^{\varepsilon_3 t}\partial_u.$$

The only nonzero bracket is

$$[X_1, X_3] = \varepsilon_3 X_3.$$

Thus, with respect to the adjoint group, an optimal system is

$$\langle X_1 \rangle, \quad \langle X_2 \rangle, \quad \langle X_3 \rangle, \quad \langle X_1 + aX_2 \rangle, \quad a \neq 0,$$

and

$$\langle X_2 + \delta X_3 \rangle, \quad \delta = \pm 1.$$

Here, the constants Φ_0, F_0, α, and k have been removed by equivalence transformations, with only normalized signs retained.

For all remaining nonconstant choices of $\Phi(u)$ and $F(u)$, the maximal Lie invariance algebra coincides with the kernel algebra, and the optimal system is therefore the principal one listed in Case 0.

The proof follows the standard adjoint-reduction procedure. For each admitted Lie algebra, one starts with a general nonzero element $X = \sum_i a_i X_i$, applies the adjoint transformations generated by the basis elements X_i, and uses these transformations to eliminate or normalize as many coefficients as possible. Since a one-dimensional subalgebra is unchanged by multiplying its generator by a nonzero constant, only essential ratios of coefficients are retained. The representatives listed in the theorem are precisely those generators that remain after all possible adjoint simplifications have been applied. Parameter-dependent exceptional cases occur when some structure constants vanish or coincide, since then certain adjoint normalizations are no longer available. Hence, every one-dimensional subalgebra is conjugate to exactly one of the listed representatives, up to the stated parameter restrictions.

6. Symmetry reductions and invariant solutions

Similarity reductions are obtained by imposing invariance under representatives of the optimal systems. This reduces the original PDEs to ordinary differential equations in invariant variables and is a standard consequence of the Lie reduction method [12–14]. Related formulations of symmetry reduction and invariant-solution construction are also discussed in [16]. The resulting reduced equations are then analyzed to construct exact invariant solutions for the classified nonlinear models.

Similar reduction procedures have been applied recently to nonlinear diffusion and reaction–diffusion equations [17, 19, 20], as well as to fractional diffusion and quasilinear evolution equations [18, 21].

In this section, we use the one-dimensional optimal systems of subalgebras obtained in the previous section to reduce PDE (1.3). We present the reductions corresponding to the most relevant representatives of the optimal systems. The complete reduction is presented in Table 1.

Table 1. Reduced ODEs.

Case	Symmetry	Similarity form	Reduced ODE
0	$X^3 = \partial_t + \partial_x$	$\xi = x - t$ $u = f(\xi)$	$f' + \Phi'(f)(f')^{n+1} + n\Phi(f)(f')^{n-1}f'' + F(f) = 0$
0	$X^2 = \partial_x$	$\xi = t$ $u = f(\xi)$	$f' - F(f) = 0$
0	$X^1 = \partial_t$	$\xi = x$ $u = f(\xi)$	$\Phi'(f)(f')^{n+1} + n\Phi(f)(f')^{n-1}f'' + F(f) = 0$
I-(1)	$X^4 = \partial_t + a\partial_x$, $a \neq 0$	$\xi = t - x/a$ $u = f(\xi)$	$f' + \varepsilon_1 \frac{(-1)^n}{a^{n+1}}(f')^{n-1}\left[mf^{m-1}(f')^2 + nf^m f''\right] - \varepsilon_2 f^r = 0$
I-(1)	$X^3 = X_3$	$\xi = t\, x^{\frac{(r-1)(n+1)}{m+n-r}}$ $u = x^{\frac{n+1}{m+n-r}} f(\xi)$	$f' - f^r - \frac{(n+1)^n f^{m-1}}{(m+n-r)^{n+1}}\left[f + (r-1)\xi f'\right]^{n-1}\mathcal{B} = 0$ $\mathcal{B} = (r-1)^2(n+1)\xi^2\left[m(f')^2 + nff''\right]$ $+(r-1)\xi f f'\left[n^2 r + (m+2r+1)n + 2m\right]$ $+f^2\left[(r+1)n + m\right]$
I-(1)	$X^2 = \partial_x$	$\xi = t$ $u = f(\xi)$	$f' - \varepsilon_2 f^r = 0$
I-(1)	$X^1 = \partial_t$	$\xi = x$ $u = f(\xi)$	$\varepsilon_1\left[mf^{m-1}(f')^{n+1} + nf^m(f')^{n-1}f''\right] + \varepsilon_2 f^r = 0$
I-(2)	$X^5 = X_2 + \delta X_3$, $\delta = \pm 1$	$\xi = x + \frac{\delta}{\varepsilon_3}e^{-\varepsilon_3 t}$ $u = e^{-\frac{\varepsilon_3 t}{m+n-1}} f(\xi),\ m + n \neq 1$	$\delta f' + \varepsilon_1\left[mf^{m-1}(f')^{n+1} + nf^m(f')^{n-1}f''\right] + \varepsilon_2 f^{m+n} = 0$

Continued on next page

Case	Symmetry	Similarity form	Reduced ODE
I-(2)	$X^4 = X_1 + \delta X_2$, $\delta = \pm 1$	$\xi = x - \delta t$ $u = f(\xi)$	$\delta f' + \varepsilon_1 \left[m f^{m-1} (f')^{n+1} + n f^m (f')^{n-1} f'' \right]$ $+\varepsilon_2 f^{m+n} - \frac{\varepsilon_3}{m+n-1} f = 0$
I-(2)	$X^3 = X_3$	$\xi = x$ $u = e^{-\frac{\varepsilon_3 t}{m+n-1}} f(\xi)$	$\varepsilon_1 \left[m f^{m-1} (f')^{n+1} + n f^m (f')^{n-1} f'' \right] + \varepsilon_2 f^{m+n} = 0$
I-(3)	$X^5 = X_2 + \delta X_3$, $\delta = \pm 1$	$\xi = t$ $u = e^{\delta x e^{\varepsilon_2 t}} f(\xi)$	$f' - \varepsilon_2 f \ln \lvert f \rvert - \varepsilon_1 \left(\delta e^{\varepsilon_2 \xi} \right)^{n+1} f = 0$
I-(3)	$X^4 = X_1 + \delta X_2$, $\delta = \pm 1$	$\xi = x - \delta t, \quad u = f(\xi)$ $m + n = 1$	$\delta f' + \varepsilon_1 \left[m f^{m-1} (f')^{n+1} + n f^m (f')^{n-1} f'' \right] + \varepsilon_2 f \ln \lvert f \rvert = 0$
I-(4)	$X^8 = X_2 + \delta X_4$, $\delta = \pm 1$	$\xi = x - \dfrac{\delta e^{\varepsilon_2 (m+n-1) t}}{\varepsilon_2 (m + n - 1)}$ $u = e^{\varepsilon_2 t} f(\xi),\ m + n \neq 1$	$\delta f' + \varepsilon_1 \left[m f^{m-1} (f')^{n+1} + n f^m (f')^{n-1} f'' \right] = 0$
I-(4)	$X^7 = X_4$	$\xi = x$ $u = e^{\varepsilon_2 t} f(\xi)$	$m f^{m-1} (f')^{n+1} + n f^m (f')^{n-1} f'' = 0$
I-(4)	$X^4 = X_1 + \delta X_2$, $\delta = \pm 1$	$\xi = x - \delta t$ $u = f(\xi)$	$\delta f' + \varepsilon_1 \left[m f^{m-1} (f')^{n+1} + n f^m (f')^{n-1} f'' \right] + \varepsilon_2 f = 0$
I-(4)	$X^3 = X_3$	$\xi = t$ $u = x^{\frac{n+1}{m+n-1}} f(\xi)$	$f' - \varepsilon_1 \left(\dfrac{n+1}{m+n-1} \right)^n \left[m \left(\dfrac{n+1}{m+n-1} \right) \right.$ $\left. +n \left(\dfrac{n+1}{m+n-1} - 1 \right) \right] f^{m+n} - \varepsilon_2 f = 0$
I-(4)	$X^1 = X_1 + a X_3$	$\xi = t - \dfrac{n+1}{a(m+n-1)} \ln x$, $a \neq 0$ $u = x^{\frac{n+1}{m+n-1}} f(\xi)$ $p = \dfrac{n+1}{m+n-1}$	$f' - \varepsilon_1 \Big[m f^{m-1} \left(p f - \dfrac{p}{a} f' \right)^{n+1} + n f^m \left(p f - \dfrac{p}{a} f' \right)^{n-1}$ $\left(p(p-1) f - \dfrac{p(2p-1)}{a} f' + \dfrac{p^2}{a^2} f'' \right) \Big] - \varepsilon_2 f = 0$
I-(5)	$X^4 = X_1 + \delta X_2 + a X_3$, $\delta = \pm 1$	$\xi = x - \delta t$ $u = e^{at} f(\xi)$	$\delta f' + \varepsilon_1 \left[(1-n) f^{-n} (f')^{n+1} + n f^{1-n} (f')^{n-1} f'' \right]$ $+(\varepsilon_2 - a) f = 0$

