AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(9): 27947–27968.
DOI: 10.3934/math.20261116
Received: 20 May 2026
Revised: 03 August 2026
Accepted: 17 August 2026
Published: 02 September 2026

Research article

On the structure and spectral properties of an extension of normal operators

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Abstract: Let $\mathbb{H}$ be a complex Hilbert space, and let $\mathcal{P}$ be a non-zero positive semi-definite operator on $\mathbb{H}$. Denote by $\mathscr{L}_{\mathcal{P}}(\mathbb{H})$ the class of bounded linear operators on $\mathbb{H}$ admitting a $\mathcal{P}$-adjoint, and by $\mathrm{Q}^{\sharp_{\mathcal{P}}}$ the distinguished $\mathcal{P}$-adjoint of $\mathrm{Q} \in \mathscr{L}_{\mathcal{P}}(\mathbb{H})$. In this paper, we study the algebraic and spectral properties of $\langle\mathcal{P}\rangle$-normal operators. By definition, these operators satisfy the relation $\mathcal{P}\mathrm{Q}\mathrm{Q}^{\sharp_{\mathcal{P}}} = \mathcal{P}\mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{Q}$. We establish equivalent metric and algebraic characterizations for this class. We prove that every $\langle\mathcal{P}\rangle$-normal operator is $\mathcal{P}$-normaloid. To overcome the restrictive closed range assumption of $\mathcal{P}$ in classical spectral theory, we introduce a topological formulation of $\mathcal{P}$-invertibility and the $\mathcal{P}$-spectrum. Using this framework, without assuming that the range of $\mathcal{P}$ is closed, we generalize several $\mathcal{P}$-numerical radius inequalities and spectral inclusion theorems. Finally, we establish the null space symmetry and prove that the $\mathcal{P}$-eigenspaces of these operators are mutually $\mathcal{P}$-orthogonal.

Keywords: positive operator; semi-inner product; $\langle\mathcal{P}\rangle$-normal operator; $\mathcal{P}$-normaloid; $\mathcal{P}$-numerical radius; topological $\mathcal{P}$-spectrum
Mathematics Subject Classification: 47B65, 47A12, 46C05, 47A10

1. Introduction

The concept of normal operators plays a fundamental role in classical Hilbert space theory [1–3]. Due to its importance, many researchers have generalized the classical idea of normality. Well-known extensions include polynomially normal operators [4, 5], various classes of n-normal operators [6–8], and C-normal operators [9].

Another natural extension arises from the use of a semi-inner product. Let $\mathcal{P}$ be a non-zero positive operator acting on a complex Hilbert space $\mathbb{H}$.

Building on this framework, Arias, Corach, and Gonzalez developed metric properties and partial isometries in semi-Hilbertian spaces [10–12]. Subsequently, Baklouti, Feki, and Sid Ahmed extended classical operator concepts such as the joint numerical range [13] and joint normality [14] to this setting.

The semi-inner product structure naturally induces an adjoint operation. Arias et al. utilized this generalized adjoint to introduce the class of $\mathcal{P}$-normal operators [12]. Assume that Q is a bounded linear operator on $\mathbb{H}$ admitting a $\mathcal{P}$-adjoint. Then Q is said to be $\mathcal{P}$-normal if its distinguished $\mathcal{P}$-adjoint $Q^{\sharp_{\mathcal{P}}}$ satisfies the commutation relation $QQ^{\sharp_{\mathcal{P}}} = Q^{\sharp_{\mathcal{P}}}Q$. This class of operators was later investigated by Saddi in [15]. Following this, various extensions of $\mathcal{P}$-normality were explored, such as (α,β)-$\mathcal{P}$-normal operators [16] and (n,m)-$\mathcal{P}$-normal operators [17, 18].

Recently, Sen, Birbonshi, and Paul introduced a broader class known as $\langle\mathcal{P}\rangle$-normal operators [19]. Since this is the main class of operators that we investigate, we formally state its definition here. A bounded linear operator Q on $\mathbb{H}$ that admits a $\mathcal{P}$-adjoint is called $\langle\mathcal{P}\rangle$-normal if it satisfies the condition:

$$\mathcal{P}QQ^{\sharp_{\mathcal{P}}} = \mathcal{P}Q^{\sharp_{\mathcal{P}}}Q.$$

The set of all $\langle\mathcal{P}\rangle$-normal operators is denoted by $\mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$. This new class weakens the strict commutation required in standard $\mathcal{P}$-normality. Thus, every $\mathcal{P}$-normal operator is $\langle\mathcal{P}\rangle$-normal, but the converse is not generally true (see [19]).

The motivation for studying this new class relies on the concept of $\mathcal{P}$-commutativity. Two operators Q and R are said to $\mathcal{P}$-commute if $\mathcal{P}QR = \mathcal{P}RQ$. The connection between $\mathcal{P}$-bounded operators and operators acting on the Hilbert space $\mathbf{R}(\sqrt{\mathcal{P}})$ plays a crucial role. Here, $\sqrt{\mathcal{P}}$ denotes the unique positive square root of $\mathcal{P}$. The Hilbert space $\mathbf{R}(\sqrt{\mathcal{P}})$ is constructed by completing the canonical quotient space associated with $\mathcal{P}$. More precisely, it is the range of $\sqrt{\mathcal{P}}$ equipped with a suitable inner product that provides a rigorous Hilbert space structure. By using $\mathcal{P}$-commutativity and the canonical space $\mathbf{R}(\sqrt{\mathcal{P}})$, we establish a direct equivalence between the $\langle\mathcal{P}\rangle$-normality of an operator and the classical normality of its induced operator. This equivalence provides a flexible framework for investigating the algebraic properties of these operators.

Another main target of this paper is to investigate spectral theory in semi-Hilbertian spaces. Baklouti and Namouri [20] introduced the concept of $\mathcal{P}$-invertibility. By their definition, an operator Q is $\mathcal{P}$-invertible if there exists a bounded operator R such that $\mathcal{P}QR = \mathcal{P}RQ = \mathcal{P}$. This strict algebraic definition demands that the resolvent operator maps the dense subspace $\mathrm{Im}(\mathcal{P})$ exactly back into itself. Consequently, the associated $\mathcal{P}$-spectrum coincides with the classical spectrum of the induced operator only when the range of $\mathcal{P}$ is closed.

To resolve this limitation, we introduce a topological $\mathcal{P}$-invertibility. We define an operator to be topologically $\mathcal{P}$-invertible if it is bounded below with respect to the $\mathcal{P}$-seminorm and has a $\mathcal{P}$-dense range. By applying this topological approach, we construct a topological $\mathcal{P}$-spectrum. This new spectrum without assuming that the range of $\mathcal{P}$ is closed, coincides with the classical spectrum on the Hilbert space $\mathbf{R}(\sqrt{\mathcal{P}})$.

The rest of this paper is organized as follows. Section 2 establishes the preliminary framework and fixes the notation. Section 3 presents our main results. First, we provide metric and algebraic characterizations of $\langle\mathcal{P}\rangle$-normal operators. Without assuming that the range of $\mathcal{P}$ is closed, this demonstrates that the class satisfies the $\mathcal{P}$-normaloid property. Next, we formulate a topological approach to $\mathcal{P}$-invertibility. By doing so, we construct a robust $\mathcal{P}$-spectrum that perfectly coincides with the classical spectrum of the induced operator. Applying this spectral equality, we generalize several $\mathcal{P}$-numerical radius inequalities and spectral inclusion theorems, removing the restrictive closed-range hypothesis. This generalizes key results established by Saddi [15] and by Sen, Birbonshi, and Paul [19]. Finally, we investigate the geometric structure of these operators. We establish the strict orthogonality and symmetry of their generalized $\mathcal{P}$-eigenspaces (defined as the $\mathcal{P}$-kernels associated with the translated operator $\mathrm{Q} - \xi I_{\mathbb{H}}$), and we prove that their $\mathcal{P}$-ascent is exactly one.

2. Preliminary results and notions

In this section, we review the fundamental concepts of semi-Hilbertian spaces. Let $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space equipped with the standard norm $\|\cdot\|$. Let $\mathscr{L}(\mathbb{H})$ be the set of all bounded linear operators on $\mathbb{H}$, and let $I_{\mathbb{H}}$ be the identity operator. Recall that an operator $\mathrm{U} \in \mathscr{L}(\mathbb{H})$ is unitary if $\mathrm{U}\mathrm{U}^* = \mathrm{U}^*\mathrm{U} = I_{\mathbb{H}}$. Let $\mathscr{L}^+(\mathbb{H})$ be the set of all non-zero positive semi-definite operators on $\mathbb{H}$. Throughout this paper, assume that $\mathcal{P} \in \mathscr{L}^+(\mathbb{H})$. The operator $\mathcal{P}$ induces a positive semi-definite sesquilinear form given by

$$\langle \mathrm{y}, \mathrm{z}\rangle_{\mathcal{P}} = \langle \mathcal{P}\mathrm{y}, \mathrm{z}\rangle, \quad \forall\, \mathrm{y}, \mathrm{z} \in \mathbb{H}.$$

This form defines a semi-norm $\|\mathrm{y}\|_{\mathcal{P}} = \sqrt{\langle \mathrm{y}, \mathrm{y}\rangle_{\mathcal{P}}}$. The corresponding unit $\mathcal{P}$-sphere is $\mathbb{S}^1_{\mathcal{P}} = \{\mathrm{y} \in \mathbb{H} : \|\mathrm{y}\|_{\mathcal{P}} = 1\}$. We use $\mathbb{N}$ for the set of positive integers and $\mathbb{N}_0$ for the set of non-negative integers. For any subset $S \subseteq \mathbb{H}$, its $\mathcal{P}$-orthogonal complement is defined as:

$$S^{\perp_{\mathcal{P}}} = \{\mathrm{y} \in \mathbb{H} : \langle \mathrm{y}, \mathrm{z}\rangle_{\mathcal{P}} = 0, \quad \forall\, \mathrm{z} \in S\}.$$

An operator $\mathrm{R} \in \mathscr{L}(\mathbb{H})$ is called a $\mathcal{P}$-adjoint of $\mathrm{Q} \in \mathscr{L}(\mathbb{H})$ if $\langle \mathrm{Q}\mathrm{y}, \mathrm{z}\rangle_{\mathcal{P}} = \langle \mathrm{y}, \mathrm{R}\mathrm{z}\rangle_{\mathcal{P}}$ for all $\mathrm{y}, \mathrm{z} \in \mathbb{H}$. This is algebraically equivalent to $\mathcal{P}\mathrm{R} = \mathrm{Q}^*\mathcal{P}$. By Douglas's theorem [21], the set of operators that possess a $\mathcal{P}$-adjoint is given by:

$$\mathscr{L}_{\mathcal{P}}(\mathbb{H}) = \{\mathrm{Q} \in \mathscr{L}(\mathbb{H}) : \mathrm{Im}(\mathrm{Q}^*\mathcal{P}) \subseteq \mathrm{Im}(\mathcal{P})\}.$$

For any $\mathrm{Q} \in \mathscr{L}_{\mathcal{P}}(\mathbb{H})$, the equation $\mathcal{P}\mathrm{R} = \mathrm{Q}^*\mathcal{P}$ admits a unique reduced solution denoted by $\mathrm{Q}^{\sharp_{\mathcal{P}}}$, which satisfies $\mathrm{Im}(\mathrm{Q}^{\sharp_{\mathcal{P}}}) \subseteq \overline{\mathrm{Im}(\mathcal{P})}$. The Moore-Penrose inverse of $\mathcal{P}$, denoted by $\mathcal{P}^\dagger$, is defined as the unique linear extension of $(\mathcal{P}|_{\mathrm{Ker}(\mathcal{P})^\perp})^{-1}$ to its domain $\mathcal{D}(\mathcal{P}^\dagger) = \mathrm{Im}(\mathcal{P}) + \mathrm{Im}(\mathcal{P})^\perp$ that satisfies $\mathrm{Ker}(\mathcal{P}^\dagger) = \mathrm{Im}(\mathcal{P})^\perp$. Following [11], $\mathcal{P}^\dagger$ is the unique solution of the following four equations:

$$(1)\ \mathcal{P}\mathrm{X}\mathcal{P} = \mathcal{P}, \quad (2)\ \mathrm{X}\mathcal{P}\mathrm{X} = \mathrm{X}, \quad (3)\ \mathrm{X}\mathcal{P} = \Pi_{\mathcal{P}}, \quad (4)\ \mathcal{P}\mathrm{X} = \Pi_{\mathcal{P}}|_{\mathcal{D}(\mathcal{P}^\dagger)},$$

where $\Pi_{\mathcal{P}}$ is the orthogonal projection onto $\overline{\mathrm{Im}(\mathcal{P})}$. Thus, the reduced solution can be explicitly written as $\mathrm{Q}^{\sharp_{\mathcal{P}}} = \mathcal{P}^\dagger\mathrm{Q}^*\mathcal{P}$. We have $(\mathrm{Q}^{\sharp_{\mathcal{P}}})^{\sharp_{\mathcal{P}}} = \Pi_{\mathcal{P}}\mathrm{Q}\Pi_{\mathcal{P}}$.

An operator $\mathrm{Q} \in \mathscr{L}(\mathbb{H})$ is *$\mathcal{P}$-bounded* if there exists a constant $\xi > 0$ such that $\|\mathrm{Q}\mathrm{y}\|_{\mathcal{P}} \le \xi\|\mathrm{y}\|_{\mathcal{P}}$ for all $\mathrm{y} \in \mathbb{H}$. The space of all $\mathcal{P}$-bounded operators is denoted by $\mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$. For $\mathrm{Q} \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$, the $\mathcal{P}$-operator seminorm is defined by [22]:

$$\|\mathrm{Q}\|_{\mathcal{P}} = \sup_{\mathrm{y} \in \mathbb{S}^1_{\mathcal{P}}} \|\mathrm{Q}\mathrm{y}\|_{\mathcal{P}} = \sup_{\mathrm{y},\mathrm{z} \in \mathbb{S}^1_{\mathcal{P}}} \left|\langle \mathrm{Q}\mathrm{y}, \mathrm{z}\rangle_{\mathcal{P}}\right|.$$

For any $Q, R \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$, we have $\|QR\|_{\mathcal{P}} \leq \|Q\|_{\mathcal{P}}\|R\|_{\mathcal{P}}$.

