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Research article

Bott–Duffin Drazin inverse and its application

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Abstract: The paper introduces the Bott–Duffin Drazin inverse (in short, BDD inverse) of a complex square matrix, which is a generalization of the Bott–Duffin inverse and the Bott–Duffin group inverse. We study some properties, characterizations and representations of the BDD inverse. Finally, we further explore its applications in solving a class of restricted linear equations in electrical network theory.

Keywords: Bott–Duffin inverse; Bott–Duffin Drazin inverse; Drazin inverse; constraint linear equations
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1. Introduction

In electrical network theory, one considers a network of n two-terminal branches. Let $\mathbf{x}$ denote the vector of branch voltages and $\mathbf{y}$ the vector of branch currents. Kirchhoff's voltage law requires that $\mathbf{x}$ belong to a subspace L, and Kirchhoff's current law requires that $\mathbf{y}$ belong to its orthogonal complement $L^{\perp}$. With the branch constitutive relations and external sources, the fundamental equation of network theory is described by the following constrained linear equation [1]:

$$A\mathbf{x} + \mathbf{y} = \beta, \ \mathbf{x} \in L, \ \mathbf{y} \in L^{\perp}, \tag{1.1}$$

where $\mathbf{A}$ is the branch admittance matrix over the complex field and β the external excitation vector. To solve this system without explicitly eliminating the constraints, Bott and Duffin [1] introduced the constrained inverse $A_{(L)}^{(-1)}$, defined by

$$A_{(L)}^{(-1)} = P_L(AP_L + P_{L^{\perp}})^{-1},$$

where P_L and $P_{L^\perp}$ are the orthogonal projections onto L and $L^\perp$, respectively, provided that $AP_L + P_{L^\perp}$ is nonsingular. This inverse is now commonly called the Bott–Duffin inverse (BD inverse). In this case, the consistency of Eq (1.1) is equivalent to the consistency of the linear equation

$$(AP_L + P_{L^\perp})\mathbf{z} = \beta,$$

where $\mathbf{z} = \mathbf{x} + \mathbf{y}$. Under this condition, the unique solution $(\mathbf{x}, \mathbf{y})$ of (1.1) is given by

$$\mathbf{x} = A_{(L)}^{(-1)}\beta,\ \mathbf{y} = (I_n - AA_{(L)}^{(-1)})\beta.$$

Since then, the BD inverse has attracted considerable attention: Ben-Israel and Charnes [2] established a deep mathematical connection between the theory of generalized inverses and the BD inverse; further characterizations, representations, and perturbation analyses can be found in [3–5].

However, in many physically situations, such as degenerate networks, resonant conditions, or singular parameter regions, the aforementioned nonsingularity condition fails, and the BD inverse is no longer applicable. This naturally prompted the study of generalized BD inverses.

In 1990, Chen [6] defined the generalized BD inverse (GBD inverse) using the Moore–Penrose inverse (MP inverse):

$$A_{(L)}^{(\dagger)} = P_L(AP_L + P_{L^\perp})^\dagger,$$

where $(\cdot)^\dagger$ denotes the MP inverse of a matrix. Although the GBD inverse always exists and extends the BD inverse to the case where $AP_L + P_{L^\perp}$ is singular, the direct study of its properties is rather difficult. To overcome this difficulty, Chen [6] further introduced L-positive semidefinite (L-p.s.d.) matrices and applied the GBD inverse of such matrices to a class of equality-constrained quadratic programming problems. For further characterizations and properties of the GBD inverse, we refer the reader to [7,8].

Since Penrose [9] concisely defined the MP inverse $A^\dagger$ of any matrix A by means of four matrix equations in 1955, the theory of generalized inverses has flourished. The MP inverse $A^\dagger$ is the unique matrix satisfying the following conditions:

$$AA^\dagger A = A,\ A^\dagger AA^\dagger = A^\dagger,\ (AA^\dagger)^* = AA^\dagger,\ (A^\dagger A)^* = A^\dagger A,$$

where $(\cdot)^*$ denotes the conjugate transpose of a matrix. Subsequently, motivated by various practical problems and theoretical needs, numerous generalized inverses have emerged, such as the Drazin inverse [10], the group inverse [11], the core inverse [12], and the core–EP inverse [13], among others, and the theory of generalized inverses has been progressively developed [14–16].

Interestingly, following Chen's idea of generalizing the BD inverse via the MP inverse, Zhang et al. [17] defined the Bott–Duffin group inverse (BDG inverse) $A_{(L)}^{(\#)}$ by means of the group inverse:

$$A_{(L)}^{(\#)} = P_L(AP_L + P_{L^\perp})^\#,$$

where $AP_L + P_{L^\perp}$ is group invertible, with its group inverse $(AP_L + P_{L^\perp})^\#$. The properties of the BDG inverse and its various characterizations and representations were also systematically discussed in [17]. The existence of the group inverse is equivalent to that of the core inverse. Recently, Zhou and Zheng et al. [18,19] defined the Bott–Duffin core inverse (BDC inverse) $A_{(L)}^{(\circledast)}$ using the core inverse:

$$A_{(L)}^{(\circledast)} = P_L(AP_L + P_{L^\perp})^{\circledast},$$

where $(\cdot)^{\#}$ denotes the core inverse of a matrix, and $AP_L + P_{L^\perp}$ is core invertible. The core–EP inverse is a generalization of the core inverse to arbitrary square matrices [13]. In 2026, Kara and Mosić [20] proposed the Bott–Duffin core–EP inverse $A_{(L)}^{(\dagger)}$ (BDCEP inverse) based on the BDC inverse:

$$A_{(L)}^{(\dagger)} = P_L(AP_L + P_{L^\perp})^{\dagger},$$

where $(\cdot)^{\dagger}$ denotes the core–EP inverse of a matrix. The BDCEP inverse was applied to solve a class of minimization problems [20].