Continued on next page

Case	Symmetry	Similarity form	Reduced ODE
I-(5)	$X^3 = X_2 + aX_3$	$\xi = t$ $u = e^{ax} f(\xi)$	$f' - \left(\varepsilon_1 a^{n+1} + \varepsilon_2\right) f = 0$
I-(5)	$X^2 = X_1 + aX_3$	$\xi = x$ $u = e^{at} f(\xi)$	$\varepsilon_1 \left[(1-n) f^{-n} (f')^{n+1} \ +n f^{1-n} (f')^{n-1} f''\right]$ $+(\varepsilon_2 - a) f = 0$
I-(5)	$X^1 = X_4 + aX_3$	$\xi = t x^{-(n+1)}$, $u = x^a e^{\varepsilon_2 t} f(\xi)$	$f' - \varepsilon_1 (a-n) f^{1-n} \left(af - (n+1)\xi f'\right)^n$ $+\varepsilon_1 (n+1)\xi \Big[(1-n) f^{-n} f' \left(af - (n+1)\xi f'\right)^n$ $+n f^{1-n} \left(af - (n+1)\xi f'\right)^{n-1}$ $\left((a-n-1) f' - (n+1)\xi f''\right)\Big] = 0$
II-(1)	$X^4 = X_1 + \delta X_2$, $\delta = \pm 1$	$\xi = x - \delta t$ $u = f(\xi)$	$\delta f' + \varepsilon_1 e^f \left[(f')^{n+1} + n (f')^{n-1} f''\right] + \varepsilon_2 e^{qf} = 0$
II-(1)	$X^3 = X_3$	$\xi = t x^{\frac{q(n+1)}{1-q}}$ $u = f(\xi) + p \ln x, \quad q \neq 1$ $p = \dfrac{n+1}{1-q}$ $s = \dfrac{q(n+1)}{1-q}$	$f' - \varepsilon_1 e^f \Big[(p - n + s\xi f')\,(p + s\xi f')^n$ $+n s^2 \xi\,(p + s\xi f')^{n-1}\,(f' + \xi f'')\Big] - \varepsilon_2 e^{qf} = 0$
II-(2)	$X^5 = X_2 + \delta X_3$, $\delta = \pm 1$	$\xi = x + \dfrac{\delta}{\varepsilon_3} e^{-\varepsilon_3 t}$ $u = f(\xi) - \varepsilon_3 t$	$\delta f' + \varepsilon_1 e^f \left[(f')^{n+1} + n (f')^{n-1} f''\right] + \varepsilon_2 e^f = 0$
II-(2)	$X^4 = X_1 + \delta X_2$, $\delta = \pm 1$	$\xi = x - \delta t$ $u = f(\xi)$	$\delta f' + \varepsilon_1 e^f \left[(f')^{n+1} + n (f')^{n-1} f''\right] + \varepsilon_2 e^f - \varepsilon_3 = 0$
II-(2)	$X^3 = X_3$	$\xi = x$ $u = f(\xi) - \varepsilon_3 t$	$\varepsilon_1 \left[(f')^{n+1} + n (f')^{n-1} f''\right] + \varepsilon_2 = 0$

(0) Principal algebra. Consider the generator $\langle X_1 + X_2 \rangle$ from the principal case for which the respective invariant solution of (1.3) is thus of the form

$$u(x,t) = f(\xi), \quad \text{where} \quad \xi = x - t,$$

that satisfies the reduced ODE in f given by

$$n\Phi(f)(f')^{n-1} f'' + \Phi'(f)(f')^{n+1} + F(f) + f' = 0.$$

(I-(1)) Power diffusivity with a power reaction. The generator $\langle X_1 + X_2\rangle$ has the invariant solution of the form

$$u(x,t) = f(\xi), \quad \text{where} \quad \xi = x - t,$$

that satisfies the reduced ODE in f given by

$$f' + \varepsilon_1 (f')^{n-1}\left[m f^{m-1}(f')^2 + n f^m f''\right] + \varepsilon_2 f^r = 0.$$

(I-(2)) Power diffusivity with resonant reaction.

For the reduction under $\langle X_2 + \delta X_3\rangle$, $\delta = \pm 1$, the generator, where, in normalized form, we have

$$\Phi(u) = \varepsilon_1 u^m, \qquad F(u) = \varepsilon_2 u^{m+n} - \frac{\varepsilon_3}{m+n-1} u, \qquad m+n \neq 1,$$

has the invariant solution

$$u(x,t) = e^{-\frac{\varepsilon_3 t}{m+n-1}} f(\xi), \qquad \xi = x + \frac{\delta}{\varepsilon_3} e^{-\varepsilon_3 t},$$

provided that it satisfies the ODE

$$\delta f' + \varepsilon_1\left[m f^{m-1}(f')^{n+1} + n f^m (f')^{n-1} f''\right] + \varepsilon_2 f^{m+n} = 0.$$

For the reduction under $X_1 + X_2$, the generator, where, in normalized form, we have

$$\Phi(u) = \varepsilon_1 u^m, \qquad F(u) = \varepsilon_2 u^{m+n} - \frac{\varepsilon_3}{m+n-1} u, \qquad m+n \neq 1,$$

gives the form of an invariant solution

$$u(x,t) = f(\xi), \qquad \xi = x - t,$$

that satisfies the reduced ODE

$$f' + \varepsilon_1\left[m f^{m-1}(f')^{n+1} + n f^m (f')^{n-1} f''\right] + \varepsilon_2 f^{m+n} - \frac{\varepsilon_3}{m+n-1} f = 0.$$

Similarly, the generator $\langle X_3\rangle$ has invariant solution

$$u(x,t) = e^{-\frac{\varepsilon_3 t}{m+n-1}} f(\xi), \quad \xi = x,$$

that satisfies the ODE

$$\varepsilon_1\left[m f^{m-1}(f')^{n+1} + n f^m (f')^{n-1} f''\right] + \varepsilon_2 f^{m+n} = 0.$$

(I-(3)) Power diffusivity with logarithmic reaction. For

$$\Phi(u) = \varepsilon_1 u^m, \qquad F(u) = \varepsilon_2 u \ln|u|, \qquad m+n = 1,$$

where $\varepsilon_i = \pm 1$. For the generator

$$\langle X_2 + \delta X_3\rangle, \qquad \delta = \pm 1,$$

with

$$X_3 = e^{\varepsilon_2 t} u \partial_u,$$

the invariant solution is

$$u(x,t) = e^{\delta x e^{\varepsilon_2 t}} f(\xi), \qquad \xi = t,$$

that satisfies the reduced ODE in f given by

$$f' - \varepsilon_2 f \ln|f| - \varepsilon_1 (\delta e^{\varepsilon_2 \xi})^{n+1} f = 0. \tag{6.1}$$

On the chosen real branch of $(u_x)^n$, Eq (6.1) has the solution

$$f(\xi) = \exp\left(C_1 e^{\varepsilon_2 \xi} + \frac{\varepsilon_1 \delta^{n+1}}{\varepsilon_2 n} e^{\varepsilon_2 (n+1)\xi} \right).$$

Thus, the invariant solution of (1.3) in variables x, t is

$$u(x,t) = \exp\left(\delta x e^{\varepsilon_2 t} + C_1 e^{\varepsilon_2 t} + \frac{\varepsilon_1 \delta^{n+1}}{\varepsilon_2 n} e^{\varepsilon_2 (n+1) t} \right).$$

Since $u > 0$ is strictly positive, the invariant solution corresponds to the logarithmic reaction case remains in the admissible domain of the logarithmic reaction term $\ln|u|$. For each fixed t, the spatial profile is exponential in x, with growth or decay determined by the sign of ε. The temporal dependence is governed by the exponential factors $e^{\varepsilon_2 t}$ and $e^{\varepsilon_2 (n+1)t}$, so the parameter ε_2 controls whether the solution amplifies rapidly or approaches a bounded limiting profile. In particular, for $\varepsilon_2 < 0$, the time-dependent exponential terms decay as $t \to \infty$, and the solution tends to the constant state

$$u(x,t) \to C_0.$$

Figure 1 illustrates the corresponding solution surface for representative parameter values.