We formally define the $\mathcal{P}$-kernel for an operator $Q \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$ as

$$\mathrm{Ker}_{\mathcal{P}}(Q) = \{y \in \mathbb{H} : \|Qy\|_{\mathcal{P}} = 0\}.$$

By applying the properties of the semi-norm, we establish $\mathrm{Ker}_{\mathcal{P}}(Q) = \mathrm{Ker}(\sqrt{\mathcal{P}}Q) = \mathrm{Ker}(\mathcal{P}Q)$. Since $Q \in \mathscr{L}_{\mathcal{P}}(\mathbb{H})$, it follows that $\mathrm{Ker}(\mathcal{P}) \subseteq \mathrm{Ker}_{\mathcal{P}}(Q)$.

Following [10], an operator $Q \in \mathscr{L}(\mathbb{H})$ is called $\langle\mathcal{P}\rangle$-*self-adjoint* if $\mathcal{P}Q$ is self-adjoint. This means $(\mathcal{P}Q)^* = \mathcal{P}Q$. The set of all such operators is denoted by $\mathbf{SA}_{\langle\mathcal{P}\rangle}(\mathbb{H})$. It is clear that $\mathbf{SA}_{\langle\mathcal{P}\rangle}(\mathbb{H}) \subseteq \mathscr{L}_{\mathcal{P}}(\mathbb{H})$. Furthermore, $Q \in \mathbf{SA}_{\langle\mathcal{P}\rangle}(\mathbb{H})$ implies $\mathcal{P}Q = \mathcal{P}Q^{\sharp_{\mathcal{P}}}$.

As introduced in Section 1, the set of all $\langle\mathcal{P}\rangle$-normal operators is denoted by $\mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$. Following [23], an operator $Q \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$ is called $\mathcal{P}$-*normaloid* if

$$r_{\mathcal{P}}(Q) = \|Q\|_{\mathcal{P}},$$

where $r_{\mathcal{P}}(Q)$ denotes the $\mathcal{P}$-spectral radius of Q, defined in [23] by

$$r_{\mathcal{P}}(Q) = \lim_{j\to\infty} \|Q^j\|_{\mathcal{P}}^{\frac{1}{j}} = \inf_{j\in\mathbb{N}} \|Q^j\|_{\mathcal{P}}^{\frac{1}{j}}.$$

This is equivalent to the condition $\|Q^j\|_{\mathcal{P}} = \|Q\|_{\mathcal{P}}^j$ for all $j \in \mathbb{N}_0$. The set of all $\mathcal{P}$-normaloid operators is denoted by $\mathbf{NO}_{\mathcal{P}}(\mathbb{H})$.

To simplify notation, we define the classical commutator as $[Q, R] = QR - RQ$. The $\mathcal{P}$-commutator is defined as

$$[Q, R]_{\mathcal{P}} = \mathcal{P}QR - \mathcal{P}RQ.$$

Two operators Q and R are said to $\mathcal{P}$-commute if $[Q, R]_{\mathcal{P}} = \mathbf{0}$, where $\mathbf{0}$ denotes the zero operator.

Following [13, 22], we define the $\mathcal{P}$-numerical range $W_{\mathcal{P}}(Q)$ and the $\mathcal{P}$-numerical radius $\omega_{\mathcal{P}}(Q)$ for $Q \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$ as follows:

$$W_{\mathcal{P}}(Q) = \{\langle Qy, y\rangle_{\mathcal{P}} : y \in \mathbb{S}^1_{\mathcal{P}}\} \quad \text{and} \quad \omega_{\mathcal{P}}(Q) = \sup\{|\zeta| : \zeta \in W_{\mathcal{P}}(Q)\}.$$

By definition, $W_{\mathcal{P}}(Q)$ is a convex subset of $\mathbb{C}$ [13]. For any $\xi, \eta \in \mathbb{C}$, we have $W_{\mathcal{P}}(\xi I_{\mathbb{H}} + \eta Q) = \xi + \eta W_{\mathcal{P}}(Q)$. Furthermore, it is well established [13, 23] that the $\mathcal{P}$-spectral radius, $\mathcal{P}$-numerical radius, and $\mathcal{P}$-seminorm satisfy the fundamental inequalities

$$r_{\mathcal{P}}(Q) \leq \omega_{\mathcal{P}}(Q) \leq \|Q\|_{\mathcal{P}} \leq 2\omega_{\mathcal{P}}(Q) \tag{2.1}$$

for $Q \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$. Recent investigations on norm and numerical radius inequalities in both Hilbert and semi-Hilbert spaces can be found in [24–26] and the references therein, as well as in [27–29].

Following [20], an operator $Q \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$ is said to be $\mathcal{P}$-invertible if there exists $R \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$ such that $\mathcal{P}QR = \mathcal{P}RQ = \mathcal{P}$. The classical $\mathcal{P}$-spectrum is defined by

$$\sigma^{\mathrm{BN}}_{\mathcal{P}}(Q) = \{\xi \in \mathbb{C} : \xi I_{\mathbb{H}} - Q \text{ is not } \mathcal{P}\text{-invertible}\}.$$

Recent investigations involving this concept can be found in [30–32] and the references therein. In addition, according to [33], the $\mathcal{P}$-approximate point spectrum $\sigma_{\pi,\mathcal{P}}(Q)$ consists of all $\xi \in \mathbb{C}$ for which there exists a sequence $\{y_j\}_{j\in\mathbb{N}} \subseteq \mathbb{S}^1_{\mathcal{P}}$ such that

$$\lim_{j\to\infty} \|(Q - \xi I_{\mathbb{H}})y_j\|_{\mathcal{P}} = 0.$$

It is known that $\sigma_{\pi,\mathcal{P}}(\mathrm{Q}) \subseteq \sigma_{\mathcal{P}}^{\mathrm{BN}}(\mathrm{Q})$ [33].

We use the standard setup by de Branges and Rovnyak [34]. The semi-inner product $\langle\cdot,\cdot\rangle_{\mathcal{P}}$ gives a seminorm. This induces a true inner product on the quotient space $\mathbb{H}/\operatorname{Ker}(\mathcal{P})$. We define it as $[\overline{\mathrm{y}},\overline{\mathrm{z}}]_{\mathcal{P}} = \langle\mathcal{P}\mathrm{y},\mathrm{z}\rangle$ for any classes $\overline{\mathrm{y}},\overline{\mathrm{z}} \in \mathbb{H}/\operatorname{Ker}(\mathcal{P})$. We complete $\mathbb{H}/\operatorname{Ker}(\mathcal{P})$ with respect to this inner product. This provides a space that is isometrically isomorphic to the Hilbert space $\operatorname{Im}(\sqrt{\mathcal{P}})$. This space has the inner product $(\sqrt{\mathcal{P}}\mathrm{y},\sqrt{\mathcal{P}}\mathrm{z})_{\mathbf{R}(\sqrt{\mathcal{P}})} = \langle\Pi_{\mathcal{P}}\mathrm{y},\Pi_{\mathcal{P}}\mathrm{z}\rangle$ for all $\mathrm{y},\mathrm{z} \in \mathbb{H}$.

Next, we denote the Hilbert space $\left(\operatorname{Im}(\sqrt{\mathcal{P}}),(\cdot,\cdot)_{\mathbf{R}(\sqrt{\mathcal{P}})}\right)$ as $\mathbf{R}(\sqrt{\mathcal{P}})$. Due to range inclusion, we obtain the following useful rule:

$$\|\mathcal{P}\mathrm{y}\|_{\mathbf{R}(\sqrt{\mathcal{P}})} = \|\mathrm{y}\|_{\mathcal{P}}, \quad \forall\, \mathrm{y} \in \mathbb{H}. \tag{2.2}$$

We define the map $\Phi_{\mathcal{P}} : \mathbb{H} \to \mathbf{R}(\sqrt{\mathcal{P}})$ by $\Phi_{\mathcal{P}}\mathrm{y} = \mathcal{P}\mathrm{y}$.

The following lemma provides fundamental properties, as established in [10, 23, 35].

Lemma 2.1. *An operator* $\mathrm{Q} \in \mathscr{L}(\mathbb{H})$ *is in* $\mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$ *if and only if there exists a unique bounded operator* $\widehat{\mathrm{Q}}^{\mathcal{P}} \in \mathscr{L}(\mathbf{R}(\sqrt{\mathcal{P}}))$ *satisfying* $\Phi_{\mathcal{P}}\mathrm{Q} = \widehat{\mathrm{Q}}^{\mathcal{P}}\Phi_{\mathcal{P}}$.

In this situation, we have:

(1) $\|\mathrm{Q}\|_{\mathcal{P}} = \|\widehat{\mathrm{Q}}^{\mathcal{P}}\|_{\mathscr{L}(\mathbf{R}(\sqrt{\mathcal{P}}))}$ *and* $\omega_{\mathcal{P}}(\mathrm{Q}) = \omega(\widehat{\mathrm{Q}}^{\mathcal{P}})$.

(2) $W_{\mathcal{P}}(\mathrm{Q}) \subseteq W(\widehat{\mathrm{Q}}^{\mathcal{P}}) \subseteq \overline{W_{\mathcal{P}}(\mathrm{Q})}$. *Therefore, we obtain* $\overline{W_{\mathcal{P}}(\mathrm{Q})} = \overline{W(\widehat{\mathrm{Q}}^{\mathcal{P}})}$.

(3) If in addition $\mathrm{Q} \in \mathscr{L}_{\mathcal{P}}(\mathbb{H})$, *we have* $\widehat{\mathrm{Q}^{\sharp_{\mathcal{P}}}}^{\mathcal{P}} = (\widehat{\mathrm{Q}}^{\mathcal{P}})^*$ *and* $\widehat{(\mathrm{Q}^{\sharp_{\mathcal{P}}})^{\sharp_{\mathcal{P}}}}^{\mathcal{P}} = \widehat{\mathrm{Q}}^{\mathcal{P}}$.

In the next result, we establish properties involving the orthogonal projection onto $\overline{\operatorname{Im}(\mathcal{P})}$ and the invariance of the null space.

Lemma 2.2. *Let* $\mathrm{Q},\mathrm{R} \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$. *Then*

(1) $\operatorname{Ker}(\mathcal{P})$ *is an invariant subspace for* Q.

(2) $\sqrt{\mathcal{P}}\mathrm{Q}\Pi_{\mathcal{P}} = \sqrt{\mathcal{P}}\mathrm{Q}$.

(3) *The following assertions are equivalent:*

  (i) $[\mathrm{Q},\mathrm{R}]_{\mathcal{P}} = \mathbf{0}$.

  (ii) $[\mathrm{Q},\mathrm{R}]_{\Pi_{\mathcal{P}}} = \mathbf{0}$.

  (iii) $[\widehat{\mathrm{Q}}^{\mathcal{P}},\widehat{\mathrm{R}}^{\mathcal{P}}] = \mathbf{0}$, *where* $\widehat{\mathrm{Q}}^{\mathcal{P}}$ *and* $\widehat{\mathrm{R}}^{\mathcal{P}}$ *are the unique operators on* $\mathbf{R}(\sqrt{\mathcal{P}})$ *induced by* $\Phi_{\mathcal{P}}\mathrm{Q} = \widehat{\mathrm{Q}}^{\mathcal{P}}\Phi_{\mathcal{P}}$ *and* $\Phi_{\mathcal{P}}\mathrm{R} = \widehat{\mathrm{R}}^{\mathcal{P}}\Phi_{\mathcal{P}}$.

Proof. (1) Assume that $\mathrm{y} \in \operatorname{Ker}(\mathcal{P})$. Since $\mathrm{Q} \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$, there exists a constant $c > 0$ such that $\|\mathrm{Q}\mathrm{y}\|_{\mathcal{P}} \le c\|\mathrm{y}\|_{\mathcal{P}}$. Since $\mathrm{y} \in \operatorname{Ker}(\mathcal{P})$, we have $\|\mathrm{y}\|_{\mathcal{P}} = 0$. This implies that $\|\mathrm{Q}\mathrm{y}\|_{\mathcal{P}} = 0$. Consequently, we obtain $\mathrm{Q}\mathrm{y} \in \operatorname{Ker}(\mathcal{P})$. Therefore, $\mathrm{Q}(\operatorname{Ker}(\mathcal{P})) \subseteq \operatorname{Ker}(\mathcal{P})$.

(2) Assume that $\mathrm{Q} \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$. Then it follows that $\operatorname{Im}(\mathrm{Q}^*\sqrt{\mathcal{P}}) \subseteq \operatorname{Im}(\sqrt{\mathcal{P}})$. Taking the closure yields $\operatorname{Im}(\mathrm{Q}^*\sqrt{\mathcal{P}}) \subseteq \overline{\operatorname{Im}(\mathcal{P})}$. Passing to the orthogonal complement, we deduce $\operatorname{Ker}(\sqrt{\mathcal{P}}\mathrm{Q}) \supseteq \operatorname{Ker}(\mathcal{P})$. By definition, $\Pi_{\mathcal{P}}$ is the orthogonal projection onto $\operatorname{Ker}(\mathcal{P})^{\perp}$. Thus, the complementary operator $I_{\mathbb{H}}-\Pi_{\mathcal{P}}$ is the orthogonal projection onto $\operatorname{Ker}(\mathcal{P})$. It follows that $\sqrt{\mathcal{P}}\mathrm{Q}(I_{\mathbb{H}} - \Pi_{\mathcal{P}}) = \mathbf{0}$. Therefore, we obtain $\sqrt{\mathcal{P}}\mathrm{Q}\Pi_{\mathcal{P}} = \sqrt{\mathcal{P}}\mathrm{Q}$.

(3) First, we prove (i) $\iff$ (ii). Since $\mathcal{P} = \sqrt{\mathcal{P}}\sqrt{\mathcal{P}}$ and $\operatorname{Ker}(\sqrt{\mathcal{P}}) = \operatorname{Ker}(\mathcal{P})$, the relation $\mathcal{P}[\mathrm{Q},\mathrm{R}] = \mathbf{0}$ is equivalent to $\sqrt{\mathcal{P}}[\mathrm{Q},\mathrm{R}] = \mathbf{0}$. Moreover, because $\Pi_{\mathcal{P}}$ is the orthogonal projection onto $\operatorname{Ker}(\mathcal{P})^{\perp}$, we

have $\mathrm{Ker}(\Pi_{\mathcal{P}}) = \mathrm{Ker}(\mathcal{P}) = \mathrm{Ker}(\sqrt{\mathcal{P}})$. Hence, $\sqrt{\mathcal{P}}[\mathrm{Q},\mathrm{R}] = \mathbf{0}$ is equivalent to $\mathrm{Im}([\mathrm{Q},\mathrm{R}]) \subseteq \mathrm{Ker}(\sqrt{\mathcal{P}})$ and therefore to $\mathrm{Im}([\mathrm{Q},\mathrm{R}]) \subseteq \mathrm{Ker}(\Pi_{\mathcal{P}})$, which in turn is equivalent to $\Pi_{\mathcal{P}}[\mathrm{Q},\mathrm{R}] = \mathbf{0}$. Consequently, $[\mathrm{Q},\mathrm{R}]_{\mathcal{P}} = \mathbf{0}$ if and only if $[\mathrm{Q},\mathrm{R}]_{\Pi_{\mathcal{P}}} = \mathbf{0}$.