It is well–known that the Drazin inverse is a generalization of the group inverse and always exists. The Drazin inverse A^D of an $n \times n$ matrix A is the unique matrix satisfying the following conditions:

$$A^D A A^D = A^D,\ AA^D = A^D A,\ A^D A^{k+1} = A^k, \tag{1.2}$$

where k is the index of A. When $k = 1$, the Drazin inverse reduces to the group inverse. Motivated by the works above, in this paper, we further generalize the BDG inverse by means of the Drazin inverse and introduce a new generalized inverse of complex matrices—the Bott–Duffin Drazin inverse (BDD inverse) of A with respect to the subspace L:

$$A_{(L)}^{(D)} = P_L(AP_L + P_{L^\perp})^D. \tag{1.3}$$

Clearly, when $AP_L + P_{L^\perp}$ is nonsingular, the BDD inverse coincides with the BD inverse; when $\mathrm{Ind}(AP_L + P_{L^\perp}) = 1$, the BDD inverse coincides with the BDG inverse. In this paper, we investigate the properties, characterizations, representations, and applications of the BDD inverse. The main contributions are summarized as follows:

- We establish fundamental properties of the BDD inverse. In particular, we show that the BDD inverse is an outer inverse with prescribed range and null space, and is a special type of (B,C)–inverse.
- We derive various characterizations and representations of the BDD inverse using range spaces, null spaces, projectors, matrix equations, and rank equalities.
- We apply the BDD inverse to solve a class of constrained linear equations arising in electrical network theory.

The remainder of this paper is organized as follows. Section 2 reviews some necessary notation and auxiliary lemmas. In Section 3, we establish equivalent conditions for $\mathrm{Ind}(AP_L + P_{L^\perp}) = k$ and discuss the fundamental properties of the BDD inverse. Sections 4 and 5 are devoted to characterizations and representations of the BDD inverse, respectively. Finally, in Section 6, we apply the BDD inverse to solve constrained linear equations.

2. Premliminaries

Some necessary notations and lemmas are recalled. The notation $\mathbb{C}^{m\times n}$ represents the set of $m \times n$ complex matrices. The $L \leq \mathbb{C}^n$ means L is a subspace of $\mathbb{C}^n$, and $\dim(L)$ stands for the dimension of L. For $S, T \leq \mathbb{C}^n$ such that $S \oplus T = \mathbb{C}^n$, $P_{S,T}$ stands for the oblique projector onto S along T. The orthogonal projector onto a subspace S is indicated by P_S, that is, $P_S = P_{S,S^\perp}$. The $\mathrm{rank}(A)$ is the rank of $A \in \mathbb{C}^{m\times n}$. The smallest non–negative integer k for which $\mathrm{rank}(A^k) = \mathrm{rank}(A^{k+1})$ is called the index of $A \in \mathbb{C}^{n\times n}$ and is denoted by $\mathrm{Ind}(A)$. For $A \in \mathbb{C}^{m\times n}$, the symbols A^*, A^{-1}, $\mathcal{R}(A)$, and $\mathcal{N}(A)$ stand for the conjugate transpose, the inverse ($m = n$), the range space, and the null space of A, respectively. Moreover, I_n represents the identity matrix in $\mathbb{C}^{n\times n}$, and O refers to the null matrix with appropriate orders. We use the symbol A^π to represent $I_n - AA^D$. For $A \in \mathbb{C}^{m\times n}$, if $X \in \mathbb{C}^{n\times m}$ satisfies the following conditions:

$$XAX = X,\ \mathcal{R}(X) = S \text{ and } \mathcal{N}(X) = T,$$

where $S \leq \mathbb{C}^n$ and $T \leq \mathbb{C}^m$, then X exists as the unique solution defined by $A^{(2)}_{S,T}$ (see [14–16]).

Lemma 2.1. [21, 22] *Let* $M = \begin{bmatrix} A & B \\ O & E \end{bmatrix}$, *and* $N = \begin{bmatrix} E & O \\ B & A \end{bmatrix}$, *where A and E are square matrices such that $r = \mathrm{Ind}(A)$ and $s = \mathrm{Ind}(E)$. Then,*

$$M^D = \begin{bmatrix} A^D & X \\ O & E^D \end{bmatrix} \text{ and } N^D = \begin{bmatrix} E^D & O \\ X & A^D \end{bmatrix},$$

where

$$X = \sum_{i=0}^{s-1} (A^{(i+2)})^D B E^i E^\pi + A^\pi \sum_{i=0}^{r-1} A^i B (E^{(i+2)})^D - A^D B E^D.$$

Lemma 2.2. [23] *Let* $M = \begin{bmatrix} A & O \\ C & O \end{bmatrix}$, *and* $N = \begin{bmatrix} A & E \\ O & O \end{bmatrix} \in \mathbb{C}^{n\times n}$, *where A is a square matrix. Then,*

$$M^D = \begin{bmatrix} A^D & O \\ C(A^D)^2 & O \end{bmatrix} \text{ and } N = \begin{bmatrix} A^D & (A^D)^2 E \\ O & O \end{bmatrix}.$$

Lemma 2.3. [24] *Let $M,\ N \in \mathbb{C}^{n\times n}$, and $MN = NM = O$. Then,*

$$(M + N)^D = M^D + N^D.$$

Lemma 2.4. [21, Theorem 7.8.4] *Let $M,\ N \in \mathbb{C}^{n\times n}$. Then,*

$$(MN)^D = M((NM)^2)^D N.$$

3. The properties of the BDD inverse

An appropriate matrix factorization of $A \in \mathbb{C}^{n\times n}$ associated with $P_L \in \mathbb{C}^{n\times n}$ is provided to investigate properties of the BDD inverse. Indeed, because P_L is an orthogonal idempotent matrix, then there exists an unitary matrix $U \in \mathbb{C}^{n\times n}$ satisfying

$$P_L = U \begin{bmatrix} I_l & O \\ O & O \end{bmatrix} U^*, \tag{3.1}$$

where $l = \dim(L)$. Denote

$$A = U \begin{bmatrix} A_L & B_L \\ C_L & E_L \end{bmatrix} U^*, \tag{3.2}$$

where $A_L \in \mathbb{C}^{l\times l}$, $B_L \in \mathbb{C}^{l\times(n-l)}$, $C_L \in \mathbb{C}^{(n-l)\times l}$, and $E_L \in \mathbb{C}^{(n-l)\times(n-l)}$. Then, Eq (3.2) is called the decomposition of A with respect to P_L.

Theorem 3.1. *Let P_L and A be given by* (3.1) *and* (3.2)*, respectively. Then,*

$$A_{(L)}^{(D)} = U\begin{bmatrix} {A_L}^D & O \\ O & O \end{bmatrix} U^*. \tag{3.3}$$

Proof. Using (3.1) and (3.2), we get

$$AP_L + P_{L^\perp} = U\begin{bmatrix} A_L & O \\ C_L & I_{n-l} \end{bmatrix} U^*. \tag{3.4}$$

It follows from (3.4) and Lemma 2.1 that

$$(AP_L + P_{L^\perp})^D = U\begin{bmatrix} {A_L}^D & O \\ X_1 & I_{n-l} \end{bmatrix} U^*,$$

where

$$X_1 = C_L(\sum_{i=0}^{m-1} {A_L}^i {A_L}^\pi - {A_L}^D),$$

and $m = \mathrm{Ind}(A_L)$. By using (1.3), we can get (3.3). □

Several equivalent conditions for $\mathrm{Ind}(AP_L + P_{L^\perp}) = k$ are given as follows.