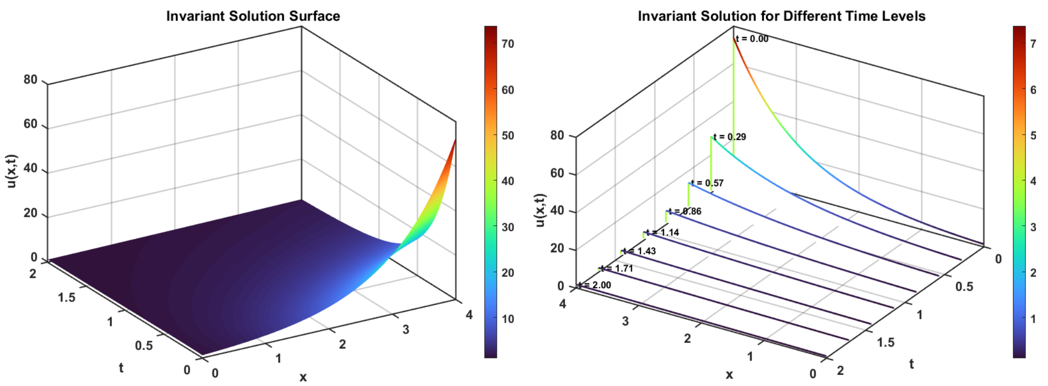

Figure 1. Surface plot of the invariant solution in Case I-(3). The parameter values used are $C_0 = 1, C_1 = 0.8, \delta = 1, \varepsilon_1 = 1, \varepsilon_2 = -1$, and $n = 2$.

(I-(4)) Power diffusivity with a linear reaction, $m + n \neq 1$**.** For the generator

$$X_4 = e^{\varepsilon_2 (1-m-n)t} \partial_t + \varepsilon_2 e^{\varepsilon_2 (1-m-n)t} u \partial_u.$$

The generator $\langle X_4 \rangle$ has the invariant solution of the form

$$u(x,t) = e^{\varepsilon_2 t} f(\xi), \qquad \xi = x,$$

that satisfies the reduced ODE in f given by

$$m f^{m-1}(f')^{n+1} + n f^m (f')^{n-1} f'' = (f^m (f')^n)' = 0. \tag{6.2}$$

For $m + n \neq 0$, this gives

$$f(\xi) = (C_1\xi + C_2)^{\frac{n}{m+n}} \implies u(x,t) = e^{\varepsilon_2 t}\,(C_1 x + C_2)^{\frac{n}{m+n}}.$$

If $m + n = 0$, the corresponding solution is

$$f(\xi) = C_1 e^{C_2 \xi} \implies u(x,t) = C_1 e^{\varepsilon_2 t + C_2 x}.$$

This invariant solution, corresponding to the power diffusivity with linear reaction, separates the temporal and spatial effects into two distinct factors. The factor $e^{\varepsilon_2 t}$ determines the temporal amplification or decay of the profile, while the spatial factor is a power-law function of $C_1 x + C_2$. Hence, the parameter ε_2 controls the time evolution of the solution, whereas the exponent $n/(m+n)$ determines the curvature and spatial growth rate of the surface. For $\varepsilon_2 = 1$, the solution grows exponentially in time, while for $\varepsilon_2 = -1$, it decays to zero as $t \to \infty$. The constant C_2 shifts the spatial profile along the x-axis, and the condition $m + n \neq 1$ distinguishes this case from the resonant linear reaction case treated separately. When $m + n = 0$, the spatial power law is replaced by an exponential profile $C_1 e^{C_2 x}$. Figure 2 illustrates the corresponding solution surface for representative parameter values.

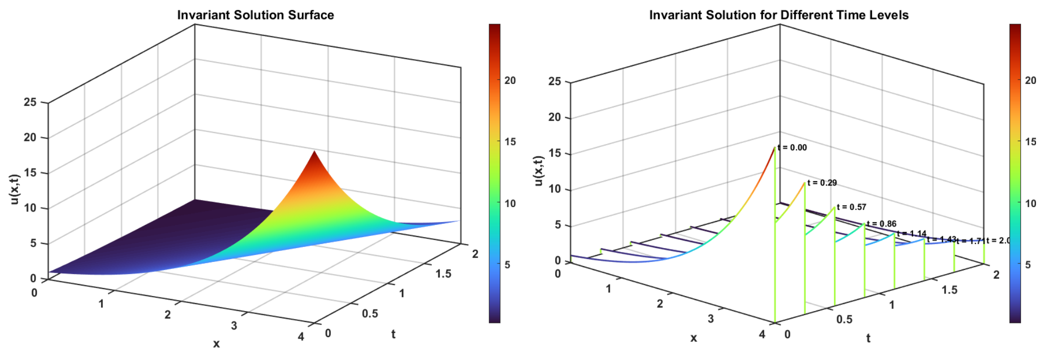

Figure 2. Surface plot of the invariant solution in Case I-(4), when $m + n = 0$. The parameter values used are $C_1 = 1, C_2 = 0.8$, and $\varepsilon_2 = -1$.

(I-(5)) Power diffusivity with a linear reaction, $m + n = 1$**.** The reduced ODE corresponding to the symmetry generator

$$\langle X_2 + a X_3 \rangle,$$

where $X_2 = \partial_x, X_3 = u\partial_u$ is given by

$$f' = \left(\varepsilon_1 a^{n+1} + \varepsilon_2\right) f.$$

This reduced ODE can be solved further and has the following solution:

$$f(\xi) = C_1 e^{\left(\varepsilon_1 a^{n+1}+\varepsilon_2\right)\xi},$$

where $\xi = t$.

Finally, the solution of the PDE (1.3) can be obtained using the similarity variables

$$u(x,t) = e^{ax} f(\xi), \qquad \xi = t,$$

and thus we get

$$f(\xi) = C_1 \exp\left[\left(\varepsilon_1 a^{n+1} + \varepsilon_2\right)\xi\right].$$

Therefore

$$u(x,t) = C_1 \exp\left[ax + \left(\varepsilon_1 a^{n+1} + \varepsilon_2\right)t\right].$$

This invariant solution corresponds to the power diffusivity with linear reaction in the resonant case $m+n = 1$. Unlike the preceding case $m+n \neq 1$, this solution has a purely exponential structure in both the spatial and temporal variables. More precisely, the shifted quantity u separates into the product of a spatial exponential factor e^{ax} and a temporal exponential factor

$$e^{(\varepsilon_1 a^{n+1}+\varepsilon_2)t}.$$

The parameter a controls the spatial growth or decay rate, while the coefficient $\varepsilon_1 a^{n+1} + \varepsilon_2$ determines the temporal amplification of the solution. Since

$$u(x,t) = C_1 e^{ax+(\varepsilon_1 a^{n+1}+\varepsilon_2)t},$$

the sign of C_1 determines whether the solution is positive or negative. For $C_1 > 0$, the solution remains in the positive branch $u > 0$ and grows exponentially whenever $ax + (\varepsilon_1 a^{n+1} + \varepsilon_2)t$. Figure 3 shows the corresponding solution surface for representative parameter values.

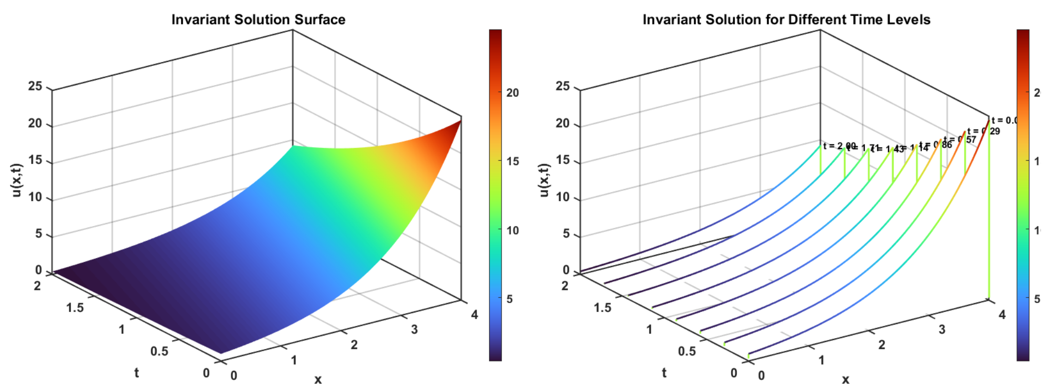

Figure 3. Surface plot of the invariant solution in Case I-(5). The parameter values used are $C_1 = 1, a = 0.8, n = 2, \varepsilon_1 = 1$, and $\varepsilon_2 = -1$.