Next, we prove $(i) \iff (iii)$. Applying the relations $\Phi_{\mathcal{P}}\mathrm{Q} = \widehat{\mathrm{Q}}^{\mathcal{P}}\Phi_{\mathcal{P}}$ and $\Phi_{\mathcal{P}}\mathrm{R} = \widehat{\mathrm{R}}^{\mathcal{P}}\Phi_{\mathcal{P}}$, we obtain $\mathcal{P}\mathrm{Q} = \widehat{\mathrm{Q}}^{\mathcal{P}}\mathcal{P}$ and $\mathcal{P}\mathrm{R} = \widehat{\mathrm{R}}^{\mathcal{P}}\mathcal{P}$. Thus, we obtain:

$$\mathcal{P}\mathrm{QR} - \mathcal{P}\mathrm{RQ} = (\widehat{\mathrm{Q}}^{\mathcal{P}}\widehat{\mathrm{R}}^{\mathcal{P}} - \widehat{\mathrm{R}}^{\mathcal{P}}\widehat{\mathrm{Q}}^{\mathcal{P}})\mathcal{P}.$$

Thus, $[\mathrm{Q},\mathrm{R}]_{\mathcal{P}} = \mathbf{0}$ if and only if $[\widehat{\mathrm{Q}}^{\mathcal{P}},\widehat{\mathrm{R}}^{\mathcal{P}}]\mathcal{P} = \mathbf{0}$. Since $\mathrm{Im}(\mathcal{P})$ is dense in $\mathbf{R}(\sqrt{\mathcal{P}})$, this is true if and only if $[\widehat{\mathrm{Q}}^{\mathcal{P}},\widehat{\mathrm{R}}^{\mathcal{P}}] = \mathbf{0}$. This completes the proof. □

In the following lemma, without assuming that the range of $\mathcal{P}$ is closed, we establish important algebraic identities. The first two items have already been stated in [36]. For the sake of completeness, we include the proofs.

Lemma 2.3. *Let* $\mathrm{Q} \in \mathscr{L}_{\mathcal{P}}(\mathbb{H})$. *Then the following algebraic identities hold:*

(i) $\mathcal{P}\Pi_{\mathcal{P}} = \Pi_{\mathcal{P}}\mathcal{P} = \mathcal{P}$.
(ii) $\Pi_{\mathcal{P}}\mathrm{Q}^{\sharp_{\mathcal{P}}} = \mathrm{Q}^{\sharp_{\mathcal{P}}}\Pi_{\mathcal{P}} = \mathrm{Q}^{\sharp_{\mathcal{P}}}$.
(iii) $\mathcal{P}\mathrm{Q}\Pi_{\mathcal{P}} = \mathcal{P}\Pi_{\mathcal{P}}\mathrm{Q} = \mathcal{P}\mathrm{Q}$.

Proof. (i) By definition, we have $\Pi_{\mathcal{P}}|_{\overline{\mathrm{Im}(\mathcal{P})}} = I_{\mathbb{H}}$, which implies $\Pi_{\mathcal{P}}\mathcal{P} = \mathcal{P}$. By taking the adjoint, we obtain $\mathcal{P}\Pi_{\mathcal{P}} = \mathcal{P}$. Thus, $\mathcal{P}\Pi_{\mathcal{P}} = \Pi_{\mathcal{P}}\mathcal{P} = \mathcal{P}$.

(ii) Since $\mathrm{Im}(\mathrm{Q}^{\sharp_{\mathcal{P}}}) \subseteq \overline{\mathrm{Im}(\mathcal{P})}$, we have $\Pi_{\mathcal{P}}\mathrm{Q}^{\sharp_{\mathcal{P}}} = \mathrm{Q}^{\sharp_{\mathcal{P}}}$. Let $\mathrm{y} \in \mathbb{H}$. Then $\mathrm{y} = \Pi_{\mathcal{P}}\mathrm{y} + (I_{\mathbb{H}} - \Pi_{\mathcal{P}})\mathrm{y}$. Since $(I_{\mathbb{H}} - \Pi_{\mathcal{P}})\mathrm{y} \in \mathrm{Ker}(\mathcal{P})$, it follows that $\mathcal{P}(I_{\mathbb{H}} - \Pi_{\mathcal{P}})\mathrm{y} = \mathbf{0}$. By using the relation $\mathrm{Q}^*\mathcal{P} = \mathcal{P}\mathrm{Q}^{\sharp_{\mathcal{P}}}$, we deduce that $\mathcal{P}\mathrm{Q}^{\sharp_{\mathcal{P}}}(I_{\mathbb{H}} - \Pi_{\mathcal{P}})\mathrm{y} = \mathbf{0}$. Consequently, $\mathrm{Q}^{\sharp_{\mathcal{P}}}(I_{\mathbb{H}} - \Pi_{\mathcal{P}})\mathrm{y} \in \mathrm{Ker}(\mathcal{P}) \cap \overline{\mathrm{Im}(\mathcal{P})} = \{\mathbf{0}\}$. Therefore,

$$\mathrm{Q}^{\sharp_{\mathcal{P}}}(I_{\mathbb{H}} - \Pi_{\mathcal{P}})\mathrm{y} = \mathbf{0},$$

which yields $\mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{y} = \mathrm{Q}^{\sharp_{\mathcal{P}}}\Pi_{\mathcal{P}}\mathrm{y}$.

(iii) By part (i), we have $\mathcal{P}\Pi_{\mathcal{P}}\mathrm{Q} = \mathcal{P}\mathrm{Q}$. Let $\mathrm{y} \in \mathbb{H}$. Since $(I_{\mathbb{H}} - \Pi_{\mathcal{P}})\mathrm{y} \in \mathrm{Ker}(\mathcal{P})$, Lemma 2.2 implies that $\mathrm{Q}(I_{\mathbb{H}} - \Pi_{\mathcal{P}})\mathrm{y} \in \mathrm{Ker}(\mathcal{P})$. Hence, $\mathcal{P}\mathrm{Q}(I_{\mathbb{H}} - \Pi_{\mathcal{P}})\mathrm{y} = \mathbf{0}$, which yields $\mathcal{P}\mathrm{Q} = \mathcal{P}\mathrm{Q}\Pi_{\mathcal{P}}$. This completes the proof. □

3. Main results

This section presents our main results. First, we provide metric and algebraic characterizations of $\langle\mathcal{P}\rangle$-normality. Although the equivalences in Theorem 3.1 (except part 3) and Theorem 3.3 recently appeared in [19] (Theorems 2.16 and 2.17), our approach provides independent proofs based solely on the $\mathcal{P}$-commutator $[\cdot,\cdot]_{\mathcal{P}}$ and the absorption properties of $\Pi_{\mathcal{P}}$.

For $\mathrm{Q} \in \mathscr{L}_{\mathcal{P}}(\mathbb{H})$, we define the *$\mathcal{P}$-real part* and the *$\mathcal{P}$-imaginary part* of Q by

$$\Re_{\mathcal{P}}(\mathrm{Q}) = \frac{1}{2}(\mathrm{Q} + \mathrm{Q}^{\sharp_{\mathcal{P}}}), \quad \Im_{\mathcal{P}}(\mathrm{Q}) = \frac{1}{2i}(\mathrm{Q} - \mathrm{Q}^{\sharp_{\mathcal{P}}}).$$

In the next result, we present equivalent characterizations of $\langle\mathcal{P}\rangle$-normal operators.

Theorem 3.1. *Let* $\mathrm{Q} \in \mathscr{L}_{\mathcal{P}}(\mathbb{H})$. *The following assertions are equivalent:*

(1) $\mathrm{Q} \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$.
(2) $\|\mathrm{Qy}\|_{\mathcal{P}} = \|\mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{y}\|_{\mathcal{P}}$ *for all* $\mathrm{y} \in \mathbb{H}$.
(3) $[\Re_{\mathcal{P}}(\mathrm{Q}), \Im_{\mathcal{P}}(\mathrm{Q})]_{\mathcal{P}} = \mathbf{0}$.
(4) $\widehat{\mathrm{Q}}^{\mathcal{P}}$ *is a normal operator on the Hilbert space* $\mathbf{R}(\sqrt{\mathcal{P}})$.
(5) $\mathrm{Q}^{\sharp_{\mathcal{P}}} \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$.
(6) *For any* $\xi \in \mathbb{C}$, *the translated operator* $\mathrm{Q}_{\xi} = \mathrm{Q} - \xi I_{\mathbb{H}}$ *belongs to* $\mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$.

Proof. First, we prove (1) $\Longleftrightarrow$ (2). By definition, we have $\mathcal{P}\mathrm{Q} = (\mathrm{Q}^{\sharp_{\mathcal{P}}})^*\mathcal{P}$. Let $\mathrm{y} \in \mathbb{H}$. We have:

$$\begin{aligned}\|\mathrm{Qy}\|_{\mathcal{P}}^2 - \|\mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{y}\|_{\mathcal{P}}^2 &= \langle \mathcal{P}\mathrm{Qy}, \mathrm{Qy}\rangle - \langle \mathcal{P}\mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{y}, \mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{y}\rangle \\ &= \langle \mathrm{y}, \mathcal{P}\mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{Qy}\rangle - \langle \mathrm{y}, \mathcal{P}\mathrm{Q}\mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{y}\rangle \\ &= -\langle \mathrm{y}, [\mathrm{Q}, \mathrm{Q}^{\sharp_{\mathcal{P}}}]_{\mathcal{P}}\mathrm{y}\rangle.\end{aligned}$$

Since $[\mathrm{Q}, \mathrm{Q}^{\sharp_{\mathcal{P}}}]_{\mathcal{P}}$ is self-adjoint, we deduce that $\|\mathrm{Qy}\|_{\mathcal{P}} = \|\mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{y}\|_{\mathcal{P}}$ for all $\mathrm{y} \in \mathbb{H}$ if and only if $[\mathrm{Q}, \mathrm{Q}^{\sharp_{\mathcal{P}}}]_{\mathcal{P}} = \mathbf{0}$. This proves (1) $\Longleftrightarrow$ (2).

Next, we prove (1) $\Longleftrightarrow$ (3). By expanding the products, we obtain:

$$\begin{aligned}\Re_{\mathcal{P}}(\mathrm{Q})\Im_{\mathcal{P}}(\mathrm{Q}) &= \frac{1}{4i}\left(\mathrm{Q}^2 - \mathrm{Q}\mathrm{Q}^{\sharp_{\mathcal{P}}} + \mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{Q} - (\mathrm{Q}^{\sharp_{\mathcal{P}}})^2\right), \\ \Im_{\mathcal{P}}(\mathrm{Q})\Re_{\mathcal{P}}(\mathrm{Q}) &= \frac{1}{4i}\left(\mathrm{Q}^2 + \mathrm{Q}\mathrm{Q}^{\sharp_{\mathcal{P}}} - \mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{Q} - (\mathrm{Q}^{\sharp_{\mathcal{P}}})^2\right).\end{aligned}$$

It follows that:

$$[\Re_{\mathcal{P}}(\mathrm{Q}), \Im_{\mathcal{P}}(\mathrm{Q})]_{\mathcal{P}} = \frac{1}{2i}\left(\mathcal{P}\mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{Q} - \mathcal{P}\mathrm{Q}\mathrm{Q}^{\sharp_{\mathcal{P}}}\right).$$

Thus, we deduce that $[\Re_{\mathcal{P}}(\mathrm{Q}), \Im_{\mathcal{P}}(\mathrm{Q})]_{\mathcal{P}} = \mathbf{0}$ if and only if $\mathcal{P}\mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{Q} = \mathcal{P}\mathrm{Q}\mathrm{Q}^{\sharp_{\mathcal{P}}}$. This proves (1) $\Longleftrightarrow$ (3).

Now, we prove (1) $\Longleftrightarrow$ (4). Applying Lemma 2.2, we have $[\mathrm{Q}, \mathrm{Q}^{\sharp_{\mathcal{P}}}]_{\mathcal{P}} = \mathbf{0}$ if and only if $[\widehat{\mathrm{Q}}^{\mathcal{P}}, \widehat{\mathrm{Q}^{\sharp_{\mathcal{P}}}}^{\mathcal{P}}] = \mathbf{0}$. Lemma 2.1(3) states that $\widehat{\mathrm{Q}^{\sharp_{\mathcal{P}}}}^{\mathcal{P}} = (\widehat{\mathrm{Q}}^{\mathcal{P}})^*$. Hence, this holds if and only if $[\widehat{\mathrm{Q}}^{\mathcal{P}}, (\widehat{\mathrm{Q}}^{\mathcal{P}})^*] = \mathbf{0}$. This means $\widehat{\mathrm{Q}}^{\mathcal{P}}$ is normal on $\mathbf{R}(\sqrt{\mathcal{P}})$. This proves (1) $\Longleftrightarrow$ (4).

Next, we prove (1) $\Longleftrightarrow$ (5). By definition, we have $(\mathrm{Q}^{\sharp_{\mathcal{P}}})^{\sharp_{\mathcal{P}}} = \Pi_{\mathcal{P}}\mathrm{Q}\Pi_{\mathcal{P}}$. Thus, the $\mathcal{P}$-commutator for $\mathrm{Q}^{\sharp_{\mathcal{P}}}$ expands to:

$$[\mathrm{Q}^{\sharp_{\mathcal{P}}}, (\mathrm{Q}^{\sharp_{\mathcal{P}}})^{\sharp_{\mathcal{P}}}]_{\mathcal{P}} = [\mathrm{Q}^{\sharp_{\mathcal{P}}}, \Pi_{\mathcal{P}}\mathrm{Q}\Pi_{\mathcal{P}}]_{\mathcal{P}}.$$

Applying Lemma 2.3, we verify:

$$\mathcal{P}\mathrm{Q}^{\sharp_{\mathcal{P}}}\Pi_{\mathcal{P}}\mathrm{Q}\Pi_{\mathcal{P}} - \mathcal{P}\Pi_{\mathcal{P}}\mathrm{Q}\Pi_{\mathcal{P}}\mathrm{Q}^{\sharp_{\mathcal{P}}} = \mathcal{P}\mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{Q} - \mathcal{P}\mathrm{Q}\mathrm{Q}^{\sharp_{\mathcal{P}}} = [\mathrm{Q}^{\sharp_{\mathcal{P}}}, \mathrm{Q}]_{\mathcal{P}}.$$

Thus, $\mathrm{Q}^{\sharp_{\mathcal{P}}} \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$ if and only if $[\mathrm{Q}^{\sharp_{\mathcal{P}}}, \mathrm{Q}]_{\mathcal{P}} = \mathbf{0}$. This proves (1) $\Longleftrightarrow$ (5).