Theorem 3.2. *Let $A \in \mathbb{C}^{n\times n}$, $L \le \mathbb{C}^n$ with* $\dim(L) = l$ *and k be some positive integer. Then, the following statements are equivalent:*
(a) $\mathrm{Ind}(AP_L + P_{L^\perp}) = k$;
(b) $\mathrm{Ind}(P_LA + P_{L^\perp}) = k$;
(c) $\mathrm{Ind}(P_LAP_L + P_{L^\perp}) = k$;
(d) $\mathrm{Ind}(P_LAP_L) = k$;
(e) $\mathrm{Ind}(P_LA^*P_L) = k$.

Proof. (a) ⇔ (b) It follows from (3.4) that

$$\mathrm{rank}(AP_L + P_{L^\perp}) = \mathrm{rank}\begin{bmatrix} A_L & O \\ C_L & I_{n-l} \end{bmatrix} = \mathrm{rank}(A_L) + n - l.$$

Through mathematical induction, we get

$$\mathrm{rank}((AP_L + P_{L^\perp})^t) = \mathrm{rank}({A_L}^t) + n - l,$$

for any positive integer t. Therefore,

$$\mathrm{Ind}(AP_L + P_{L^\perp}) = k \Leftrightarrow \mathrm{Ind}(A_L) = k.$$

Similarly, we can conclude that

$$\mathrm{Ind}(P_LA + P_{L^\perp}) = k \Leftrightarrow \mathrm{Ind}(A_L) = k \Leftrightarrow \mathrm{Ind}(AP_L + P_{L^\perp}) = k.$$

The rest of the proof follows similarly. □

For clarity, the following assumptions will be maintained throughout this and subsequent sections:

$$A \in \mathbb{C}^{n\times n},\ L \leq \mathbb{C}^n,\ k = \mathrm{Ind}(AP_L + P_{L^\perp}),\ S = \mathcal{R}((P_LAP_L)^k),\ T = \mathcal{N}((P_LAP_L)^k).$$

Several basic properties of the BDD inverse are subsequently derived.

Theorem 3.3. *Let* $A \in \mathbb{C}^{n\times n}$, $L \leq \mathbb{C}^n$, *and* $\mathrm{Ind}(AP_L + P_{L^\perp}) = k$, $S = \mathcal{R}((P_LAP_L)^k)$, *and* $T = \mathcal{N}((P_LAP_L)^k)$. *Then:*
(a) $A^{(D)}_{(L)} = P_LA^{(D)}_{(L)} = A^{(D)}_{(L)}P_L = P_LA^{(D)}_{(L)}P_L$;
(b) $\mathcal{R}(A^{(D)}_{(L)}) = S$ *and* $\mathcal{N}(A^{(D)}_{(L)}) = T$;
(c) $A^{(D)}_{(L)} = A^{(D)}_{(L)}AA^{(D)}_{(L)} = A^{(D)}_{(L)}AP_LA^{(D)}_{(L)} = A^{(D)}_{(L)}P_LAA^{(D)}_{(L)} = A^{(D)}_{(L)}P_LAP_LA^{(D)}_{(L)}$;
(d) $AA^{(D)}_{(L)} = P_{AS,T}$ *and* $A^{(D)}_{(L)}A = P_{S,(A^*T^\perp)^\perp}$;
(e) $A^{(D)}_{(L)}AP_L = P_LAA^{(D)}_{(L)} = P_{S,T}$;
(f) $A^{(D)}_{(L)} = A^{(2)}_{S,T} = (AP_L)^{(2)}_{S,T} = (P_LA)^{(2)}_{S,T} = (P_LAP_L)^{(2)}_{S,T}$;
(g) $(A^*)^{(D)}_{(L)} = (A^{(D)}_{(L)})^*$.

Proof. (a) Premultiplying (1.3) with P_L, we have $A^{(D)}_{(L)} = P_LA^{(D)}_{(L)}$. Using (3.1) and (3.3), we have

$$A^{(D)}_{(L)}P_L = U\begin{bmatrix} {A_L}^D & O \\ O & O \end{bmatrix}\begin{bmatrix} I_l & O \\ O & O \end{bmatrix}U^* = U\begin{bmatrix} {A_L}^D & O \\ O & O \end{bmatrix}U^* = A^{(D)}_{(L)}.$$

It follows from $A^{(D)}_{(L)} = P_LA^{(D)}_{(L)} = A^{(D)}_{(L)}P_L$ that $A^{(D)}_{(L)} = P_LA^{(D)}_{(L)}P_L$.
(b) By (1.3), we get

$$\mathcal{R}(A^{(D)}_{(L)}) = P_L\mathcal{R}((AP_L + P_{L^\perp})^D) = P_L\mathcal{R}((AP_L + P_{L^\perp})^k) = \mathcal{R}((P_LAP_L)^k) = S.$$

Then, $\mathrm{rank}(A^{(D)}_{(L)}) = \mathrm{rank}((P_LAP_L)^k)$. From (1.2) and (1.3), we have

$$\begin{aligned}\mathcal{N}(A^{(D)}_{(L)}) &= \mathcal{N}(P_L(AP_L + P_{L^\perp})^D)\\ &\subseteq \mathcal{N}((P_LAP_L)^{k+1}(AP_L + P_{L^\perp})^D)\\ &= \mathcal{N}(P_L(AP_L + P_{L^\perp})^{k+1}(AP_L + P_{L^\perp})^D)\\ &= \mathcal{N}(P_L(AP_L + P_{L^\perp})^k)\\ &= \mathcal{N}((P_LAP_L)^k).\end{aligned}$$

Because

$$\mathcal{N}(A^{(D)}_{(L)})\subseteq\mathcal{N}((P_LAP_L)^k) = T,\ \text{ and }\ \mathrm{rank}(A^{(D)}_{(L)}) = \mathrm{rank}((P_LAP_L)^k),$$

it is easily obtained that $\mathcal{N}(A^{(D)}_{(L)}) = T$.
(c) From (3.2) and (3.3), we infer that

$$A^{(D)}_{(L)}AA^{(D)}_{(L)} = U\begin{bmatrix} {A_L}^D & O \\ O & O \end{bmatrix}\begin{bmatrix} A_L & B_L \\ C_L & E_L \end{bmatrix}\begin{bmatrix} {A_L}^D & O \\ O & O \end{bmatrix}U^* = U\begin{bmatrix} {A_L}^D & O \\ O & O \end{bmatrix}U^* = A^{(D)}_{(L)}.$$