7. Conservation laws

Conservation laws are important in studying the invariance property of the surface generated by the solution of PDEs. We use the direct multiplier method to construct conservation laws. This method characterizes conservation laws through multipliers whose product with the given differential equation is a total divergence. It is particularly useful for nonvariational PDEs, since it does not require the equation to arise from a Lagrangian formulation [6,7,16]. Multiplier-based conservation laws have also been used extensively in recent Lie-symmetry investigations of nonlinear diffusion-type and reaction–diffusion equations, often together with optimal systems and similarity reductions [17, 19, 20]. Related applications combining conservation laws with symmetry reductions and exact invariant solutions can also be found in [18, 21].

The Euler operator is defined as

$$\mathscr{E} = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - D_t \frac{\partial}{\partial u_t} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_t \frac{\partial}{\partial u_{xt}} + \cdots ,$$

where D_x and D_t are the total derivative operators. A conserved flow of PDE (1.3) is a vector of the form $T = (T^t, T^x)$ such that

$$D_t T^t + D_x T^x = 0 \tag{7.1}$$

holds for all solutions of (1.3). Equation (7.1) is called the local conservation law. The multipliers $\mathscr{M}$ of the PDE (1.3) have the property

$$D_t T^t + D_x T^x = \mathscr{M} \left[u_t - n\Phi(u)(u_x)^{n-1} u_{xx} - \Phi'(u)(u_x)^{n+1} - F(u) \right], \tag{7.2}$$

where multipliers $\mathscr{M}$ are dependent on the independent variable x, t and dependent variable u as well as their derivatives $u_x, u_t, \cdots$ accordingly. For zero-order multipliers of the form considered here, a nontrivial multiplier exists when $F(u) = u$.

The determining equations for the multipliers are obtained by taking the variational derivative of (7.2), which can be calculated by

$$\mathscr{E} \left[\mathscr{M} \left(u_t - n\Phi(u)(u_x)^{n-1} u_{xx} - \Phi'(u)(u_x)^{n+1} - F(u) \right) \right] = 0, \tag{7.3}$$

where $\mathscr{E}$ is the variational derivative operator that is also known as the Euler operator. From (7.3), we get the following set of determining equations:

$$\mathscr{M}_{u_t u_t} = 0, \quad \mathscr{M}_{u_t} = 0, \quad \mathscr{M}_{u_x} = 0, \quad \mathscr{M}_u = 0, \quad \mathscr{M}_x = 0, \quad \mathscr{M}_t + \mathscr{M} = 0.$$

Solving the set of equations above provide a unique multiplier given by

$$\mathscr{M} = c_1 e^{-t}.$$

The local conservation laws of (1.3) for arbitrary functions $\Phi(u)$ and for the diffusivities $\Phi(u) = \varepsilon_1 u^m$, $\Phi(u) = \varepsilon_1 u^{1-n}$, and $\Phi(u) = \varepsilon_1 e^u$, are, respectively, given in the form of conserved vectors

$$\begin{aligned} T_{\Phi(u)} &= (T^t, T^x) = (e^{-t}u, -e^{-t}\Phi(u)u_x^n), \\ T_{\Phi(u)=\varepsilon_1 u^m} &= (T^t, T^x) = (e^{-t}u, -\varepsilon_1 e^{-t} u^m u_x^n), \\ T_{\Phi(u)=\varepsilon_1 u^{1-n}} &= (T^t, T^x) = (e^{-t}u, -\varepsilon_1 e^{-t} u^{1-n} u_x^n), \\ T_{\Phi(u)=\varepsilon_1 e^u} &= (T^t, T^x) = (e^{-t}u, -\varepsilon_1 e^{u-t} u_x^n). \end{aligned}$$

8. Conclusions and future directions

In this paper, we investigated the Lie symmetry structure of the nonlinear evolution equation

$$u_t = \left(\Phi(u)(u_x)^n\right)_x + F(u), \qquad n \neq -1, 0, 1,$$

with the arbitrary state-dependent diffusivity $\Phi(u)$ and the reaction term $F(u)$. The kernel Lie algebra was identified, and a complete group classification was obtained for all nonconstant functions $\Phi(u)$ and $F(u)$. The classification revealed seven inequivalent symmetry extension families, including power, exponential, logarithmic, and linear reaction structures. For each admitted extension, the corresponding symmetry generators were constructed explicitly.

Using the classified symmetry algebras, one-dimensional optimal systems of subalgebras were established and used to derive similarity reductions of the governing equation. Several reduced ODEs were obtained, and explicit invariant solutions were constructed in closed form for selected cases. In addition, conservation laws were derived through the multiplier approach for the admissible reaction term. The results provide a systematic symmetry-based characterization of this class of quasilinear diffusion-reaction equations and furnish a foundation for further investigations, including nonclassical symmetries, potential symmetries, conservation law classifications, and higher-dimensional generalizations.

The explicit invariant solutions obtained in this work have been analytically validated by direct substitution into the corresponding reduced equations and, through the invariant ansatz, into the original equation. A natural continuation is to study their numerical stability and long-time behavior using structure-preserving discretization methods. Further directions include extending the present classification to higher-dimensional, fractional-order, and coupled quasilinear diffusion-reaction models, as well as developing computer-assisted or AI-assisted symbolic tools for automatic symmetry classification, including the generation of determining equations and equivalence transformations.

A natural direction for future work is to extend the present classification to fractional diffusion-reaction equations. Guo et al. [22] studied, under suitable assumptions, a singular Caputo-Hadamard fractional boundary value problem of the form

$${}^{H}D^{\beta}_{1+}\left(\phi_p\left({}^{CH}D^{\gamma}_{1+}u\right)\right)(t) + f(t, u(t), u'(t)) = 0, \qquad \phi_p(z) = |z|^{p-2}z,$$

subject to infinite-point nonlocal boundary conditions. By constructing Green functions and an operator on a suitable cone, they obtained a unique positive solution together with an iterative approximation procedure, convergence results, and error estimates. Motivated by this framework, one possible fractional extension of the present equation is

$${}^{C}D^{\alpha}_{t}u = \left(\Phi(u)(u_x)^n\right)_x + F(u), \qquad 0 < \alpha < 1.$$

It would be of interest to determine whether analogs of the power and exponential constitutive forms identified in the present classification remain distinguished in the fractional setting, and to develop the corresponding equivalence transformations, symmetry reductions, and invariant solutions.

Author contributions

Khalid Ali Alanezy: Conceptualization, methodology, formal analysis, validation, supervision, writing—original draft, writing—review and editing; Ali Raza: Conceptualization, methodology, formal analysis, validation, writing—original draft, writing—review and editing. Both authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used AI tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Appendix

This appendix provides a detailed proof of the construction of the one-dimensional optimal systems. Each admitted Lie algebra is treated separately by first listing its full commutator table and then computing the corresponding adjoint table. These adjoint actions are then used to simplify a general linear combination of the basis generators and obtain the inequivalent representatives of the optimal system.