Finally, we prove (1) $\Longleftrightarrow$ (6). By definition, we obtain $(\mathrm{Q} - \xi I_{\mathbb{H}})^{\sharp_{\mathcal{P}}} = \mathrm{Q}^{\sharp_{\mathcal{P}}} - \overline{\xi}\Pi_{\mathcal{P}}$. Applying the linearity of the commutator, we obtain:

$$\begin{aligned}&[\mathrm{Q} - \xi I_{\mathbb{H}}, \mathrm{Q}^{\sharp_{\mathcal{P}}} - \overline{\xi}\Pi_{\mathcal{P}}]_{\mathcal{P}} \\ &= [\mathrm{Q}, \mathrm{Q}^{\sharp_{\mathcal{P}}}]_{\mathcal{P}} - \overline{\xi}[\mathrm{Q}, \Pi_{\mathcal{P}}]_{\mathcal{P}} - \xi[I_{\mathbb{H}}, \mathrm{Q}^{\sharp_{\mathcal{P}}}]_{\mathcal{P}} + |\xi|^2[I_{\mathbb{H}}, \Pi_{\mathcal{P}}]_{\mathcal{P}}.\end{aligned}$$

Applying Lemma 2.3, we find $[\mathrm{Q}, \Pi_{\mathcal{P}}]_{\mathcal{P}} = \mathbf{0}$. Furthermore, we have $[I_{\mathbb{H}}, \mathrm{Q}^{\sharp_{\mathcal{P}}}]_{\mathcal{P}} = \mathbf{0}$ and $[I_{\mathbb{H}}, \Pi_{\mathcal{P}}]_{\mathcal{P}} = \mathbf{0}$. Consequently, we deduce $[\mathrm{Q}_{\xi}, \mathrm{Q}_{\xi}^{\sharp_{\mathcal{P}}}]_{\mathcal{P}} = [\mathrm{Q}, \mathrm{Q}^{\sharp_{\mathcal{P}}}]_{\mathcal{P}}$. Therefore, $\mathrm{Q}_{\xi} \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$ if and only if $\mathrm{Q} \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$. This completes the proof. $\square$

The following result shows that $\langle\mathcal{P}\rangle$-normality is preserved under unitary equivalence.

Proposition 3.1. *Let* $\mathrm{Q} \in \mathscr{L}_{\mathcal{P}}(\mathbb{H})$*, and let* $\mathrm{U} \in \mathscr{L}(\mathbb{H})$ *be unitary. Then,* $\mathrm{Q} \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$ *if and only if* $\mathrm{S} \in \mathbf{N}_{\langle\widetilde{\mathcal{P}}\rangle}(\mathbb{H})$*, where* $\widetilde{\mathcal{P}} = \mathrm{U}\mathcal{P}\mathrm{U}^*$ *and* $\mathrm{S} = \mathrm{UQU}^*$.

Proof. Assume that U is a unitary operator. First, we determine the distinguished $\widetilde{\mathcal{P}}$-adjoint of S. By definition, $\mathrm{S}^{\sharp_{\widetilde{\mathcal{P}}}}$ must satisfy

$$\widetilde{\mathcal{P}}\mathrm{S}^{\sharp_{\widetilde{\mathcal{P}}}} = \mathrm{S}^*\widetilde{\mathcal{P}}.$$

By substituting the definitions of S and $\widetilde{\mathcal{P}}$, we obtain

$$\mathrm{S}^*\widetilde{\mathcal{P}} = (\mathrm{UQU}^*)^*(\mathrm{U}\mathcal{P}\mathrm{U}^*) = \mathrm{UQ}^*\mathcal{P}\mathrm{U}^*.$$

Since $\mathrm{Q}^*\mathcal{P} = \mathcal{P}\mathrm{Q}^{\sharp_{\mathcal{P}}}$, we substitute this relation to obtain

$$\mathrm{S}^*\widetilde{\mathcal{P}} = \mathrm{U}\mathcal{P}\mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{U}^* = (\mathrm{U}\mathcal{P}\mathrm{U}^*)(\mathrm{UQ}^{\sharp_{\mathcal{P}}}\mathrm{U}^*) = \widetilde{\mathcal{P}}(\mathrm{UQ}^{\sharp_{\mathcal{P}}}\mathrm{U}^*).$$

By assumption, $\mathrm{Im}(\mathrm{Q}^{\sharp_{\mathcal{P}}}) \subseteq \overline{\mathrm{Im}(\mathcal{P})}$. Thus, the range condition holds, since $\mathrm{Im}(\mathrm{UQ}^{\sharp_{\mathcal{P}}}\mathrm{U}^*) \subseteq \mathrm{U}(\overline{\mathrm{Im}(\mathcal{P})}) = \overline{\mathrm{Im}(\mathrm{U}\mathcal{P}\mathrm{U}^*)} = \overline{\mathrm{Im}(\widetilde{\mathcal{P}})}$. Consequently, we deduce

$$\mathrm{S}^{\sharp_{\widetilde{\mathcal{P}}}} = \mathrm{UQ}^{\sharp_{\mathcal{P}}}\mathrm{U}^*.$$

Next, we evaluate the products scaled by $\widetilde{\mathcal{P}}$. We have

$$\widetilde{\mathcal{P}}\mathrm{SS}^{\sharp_{\widetilde{\mathcal{P}}}} = (\mathrm{U}\mathcal{P}\mathrm{U}^*)(\mathrm{UQU}^*)(\mathrm{UQ}^{\sharp_{\mathcal{P}}}\mathrm{U}^*) = \mathrm{U}\mathcal{P}\mathrm{QQ}^{\sharp_{\mathcal{P}}}\mathrm{U}^*.$$

Similarly, we obtain

$$\widetilde{\mathcal{P}}\mathrm{S}^{\sharp_{\widetilde{\mathcal{P}}}}\mathrm{S} = (\mathrm{U}\mathcal{P}\mathrm{U}^*)(\mathrm{UQ}^{\sharp_{\mathcal{P}}}\mathrm{U}^*)(\mathrm{UQU}^*) = \mathrm{U}\mathcal{P}\mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{QU}^*.$$

Since U is invertible, it follows that $\widetilde{\mathcal{P}}\mathrm{SS}^{\sharp_{\widetilde{\mathcal{P}}}} = \widetilde{\mathcal{P}}\mathrm{S}^{\sharp_{\widetilde{\mathcal{P}}}}\mathrm{S}$ if and only if $\mathcal{P}\mathrm{QQ}^{\sharp_{\mathcal{P}}} = \mathcal{P}\mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{Q}$. This implies that $\mathrm{S} \in \mathbf{N}_{\langle\widetilde{\mathcal{P}}\rangle}(\mathbb{H})$ if and only if $\mathrm{Q} \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$. This completes the proof. □

The next result characterizes the powers of a $\langle\mathcal{P}\rangle$-normal operator.

Theorem 3.2. *Let* $\mathrm{Q} \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$ *and* $j \in \mathbb{N}$*. Then* $\mathrm{Q}^j \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$.

Proof. Assume that $\mathrm{Q} \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$. Then $\mathrm{Q} \in \mathscr{L}_{\mathcal{P}}(\mathbb{H})$, which implies that $\mathrm{Q}^j \in \mathscr{L}_{\mathcal{P}}(\mathbb{H})$ and $\widehat{\mathrm{Q}^j}^{\mathcal{P}} = (\widehat{\mathrm{Q}}^{\mathcal{P}})^j$ for any integer $j \in \mathbb{N}$ (this can be easily proved by induction). By Theorem 3.1, $\mathrm{Q} \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$ implies $\widehat{\mathrm{Q}}^{\mathcal{P}}$ is normal on $\mathbf{R}(\sqrt{\mathcal{P}})$. Then $\widehat{\mathrm{Q}^j}^{\mathcal{P}} = (\widehat{\mathrm{Q}}^{\mathcal{P}})^j$ is normal on $\mathbf{R}(\sqrt{\mathcal{P}})$, and, again by Theorem 3.1, $\mathrm{Q}^j \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$. This completes the proof. □

In the following remark, we note that the converse of Theorem 3.2 is generally false.

Remark 3.1. *If* $\mathrm{Q}^j \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$ *for some* $j > 1$*, then* Q *is not necessarily in* $\mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$.

The next example illustrates this remark.

Example 3.1. *Let* $\mathbb{H} = \mathbb{C}^2$ *and* $\mathcal{P} = I_{\mathbb{H}}$. *In this case, a* $\langle\mathcal{P}\rangle$*-normal operator is exactly a standard normal operator. Let* $Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. *Direct calculation yields* $QQ^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $Q^*Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. *Since* $QQ^* \neq Q^*Q$, Q *is not* $\langle\mathcal{P}\rangle$*-normal. On the other hand, we have* $Q^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$. *Since the zero operator is normal, we obtain* $Q^2 \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$. *Therefore, the power of an operator can be* $\langle\mathcal{P}\rangle$*-normal even if the operator itself is not.*

In the next result, we show that every $\langle\mathcal{P}\rangle$-normal operator is $\mathcal{P}$-normaloid.

Theorem 3.3. *Let* $Q \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$. *Then* $Q \in \mathbf{NO}_{\mathcal{P}}(\mathbb{H})$.

Proof. Let $y \in \mathbb{H}$. Applying the Cauchy-Schwarz inequality, we obtain

$$\|Qy\|_{\mathcal{P}}^2 = \langle Qy, Qy\rangle_{\mathcal{P}} = \langle Q^{\sharp_{\mathcal{P}}}Qy, y\rangle_{\mathcal{P}} \leq \|Q^{\sharp_{\mathcal{P}}}Qy\|_{\mathcal{P}}\|y\|_{\mathcal{P}}. \tag{3.1}$$

By expanding the squared $\mathcal{P}$-seminorm, we obtain

$$\begin{aligned}\|Q^{\sharp_{\mathcal{P}}}Qy\|_{\mathcal{P}}^2 &= \langle Q^{\sharp_{\mathcal{P}}}Qy, Q^{\sharp_{\mathcal{P}}}Qy\rangle_{\mathcal{P}} \\ &= \langle QQ^{\sharp_{\mathcal{P}}}Qy, Qy\rangle_{\mathcal{P}} \\ &= \langle \mathcal{P}QQ^{\sharp_{\mathcal{P}}}Qy, Qy\rangle.\end{aligned}$$

Since $Q \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$, we deduce that $\mathcal{P}QQ^{\sharp_{\mathcal{P}}} = \mathcal{P}Q^{\sharp_{\mathcal{P}}}Q$. By using this rule, we obtain

$$\begin{aligned}\langle \mathcal{P}(QQ^{\sharp_{\mathcal{P}}})Qy, Qy\rangle &= \langle \mathcal{P}(Q^{\sharp_{\mathcal{P}}}Q)Qy, Qy\rangle \\ &= \langle \mathcal{P}Q^{\sharp_{\mathcal{P}}}Q^2y, Qy\rangle.\end{aligned}$$

It follows that $\|Q^{\sharp_{\mathcal{P}}}Qy\|_{\mathcal{P}} = \|Q^2y\|_{\mathcal{P}}$. Thus, by (3.1), we obtain

$$\|Qy\|_{\mathcal{P}}^2 \leq \|Q^2y\|_{\mathcal{P}}\|y\|_{\mathcal{P}}. \tag{3.2}$$

Taking the supremum over all $y \in \mathbb{S}_{\mathcal{P}}^1$, we verify that $\|Q\|_{\mathcal{P}}^2 \leq \|Q^2\|_{\mathcal{P}}$. Since the $\mathcal{P}$-seminorm is submultiplicative, we confirm that $\|Q^2\|_{\mathcal{P}} \leq \|Q\|_{\mathcal{P}}^2$. Therefore, we deduce

$$\|Q^2\|_{\mathcal{P}} = \|Q\|_{\mathcal{P}}^2.$$

By assumption, $Q \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$. Then Theorem 3.2 implies that $Q^k \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$ for any integer $k \geq 1$. Applying induction, we obtain

$$\|Q^{2^s}\|_{\mathcal{P}} = \|Q\|_{\mathcal{P}}^{2^s}, \quad \forall s \in \mathbb{N}_0.$$

Finally, we deduce

$$\begin{aligned}r_{\mathcal{P}}(Q) &= \lim_{j\to\infty} \|Q^j\|_{\mathcal{P}}^{\frac{1}{j}} \\ &= \lim_{s\to\infty} \|Q^{2^s}\|_{\mathcal{P}}^{\frac{1}{2^s}} \\ &= \lim_{s\to\infty} \left(\|Q\|_{\mathcal{P}}^{2^s}\right)^{\frac{1}{2^s}} \\ &= \|Q\|_{\mathcal{P}}.\end{aligned}$$

This completes the proof. □

The following result provides a consequence for the powers of an operator.

Corollary 3.1. *Let* $Q \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$*. Then for all integers* $j \in \mathbb{N}$*, the following equality holds:*

$$\omega_{\mathcal{P}}(Q^j) = \omega_{\mathcal{P}}(Q)^j.$$

Proof. Assume that $Q \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$. This implies that $r_{\mathcal{P}}(Q) = \|Q\|_{\mathcal{P}}$. By Theorem 3.3, Q is $\mathcal{P}$-normaloid. By using (2.1), we obtain

$$r_{\mathcal{P}}(Q) = \omega_{\mathcal{P}}(Q) = \|Q\|_{\mathcal{P}}.$$

Let $j \in \mathbb{N}$. By Theorem 3.2, we have $Q^j \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$. By using the same argument, we obtain $r_{\mathcal{P}}(Q^j) = \omega_{\mathcal{P}}(Q^j) = \|Q^j\|_{\mathcal{P}}$. By definition, we have $r_{\mathcal{P}}(Q^j) = r_{\mathcal{P}}(Q)^j$. Therefore, we deduce $\omega_{\mathcal{P}}(Q^j) = r_{\mathcal{P}}(Q^j) = r_{\mathcal{P}}(Q)^j = \omega_{\mathcal{P}}(Q)^j$. This completes the proof. □

In the next result, we evaluate the $\mathcal{P}$-operator seminorm of the translated operator.