Using (a) and $A^{(D)}_{(L)} = A^{(D)}_{(L)} A A^{(D)}_{(L)}$, we directly get

$$A^{(D)}_{(L)} = A^{(D)}_{(L)} A A^{(D)}_{(L)} = A^{(D)}_{(L)} A P_L A^{(D)}_{(L)} = A^{(D)}_{(L)} P_L A P_L A^{(D)}_{(L)} = A^{(D)}_{(L)} P_L A A^{(D)}_{(L)}.$$

(d) From the first equation in (c), we obtain $AA^{(D)}_{(L)} AA^{(D)}_{(L)} = AA^{(D)}_{(L)}$. Also, (b) implies that

$$\mathcal{R}(AA^{(D)}_{(L)}) = A\mathcal{R}(A^{(D)}_{(L)}) = A \text{ and } \mathcal{N}(AA^{(D)}_{(L)}) = \mathcal{N}(A^{(D)}_{(L)}) = T.$$

Thus, $AA^{(D)}_{(L)} = P_{AS,T}$. Similarly, we have $A^{(D)}_{(L)} A = P_{S,(A^*T^\perp)^\perp}$.

(e) By (c), we have

$$P_L AA^{(D)}_{(L)} P_L AA^{(D)}_{(L)} = P_L AA^{(D)}_{(L)}$$

and

$$\mathcal{N}(A^{(D)}_{(L)}) = \mathcal{N}(A^{(D)}_{(L)} P_L AA^{(D)}_{(L)}) \supseteq \mathcal{N}(P_L AA^{(D)}_{(L)}).$$

Because $\mathcal{N}(P_L AA^{(D)}_{(L)}) \supseteq \mathcal{N}(A^{(D)}_{(L)})$ and (b), we get

$$\mathcal{N}(P_L AA^{(D)}_{(L)}) = \mathcal{N}(A^{(D)}_{(L)}) = T \text{ and } \operatorname{rank}(P_L AA^{(D)}_{(L)}) = \operatorname{rank}((P_L A P_L)^k).$$

$$\mathcal{R}(P_L AA^{(D)}_{(L)}) = P_L A P_L \mathcal{R}((AP_L + P_{L^\perp})^D) = P_L A P_L \mathcal{R}((AP_L + P_{L^\perp})^k) = \mathcal{R}((P_L A P_L)^{k+1}).$$

Considering $\mathcal{R}(P_L AA^{(D)}_{(L)}) \leq \mathcal{R}((P_L A P_L)^k)$ and $\operatorname{rank}(P_L AA^{(D)}_{(L)}) = \operatorname{rank}((P_L A P_L)^k)$, it implies that $\mathcal{R}(P_L AA^{(D)}_{(L)}) = S$. Hence, $P_L AA^{(D)}_{(L)} = P_{S,T}$. Analogously, we can prove $A^{(D)}_{(L)} A P_L = P_{S,T}$.

(f) It is apparent from (a)–(c).

(g) By (f), we have $A^{(D)}_{(L)} = A^{(2)}_{S,T}$. Thus,

$$(A^*)^{(D)}_{(L)} = (A^*)^{(2)}_{\mathcal{R}((P_L A^* P_L)^k),\mathcal{N}((P_L A^* P_L)^k)} = (A^*)^{(2)}_{T^\perp,S^\perp} = (A^{(2)}_{S,T})^* = (A^{(D)}_{(L)})^*.$$

This completes the proof. □

In the following theorem, $A^{(D)}_{(L)}$ is expressed in terms of the Drazin inverses of certain matrices.

Theorem 3.4. *Let $A \in \mathbb{C}^{n\times n}$, $L \leq \mathbb{C}^n$, and let $k = \operatorname{Ind}(AP_L + P_{L^\perp})$, $S = \mathcal{R}((P_L A P_L)^k)$ and $T = \mathcal{N}((P_L A P_L)^k)$. Then, the following statements hold:*

$$\begin{aligned}
A^{(D)}_{(L)} &= (P_L A P_L)^D = P_L(P_L A P_L)^D = (P_L A P_L)^D P_L \\
&= P_L(P_L A P_L + P_{L^\perp})^D = (P_L A P_L + P_{L^\perp})^D P_L \\
&= (P_L A P_L + P_{L^\perp})^D - P_{L^\perp} = (P_L A + P_{L^\perp})^D P_L \\
&= P_L(AP_L)^D = (P_L A)^D P_L = P_L((AP_L)^2)^D A P_L \\
&= P_L A((P_L A)^2)^D P_L.
\end{aligned}$$

Proof. Notice that for $\mathrm{Ind}(B) = k$, we have $B^D = B^{(2)}_{\mathcal{R}(B^k),\mathcal{N}(B^k)}$ (see [15]). Using the fourth equation in Theorem 3.2(f), we have

$$A^{(D)}_{(L)} = (P_L A P_L)^{(2)}_{S,T} = (P_L A P_L)^D.$$

From Theorem 3.3(a), we know that

$$A^{(D)}_{(L)} = P_L A^{(D)}_{(L)} = P_L(P_L A P_L)^D = (P_L A P_L)^D P_L.$$

Furthermore, using (1.3), we have that

$$A^{(D)}_{(L)} = (P_L A P_L)^D = (P_L(P_L A P_L)P_L)^D = P_L(P_L A P_L + P_{L^\perp})^D.$$

By Lemma 2.3, we can verify that

$$(P_L A P_L + P_{L^\perp})^D = (P_L A P_L)^D + P_{L^\perp}.$$

Therefore,

$$A^{(D)}_{(L)} = (P_L A P_L)^D = (P_L A P_L + P_{L^\perp})^D - P_{L^\perp}.$$

By Theorem 3.3(a), we can deduce that

$$A^{(D)}_{(L)} = A^{(D)}_{(L)} P_L = ((P_L A P_L + P_{L^\perp})^D - P_{L^\perp})P_L = (P_L A P_L + P_{L^\perp})^D P_L.$$