Proof.
Case 0: Optimal system for the principal case. Consider a general vector $X \in \mathscr{L}_2$ in the Lie algebra

$$X = a_1X_1 + a_2X_2,$$

where a_1 and a_2 are constants. So every one-dimensional subalgebra in this case has the form given by

$$\langle X \rangle = \langle a_1X_1 + a_2X_2 \rangle .$$

Since $[X_1, X_2] = 0$, the commutator table is

$[X_i, X_j]$	$\mathbf{X_1}$	$\mathbf{X_2}$
$\mathbf{X_1}$	0	0
$\mathbf{X_2}$	0	0

The adjoint action is given by

$$\mathrm{Ad}(e^{\epsilon X_i})X_j = X_j - \epsilon[X_i, X_j] + \frac{\epsilon^2}{2}[X_i, [X_i, X_j]] - \cdots .$$

Since all commutators are zero, we get

$$\mathrm{Ad}(e^{\epsilon X_1})X_1 = X_1, \quad \mathrm{Ad}(e^{\epsilon X_1})X_2 = X_2, \quad \mathrm{Ad}(e^{\epsilon X_2})X_1 = X_1, \quad \mathrm{Ad}(e^{\epsilon X_2})X_2 = X_2.$$

Therefore, the adjoint action leaves every generator unchanged. Hence, $\mathrm{Ad}(g)X = X$ for every group element g. The possible cases are as follows.
Case 0-(1): $a_1 \neq 0, a_2 = 0$. Under the adjoint action for arbitrary ϵ on X, we have

$$X = a_1X_1 \sim X_1.$$

Thus, one representative of the optimal system is

$$\widetilde{X}^1 = \langle X_1 \rangle .$$

Case 0-(2): $a_1 = 0, a_2 \neq 0$. Under the adjoint action for arbitrary ϵ on X, we get

$$X = a_2X_2 \sim X_2.$$

Thus, the second representative of an optimal system is

$$\widetilde{X}^2 = \langle X_2 \rangle .$$

Case 0-(3): $a_1 \neq 0, a_2 \neq 0$. Under the adjoint action for arbitrary ϵ on X, we get

$$X = a_1X_1 + a_2X_2 \sim X_1 + \frac{a_2}{a_1}X_2 \sim X_1 + a\,X_2,$$

where $a = a_2/a_1$. Thus, the third representative of an optimal system is

$$\widetilde{X}^3 = \langle X_1 + aX_2 \rangle .$$

Case I-(1): Optimal system for power diffusivity. Let $A = (n + 1)(1 - r)$ and $B = m + n - r$. Then X_3 can be written as

$$X_3 = At\partial_t + Bx\partial_x + (n + 1)u\partial_u,$$

with nonzero commutators

$$[X_1, X_3] = AX_1, \quad [X_2, X_3] = BX_2.$$

The commutator table is given by

$[X_i, X_j]$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$
$\mathbf{X_1}$	0	0	AX_1
$\mathbf{X_2}$	0	0	BX_2
$\mathbf{X_3}$	$-AX_1$	$-BX_2$	0

The adjoint actions are required to find the optimal system of subalgebras given by

$$\begin{aligned}
&\mathrm{Ad}(e^{\epsilon X_1})X_3 = X_3 - \epsilon AX_1,\\
&\mathrm{Ad}(e^{\epsilon X_2})X_3 = X_3 - \epsilon BX_2,\\
&\mathrm{Ad}(e^{\epsilon X_3})X_1 = e^{A\epsilon}X_1, \quad \mathrm{Ad}(e^{\epsilon X_3})X_2 = e^{B\epsilon}X_2.
\end{aligned}$$

The corresponding adjoint representation is constructed and presented in the following table.

Ad	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$
$\mathbf{X_1}$	X_1	X_2	$X_3 - \epsilon A\, X_1$
$\mathbf{X_2}$	X_1	X_2	$X_3 - \epsilon B\, X_2$
$\mathbf{X_3}$	$e^{\epsilon A}X_1$	$e^{\epsilon B}X_2$	X_3

Consider a general element $X \in \mathscr{L}_3$ in the Lie algebra

$$X = a_1X_1 + a_2X_2 + a_3X_3,$$

where a_1, a_2, and a_3 are constants.

Case I-(1a): $a_3 \neq 0, A \neq 0$**,** $B \neq 0$**.** Under the adjoint action of X_1 for arbitrary ϵ_1 on X, we get

$$\mathrm{Ad}\left(e^{\epsilon_1 X_1}\right) X = (a_1 - a_3\epsilon_1 A)\, X_1 + a_2X_2 + X_3.$$

Choosing $\epsilon_1 = \frac{a_1}{a_3 A}, A \neq 0, a_3 \neq 0$, we get

$$\mathrm{Ad}\left(e^{\epsilon_1 X_1}\right) X = \widetilde{X} = a_2X_2 + X_3.$$

Similarly, by the adjoint action of X_2 for arbitrary ϵ_2 on $\widetilde{X}$, we have

$$\mathrm{Ad}\left(e^{\epsilon_2 X_2}\right)\widetilde{X} = (a_2 - \epsilon_2 B)X_2 + X_3.$$

By choosing $\epsilon_2 = \frac{a_2}{B}, B \neq 0$, we obtain

$$\widetilde{X}^1 = X_3.$$

Case I-(1b): $a_3 = 0, a_1 \neq 0$**,** $a_2 \neq 0$**.** Under the adjoint action of X_3 for arbitrary ϵ on X, we get

$$\begin{aligned}
\widetilde{X} &= a_1 X_1 + a_2 X_2,\\
\mathrm{Ad}(e^{\epsilon X_3})\widetilde{X} &= a_1 e^{A\epsilon} X_1 + a_2 e^{B\epsilon} X_2,\\
\mathrm{Ad}(e^{\epsilon X_3})\widetilde{X} &= X_1 + \frac{a_2}{a_1} e^{(B-A)\epsilon} X_2.
\end{aligned}$$

Choose

$$a = \frac{a_2}{a_1} e^{(B-A)\epsilon},$$

we have

$$\widetilde{X}^2 = X_1 + aX_2.$$

Case I-(1c): $a_3 = 0, a_1 \neq 0$**,** $a_2 = 0$**.** Under the adjoint action, we get

$$\widetilde{X}^3 = X_1.$$

Case I-(1d): $a_3 = 0, a_1 = 0$**,** $a_2 \neq 0$**.** Under the adjoint action, we get

$$\widetilde{X}^4 = X_2.$$

Case I-(2): Optimal system for power diffusivity. The commutator table in this case with nonzero commutator $[X_1, X_3] = kX_3$ is

$[X_i, X_j]$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$
$\mathbf{X_1}$	0	0	$k X_3$
$\mathbf{X_2}$	0	0	0
$\mathbf{X_3}$	$-k X_3$	0	0

Using the adjoint action representation

$$\mathrm{Ad}(e^{\epsilon X_i})X_j = X_j - \epsilon[X_i, X_j] + \frac{\epsilon^2}{2!}[X_i, [X_i, X_j]] - \cdots,$$

we obtain the corresponding adjoint actions given by

$$\begin{aligned}
&\mathrm{Ad}(e^{\epsilon X_1})X_1 = X_1, &&\mathrm{Ad}(e^{\epsilon X_1})X_2 = X_2, &&\mathrm{Ad}(e^{\epsilon X_1})X_3 = e^{k\epsilon}X_3,\\
&\mathrm{Ad}(e^{\epsilon X_2})X_1 = X_1, &&\mathrm{Ad}(e^{\epsilon X_2})X_2 = X_2, &&\mathrm{Ad}(e^{\epsilon X_2})X_3 = X_3,\\
&\mathrm{Ad}(e^{\epsilon X_3})X_1 = X_1 - k\epsilon X_3, &&\mathrm{Ad}(e^{\epsilon X_3})X_2 = X_2, &&\mathrm{Ad}(e^{\epsilon X_3})X_3 = X_3,
\end{aligned}$$

and presented in the following table:

Ad	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$
$\mathbf{X_1}$	X_1	X_2	$e^{\epsilon k}X_3$
$\mathbf{X_2}$	X_1	X_2	X_3
$\mathbf{X_3}$	$X_1 - \epsilon k\, X_3$	X_2	X_3

For a general element $X \in \mathscr{L}_3$ in the Lie algebra

$$X = a_1X_1 + a_2X_2 + a_3X_3,$$

the adjoint action of X_3 gives

$$\mathrm{Ad}\left(e^{\epsilon X_3}\right)\widetilde{X} = a_1X_1 + a_2X_2 + \left(a_3 - k\epsilon a_1\right)X_3.$$

If $a_1 \neq 0$, choose

$$\epsilon = \frac{a_3}{ka_1}, \quad a_1 \neq 0.$$

Therefore, we have

$$\widetilde{X}^1 = X_1 + a_2X_2, \quad a_2 \neq 0,$$
$$\widetilde{X}^2 = X_1, \quad a_2 = 0.$$

If $a_1 = 0$, then we get

$$\widetilde{X} = a_2X_2 + a_3X_3.$$

This gives the remaining cases

$$\widetilde{X}^3 = X_2, \quad \widetilde{X}^4 = X_3, \quad \widetilde{X}^5 = X_2 + aX_3.$$

Case I-(3): Optimal system for power diffusivity. Since the commutator relations and the corresponding adjoint representation for this case are identical to those obtained in the preceding case, the classification follows in the same manner. Consequently, the one-dimensional optimal system of subalgebras is also the same as that of the previous case.