Corollary 3.2. *Let* $Q \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$*, and let* $\xi \in \mathbb{C}$*. Then the* $\mathcal{P}$*-operator seminorm of the translated operator strictly matches its* $\mathcal{P}$*-numerical radius:*

$$\|Q - \xi I_{\mathbb{H}}\|_{\mathcal{P}} = \omega_{\mathcal{P}}(Q - \xi I_{\mathbb{H}}).$$

Proof. Since Q is $\langle\mathcal{P}\rangle$-normal, Theorem 3.1(6) implies that the translated operator $Q_\xi = Q - \xi I_{\mathbb{H}}$ is also $\langle\mathcal{P}\rangle$-normal. By applying Theorem 3.3, we know that every $\langle\mathcal{P}\rangle$-normal operator is $\mathcal{P}$-normaloid, meaning $r_{\mathcal{P}}(Q_\xi) = \|Q_\xi\|_{\mathcal{P}}$. From the fundamental inequalities in (2.1), we have $r_{\mathcal{P}}(Q_\xi) \le \omega_{\mathcal{P}}(Q_\xi) \le \|Q_\xi\|_{\mathcal{P}}$. This establishes the exact equality $\|Q - \xi I_{\mathbb{H}}\|_{\mathcal{P}} = \omega_{\mathcal{P}}(Q - \xi I_{\mathbb{H}})$. This completes the proof. □

The next result is a direct consequence for operators with a zero $\mathcal{P}$-spectral radius.

Corollary 3.3. *Let* $Q \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$*. Then* $r_{\mathcal{P}}(Q) = 0$ *if and only if* $\mathcal{P}Q = \mathbf{0}$.

Proof. Since $Q \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$, we have $r_{\mathcal{P}}(Q) = \|Q\|_{\mathcal{P}}$. Thus, $r_{\mathcal{P}}(Q) = 0$ if and only if $\|Q\|_{\mathcal{P}} = 0$, which is equivalent to $\mathcal{P}Q = \mathbf{0}$. □

It is known from the literature that $\mathcal{P}$-normal operators are not in general $\mathcal{P}$-self-adjoint [23]. However, as we establish in the next result, the generalized class of $\langle\mathcal{P}\rangle$-normal operators without assuming that the range of $\mathcal{P}$ is closed, unconditionally contains the set of all $\langle\mathcal{P}\rangle$-self-adjoint operators. This robust hierarchical inclusion significantly increases the importance of investigating this class of operators.

In the following theorem, without assuming that the range of $\mathcal{P}$ is closed, we provide a hierarchy among classes of operators.

Theorem 3.4. *The following inclusions hold for any operator acting on the semi-Hilbertian space* $\mathbb{H}$*:*

$$\mathscr{L}_{\mathcal{P}}^{+}(\mathbb{H}) \subseteq \mathbf{SA}_{\langle\mathcal{P}\rangle}(\mathbb{H}) \subseteq \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H}) \subseteq \mathbf{NO}_{\mathcal{P}}(\mathbb{H}).$$

Proof. First, let $Q \in \mathscr{L}_{\mathcal{P}}^{+}(\mathbb{H})$. By definition, $\mathcal{P}Q$ is a positive semi-definite operator in the classical sense. Since every positive semi-definite operator is classically self-adjoint, we obtain $(\mathcal{P}Q)^* = \mathcal{P}Q$. Thus, we have $Q^*\mathcal{P} = \mathcal{P}Q$. Since the $\mathcal{P}$-adjoint guarantees $Q^*\mathcal{P} = \mathcal{P}Q^{\sharp_{\mathcal{P}}}$, we deduce $\mathcal{P}Q = \mathcal{P}Q^{\sharp_{\mathcal{P}}}$. This means that $Q \in \mathbf{SA}_{\langle\mathcal{P}\rangle}(\mathbb{H})$. Hence, the inclusion $\mathscr{L}_{\mathcal{P}}^{+}(\mathbb{H}) \subseteq \mathbf{SA}_{\langle\mathcal{P}\rangle}(\mathbb{H})$ holds.

Next, let $Q \in \mathbf{SA}_{\langle\mathcal{P}\rangle}(\mathbb{H})$. Then we have $\mathcal{P}Q = \mathcal{P}Q^{\sharp_{\mathcal{P}}}$. Multiplying by Q on the right, we obtain $\mathcal{P}Q^2 = \mathcal{P}Q^{\sharp_{\mathcal{P}}}Q$. Since $\mathcal{P}Q = Q^*\mathcal{P}$, we also obtain $\mathcal{P}QQ^{\sharp_{\mathcal{P}}} = Q^*\mathcal{P}Q^{\sharp_{\mathcal{P}}} = Q^*(\mathcal{P}Q) = (Q^*\mathcal{P})Q = \mathcal{P}Q^2$. Therefore, we have $\mathcal{P}QQ^{\sharp_{\mathcal{P}}} = \mathcal{P}Q^{\sharp_{\mathcal{P}}}Q$. This indicates that $Q \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$. Hence, the inclusion $\mathbf{SA}_{\langle\mathcal{P}\rangle}(\mathbb{H}) \subseteq \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$ holds.

Finally, let $Q \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$. Since Q is $\langle\mathcal{P}\rangle$-normal, Theorem 3.3 confirms that Q is $\mathcal{P}$-normaloid. Thus, the inclusion $\mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H}) \subseteq \mathbf{NO}_{\mathcal{P}}(\mathbb{H})$ holds. This completes the proof. □

In the next proposition, without assuming that the range of $\mathcal{P}$ is closed, we consolidate several generalized inequalities for the $\mathcal{P}$-numerical radius and operator norm, extending the results of Saddi [15] to the broader class of $\langle\mathcal{P}\rangle$-normal operators.

Proposition 3.2. *Let* $Q \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$*, let* $\xi, \eta \in \mathbb{C}$*, and let* $p \geq 2$*. Then the following properties hold:*

(1) For all $y \in \mathbb{H}$, $|\langle Q^2y, y\rangle_{\mathcal{P}}| \leq \|Qy\|_{\mathcal{P}}^2$.

(2) For every $\mathcal{P}$-unit vector $y \in \mathbb{S}_{\mathcal{P}}^1$,

$$|\langle Qy, y\rangle_{\mathcal{P}}|^2 \leq \frac{1}{2}\left(\|Qy\|_{\mathcal{P}}^2 + |\langle Q^2y, y\rangle_{\mathcal{P}}|\right) \leq \|Qy\|_{\mathcal{P}}^2.$$

(3) For any $y \in \mathbb{H}$,

$$0 \leq \|Qy\|_{\mathcal{P}}^2 - |\langle Q^2y, y\rangle_{\mathcal{P}}| \leq \frac{2}{(1+|\xi|)^2}\|Qy - \xi Q^{\sharp_{\mathcal{P}}}y\|_{\mathcal{P}}^2.$$

(4) For $\xi \neq 0$ *and* $y \in \mathbb{S}_{\mathcal{P}}^1$,

$$0 \leq \|Qy\|_{\mathcal{P}}^4 - |\langle Q^2y, y\rangle_{\mathcal{P}}|^2 \leq \frac{1}{|\xi|^2}\|Q\|_{\mathcal{P}}^2\|Q - \xi Q^{\sharp_{\mathcal{P}}}\|_{\mathcal{P}}^2.$$

(5) The following mixed operator norm inequality holds:

$$2(|\xi|^p + |\eta|^p)\|Q\|_{\mathcal{P}}^p \leq \|\xi Q + \eta Q^{\sharp_{\mathcal{P}}}\|_{\mathcal{P}}^p + \|\xi Q - \eta Q^{\sharp_{\mathcal{P}}}\|_{\mathcal{P}}^p.$$

Proof. By hypothesis, $Q \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$. By Theorem 3.1(2), we have $\|Q^{\sharp_{\mathcal{P}}}y\|_{\mathcal{P}} = \|Qy\|_{\mathcal{P}}$ for all $y \in \mathbb{H}$. We use this property throughout the proof.

(1) Let $y \in \mathbb{H}$. By applying the Cauchy-Schwarz inequality, we obtain

$$|\langle Q^2y, y\rangle_{\mathcal{P}}| = |\langle Qy, Q^{\sharp_{\mathcal{P}}}y\rangle_{\mathcal{P}}| \leq \|Qy\|_{\mathcal{P}}\|Q^{\sharp_{\mathcal{P}}}y\|_{\mathcal{P}}.$$

By using Theorem 3.1(2), we obtain

$$|\langle Q^2y, y\rangle_{\mathcal{P}}| \leq \|Qy\|_{\mathcal{P}}^2. \tag{3.3}$$

(2) Let $y \in \mathbb{S}_{\mathcal{P}}^1$. The generalized Buzano inequality established in [15] states that for any $x, z \in \mathbb{H}$ and $e \in \mathbb{S}_{\mathcal{P}}^1$, we have

$$|\langle x, e\rangle_{\mathcal{P}}\langle e, z\rangle_{\mathcal{P}}| \leq \frac{1}{2}(\|x\|_{\mathcal{P}}\|z\|_{\mathcal{P}} + |\langle x, z\rangle_{\mathcal{P}}|).$$

Let $e = y$, $x = Qy$, and $z = Q^{\sharp_{\mathcal{P}}}y$. By substituting these vectors, we obtain

$$|\langle Qy, y\rangle_{\mathcal{P}}|^2 \leq \frac{1}{2}(\|Qy\|_{\mathcal{P}}\|Q^{\sharp_{\mathcal{P}}}y\|_{\mathcal{P}} + |\langle Q^2y, y\rangle_{\mathcal{P}}|).$$

Applying Theorem 3.1(2), this simplifies to the desired lower bound. Furthermore, (3.3) directly provides the upper bound.

(3) The first inequality follows from the Cauchy–Schwarz inequality and Theorem 3.1(2).

Assume that $\xi = 0$. Then the inequality follows from (3.3). Suppose that $\xi \neq 0$, and let $x, z \notin \mathrm{Ker}(\mathcal{P})$. We apply the generalized Dunkl–Williams inequality established in [15]:

$$\frac{\|x\|_{\mathcal{P}}\|z\|_{\mathcal{P}} - |\langle x, z\rangle_{\mathcal{P}}|}{\|x\|_{\mathcal{P}}\|z\|_{\mathcal{P}}} \leq \frac{2\|x - z\|_{\mathcal{P}}^2}{(\|x\|_{\mathcal{P}} + \|z\|_{\mathcal{P}})^2}.$$

Set $x = Qy$ and $z = \xi Q^{\sharp_{\mathcal{P}}}y$. By Theorem 3.1(2), we have

$$\|z\|_{\mathcal{P}} = |\xi|\, \|Qy\|_{\mathcal{P}}.$$

Substituting these expressions into the above inequality yields the desired result.

(4) Let $y \in \mathbb{S}^1_{\mathcal{P}}$. Let $x = Qy$ and $z = Q^{\sharp_{\mathcal{P}}}y$. The generalized Dragomir inequality, as stated in [15], guarantees:

$$0 \leq \|x\|_{\mathcal{P}}^2\|z\|_{\mathcal{P}}^2 - |\langle x, z\rangle_{\mathcal{P}}|^2 \leq \frac{1}{|\xi|^2}\|x\|_{\mathcal{P}}^2\|x - \xi z\|_{\mathcal{P}}^2.$$

Substituting the chosen vectors and applying Theorem 3.1(2), we find $\|z\|_{\mathcal{P}} = \|x\|_{\mathcal{P}} = \|Qy\|_{\mathcal{P}}$. Thus, we obtain:

$$\begin{aligned} 0 \leq \|Qy\|_{\mathcal{P}}^4 - |\langle Q^2y, y\rangle_{\mathcal{P}}|^2 &\leq \frac{1}{|\xi|^2}\|Qy\|_{\mathcal{P}}^2\|Qy - \xi Q^{\sharp_{\mathcal{P}}}y\|_{\mathcal{P}}^2 \\ &\leq \frac{1}{|\xi|^2}\|Q\|_{\mathcal{P}}^2\|Q - \xi Q^{\sharp_{\mathcal{P}}}\|_{\mathcal{P}}^2. \end{aligned}$$

This finishes the proof.

(5) Let $y \in \mathbb{S}^1_{\mathcal{P}}$. By applying classical analysis, we obtain

$$2(\|x\|_{\mathcal{P}}^p + \|z\|_{\mathcal{P}}^p) \leq \|x + z\|_{\mathcal{P}}^p + \|x - z\|_{\mathcal{P}}^p.$$

Let $x = \xi Qy$ and $z = \eta Q^{\sharp_{\mathcal{P}}}y$. Applying Theorem 3.1(2), we deduce that $\|x\|_{\mathcal{P}} = |\xi|\|Qy\|_{\mathcal{P}}$ and $\|z\|_{\mathcal{P}} = |\eta|\|Qy\|_{\mathcal{P}}$. By substituting these into the inequality, we yield

$$2(|\xi|^p + |\eta|^p)\|Qy\|_{\mathcal{P}}^p \leq \|\xi Qy + \eta Q^{\sharp_{\mathcal{P}}}y\|_{\mathcal{P}}^p + \|\xi Qy - \eta Q^{\sharp_{\mathcal{P}}}y\|_{\mathcal{P}}^p.$$

By taking the supremum over $y \in \mathbb{S}^1_{\mathcal{P}}$, we verify the final operator norm inequality.

This completes the proof. □

The next result establishes exact equalities and bounds for $\langle\mathcal{P}\rangle$-normal operators.