Based on (3.1), (3.2), and Lemma 2.1, we infer that

$$\begin{aligned}
(P_L A + P_{L^\perp})^D P_L &= U\begin{bmatrix} A_L & B_L \\ O & I_{n-l}\end{bmatrix}^D \begin{bmatrix} I_l & O \\ O & O\end{bmatrix} U^* \\
&= U\begin{bmatrix} {A_L}^D & X_2 \\ O & I_{n-l}\end{bmatrix}\begin{bmatrix} I_l & O \\ O & O\end{bmatrix} U^* \\
&= U\begin{bmatrix} {A_L}^D & O \\ O & O\end{bmatrix} U^* = A^{(D)}_{(L)},
\end{aligned}$$

where

$$X_2 = {A_L}^\pi \sum_{i=0}^{k-1} {A_L}^i B_L - {A_L}^D B_L.$$

Let P_L, A, and $A^{(D)}_{(L)}$ be expressed as in (3.1)–(3.3), respectively. From Lemma 2.2, we derive

$$\begin{aligned}
P_L(AP_L)^D &= U\begin{bmatrix} I_l & O \\ O & O\end{bmatrix}\begin{bmatrix} A_L & O \\ C_L & O\end{bmatrix}^D U^* \\
&= U\begin{bmatrix} I_l & O \\ O & O\end{bmatrix}\begin{bmatrix} {A_L}^D & O \\ C_L({A_L}^D)^2 & O\end{bmatrix} U^* \\
&= U\begin{bmatrix} {A_L}^D & O \\ O & O\end{bmatrix} U^* = A^{(D)}_{(L)}
\end{aligned}$$

and

$$
\begin{aligned}
(P_LA)^DP_L &= U\begin{bmatrix} A_L & B_L \\ O & O \end{bmatrix}^D \begin{bmatrix} I_l & O \\ O & O \end{bmatrix} U^* \\
&= U\begin{bmatrix} A_L{}^D & (A_L{}^D)^2B_L \\ O & O \end{bmatrix}\begin{bmatrix} I_l & O \\ O & O \end{bmatrix} U^* \\
&= U\begin{bmatrix} A_L{}^D & O \\ O & O \end{bmatrix} U^* = A_{(L)}^{(D)}.
\end{aligned}
$$

From Lemma 2.4 and $A_{(L)}^{(D)} = (P_LA)^DP_L = P_L(AP_L)^D$, we obtain

$$
\begin{aligned}
A_{(L)}^{(D)} &= (P_LA)^DP_L = P_L((AP_L)^2)^DAP_L, \\
A_{(L)}^{(D)} &= P_L(P_LA)^D = P_LA((P_LA)^2)^DP_L.
\end{aligned}
$$

□

The following example illustrates the computation of $A_{(L)}^{(D)}$ by means of the definition, Theorems 3.3 and 3.4.

Example 3.5. We consider the following matrix $A \in \mathbb{C}^{n\times n}$ ([20, Example 2.2]):

$$
A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \text{ and } P_L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
$$

We get

$$
AP_L = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad AP_L + P_{L^\perp} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},
$$

$$
P_LAP_L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad P_LAP_L + P_{L^\perp} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},
$$

and $\mathrm{Ind}(AP_L + P_{L^\perp}) = 2$. So, because $AP_L + P_{L^\perp}$ is singular, the BD inverse of A with respect to L does not exist. However, $A_{(L)}^{(D)}$ always exists, and we have

$$
A_{(L)}^{(D)} = P_L(AP_L + P_{L^\perp})^D = (P_LAP_L)^D = (P_LAP_L + P_{L^\perp})^D - P_{L^\perp} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{27} & \frac{2}{27} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{9} & \frac{2}{9} \\ 0 & 0 & 0 & \frac{1}{9} & \frac{1}{9} \end{bmatrix}.
$$

We now consider the relationship between $A^{(D)}_{(L)}$ and the (B,C)–inverse. In [25, Definition 4.1], Benítez et al. provided the following definition, which simultaneously extends the concept of the (B,C)–inverse from ring elements [26] to rectangular matrices as well as the notion of invertibility along an element. Letting $A \in \mathbb{C}^{n\times m}$, a matrix **A** is said to be (B,C)–inverse if there exists a matrix $Y \in \mathbb{C}^{n\times m}$ such that

$$YAB = B, \quad CAY = C, \quad \mathcal{R}(Y)\subseteq\mathcal{R}(B) \text{ and } \mathcal{N}(C)\subseteq\mathcal{N}(Y). \tag{3.5}$$

If such a matrix Y exists, then it is unique and is denoted by $A^{\|(B,C)}$. Many existence criteria and properties of the (B,C)–inverse can be found in, for example, [27–29]. From the definition of the (B,C)–inverse, it follows that $\mathcal{R}(Y) = \mathcal{R}(B)$, and $\mathcal{N}(Y) = \mathcal{N}(C)$. The (B,C)–inverse of **A** generalizes several classical generalized inverses. By [26, p. 1910], the Moore–Penrose inverse of **A** coincides with the (A^*, A^*)–inverse of **A**, and A^D is the (A^k, A^k)–inverse of **A**, where $k = \mathrm{Ind}(A)$.

Theorem 3.6. *Let $A \in \mathbb{C}^{n\times n}$, $L \leq \mathbb{C}^n$ with $k = \mathrm{Ind}(P_LAP_L)$. Then,*

$$A^{(D)}_{(L)} = A^{\|(P_L(AP_L)^k, P_L(AP_L)^k)}.$$

Proof. It suffices to verify that $A^{(D)}_{(L)}$ satisfies condition (3.5). From items (b) and (d) in Theorem 3.3, we have

$$\begin{aligned}
&A^{(D)}_{(L)}AP_L(AP_L)^k = P_{S,(A^*T^\perp)^\perp}(P_LAP_L)^k = (P_LAP_L)^k = P_L(AP_L)^k,\\
&P_L(AP_L)^kAA^{(D)}_{(L)} = (P_LAP_L)^kP_{AS,T} = (P_LAP_L)^k = P_L(AP_L)^k,\\
&\mathcal{R}(P_L(AP_L)^k) = \mathcal{R}((P_LAP_L)^k) = \mathcal{R}(A^{(D)}_{(L)}),\\
&\mathcal{N}(P_L(AP_L)^k) = \mathcal{N}((P_LAP_L)^k) = \mathcal{N}(A^{(D)}_{(L)}).
\end{aligned}$$

□

4. Some characterizations of the BDD inverse

Several characterizations of the BDD inverse are established in this section. First, the BDD inverse is characterized based on the range and null space of related matrices.

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$, and $X \in \mathbb{C}^{n\times n}$. Then, the following statements are equivalent:*
(a) $X = A^{(D)}_{(L)}$;
(b) $\mathcal{R}(X) = S$, $AX = P_{AS,T}$;
(c) $\mathcal{R}(X) = S$, $P_LAX = P_{S,T}$;
(d) $\mathcal{R}(X) = S$, $XA = P_{S,(A^*T^\perp)^\perp}$, $XP_{T^\perp} = X$.