Case I-(4): Optimal system for power diffusivity. Let $A = m + n - 1, \quad B = n + 1$. Then X_3 and X_4 can be written as

$$X_3 = Ax\partial_x + Bu\partial_u,$$
$$X_4 = e^{-\varepsilon_2 At}\partial_t + \varepsilon_2 e^{-\varepsilon_2 At}u\partial_u,$$

with the nonzero commutators

$$[X_1, X_4] = -AX_4, \quad [X_2, X_3] = AX_2,$$

which provides the commutator table for this case

$[X_i, X_j]$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$
$\mathbf{X_1}$	0	0	0	$-A\,X_4$
$\mathbf{X_2}$	0	0	$A\,X_2$	0
$\mathbf{X_3}$	0	$-A\,X_2$	0	0
$\mathbf{X_4}$	$A\,X_4$	0	0	0

Using the adjoint action representation

$$\mathrm{Ad}(e^{\epsilon X_i})X_j = X_j - \epsilon[X_i, X_j] + \frac{\epsilon^2}{2!}[X_i, [X_i, X_j]] - \cdots,$$

we obtain the corresponding adjoint actions presented in table.

Ad	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$
$\mathbf{X_1}$	X_1	X_2	X_3	$e^{\epsilon A}X_4$
$\mathbf{X_2}$	X_1	X_2	$X_3 - \epsilon A\,X_2$	X_4
$\mathbf{X_3}$	X_1	$e^{\epsilon A}X_2$	X_3	X_4
$\mathbf{X_4}$	$X_1 - \epsilon A\,X_4$	X_2	X_3	X_4

Now take the general element $X \in \mathscr{L}_4$ in the Lie algebra

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4,$$

under the adjoint action of X_4, we have

$$\mathrm{Ad}(e^{\epsilon X_4})X = a_1(X_1 - A\,\epsilon X_4) + a_2X_2 + a_3X_3 + a_4X_4.$$

Equivalently, we have

$$\mathrm{Ad}(e^{\epsilon X_4})X = a_1X_1 + a_2X_2 + a_3X_3 + (a_4 - A\,\epsilon\,a_1)X_4.$$

If $a_1 \neq 0$, choose

$$\epsilon = \frac{a_4}{Aa_1}.$$

Then

$$a_4 - A\epsilon a_1 = 0.$$

Thus, the coefficient of X_4 is removed.

Similarly, using the adjoint action of X_2, we obtain

$$\mathrm{Ad}(e^{\epsilon X_2})\widetilde{X} = a_1X_1 + (a_2 - A\,\epsilon\, a_3)X_2 + a_3X_3 + a_4X_4.$$

If $a_3 \neq 0$, choose

$$\epsilon = \frac{a_2}{Aa_3}.$$

Then

$$a_2 - A\epsilon a_3 = 0.$$

Thus, the coefficient of X_2 is removed.

Case I-(4a): $a_1 \neq 0, a_3 \neq 0$**.** Using the adjoint action of X_4 and X_2, we have

$$\widetilde{X} \sim X_1 + aX_3.$$

Hence, we have

$$X^1 = X_1 + aX_3.$$

Case I-(4b): $a_1 \neq 0, a_3 = 0$**.** Using the adjoint action of X_4, we obtain

$$\widetilde{X} \sim X_1 + aX_2.$$

Hence, we have

$$X^2 = X_1 + aX_2.$$

Now X_3 acts on X_2 as

$$\mathrm{Ad}(e^{\epsilon X_3})X_2 = e^{A\,\epsilon}X_2.$$

Therefore, if $a \neq 0$ and for $\epsilon = \frac{1}{A}ln|\pm\frac{1}{a}|$, we get

$$X^2 = X_1, \quad X^3 = X_1 \pm X_2.$$

Case I-(4c): $a_1 = 0, a_3 \neq 0$**.** Using the adjoint action of X_2, we obtain

$$\widetilde{X} \sim X_3 + aX_4.$$

Hence,

$$X^3 = X_3 + aX_4.$$

Now X_1 acts on X_4 as

$$\mathrm{Ad}(e^{\epsilon X_1})X_4 = e^{A\,\epsilon}X_4.$$

Hence, if $a \neq 0$ and for $\epsilon = \frac{1}{A}ln|\pm\frac{1}{a}|$, we obtain

$$X^4 = X_3, \quad X^5 = X_3 \pm X_4.$$

Case I-(4d): $a_1 = 0, a_3 = 0$**.** In this case

$$\widetilde{X} = a_2X_2 + a_4X_4.$$

If $a_2 \neq 0$, then

$$\widetilde{X} \sim X_2 + aX_4.$$

Hence,

$$X^6 = X_2 + aX_4.$$

When $a_2 = 0, a_4 \neq 0$, we obtain

$$X^7 = X_4.$$

When $a_2 \neq 0, a_4 = 0$, we obtain

$$X^8 = X_2.$$

Case I-(5): Optimal system for power diffusivity. Let $B = n + 1$. Then X_4 becomes

$$X_4 = Bt\partial_t + x\partial_x + \varepsilon_2 Bt\, u\partial_u,$$

with nonzero commutators

$$[X_1, X_4] = B(X_1 + X_3), \quad [X_2, X_4] = X_2.$$

Therefore, the commutator table is

$[X_i, X_j]$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$
$\mathbf{X_1}$	0	0	0	$B(X_1 + X_3)$
$\mathbf{X_2}$	0	0	0	X_2
$\mathbf{X_3}$	0	0	0	0
$\mathbf{X_4}$	$-B(X_1 + X_3)$	$-X_2$	0	0

Using

$$\mathrm{Ad}(e^{\epsilon X_i})X_j = X_j - \epsilon[X_i, X_j] + \frac{\epsilon^2}{2!}[X_i, [X_i, X_j]] - \cdots,$$

we obtain

$$\begin{aligned}
\mathrm{Ad}(e^{\epsilon X_1})X_4 &= X_4 - B\epsilon(X_1 + X_3),\\
\mathrm{Ad}(e^{\epsilon X_2})X_4 &= X_4 - \epsilon X_2,\\
\mathrm{Ad}(e^{\epsilon X_4})X_2 &= e^{\epsilon} X_2,
\end{aligned}$$

and

$$\mathrm{Ad}(e^{\epsilon X_4})X_1 = e^{B\epsilon} X_1 + (e^{B\epsilon} - 1)X_3.$$

The adjoint table is therefore

$\mathrm{Ad}(e^{\epsilon X_i})X_j$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$
$\mathbf{X_1}$	X_1	X_2	X_3	$X_4 - B\,\epsilon\,(X_1 + X_3)$
$\mathbf{X_2}$	X_1	X_2	X_3	$X_4 - \epsilon X_2$
$\mathbf{X_3}$	X_1	X_2	X_3	X_4
$\mathbf{X_4}$	$e^{B\,\epsilon}X_1 + (e^{B\,\epsilon} - 1)X_3$	$e^{\epsilon}X_2$	X_3	X_4

Now take the general element $X \in \mathscr{L}_4$ in the Lie algebra

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4.$$

We begin with the action of X_2 as follows:

$$\mathrm{Ad}(e^{\epsilon X_2})X = a_1X_1 + (a_2 - \epsilon\,a_4)X_2 + a_3X_3 + a_4X_4.$$

If $a_4 \neq 0$, choose $\epsilon = a_2/a_4$. Then

$$a_2 - \epsilon a_4 = 0.$$

Thus

$$\widetilde{X} \sim a_1X_1 + a_3X_3 + X_4.$$

Next, under the action of X_1, we have

$$\mathrm{Ad}(e^{\epsilon X_1})\widetilde{X} = (a_1 - B\,\epsilon\,a_4)X_1 + (a_3 - B\,\epsilon\,a_4)X_3 + X_4.$$