Corollary 3.4. *Let $Q \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$. Then the following properties hold:*

(1) The generalized Dragomir inequality is an exact equality:

$$\omega_{\mathcal{P}}^2(Q) = \frac{1}{2}\left(\|Q\|_{\mathcal{P}}^2 + \omega_{\mathcal{P}}(Q^2)\right).$$

(2) The $\mathcal{P}$-seminorm is bounded by its $\mathcal{P}$-Cartesian components:

$$\|Q\|_{\mathcal{P}}^2 \leq \|\Re_{\mathcal{P}}(Q)\|_{\mathcal{P}}^2 + \|\Im_{\mathcal{P}}(Q)\|_{\mathcal{P}}^2.$$

Proof. (1) By taking the supremum over $y \in \mathbb{S}_{\mathcal{P}}^1$ in Proposition 3.2(2), we obtain

$$\omega_{\mathcal{P}}^2(Q) \leq \frac{1}{2}(\|Q\|_{\mathcal{P}}^2 + \omega_{\mathcal{P}}(Q^2)).$$

Since $Q \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$, Corollary 3.1 implies that $\omega_{\mathcal{P}}^2(Q) = \|Q\|_{\mathcal{P}}^2$. Since Theorem 3.3 and Corollary 3.1 yield $\omega_{\mathcal{P}}(Q^2) = \|Q^2\|_{\mathcal{P}} = \|Q\|_{\mathcal{P}}^2$, substituting gives

$$\|Q\|_{\mathcal{P}}^2 \leq \frac{1}{2}(\|Q\|_{\mathcal{P}}^2 + \|Q\|_{\mathcal{P}}^2) = \|Q\|_{\mathcal{P}}^2.$$

Thus, the inequality collapses to an exact equality.

(2) Since $2(\|x\|_{\mathcal{P}}^2 + \|z\|_{\mathcal{P}}^2) \leq \|x + z\|_{\mathcal{P}}^2 + \|x - z\|_{\mathcal{P}}^2$, setting $x = Qy$ and $z = Q^{\sharp_{\mathcal{P}}}y$ gives

$$4\|Q\|_{\mathcal{P}}^2 \leq \|Q + Q^{\sharp_{\mathcal{P}}}\|_{\mathcal{P}}^2 + \|Q - Q^{\sharp_{\mathcal{P}}}\|_{\mathcal{P}}^2.$$

Substituting $Q + Q^{\sharp_{\mathcal{P}}} = 2\Re_{\mathcal{P}}(Q)$ and $Q - Q^{\sharp_{\mathcal{P}}} = 2i\Im_{\mathcal{P}}(Q)$, we deduce

$$4\|Q\|_{\mathcal{P}}^2 \leq 4\|\Re_{\mathcal{P}}(Q)\|_{\mathcal{P}}^2 + 4\|\Im_{\mathcal{P}}(Q)\|_{\mathcal{P}}^2.$$

Dividing by 4 completes the proof. □

As discussed in the introduction, the algebraic definition of $\mathcal{P}$-invertibility introduced by Baklouti and Namouri [20] is often too restrictive, particularly when the range of $\mathcal{P}$ is not closed. To overcome this limitation, we adopt a topological approach based on the $\mathcal{P}$-seminorm. We redefine $\mathcal{P}$-invertibility by requiring the operator to be $\mathcal{P}$-bounded below with $\mathcal{P}$-dense range. This ensures that $\mathcal{P}$-invertibility on $\mathbb{H}$ coincides with the classical invertibility of the induced operator on the completion space $\mathbf{R}(\sqrt{\mathcal{P}})$.

The next definition introduces the concept of topological $\mathcal{P}$-invertibility.

Definition 3.1. *An operator $Q \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$ is topologically $\mathcal{P}$-invertible if it satisfies two conditions:*

1) $\mathcal{P}$-bounded below: There exists a constant $c > 0$ such that

$$\|Qy\|_{\mathcal{P}} \geq c\|y\|_{\mathcal{P}}, \quad \forall\, y \in \mathbb{H}.$$

2) $\mathcal{P}$-dense range: For every $z \in \mathbb{H}$, there exists a sequence $\{y_j\}_{j\in\mathbb{N}} \subseteq \mathbb{H}$ such that

$$\lim_{j\to\infty} \|Qy_j - z\|_{\mathcal{P}} = 0.$$

Using this notion, we introduce the corresponding topological spectrum in the following definition.

Definition 3.2. *The topological $\mathcal{P}$-spectrum of an operator $Q \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$, denoted by $\sigma_{\mathcal{P}}^{\text{top}}(Q)$, is defined by*

$$\sigma_{\mathcal{P}}^{\text{top}}(Q) = \{\xi \in \mathbb{C} : Q - \xi I_{\mathbb{H}} \textit{ is not topologically } \mathcal{P}\textit{-invertible}\}.$$

The next result establishes an equivalence for topological $\mathcal{P}$-invertibility.

Theorem 3.5. *Let* $\mathrm{Q} \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$*,* $\xi \in \mathbb{C}$*, and denote* $\mathrm{Q}_\xi = \mathrm{Q} - \xi I_{\mathbb{H}}$*. Then the following assertions are equivalent:*

(1) Q_ξ *is topologically* $\mathcal{P}$*-invertible.*
(2) $\widehat{\mathrm{Q}_\xi}^{\mathcal{P}}$ *is classically invertible in* $\mathscr{L}(\mathbf{R}(\sqrt{\mathcal{P}}))$*, where* $\widehat{\mathrm{Q}_\xi}^{\mathcal{P}}$ *is the unique bounded operator on* $\mathbf{R}(\sqrt{\mathcal{P}})$ *induced by* Q_ξ*.*

Proof. Since $\mathrm{Q} \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$, Lemma 2.1 guarantees the existence of a unique bounded operator $\widehat{\mathrm{Q}}^{\mathcal{P}} \in \mathscr{L}(\mathbf{R}(\sqrt{\mathcal{P}}))$ such that $\Phi_{\mathcal{P}}\mathrm{Q} = \widehat{\mathrm{Q}}^{\mathcal{P}}\Phi_{\mathcal{P}}$. We have:

$$\widehat{\mathrm{Q}_\xi}^{\mathcal{P}} = \widehat{\mathrm{Q}}^{\mathcal{P}} - \xi I_{\mathbf{R}(\sqrt{\mathcal{P}})}.$$

First, we prove (1) $\implies$ (2). Assume that Q_ξ is topologically $\mathcal{P}$-invertible. By definition, it is $\mathcal{P}$-bounded below and has a $\mathcal{P}$-dense range. Applying (2.2), we obtain $\|\mathrm{Q}_\xi \mathrm{y}\|_{\mathcal{P}} = \|\widehat{\mathrm{Q}_\xi}^{\mathcal{P}}\Phi_{\mathcal{P}}\mathrm{y}\|_{\mathbf{R}(\sqrt{\mathcal{P}})}$ and $\|\mathrm{y}\|_{\mathcal{P}} = \|\Phi_{\mathcal{P}}\mathrm{y}\|_{\mathbf{R}(\sqrt{\mathcal{P}})}$ for all $\mathrm{y} \in \mathbb{H}$. Since Q_ξ is $\mathcal{P}$-bounded below, there exists $c > 0$ such that $\|\widehat{\mathrm{Q}_\xi}^{\mathcal{P}}\mathrm{z}\|_{\mathbf{R}(\sqrt{\mathcal{P}})} \geq c\|\mathrm{z}\|_{\mathbf{R}(\sqrt{\mathcal{P}})}$ for all $\mathrm{z} \in \mathrm{Im}(\mathcal{P})$. Since $\mathrm{Im}(\mathcal{P})$ is dense in $\mathbf{R}(\sqrt{\mathcal{P}})$, this inequality extends continuously to $\mathbf{R}(\sqrt{\mathcal{P}})$. Thus, $\widehat{\mathrm{Q}_\xi}^{\mathcal{P}}$ is bounded below. On the other hand, since Q_ξ has a $\mathcal{P}$-dense range, for any $\mathrm{z} \in \mathbb{H}$, there exists a sequence $\{\mathrm{y}_j\}_{j\in\mathbb{N}} \subseteq \mathbb{H}$ such that $\lim\limits_{j\to\infty}\|\mathrm{Q}_\xi\mathrm{y}_j - \mathrm{z}\|_{\mathcal{P}} = 0$. By using $\Phi_{\mathcal{P}}$ and (2.2), we obtain

$$\lim_{j\to\infty}\|\widehat{\mathrm{Q}_\xi}^{\mathcal{P}}\Phi_{\mathcal{P}}\mathrm{y}_j - \Phi_{\mathcal{P}}\mathrm{z}\|_{\mathbf{R}(\sqrt{\mathcal{P}})} = 0.$$

This implies that the image of $\widehat{\mathrm{Q}_\xi}^{\mathcal{P}}$ is dense in $\mathrm{Im}(\mathcal{P})$. Since $\mathrm{Im}(\mathcal{P})$ is dense in $\mathbf{R}(\sqrt{\mathcal{P}})$, it follows that $\mathrm{Im}(\widehat{\mathrm{Q}_\xi}^{\mathcal{P}})$ is dense in the complete Hilbert space $\mathbf{R}(\sqrt{\mathcal{P}})$. Since $\widehat{\mathrm{Q}_\xi}^{\mathcal{P}}$ is bounded below and has a dense range, it is classically invertible. Thus, we obtain (2).

Next, we prove (2) $\implies$ (1). Assume that $\widehat{\mathrm{Q}_\xi}^{\mathcal{P}}$ is classically invertible. Then it is bounded below and has a dense range. Since it is bounded below, there exists $c > 0$ such that $\|\widehat{\mathrm{Q}_\xi}^{\mathcal{P}}\mathrm{x}\|_{\mathbf{R}(\sqrt{\mathcal{P}})} \geq c\|\mathrm{x}\|_{\mathbf{R}(\sqrt{\mathcal{P}})}$ for all $\mathrm{x} \in \mathbf{R}(\sqrt{\mathcal{P}})$. Let $\mathrm{y} \in \mathbb{H}$. This implies that $\Phi_{\mathcal{P}}\mathrm{y} \in \mathbf{R}(\sqrt{\mathcal{P}})$. By setting $\mathrm{x} = \Phi_{\mathcal{P}}\mathrm{y}$ and using (2.2), we obtain $\|\mathrm{Q}_\xi\mathrm{y}\|_{\mathcal{P}} \geq c\|\mathrm{y}\|_{\mathcal{P}}$. Thus, Q_ξ is $\mathcal{P}$-bounded below. Now, let $\mathrm{z} \in \mathbb{H}$. Thus, $\Phi_{\mathcal{P}}\mathrm{z} \in \mathbf{R}(\sqrt{\mathcal{P}})$. Then there exists a unique $\mathrm{x} \in \mathbf{R}(\sqrt{\mathcal{P}})$ such that $\Phi_{\mathcal{P}}\mathrm{z} = \widehat{\mathrm{Q}_\xi}^{\mathcal{P}}\mathrm{x}$. Since $\mathrm{Im}(\mathcal{P})$ is dense in $\mathbf{R}(\sqrt{\mathcal{P}})$, there exists a sequence $\{\mathrm{y}_j\}_{j\in\mathbb{N}} \subseteq \mathbb{H}$ such that

$$\lim_{j\to\infty}\|\Phi_{\mathcal{P}}\mathrm{y}_j - \mathrm{x}\|_{\mathbf{R}(\sqrt{\mathcal{P}})} = 0.$$

Because $\widehat{\mathrm{Q}_\xi}^{\mathcal{P}}$ is bounded, it is continuous, meaning it preserves limits. Thus, we obtain:

$$\lim_{j\to\infty}\|\widehat{\mathrm{Q}_\xi}^{\mathcal{P}}\Phi_{\mathcal{P}}\mathrm{y}_j - \widehat{\mathrm{Q}_\xi}^{\mathcal{P}}\mathrm{x}\|_{\mathbf{R}(\sqrt{\mathcal{P}})} = 0.$$

By substituting $\widehat{\mathrm{Q}_\xi}^{\mathcal{P}}\Phi_{\mathcal{P}}\mathrm{y}_j = \Phi_{\mathcal{P}}\mathrm{Q}_\xi\mathrm{y}_j$ and $\widehat{\mathrm{Q}_\xi}^{\mathcal{P}}\mathrm{x} = \Phi_{\mathcal{P}}\mathrm{z}$, we deduce:

$$\lim_{j\to\infty}\|\Phi_{\mathcal{P}}\mathrm{Q}_\xi\mathrm{y}_j - \Phi_{\mathcal{P}}\mathrm{z}\|_{\mathbf{R}(\sqrt{\mathcal{P}})} = 0.$$

Applying (2.2), we deduce that $\lim\limits_{j\to\infty}\|\mathrm{Q}_\xi\mathrm{y}_j - \mathrm{z}\|_{\mathcal{P}} = 0$. This implies that Q_ξ has a $\mathcal{P}$-dense range. Therefore, Q_ξ is topologically $\mathcal{P}$-invertible. This proves (1). This completes the proof. □

Before stating the next corollary, we introduce the following definition.

Definition 3.3. *For* $\mathrm{Q} \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$*, we define the* $\mathcal{P}$*-topological spectral radius as*

$$r_{\mathcal{P}}^{\mathrm{top}}(\mathrm{Q}) = \sup\{|\lambda| \;:\; \lambda \in \sigma_{\mathcal{P}}^{\mathrm{top}}(\mathrm{Q})\}.$$

The next result relates the topological $\mathcal{P}$-spectrum to the classical spectrum.

Corollary 3.5. *For every* $\mathrm{Q} \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$*, we have*

$$\sigma_{\mathcal{P}}^{\mathrm{top}}(\mathrm{Q}) = \sigma(\widehat{\mathrm{Q}}^{\mathcal{P}}).$$

Therefore, we have

$$r_{\mathcal{P}}^{\mathrm{top}}(\mathrm{Q}) = r_{\mathcal{P}}(\mathrm{Q}).$$

Proof. By Theorem 3.5, for any $\xi \in \mathbb{C}$, the operator Q_ξ is topologically $\mathcal{P}$-invertible if and only if $\widehat{\mathrm{Q}_\xi}^{\mathcal{P}}$ is classically invertible. Therefore, $\xi \in \sigma_{\mathcal{P}}^{\mathrm{top}}(\mathrm{Q})$ if and only if $\xi \in \sigma(\widehat{\mathrm{Q}}^{\mathcal{P}})$. This implies $\sigma_{\mathcal{P}}^{\mathrm{top}}(\mathrm{Q}) = \sigma(\widehat{\mathrm{Q}}^{\mathcal{P}})$. This completes the proof. □

The following remark highlights properties of the topological $\mathcal{P}$-spectrum.