Proof. (a) ⇒ (b). It is evident from (b) and (d) of Theorem 3.3.
(b) ⇒ (c). By (d) and (e) of Theorem 3.3, we obtain $P_LAX = P_LP_{AS,T} = P_{S,T}$.
(c) ⇒ (d). It follows $\mathcal{N}(X) \leq T$ from $P_LAX = P_{S,T}$. Further, $\mathrm{rank}(X) = \dim(\mathcal{R}(X)) = \dim(S) = n - \dim(T)$, and therefore $\mathcal{N}(X) = T$, and $XP_{T^\perp} = X$. Because $\mathcal{N}(X) = T$, and $T^\perp = (\mathcal{N}((P_LAP_L)^k))^\perp = \mathcal{R}((P_LA^*P_L)^k) \leq L$; we obtain $\mathcal{R}(X^*) \leq L$, thus, $XP_L = X$. Therefore, $XAX = XP_LAX = XP_{S,T} = X$. From $XAX = X$, it is clear that $\mathcal{R}(XA) = \mathcal{R}(X) = S$. Now, it follows from $\mathcal{N}(X) = T$ that

$$\mathcal{N}(XA) = (\mathcal{R}(A^*X^*))^\perp = (A^*\mathcal{N}(X)^\perp)^\perp = (A^*T^\perp)^\perp.$$

As a consequence, $XA = P_{S,(A^*T^\perp)^\perp}$.
$(d) \Rightarrow (a)$. Because $\mathcal{R}(X) = S$, and $XA = P_{S,(A^*T^\perp)^\perp}$, we have $XAX = X$. From $\mathrm{rank}(X) = \dim(S)$ and $XP_{T^\perp} = X$, we get $\mathcal{N}(X) = T$. Finally, by (f) in Theorem 3.3, it follows that $X = A^{(2)}_{S,T} = A^{(D)}_{(L)}$. □

Theorem 4.2. *Let $A \in \mathbb{C}^{n\times n}$, and $X \in \mathbb{C}^{n\times n}$. Then, the following statements are equivalent:*
(a) $X = A^{(D)}_{(L)}$;
(b) $\mathcal{N}(X) = T$ *and* $XA = P_{S,(A^*T^\perp)^\perp}$;
(c) $\mathcal{N}(X) = T$ *and* $XAP_L = P_{S,T}$;
(d) $\mathcal{N}(X) = T$ *and* $AX = P_{AS,T}$, $P_S X = X$.

Proof. $(a) \Rightarrow (b)$. It follows immediately from (b) and (d) in Theorem 3.3.
$(b) \Rightarrow (c)$. By (d) and (e) in Theorem 3.3, it follows that $XAP_L = P_{S,(A^*T^\perp)^\perp}P_L = P_{S,T}$.
$(c) \Rightarrow (d)$. Notice that $\mathcal{N}(X) = T$, and $XAP_L = P_{S,T}$, so we have $\mathrm{rank}(X) = n - \dim(T) = \dim(S)$ and $\mathcal{R}(X) = S$, which means that $P_S X = X$. Because $\mathcal{R}(X) = S \leq L$, we have $P_L X = X$, and thus, $XAX = XAP_L X = P_{S,T}X = X$. From $XAX = X$, it is clear that $\mathcal{N}(AX) = \mathcal{N}(X) = T$. Now, from $\mathcal{R}(X) = T$, we have $\mathcal{R}(AX) = A\mathcal{R}(X) = AS$. Hence, $AX = P_{AS,T}$.
$(d) \Rightarrow (a)$. Because $\mathcal{N}(X) = T$, and $AX = P_{AS,T}$, it follows that $XAX = X$. From $\mathrm{rank}(X) = \dim(S)$ and $P_S X = X$, we have $\mathcal{R}(X) = S$. Now, by (f) of Theorem 3.3, we get $X = A^{(D)}_{(L)}$. □

By (c) of Theorem 3.3, we know that $A^{(D)}_{(L)}$ is an outer inverse of $\mathbf{A}$. As a consequence, we obtain the following results.

Theorem 4.3. *Let $A \in \mathbb{C}^{n\times n}$, and $X \in \mathbb{C}^{n\times n}$. Then, the following statements are equivalent:*
(a) $X = A^{(D)}_{(L)}$;
(b) $XAX = X$, $XP_{T^\perp} = X$ *and* $XA = P_{S,(A^*T^\perp)^\perp}$;
(c) $XAX = X$, $XAP_L = P_{S,T}$ *and* $AX = P_{AS,T}$;
(d) $XAX = X$, $P_S X = X$ *and* $AX = P_{AS,T}$;
(e) $XAX = X$, $P_L AX = P_{S,T}$ *and* $XA = P_{S,(A^*T^\perp)^\perp}$;
(f) $XAX = X$, $P_S XP_{T^\perp} = X$ *and* $\mathrm{rank}(X) = \dim(S)$.

Proof. $(a) \Rightarrow (b)$. It is a direct consequence of Theorem 3.3(b)–(d).
$(b) \Rightarrow (c)$. From (d) and (e) from Theorem 3.3, we have $XAP_L = P_{S,(A^*T^\perp)^\perp}P_L = A^{(D)}_{(L)}AP_L = P_{S,T}$. It follows from $XAX = X$ that $AXAX = AX$, $\mathcal{R}(XA) = \mathcal{R}(X)$, and $\mathcal{N}(AX) = \mathcal{N}(X)$. Because $XA = P_{S,(A^*T^\perp)^\perp}$, we obtain $\mathcal{R}(X) = \mathcal{R}(XA) = S$, which implies $\mathrm{rank}(X) = \mathrm{rank}((P_L AP_L)^k)$ and $\mathcal{R}(AX) = A\mathcal{R}(X) = AS$. By $XP_{T^\perp} = X$, we get $\mathcal{N}(X) = T$. Thus, $AX = P_{AS,T}$.
$(c) \Rightarrow (d)$. It follows from $XAX = X$ and $AX = P_{AS,T}$ that $\mathcal{N}(X) = \mathcal{N}(AX) = T$, which implies $\mathrm{rank}(X) = n - \dim(T) = \dim(S)$. By $XAP_L = P_{S,T}$, we conclude that $S \leq \mathcal{R}(X)$. Hence, $\mathcal{R}(X) = S$, which implies $P_S X = X$.
$(d) \Rightarrow (e)$. The proof is similar to the proof of $(b) \Rightarrow (c)$.
$(e) \Rightarrow (f)$. From $XAX = X$ and $XA = P_{S,(A^*T^\perp)^\perp}$, we have $\mathcal{R}(X) = S$, which gives $P_S X = X$ and $\mathrm{rank}(X) = \dim(S)$. By $P_L AX = P_{S,T}$ and $\mathcal{R}(X) = S$, it follows that $\mathcal{N}(X) = T$, which implies $XP_{T^\perp} = X$. Therefore, $P_S XP_{T^\perp} = XP_{T^\perp} = X$.
$(f) \Rightarrow (a)$. By $P_S XP_{T^\perp} = X$ and $\mathrm{rank}(X) = \dim(S)$, we can derive that $\mathcal{R}(X) = S$, and $\mathcal{N}(X) = T$. According to Theorem 3.3(f), we obtain $X = A^{(2)}_{S,T} = A^{(D)}_{(L)}$. □