Choosing

$$\epsilon = \frac{a_1}{Ba_4},$$

we obtain

$$\widetilde{X} \sim (a_3 - a_1)\,X_3 + X_4.$$

Hence, by setting $a_3 - a_1 = a$, we get

$$\widetilde{X} \sim X_4 + aX_3.$$

Thus, the first class is

$$X^1 = X_4 + aX_3.$$

For $a_4 = 0$, we have

$$\widetilde{X} = a_1X_1 + a_2X_2 + a_3X_3.$$

Case I-(5a): $a_1 \neq 0, a_2 = 0$**.** In this case

$$\widetilde{X} = a_1X_1 + a_3X_3.$$

Dividing by a_1, we get

$$\widetilde{X} \sim X_1 + \frac{a_3}{a_1}X_3.$$

Let
$$a = \frac{a_3}{a_1}.$$
Therefore, we have
$$X^2 = X_1 + aX_3.$$
Case I-(5b): $a_1 = 0, a_2 \neq 0$. In this case
$$\widetilde{X} = a_2X_2 + a_3X_3.$$
Dividing by a_2, we get
$$\widetilde{X} \sim X_2 + \frac{a_3}{a_2}X_3.$$
Let
$$a = \frac{a_3}{a_2}.$$
Therefore,
$$X^3 = X_2 + aX_3.$$
Case I-(5c): $a_1 \neq 0, a_2 \neq 0$. In this case
$$\widetilde{X} = a_1X_1 + a_2X_2 + a_3X_3.$$
Now, under the adjoint action of X_4,
$$\begin{aligned}
\mathrm{Ad}(e^{\epsilon X_4})X_1 &= e^{B\epsilon}X_1 + (e^{B\epsilon} - 1)X_3,\\
\mathrm{Ad}(e^{\epsilon X_4})X_2 &= e^{\epsilon}X_2,\\
\mathrm{Ad}(e^{\epsilon X_4})X_3 &= X_3.
\end{aligned}$$
Thus,
$$\mathrm{Ad}(e^{\epsilon X_4})\widetilde{X} = a_1e^{B\epsilon}X_1 + a_2e^{\epsilon}X_2 + [a_3 + a_1(e^{B\epsilon} - 1)]X_3.$$
Since both $a_1 \neq 0$ and $a_2 \neq 0$, we set the coefficients of X_1 and X_2 to 1. Hence, the representative becomes
$$\widetilde{X} \sim X_1 + X_2 + aX_3.$$
Therefore,
$$X^4 = X_1 + X_2 + aX_3.$$
Case I-(5d): $a_1 = 0, a_2 = 0, a_3 \neq 0$. Thus, we get
$$\widetilde{X} \sim X_3.$$
Therefore,
$$X^5 = X_3.$$
Case II-(1): Optimal system for exponential diffusivity. The nonzero commutators in this case are
$$[X_1, X_3] = -b(n+1)X_1, \quad [X_2, X_3] = (a-b)X_2.$$

Equivalently

$$[X_3, X_1] = b(n+1)X_1, \quad [X_3, X_2] = -(a-b)X_2.$$

Thus, the commutator table is given by

$[X_i, X_j]$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$
$\mathbf{X_1}$	0	0	$-b(n+1)\,X_1$
$\mathbf{X_2}$	0	0	$(a-b)\,X_2$
$\mathbf{X_3}$	$b(n+1)\,X_1$	$-(a-b)\,X_2$	0

Using

$$\mathrm{Ad}(e^{\epsilon X_i})X_j = X_j - \epsilon[X_i, X_j] + \frac{\epsilon^2}{2!}[X_i, [X_i, X_j]] - \cdots,$$

we obtain the following adjoint actions.

$$\begin{aligned}
\mathrm{Ad}(e^{\epsilon X_1})X_3 &= X_3 + b(n+1)\,\epsilon X_1,\\
\mathrm{Ad}(e^{\epsilon X_2})X_3 &= X_3 - (a-b)\,\epsilon X_2,\\
\mathrm{Ad}(e^{\epsilon X_3})X_1 &= e^{-b(n+1)\,\epsilon}X_1,\\
\mathrm{Ad}(e^{\epsilon X_3})X_2 &= e^{(a-b)\,\epsilon}X_2.
\end{aligned}$$

Therefore, the adjoint table is

Ad	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$
$\mathbf{X_1}$	X_1	X_2	$X_3 + \epsilon\, b(n+1)\,X_1$
$\mathbf{X_2}$	X_1	X_2	$X_3 - \epsilon\,(a-b)\,X_2$
$\mathbf{X_3}$	$e^{-\epsilon\, b(n+1)}\,X_1$	$e^{\epsilon\,(a-b)}\,X_2$	X_3

Take the general element $X \in \mathscr{L}_3$ in the Lie algebra

$$\widetilde{X} = a_1X_1 + a_2X_2 + a_3X_3.$$

Case II-(1a): $a_3 \neq 0$**.** Using the action of X_1, we have

$$\begin{aligned}
\mathrm{Ad}(e^{\epsilon X_1})\widetilde{X} &= a_1X_1 + a_2X_2 + a_3(X_3 + b(n+1)\,\epsilon X_1),\\
\mathrm{Ad}(e^{\epsilon X_1})\widetilde{X} &= (a_1 + b(n+1)\,\epsilon\, a_3)X_1 + a_2X_2 + a_3X_3.
\end{aligned}$$

Choose

$$\epsilon = -\frac{a_1}{b(n+1)a_3}, \quad b(n+1) \neq 0.$$

Then the coefficient of X_1 vanishes. Next, using the action of X_2, we get

$$\mathrm{Ad}(e^{\epsilon X_2})\widetilde{X} = a_2X_2 + a_3(X_3 - (a-b)\,\epsilon X_2),$$

$$\mathrm{Ad}(e^{\epsilon X_2})\widetilde{X} = (a_2 - (a-b)\,\epsilon\, a_3)X_2 + a_3X_3.$$

Choose

$$\epsilon = \frac{a_2}{(a-b)a_3}, \quad a - b \neq 0.$$

Then the coefficient of X_2 becomes zero. Therefore, we have

$$\widetilde{X} \sim a_3X_3.$$

Thus, we obtain the first representative

$$X^1 = X_3.$$

Case II-(1b): $a_3 = 0$**.** In this case

$$\widetilde{X} = a_1X_1 + a_2X_2.$$

If $a_1 \neq 0, a_2 = 0$, then $\widetilde{X} \sim X_1$. Hence

$$X^2 = X_1.$$

If $a_1 = 0, a_2 \neq 0$, then $\widetilde{X} \sim X_2$. So

$$X^3 = X_2.$$

If $a_1 \neq 0, a_2 \neq 0$, then

$$\widetilde{X} \sim X_1 + aX_2,$$

where

$$a = \frac{a_2}{a_1}.$$

Therefore

$$X^4 = X_1 + aX_2.$$

Case II-(2): Optimal system for exponential diffusivity. The nonzero commutators are

$$[X_1, X_3] = kX_3, \quad [X_3, X_1] = -kX_3.$$

Thus, the commutator table is

$[X_i, X_j]$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$
$\mathbf{X_1}$	0	0	$k\,X_3$
$\mathbf{X_2}$	0	0	0
$\mathbf{X_3}$	$-k\,X_3$	0	0

The nontrivial adjoint actions in this case are

$$\mathrm{Ad}(e^{\epsilon X_1})X_3 = e^{-k\,\epsilon}X_3,$$
$$\mathrm{Ad}(e^{\epsilon X_3})X_1 = X_1 - \epsilon[X_3, X_1] = X_1 + k\,\epsilon X_3.$$

Therefore, the adjoint table is

Ad	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$
$\mathbf{X_1}$	X_1	X_2	$e^{-k\epsilon}X_3$
$\mathbf{X_2}$	X_1	X_2	X_3
$\mathbf{X_3}$	$X_1 + k\,\epsilon X_3$	X_2	X_3

Take the general element $X \in \mathscr{L}_3$ in the Lie algebra

$$\widetilde{X} = a_1X_1 + a_2X_2 + a_3X_3.$$

Using the action of X_3, we get

$$\mathrm{Ad}(e^{\epsilon X_3})\widetilde{X} = a_1(X_1 + k\,\epsilon X_3) + a_2X_2 + a_3X_3,$$
$$\mathrm{Ad}(e^{\epsilon X_3})\widetilde{X} = a_1X_1 + a_2X_2 + (a_3 + k\,\epsilon\,a_1)X_3.$$