Remark 3.2. *Let* $\mathrm{Q} \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$*. Then the following properties hold:*

(1) Since for $\mathrm{Q} \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$*, we have* $\sigma_{\mathcal{P}}^{\mathrm{top}}(\mathrm{Q}) = \sigma(\widehat{\mathrm{Q}}^{\mathcal{P}})$*. Therefore,* $\sigma_{\mathcal{P}}^{\mathrm{top}}(\mathrm{Q})$ *is non-empty.*
(2) If $\mathrm{Q} \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$*, then by Corollary 3.5 together with* (2.1)*, we have*

$$r_{\mathcal{P}}^{\mathrm{top}}(\mathrm{Q}) = r_{\mathcal{P}}(\mathrm{Q}) = \omega_{\mathcal{P}}(\mathrm{Q}) = \|\mathrm{Q}\|_{\mathcal{P}}.$$

The topological $\mathcal{P}$-spectrum is always non-empty. In the next result, we compare the topological $\mathcal{P}$-spectrum with the traditional one.

Theorem 3.6. *Let* $\mathrm{Q} \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$*. Then* $\sigma_{\mathcal{P}}^{\mathrm{top}}(\mathrm{Q}) \subseteq \sigma_{\mathcal{P}}^{\mathrm{BN}}(\mathrm{Q})$*. Moreover, if the range of* $\mathcal{P}$ *is closed, then* $\sigma_{\mathcal{P}}^{\mathrm{top}}(\mathrm{Q}) = \sigma_{\mathcal{P}}^{\mathrm{BN}}(\mathrm{Q})$*.*

Proof. By Corollary 3.5, we have $\sigma_{\mathcal{P}}^{\mathrm{top}}(\mathrm{Q}) = \sigma(\widehat{\mathrm{Q}}^{\mathcal{P}})$. Since $\sigma(\widehat{\mathrm{Q}}^{\mathcal{P}}) \subseteq \sigma_{\mathcal{P}}^{\mathrm{BN}}(\mathrm{Q})$, it follows that $\sigma_{\mathcal{P}}^{\mathrm{top}}(\mathrm{Q}) \subseteq \sigma_{\mathcal{P}}^{\mathrm{BN}}(\mathrm{Q})$. Assume that the range of $\mathcal{P}$ is closed. By hypothesis, this condition implies $\sigma(\widehat{\mathrm{Q}}^{\mathcal{P}}) = \sigma_{\mathcal{P}}^{\mathrm{BN}}(\mathrm{Q})$. Consequently, we deduce that $\sigma_{\mathcal{P}}^{\mathrm{top}}(\mathrm{Q}) = \sigma_{\mathcal{P}}^{\mathrm{BN}}(\mathrm{Q})$. This completes the proof. □

The next remark justifies the topological approach by showing that the equality fails without the closed range assumption.

Remark 3.3. *The equality* $\sigma_{\mathcal{P}}^{\mathrm{BN}}(\mathrm{Q}) = \sigma(\widehat{\mathrm{Q}}^{\mathcal{P}})$ *is generally not possible when the range of* $\mathcal{P}$ *is not closed. In the following example, we present a case where an operator is boundedly invertible on* $\mathbf{R}(\sqrt{\mathcal{P}})$*, but its algebraic inverse fails to map* $\mathrm{Im}(\mathcal{P})$ *back into itself.*

The following example provides a concrete case illustrating Remark 3.3.

Example 3.2. *Consider the complex Hilbert space $\mathbb{H} = \ell^2(\mathbb{N})$. Let $a_j = 1/2^j$ for all integers $j \geq 0$. We define the positive operator $\mathcal{P}$ on $\mathbb{H}$ by $\mathcal{P}((x_j)_{j\geq 0}) = (a_j^2 x_j)_{j\geq 0}$. Since $\mathcal{P}$ is injective, we deduce that $\Pi_{\mathcal{P}} = I_{\mathbb{H}}$. Since the sequence converges to zero, the range $\mathrm{Im}(\mathcal{P})$ is dense but not closed.*

We define the right shift operator Q and the left shift operator R on $\mathbb{H}$ by $\mathrm{Q}(x_0, x_1, \dots) = 2(0, x_0, x_1, \dots)$ and $\mathrm{R}(x_0, x_1, \dots) = (x_1, x_2, \dots)$. Evaluating the semi-inner product, we observe $\langle \mathrm{Qx}, \mathrm{y}\rangle_{\mathcal{P}} = \langle \mathcal{P}\mathrm{Qx}, \mathrm{y}\rangle = \sum_{j=1}^{\infty} a_j^2 (2x_{j-1})\overline{y_j}$. Since $a_j = \frac{1}{2}a_{j-1}$, it follows that $a_j^2 = \frac{1}{4}a_{j-1}^2$. By substituting this relation, we find

$$\langle \mathrm{Qx}, \mathrm{y}\rangle_{\mathcal{P}} = \sum_{j=1}^{\infty} \frac{1}{4} a_{j-1}^2 (2x_{j-1})\overline{y_j} = \frac{1}{2}\sum_{k=0}^{\infty} a_k^2 x_k \overline{y_{k+1}}.$$

Applying the definition of the left shift, we compute $\langle \mathrm{x}, \mathrm{Ry}\rangle_{\mathcal{P}} = \sum_{k=0}^{\infty} a_k^2 x_k \overline{y_{k+1}}$. Comparing these expressions, we obtain $\langle \mathrm{Qx}, \mathrm{y}\rangle_{\mathcal{P}} = \frac{1}{2}\langle \mathrm{x}, \mathrm{Ry}\rangle_{\mathcal{P}}$. Thus, we establish $\mathrm{Q}^{\sharp_{\mathcal{P}}} = \frac{1}{2}\mathrm{R}$.

Calculating the operator compositions, we observe

$$\mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{Qx} = \frac{1}{2}\mathrm{R}(2(0, x_0, x_1, \dots)) = (x_0, x_1, \dots) = \mathrm{x}.$$

Since $\mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{Q} = I_{\mathbb{H}}$, we confirm that Q is a $\mathcal{P}$-isometry. Because

$$\mathrm{Q}\mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{x} = \mathrm{Q}(\frac{1}{2}(x_1, x_2, \dots)) = (0, x_1, x_2, \dots),$$

we deduce that $\mathrm{Q}\mathrm{Q}^{\sharp_{\mathcal{P}}} \neq I_{\mathbb{H}}$.

Since $\widehat{\mathrm{Q}}^{\mathcal{P}}$ is a proper isometry on $\mathbf{R}(\sqrt{\mathcal{P}})$, its classical spectrum $\sigma(\widehat{\mathrm{Q}}^{\mathcal{P}})$ is exactly the closed unit disk $\overline{\mathbb{D}} = \{\xi \in \mathbb{C} : |\xi| \leq 1\}$. By using Corollary 3.5, we deduce that $\sigma_{\mathcal{P}}^{\mathrm{top}}(\mathrm{Q}) = \overline{\mathbb{D}}$. Next, we examine the classical BN-spectrum. The strict BN-invertibility of $\mathrm{Q} - \xi I_{\mathbb{H}}$ requires an operator $\mathrm{R} \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$ such that $\mathcal{P}(\mathrm{Q} - \xi I_{\mathbb{H}})\mathrm{R} = \mathcal{P}\mathrm{R}(\mathrm{Q} - \xi I_{\mathbb{H}}) = \mathcal{P}$. Since $\mathcal{P}$ is strictly injective, this forces

$$(\mathrm{Q} - \xi I_{\mathbb{H}})\mathrm{R} = \mathrm{R}(\mathrm{Q} - \xi I_{\mathbb{H}}) = I_{\mathbb{H}}.$$

This means the algebraic inverse R must be the exact classical inverse of $\mathrm{Q} - \xi I_{\mathbb{H}}$ on $\mathbb{H}$. Thus, we obtain $\sigma(\mathrm{Q}) \subseteq \sigma_{\mathcal{P}}^{\mathrm{BN}}(\mathrm{Q})$. Since Q is a scaled right shift on $\ell^2(\mathbb{N})$ with norm 2, its classical spectrum is $\sigma(\mathrm{Q}) = \{\xi \in \mathbb{C} : |\xi| \leq 2\}$. Substituting this, we establish $\{\xi \in \mathbb{C} : |\xi| \leq 2\} \subseteq \sigma_{\mathcal{P}}^{\mathrm{BN}}(\mathrm{Q})$. By comparing the two spectra, we confirm the strict inclusion $\sigma_{\mathcal{P}}^{\mathrm{top}}(\mathrm{Q}) \subsetneq \sigma_{\mathcal{P}}^{\mathrm{BN}}(\mathrm{Q})$. Therefore, the scalars in the annulus $1 < |\xi| \leq 2$ are artificial spectral points. This implies that the topological approach is necessary.

We now improve numerical range theorems without requiring the closed range assumption. The next theorem combines three fundamental numerical range inclusions and equalities, utilizing the topological spectrum approach. The first two results are general and do not require $\mathcal{P}$-normality, whereas the third one applies specifically to $\mathcal{P}$-normal operators.

Theorem 3.7. *Let $\mathrm{Q} \in \mathscr{L}_{\sqrt{\mathcal{P}}}(\mathbb{H})$. Then the following properties hold:*

(1) The $\mathcal{P}$-approximate point spectrum is completely contained within the closure of the $\mathcal{P}$-numerical range:

$$\sigma_{\pi,\mathcal{P}}(\mathrm{Q}) \subseteq \overline{W_{\mathcal{P}}(\mathrm{Q})}.$$

(2) The topological $\mathcal{P}$-spectrum is contained in the closure of the $\mathcal{P}$-numerical range:

$$\sigma_{\mathcal{P}}^{\text{top}}(\mathrm{Q}) \subseteq \overline{W_{\mathcal{P}}(\mathrm{Q})}.$$

(3) If $\mathrm{Q} \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$*, then:*

$$\overline{W_{\mathcal{P}}(\mathrm{Q})} = \operatorname{conv}(\sigma_{\mathcal{P}}^{\text{top}}(\mathrm{Q})),$$

where $\operatorname{conv}(\cdot)$ *denotes the convex hull of a set.*

Proof. (1) Let $\xi \in \sigma_{\pi,\mathcal{P}}(\mathrm{Q})$. By definition, there exists a sequence $\{\mathrm{y}_j\}_{j\in\mathbb{N}} \subseteq \mathbb{S}_{\mathcal{P}}^1$ such that

$$\lim_{j\to\infty} \|(\mathrm{Q} - \xi I_{\mathbb{H}})\mathrm{y}_j\|_{\mathcal{P}} = 0.$$

Evaluating the absolute difference, we establish:

$$\begin{aligned} |\langle \mathrm{Q}\mathrm{y}_j, \mathrm{y}_j\rangle_{\mathcal{P}} - \xi| &= |\langle \mathrm{Q}\mathrm{y}_j, \mathrm{y}_j\rangle_{\mathcal{P}} - \xi\langle \mathrm{y}_j, \mathrm{y}_j\rangle_{\mathcal{P}}| \\ &= |\langle (\mathrm{Q} - \xi I_{\mathbb{H}})\mathrm{y}_j, \mathrm{y}_j\rangle_{\mathcal{P}}|. \end{aligned}$$

Applying the Cauchy-Schwarz inequality yields:

$$|\langle (\mathrm{Q} - \xi I_{\mathbb{H}})\mathrm{y}_j, \mathrm{y}_j\rangle_{\mathcal{P}}| \le \|(\mathrm{Q} - \xi I_{\mathbb{H}})\mathrm{y}_j\|_{\mathcal{P}}\|\mathrm{y}_j\|_{\mathcal{P}}.$$

Since $\|\mathrm{y}_j\|_{\mathcal{P}} = 1$, we verify:

$$|\langle \mathrm{Q}\mathrm{y}_j, \mathrm{y}_j\rangle_{\mathcal{P}} - \xi| \le \|(\mathrm{Q} - \xi I_{\mathbb{H}})\mathrm{y}_j\|_{\mathcal{P}}.$$

Taking the limit as $j \to \infty$, we deduce that $\lim_{j\to\infty}\langle \mathrm{Q}\mathrm{y}_j, \mathrm{y}_j\rangle_{\mathcal{P}} = \xi$. Since $\langle \mathrm{Q}\mathrm{y}_j, \mathrm{y}_j\rangle_{\mathcal{P}} \in W_{\mathcal{P}}(\mathrm{Q})$, it follows that $\xi \in \overline{W_{\mathcal{P}}(\mathrm{Q})}$. Therefore, we conclude that $\sigma_{\pi,\mathcal{P}}(\mathrm{Q}) \subseteq \overline{W_{\mathcal{P}}(\mathrm{Q})}$.

(2) By Corollary 3.5, we have $\sigma_{\mathcal{P}}^{\text{top}}(\mathrm{Q}) = \sigma(\widehat{\mathrm{Q}}^{\mathcal{P}})$. Classical spectral theory implies that the spectrum of any bounded operator is contained within the closure of its numerical range. Thus, we have $\sigma(\widehat{\mathrm{Q}}^{\mathcal{P}}) \subseteq \overline{W(\widehat{\mathrm{Q}}^{\mathcal{P}})}$. Applying Lemma 2.1(2), we obtain $\overline{W(\widehat{\mathrm{Q}}^{\mathcal{P}})} = \overline{W_{\mathcal{P}}(\mathrm{Q})}$. Therefore, we deduce that $\sigma_{\mathcal{P}}^{\text{top}}(\mathrm{Q}) \subseteq \overline{W_{\mathcal{P}}(\mathrm{Q})}$.

(3) Assume that $\mathrm{Q} \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$. Then Theorem 3.1 implies that $\widehat{\mathrm{Q}}^{\mathcal{P}}$ is normal. By applying the Toeplitz-Hausdorff theorem, we obtain $\overline{W(\widehat{\mathrm{Q}}^{\mathcal{P}})} = \operatorname{conv}(\sigma(\widehat{\mathrm{Q}}^{\mathcal{P}}))$. Corollary 3.5 establishes that $\sigma(\widehat{\mathrm{Q}}^{\mathcal{P}}) = \sigma_{\mathcal{P}}^{\text{top}}(\mathrm{Q})$. This implies that $\operatorname{conv}(\sigma(\widehat{\mathrm{Q}}^{\mathcal{P}})) = \operatorname{conv}(\sigma_{\mathcal{P}}^{\text{top}}(\mathrm{Q}))$. Applying Lemma 2.1(2), we obtain $\overline{W_{\mathcal{P}}(\mathrm{Q})} = \overline{W(\widehat{\mathrm{Q}}^{\mathcal{P}})}$. Therefore, we deduce that $\overline{W_{\mathcal{P}}(\mathrm{Q})} = \operatorname{conv}(\sigma_{\mathcal{P}}^{\text{top}}(\mathrm{Q}))$. This completes the proof. □

The next result establishes the symmetry of $\mathcal{P}$-null spaces and proves that the $\mathcal{P}$-ascent of these operators is precisely one.