From Theorem 3.3(d), we observe that

$$X = A^{(D)}_{(L)} \Rightarrow AX = P_{AS,T}, \text{ and } XA = P_{S,(A^*T^\perp)^\perp}.$$

However, the equations

$$AX = P_{\mathcal{R}((AP_L)^{k+1}),T} \text{ and } XA = P_{S,\mathcal{N}((P_LA)^{k+1})}$$

are not enough for $X = A^{(D)}_{(L)}$, which can be confirmed by the following example.

Example 4.4. Consider

$$A = \begin{bmatrix} 3 & 3 & 3 & 2 \\ 1 & 2 & 2 & 3 \\ 2 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ P_L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ X = \begin{bmatrix} \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & 0 \\ \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & 1 \\ \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to check that $\operatorname{rank}(P_LAP_L) = \operatorname{Ind}(P_LAP_L) = 2$. By computations, we have

$$A^{(D)}_{(L)} = \begin{bmatrix} \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & 0 \\ \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & 0 \\ \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ AX = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad XA = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{2}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can immediately verify that $AX = P_{AS,T}$, and $XA = P_{S,(A^*T^\perp)^\perp}$, but $X \neq A^{(D)}_{(L)}$.

The following theorem provides characterizations of the BDD inverse in terms of $AX = P_{AS,T}$ and $XA = P_{S,(A^*T^\perp)^\perp}$.

Theorem 4.5. *Let $A \in \mathbb{C}^{n\times n}$, and $X \in \mathbb{C}^{n\times n}$. Then, the following statements are equivalent:*
(a) $X = A^{(D)}_{(L)}$*;*
(b) $AX = P_{AS,T}$*,* $XA = P_{S,(A^*T^\perp)^\perp}$ *and* $XAX = X$*;*
(c) $AX = P_{AS,T}$*,* $XA = P_{S,(A^*T^\perp)^\perp}$ *and* $\operatorname{rank}(X) = \dim(S)$*;*
(d) $AX = P_{AS,T}$*,* $XA = P_{S,(A^*T^\perp)^\perp}$ *and* $XP_{T^\perp} = X$*;*
(e) $AX = P_{AS,T}$*,* $XA = P_{S,(A^*T^\perp)^\perp}$ *and* $P_SX = X$*;*
(f) $AX = P_{AS,T}$*,* $XA = P_{S,(A^*T^\perp)^\perp}$ *and* $P_SXP_{T^\perp} = X$*.*

Proof. (a) $\Rightarrow$ (b). This follows by (d) and the first equation of (c) in Theorem 3.3.
(b) $\Rightarrow$ (c). From $XA = P_{S,(A^*T^\perp)^\perp}$ and $XAX = X$, we obtain $\mathcal{R}(X) = \mathcal{R}(XA) = S$. Hence, $\operatorname{rank}(X) = \dim(S)$.
(c) $\Rightarrow$ (d). By $AX = P_{AS,T}$ and $\operatorname{rank}(X) = \dim(S)$, it follows that $\mathcal{N}(X) = T$, which means $XP_{T^\perp} = X$.
(d) $\Rightarrow$ (e). Because $AX = P_{AS,T}$, and $XP_{T^\perp} = X$, we get $\mathcal{N}(X) = T$ and $\operatorname{rank}(X) = \dim(S)$. From $XA = P_{S,(A^*T^\perp)^\perp}$, we can infer that $\mathcal{R}(X) = S$. Hence, $P_SX = X$.
(e) $\Rightarrow$ (f). It follows from $XA = P_{S,(A^*T^\perp)^\perp}$ and $P_SX = X$ that $\mathcal{R}(X) = S$. Furthermore, from $AX = P_{AS,T}$, we can conclude that $\mathcal{N}(X) = T$, which gives $XP_{T^\perp} = X$. Therefore, we get $P_SXP_{T^\perp} = XP_{T^\perp} = X$.
(f) $\Rightarrow$ (a). Using that $XA = P_{S,(A^*T^\perp)^\perp}$ and $P_SXP_{T^\perp} = X$, we have that $XAX = XAP_SXP_{T^\perp} = P_SXP_{T^\perp} = X$. Hence, we get $\mathcal{R}(X) = \mathcal{R}(XA) = S$ and $\mathcal{N}(X) = \mathcal{N}(AX)$. Because $AX = P_{AS,T}$, we have $\mathcal{N}(X) = T$. From (f) of Theorem 3.3, we obtain $X = A^{(D)}_{(L)}$. □

As is well–known, for a nonsingular matrix $A \in \mathbb{C}^{n\times n}$, its inverse A^{-1} is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$\operatorname{rank}\begin{bmatrix} A & I_n \\ I_n & X \end{bmatrix} = \operatorname{rank}(A).$$

We generalize this fact to singular matrices A to obtain a similar result for the BDD inverse.