If $a_1 \neq 0$ and $a_2 \neq 0$, choose

$$\epsilon = -\frac{a_3}{ka_1} \implies a_3 + k\epsilon a_1 = 0.$$

Therefore

$$\widetilde{X}^1 \sim a_1X_1 + a_2X_2 \sim X_1 + aX_2.$$

If $a_1 \neq 0$ and $a_2 = 0$ then

$$\widetilde{X}^2 = X_1.$$

If $a_1 = 0$, then

$$\widetilde{X} = a_2X_2 + a_3X_3.$$

This gives the following classes:

$$\widetilde{X}^3 = X_2, \quad \widetilde{X}^4 = X_3, \quad \widetilde{X}^5 = X_2 + aX_3.$$

□

Verification of the explicit invariant solutions

We verify the explicit invariant solutions obtained above by direct substitution into the expanded form of the governing equation. The main equation is

$$u_t = \left(\Phi(u)(u_x)^n\right)_x + F(u),$$

or equivalently

$$u_t = n\Phi(u)(u_x)^{n-1}u_{xx} + \Phi'(u)(u_x)^{n+1} + F(u).$$

For the power-diffusivity cases considered below, we have

$$\Phi(u) = \varepsilon_1 u^m,$$

and hence the expanded equation becomes

$$u_t = \varepsilon_1 n u^m (u_x)^{n-1} u_{xx} + \varepsilon_1 m u^{m-1} (u_x)^{n+1} + F(u).$$

Equivalently, we have

$$u_t = \varepsilon_1 \big(u^m (u_x)^n\big)_x + F(u).$$

Case I-(3). Here

$$\Phi(u) = \varepsilon_1 u^m, \qquad F(u) = \varepsilon_2 u \ln|u|, \qquad m + n = 1.$$

The explicit invariant solution is

$$u(x,t) = \exp\Big(\delta x\, e^{\varepsilon_2 t} + C_1\, e^{\varepsilon_2 t} + \frac{\varepsilon_1 \delta^{n+1}}{\varepsilon_2 n}\, e^{\varepsilon_2 (n+1) t}\Big).$$

Set

$$S(x,t) = \delta x e^{\varepsilon_2 t} + C_1 e^{\varepsilon_2 t} + \frac{\varepsilon_1 \delta^{n+1}}{\varepsilon_2 n} e^{\varepsilon_2 (n+1) t}, \qquad u = e^S.$$

Then

$$u_x = S_x u = \delta e^{\varepsilon_2 t} u, \qquad u_{xx} = S_x^2 u = \delta^2 e^{2\varepsilon_2 t} u, \qquad u_t = S_t u.$$

Since $m + n = 1$, we obtain

$$\varepsilon_1 n u^m (u_x)^{n-1} u_{xx} + \varepsilon_1 m u^{m-1} (u_x)^{n+1} = \varepsilon_1 (m+n) \big(\delta e^{\varepsilon_2 t}\big)^{n+1} u.$$

Thus

$$\varepsilon_1 n u^m (u_x)^{n-1} u_{xx} + \varepsilon_1 m u^{m-1} (u_x)^{n+1} = \varepsilon_1 \delta^{n+1} e^{\varepsilon_2 (n+1) t} u.$$

Moreover, since $u = e^S$, we have

$$\ln u = S.$$

Therefore, the right-hand side of the PDE is

$$\varepsilon_1 \delta^{n+1} e^{\varepsilon_2 (n+1) t} u + \varepsilon_2 u S.$$

On the other hand

$$S_t = \varepsilon_2 \delta x e^{\varepsilon_2 t} + \varepsilon_2 C_1 e^{\varepsilon_2 t} + \frac{\varepsilon_1 \delta^{n+1} (n+1)}{n} e^{\varepsilon_2 (n+1) t}.$$

Also

$$\varepsilon_2 S = \varepsilon_2 \delta x e^{\varepsilon_2 t} + \varepsilon_2 C_1 e^{\varepsilon_2 t} + \frac{\varepsilon_1 \delta^{n+1}}{n} e^{\varepsilon_2 (n+1) t}.$$

Hence

$$S_t = \varepsilon_2 S + \varepsilon_1 \delta^{n+1} e^{\varepsilon_2 (n+1) t}.$$

Consequently

$$u_t = S_t u = \varepsilon_1 \delta^{n+1} e^{\varepsilon_2 (n+1) t} u + \varepsilon_2 u S,$$

which agrees with the right-hand side of the PDE. Hence, the solution of Case I-(3) satisfies the original equation.

Case I-(4): With $m+n \neq 0$. Here

$$\Phi(u)=\varepsilon_1 u^m, \qquad F(u)=\varepsilon_2 u, \qquad m+n\neq 1.$$

The explicit invariant solution is

$$u(x,t)=e^{\varepsilon_2 t}\left(C_1 x+C_2\right)^{\frac{n}{m+n}}, \qquad m+n\neq 0.$$

Set

$$u(x,t)=e^{\varepsilon_2 t} f(x), \qquad f(x)=\left(C_1 x+C_2\right)^{\frac{n}{m+n}}.$$

Then

$$u_t=\varepsilon_2 u, \qquad u_x=e^{\varepsilon_2 t} f', \qquad u_{xx}=e^{\varepsilon_2 t} f''.$$

Therefore

$$\varepsilon_1\left(u^m (u_x)^n\right)_x=\varepsilon_1 e^{\varepsilon_2 (m+n) t}\left(f^m (f')^n\right)'.$$

Let

$$p=\frac{n}{m+n}.$$

Then

$$f(x)=\left(C_1 x+C_2\right)^p, \qquad f'=pC_1\left(C_1 x+C_2\right)^{p-1}.$$

Hence

$$f^m (f')^n=(pC_1)^n\left(C_1 x+C_2\right)^{pm+n(p-1)}.$$

Since

$$pm+n(p-1)=p(m+n)-n=n-n=0,$$

we have

$$f^m (f')^n=(pC_1)^n,$$

which is constant. Therefore

$$\left(f^m (f')^n\right)'=0.$$

It follows that

$$\varepsilon_1\left(u^m (u_x)^n\right)_x=0.$$

Thus, the right-hand side of the PDE becomes

$$0+\varepsilon_2 u=\varepsilon_2 u=u_t.$$

Hence, the Case I-(4) power-law solution satisfies the original equation.

Case I-(4): With $m+n=0$. In the subcase $m+n=0$, the explicit invariant solution is

$$u(x,t)=C_1 e^{\varepsilon_2 t+C_2 x}.$$

Set

$$u(x,t)=e^{\varepsilon_2 t} f(x), \qquad f(x)=C_1 e^{C_2 x}.$$

Then

$$f'=C_2 f.$$

Since $m + n = 0$, we have $m = -n$. Hence

$$f^m (f')^n = f^{-n} (C_2 f)^n = C_2^n.$$

Thus

$$(f^m (f')^n)' = 0.$$

As before

$$u_t = \varepsilon_2 u, \qquad \varepsilon_1 \left(u^m (u_x)^n\right)_x = 0.$$

Therefore

$$u_t = \varepsilon_2 u = \varepsilon_1 \left(u^m (u_x)^n\right)_x + \varepsilon_2 u.$$

Hence, the exponential solution in Case I-(4) satisfies the original equation.

Case I-(5). Here

$$\Phi(u) = \varepsilon_1 u^m, \qquad F(u) = \varepsilon_2 u, \qquad m + n = 1.$$

The explicit invariant solution is

$$u(x, t) = C_1 \exp[a\, x + (\varepsilon_1\, a^{n+1} + \varepsilon_2)\, t].$$

Then

$$u_x = au, \qquad u_{xx} = a^2 u, \qquad u_t = \left(\varepsilon_1 a^{n+1} + \varepsilon_2\right) u.$$

Substituting into the diffusion part gives

$$\varepsilon_1 n u^m (u_x)^{n-1} u_{xx} + \varepsilon_1 m u^{m-1} (u_x)^{n+1} = \varepsilon_1 (n + m) a^{n+1} u.$$

Since $m + n = 1$, this reduces to

$$\varepsilon_1 a^{n+1} u.$$

Adding the reaction term, we obtain

$$\varepsilon_1 a^{n+1} u + \varepsilon_2 u = \left(\varepsilon_1 a^{n+1} + \varepsilon_2\right) u = u_t.$$

Therefore, the Case I-(5) solution satisfies the original equation.

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