Theorem 3.8. *Let* $\mathrm{Q} \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$*. Then its* $\mathcal{P}$*-null spaces coincide:*

$$\operatorname{Ker}_{\mathcal{P}}(\mathrm{Q}) = \operatorname{Ker}_{\mathcal{P}}(\mathrm{Q}^{\sharp_{\mathcal{P}}}) = \operatorname{Ker}_{\mathcal{P}}(\mathrm{Q}^2).$$

Proof. Let $\mathrm{y} \in \mathbb{H}$. By definition, $\mathrm{y} \in \operatorname{Ker}_{\mathcal{P}}(\mathrm{Q})$ if and only if $\|\mathrm{Q}\mathrm{y}\|_{\mathcal{P}} = 0$. Since Q is $\langle\mathcal{P}\rangle$-normal, Theorem 3.1 implies $\|\mathrm{Q}\mathrm{y}\|_{\mathcal{P}} = \|\mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{y}\|_{\mathcal{P}}$. By using this rule, we obtain $\|\mathrm{Q}\mathrm{y}\|_{\mathcal{P}} = 0$ if and only if $\|\mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{y}\|_{\mathcal{P}} = 0$. Because $\|\mathrm{Q}^{\sharp_{\mathcal{P}}}\mathrm{y}\|_{\mathcal{P}} = 0$ precisely when $\mathrm{y} \in \operatorname{Ker}_{\mathcal{P}}(\mathrm{Q}^{\sharp_{\mathcal{P}}})$, we deduce that $\operatorname{Ker}_{\mathcal{P}}(\mathrm{Q}) = \operatorname{Ker}_{\mathcal{P}}(\mathrm{Q}^{\sharp_{\mathcal{P}}})$.

Assume that $y \in \mathrm{Ker}_{\mathcal{P}}(Q)$. Then we have $\|Qy\|_{\mathcal{P}} = 0$. Since Q is $\mathcal{P}$-bounded, the submultiplicativity of the seminorm implies $\|Q^2y\|_{\mathcal{P}} \leq \|Q\|_{\mathcal{P}}\|Qy\|_{\mathcal{P}} = 0$. Consequently, we obtain $y \in \mathrm{Ker}_{\mathcal{P}}(Q^2)$. This shows that $\mathrm{Ker}_{\mathcal{P}}(Q) \subseteq \mathrm{Ker}_{\mathcal{P}}(Q^2)$.

Conversely, assume that $y \in \mathrm{Ker}_{\mathcal{P}}(Q^2)$. Then we have $\|Q^2y\|_{\mathcal{P}} = 0$. By applying (3.2), we assert:

$$\|Qy\|_{\mathcal{P}}^2 \leq \|Q^2y\|_{\mathcal{P}}\|y\|_{\mathcal{P}}.$$

By substituting $\|Q^2y\|_{\mathcal{P}} = 0$, we obtain $\|Qy\|_{\mathcal{P}}^2 \leq 0$. It follows that $\|Qy\|_{\mathcal{P}} = 0$. Thus, we deduce that $y \in \mathrm{Ker}_{\mathcal{P}}(Q)$. This confirms that $\mathrm{Ker}_{\mathcal{P}}(Q^2) \subseteq \mathrm{Ker}_{\mathcal{P}}(Q)$. This completes the proof. □

The next result establishes the symmetry of $\mathcal{P}$-kernels for translated operators.

Corollary 3.6. *Let* $Q \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$*, and let* $\xi \in \mathbb{C}$*. Then the corresponding* $\mathcal{P}$*-kernels coincide perfectly:*

$$\mathrm{Ker}_{\mathcal{P}}(Q - \xi I_{\mathbb{H}}) = \mathrm{Ker}_{\mathcal{P}}(Q^{\sharp_{\mathcal{P}}} - \overline{\xi}\Pi_{\mathcal{P}}).$$

Proof. The proof consists essentially of recalling the basic definitions and applying Theorem 3.8 to the translated operator $Q_\xi = Q - \xi I_{\mathbb{H}}$, and it is thus omitted.

In the next theorem, without assuming that the range of $\mathcal{P}$ is closed, we explore the properties of $\mathcal{P}$-eigenspaces. First, we define the $\mathcal{P}$-eigenspace associated with $\xi \in \mathbb{C}$ as:

$$\mathcal{E}_{\mathcal{P}}(Q, \xi) = \{y \in \mathbb{H} : \|Qy - \xi y\|_{\mathcal{P}} = 0\}.$$

By using the definition of the $\mathcal{P}$-kernel, we express this $\mathcal{P}$-eigenspace precisely as

$$\mathcal{E}_{\mathcal{P}}(Q, \xi) = \mathrm{Ker}_{\mathcal{P}}(Q - \xi I_{\mathbb{H}}).$$

Theorem 3.9. *Let* $Q \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$ *and* $\xi \in \mathbb{C}$*. Then the following properties hold:*

(1) If $y \in \mathcal{E}_{\mathcal{P}}(Q, \xi)$*, then* $Q^{\sharp_{\mathcal{P}}}y = \overline{\xi}\Pi_{\mathcal{P}}y$.
(2) $\mathcal{E}_{\mathcal{P}}(Q, \xi)$ *and* $\mathcal{E}_{\mathcal{P}}(Q, \xi)^{\perp_{\mathcal{P}}}$ *are invariant under both* Q *and* $Q^{\sharp_{\mathcal{P}}}$.

Proof. (1) Let $y \in \mathcal{E}_{\mathcal{P}}(Q, \xi)$. By definition, we have $\|Qy - \xi y\|_{\mathcal{P}} = 0$. Since $Q \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$, Theorem 3.1 implies the translated operator $Q_\xi = Q - \xi I_{\mathbb{H}}$ is $\langle\mathcal{P}\rangle$-normal. The norm equivalence yields $\|(Q_\xi)^{\sharp_{\mathcal{P}}}y\|_{\mathcal{P}} = \|Q_\xi y\|_{\mathcal{P}} = 0$. By substituting the adjoint, we obtain $\|(Q^{\sharp_{\mathcal{P}}} - \overline{\xi}\Pi_{\mathcal{P}})y\|_{\mathcal{P}} = 0$. Thus, we obtain $\mathcal{P}(Q^{\sharp_{\mathcal{P}}}y - \overline{\xi}\Pi_{\mathcal{P}}y) = \mathbf{0}$. Since $\mathcal{P}\Pi_{\mathcal{P}} = \mathcal{P}$, this reduces to $\mathcal{P}Q^{\sharp_{\mathcal{P}}}y = \overline{\xi}\mathcal{P}y$. Applying the Moore-Penrose pseudoinverse $\mathcal{P}^\dagger$ to both sides, we deduce that $\Pi_{\mathcal{P}}Q^{\sharp_{\mathcal{P}}}y = \overline{\xi}\Pi_{\mathcal{P}}y$. Lemma 2.3 establishes $\Pi_{\mathcal{P}}Q^{\sharp_{\mathcal{P}}} = Q^{\sharp_{\mathcal{P}}}$. Therefore, we confirm that $Q^{\sharp_{\mathcal{P}}}y = \overline{\xi}\Pi_{\mathcal{P}}y$.

(2) Assume that $y \in \mathcal{E}_{\mathcal{P}}(Q, \xi)$. This implies that $Qy - \xi y \in \mathrm{Ker}(\mathcal{P})$. Lemma 2.2 guarantees that $\mathrm{Ker}(\mathcal{P})$ is invariant under Q. This ensures that $Q(Qy - \xi y) \in \mathrm{Ker}(\mathcal{P})$. Consequently, we verify that $\|Q(Qy) - \xi(Qy)\|_{\mathcal{P}} = 0$, which proves that $Qy \in \mathcal{E}_{\mathcal{P}}(Q, \xi)$.

To establish invariance under $Q^{\sharp_{\mathcal{P}}}$, we express $Qy = \xi y + k$ for some $k \in \mathrm{Ker}(\mathcal{P})$. Since $Q \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$, we obtain:

$$\begin{aligned}\mathcal{P}QQ^{\sharp_{\mathcal{P}}}y &= \mathcal{P}Q^{\sharp_{\mathcal{P}}}Qy \\ &= \mathcal{P}Q^{\sharp_{\mathcal{P}}}(\xi y + k).\end{aligned}$$

Since $k \in \mathrm{Ker}(\mathcal{P})$, we have $k = (I_{\mathbb{H}} - \Pi_{\mathcal{P}})k$. Lemma 2.3 implies

$$Q^{\sharp_{\mathcal{P}}}k = Q^{\sharp_{\mathcal{P}}}(I_{\mathbb{H}} - \Pi_{\mathcal{P}})k = Q^{\sharp_{\mathcal{P}}}k - Q^{\sharp_{\mathcal{P}}}k = \mathbf{0}.$$

Substituting this, we obtain $\mathcal{P}Q(Q^{\sharp_{\mathcal{P}}}y) = \xi\mathcal{P}(Q^{\sharp_{\mathcal{P}}}y)$. By definition, this confirms that $Q^{\sharp_{\mathcal{P}}}y \in \mathcal{E}_{\mathcal{P}}(Q, \xi)$.

Similarly, $\langle v, Q^{\sharp_{\mathcal{P}}}z\rangle_{\mathcal{P}} = \langle Qv, z\rangle_{\mathcal{P}}$. Since $Qv \in \mathcal{E}_{\mathcal{P}}(Q, \xi)$, we find $\langle Qv, z\rangle_{\mathcal{P}} = 0$. This implies $Q^{\sharp_{\mathcal{P}}}z \in \mathcal{E}_{\mathcal{P}}(Q, \xi)^{\perp_{\mathcal{P}}}$. This completes the proof. □

The next result establishes the $\mathcal{P}$-orthogonality of generalized $\mathcal{P}$-eigenvectors.

Theorem 3.10. *Let* $Q \in \mathbf{N}_{\langle\mathcal{P}\rangle}(\mathbb{H})$. *Let* $\xi, \eta \in \mathbb{C}$ *such that* $\xi \neq \eta$. *Assume that* $y, z \in \mathbb{H}$ *satisfy* $\mathcal{P}Qy = \xi\mathcal{P}y$ *and* $\mathcal{P}Qz = \eta\mathcal{P}z$. *Then* $y \perp_{\mathcal{P}} z$.

Proof. Since $\mathcal{P}Qz = \eta\mathcal{P}z$, it follows that $z \in \mathrm{Ker}_{\mathcal{P}}(Q - \eta I_{\mathbb{H}})$. Because Q is $\langle\mathcal{P}\rangle$-normal, Corollary 3.6 implies $z \in \mathrm{Ker}_{\mathcal{P}}(Q^{\sharp_{\mathcal{P}}} - \overline{\eta}\Pi_{\mathcal{P}})$. This yields $\mathcal{P}Q^{\sharp_{\mathcal{P}}}z = \overline{\eta}\mathcal{P}\Pi_{\mathcal{P}}z$. Lemma 2.3(i) establishes $\mathcal{P}\Pi_{\mathcal{P}} = \mathcal{P}$, resulting in $\mathcal{P}\Pi_{\mathcal{P}}z = \mathcal{P}z$. Substituting this, we verify $\mathcal{P}Q^{\sharp_{\mathcal{P}}}z = \overline{\eta}\mathcal{P}z$.

Applying Q directly to y, we obtain

$$\begin{aligned}
\langle Qy, z\rangle_{\mathcal{P}} &= \langle \mathcal{P}Qy, z\rangle \\
&= \langle \xi\mathcal{P}y, z\rangle \\
&= \xi\langle y, z\rangle_{\mathcal{P}}.
\end{aligned}$$

By using the $\mathcal{P}$-adjoint and the established eigenspace symmetry, we obtain:

$$\begin{aligned}
\langle Qy, z\rangle_{\mathcal{P}} &= \langle y, Q^{\sharp_{\mathcal{P}}}z\rangle_{\mathcal{P}} = \langle \mathcal{P}y, Q^{\sharp_{\mathcal{P}}}z\rangle \\
&= \langle y, \mathcal{P}Q^{\sharp_{\mathcal{P}}}z\rangle = \langle y, \overline{\eta}\mathcal{P}z\rangle \\
&= \eta\langle y, \mathcal{P}z\rangle = \eta\langle y, z\rangle_{\mathcal{P}}.
\end{aligned}$$

Because the scalar $\overline{\eta}$ is in the second argument of the classical inner product, it emerges conjugated as η. Equating the two expressions, we obtain $\xi\langle y, z\rangle_{\mathcal{P}} = \eta\langle y, z\rangle_{\mathcal{P}}$. This implies that

$$(\xi - \eta)\langle y, z\rangle_{\mathcal{P}} = 0.$$

Since $\xi \neq \eta$, we deduce that $\langle y, z\rangle_{\mathcal{P}} = 0$. This means that $y \perp_{\mathcal{P}} z$. This completes the proof. □

4. Conclusions

In this paper, we have explored the algebraic, metric, and spectral properties of $\langle\mathcal{P}\rangle$-normal operators in semi-Hilbertian spaces. One of our main contributions is the introduction of a new topological approach to $\mathcal{P}$-invertibility and the $\mathcal{P}$-spectrum. By doing so, we successfully bypassed the traditional and highly restrictive assumption that the range of the positive operator $\mathcal{P}$ must be closed. This refined framework allowed us to establish exact equalities and generalize several fundamental $\mathcal{P}$-numerical radius inequalities and spectral inclusion theorems for this broad class of operators. Furthermore, we investigated the geometric structure of $\langle\mathcal{P}\rangle$-normal operators, ultimately proving the strict orthogonality and symmetry of their generalized $\mathcal{P}$-eigenspaces.

We believe that the topological framework introduced here provides a natural, robust, and highly flexible setting for studying operators in semi-Hilbertian spaces. Therefore, this work may serve as a strong starting point for future research. Other researchers can build upon these foundations to explore further generalizations, investigate broader classes of non-normal operators, or apply this topological spectrum to solve complex problems in advanced operator theory.

Author contributions

Noura M. Alhouiti, Mesfer H. Alqahtani and Kais Feki: Conceptualization, Visualization, Funding, Writing–review & editing, Formal analysis, Project administration, Validation, Investigation; Noura M. Alhouiti: Resources. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors sincerely thank the reviewers for their valuable comments and constructive suggestions, which significantly improved the quality of this paper. Moreover, the authors are thankful to the Deanship of Graduate Studies and Scientific Research at Najran University for funding this work under the Consortium Funding Program grant code [NU/CPL/SERC/14/4210-3].

Conflict of interest

All authors declare no conficts of interest in this paper.

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