Theorem 4.6. *Let $A \in \mathbb{C}^{n\times n}$, $L \leq \mathbb{C}^n$, $\operatorname{rank}((P_LAP_L)^k) = r$, and $\operatorname{Ind}(P_LAP_L) = k$. Also, suppose that $S = \mathcal{R}((P_LAP_L)^k)$, and $T = \mathcal{N}((P_LAP_L)^k)$. Then, there a unique matrix $W_1 \in \mathbb{C}^{n\times n}$ satisfying*

$$W_1{}^2 = W_1, \quad W_1(P_LA)^{k+1} = O, \quad (P_LA)^{k+1}W_1 = O, \quad \operatorname{rank}(W_1) = n - r; \tag{4.1}$$

a unique matrix $W_2 \in \mathbb{C}^{n\times n}$ satisfying

$$W_2{}^2 = W_2, \quad W_2(AP_L)^{k+1} = O, \quad (AP_L)^{k+1}W_2 = O, \quad \operatorname{rank}(W_2) = n - r; \tag{4.2}$$

and a unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$\operatorname{rank}\begin{bmatrix} A & I_n - W_2 \\ I_n - W_1 & X \end{bmatrix} = \operatorname{rank}(A). \tag{4.3}$$

The matrix X in (4.3) is the BDD inverse $A^{(D)}_{(L)}$ of A. Moreover,

$$W_1 = P_{(A^*T^\perp)^\perp,S} \quad \text{and} \quad W_2 = P_{T,AS}. \tag{4.4}$$

Proof. Suppose that P_L and A are expressed by (3.1) and (3.2), respectively. By $\operatorname{Ind}(P_LAP_L) = k$, we know that

$$\operatorname{rank}((P_LA)^{k+1}) = \operatorname{rank}\begin{bmatrix} A_L{}^{k+1} & A_L{}^kB_L \\ O & O \end{bmatrix} = \operatorname{rank}(A_L{}^k) = r,$$
$$\operatorname{rank}((P_LAP_L)^k) = \operatorname{rank}\begin{bmatrix} A_L{}^k & O \\ O & O \end{bmatrix} = \operatorname{rank}(A_L{}^k) = r,$$

which mean $\mathcal{R}((P_LA)^{k+1}) = \mathcal{R}((P_LAP_L)^k)$=S. Additionally, there are

$$\begin{aligned}\mathcal{N}((P_LA)^{k+1}) &= \mathcal{N}(((P_LAP_L)^k)A) = (\mathcal{R}(A^*((P_LAP_L)^k)^*))^\perp \\ &= (A^*\mathcal{R}(((P_LAP_L)^k)^*))^\perp = (A^*T^\perp)^\perp.\end{aligned}$$

As a consequence, it can be straightforwardly derived that

$$\begin{aligned}\text{the condition (4.1) holds} &\Longleftrightarrow (I_n - W_1)^2 = I_n - W_1, \quad (I_n - W_1)(P_LA)^{k+1} = (P_LA)^{k+1}, \\ &\qquad (P_LA)^{k+1} = (P_LA)^{k+1}(I_n - W_1), \quad \operatorname{rank}(I_n - W_1) = r \\ &\Longleftrightarrow (I_n - W_1)^2 = I_n - W_1, \quad \mathcal{R}(I_n - W_1) = S, \\ &\qquad \mathcal{N}(I_n - W_1) = \mathcal{N}((P_LA)^{k+1}) \\ &\Longleftrightarrow I_n - W_1 = P_{S,\mathcal{N}((P_LA)^{k+1})} = P_{S,(A^*T^\perp)^\perp} \\ &\Longleftrightarrow W_1 = P_{\mathcal{N}((P_LA)^{k+1}),S} = P_{(A^*T^\perp)^\perp,S}.\end{aligned}$$

Analogously, it can be verified that the condition (4.2) has the unique solution $W_2 = P_{T,AS}$. From this, it can be seen that (4.4) is satisfied.

Finally, it follows from (4.4) and Theorem 3.3(d) that

$$\begin{aligned}\operatorname{rank}\begin{bmatrix} A & I_n - W_2 \\ I_n - W_1 & X \end{bmatrix} &= \operatorname{rank}\begin{bmatrix} A & AA^{(D)}_{(L)} \\ A^{(D)}_{(L)}A & X \end{bmatrix} \\ &= \operatorname{rank}(A) + \operatorname{rank}(X - A^{(D)}_{(L)}).\end{aligned}$$

Thus,

$$\begin{aligned}\text{the condition (4.3) holds} &\Longleftrightarrow \operatorname{rank}(X - A^{(D)}_{(L)}) = O \\ &\Longleftrightarrow X = A^{(D)}_{(L)}.\end{aligned}$$

It follows that $X = A^{(D)}_{(L)}$ is the unique matrix satisfying (4.3). □

5. Some representations of the Bott–Duffin Drazin inverse

Some representations of the BDD inverse are given by using the matrix decomposition given in (3.1) and (3.2).

Theorem 5.1. *Let $A \in \mathbb{C}^{n\times n}$, and $L \leq \mathbb{C}^n$. Then,*

$$A^{(D)}_{(L)} = P_L(I_n - W_2)(AP_L)^D = (P_LA)^D(I_n - W_1)P_L = (I_n - W_1)A^{\dagger}(I_n - W_2),$$

where W_1 and W_2 are defined as in (4.4).

Proof. From (d) of Theorem 3.3 and (3.1)–(3.3), we deduce that

$$I_n - W_1 = U\begin{bmatrix} {A_L}^DA_L & {A_L}^DB_L \\ O & O \end{bmatrix}U^* \text{ and } I_n - W_2 = U\begin{bmatrix} A_L{A_L}^D & O \\ C_L{A_L}^D & O \end{bmatrix}U^*.$$

By Lemma 2.2, we get

$$\begin{aligned}P_L(I_n - W_2)(AP_L)^D &= U\begin{bmatrix} I_l & O \\ O & O \end{bmatrix}\begin{bmatrix} A_L{A_L}^D & O \\ C_L{A_L}^D & O \end{bmatrix}\begin{bmatrix} A_L & O \\ C_L & O \end{bmatrix}^D U^* \\ &= U\begin{bmatrix} A_L{A_L}^D & O \\ O & O \end{bmatrix}\begin{bmatrix} {A_L}^D & O \\ C_L({A_L}^D)^2 & O \end{bmatrix}U^* \\ &= U\begin{bmatrix} {A_L}^D & O \\ O & O \end{bmatrix}U^* = A^{(D)}_{(L)}, \\ (P_LA)^D(I_n - W_1)P_L &= U\begin{bmatrix} A_L & B_L \\ O & O \end{bmatrix}^D\begin{bmatrix} {A_L}^DA_L & {A_L}^DB_L \\ O & O \end{bmatrix}\begin{bmatrix} I_l & O \\ O & O \end{bmatrix}U^* \\ &= U\begin{bmatrix} {A_L}^D & ({A_L}^D)^2B_L \\ O & O \end{bmatrix}\begin{bmatrix} {A_L}^DA_L & O \\ O & O \end{bmatrix}U^* \\ &= U\begin{bmatrix} {A_L}^D & O \\ O & O \end{bmatrix}U^* = A^{(D)}_{(L)}.\end{aligned}$$

Finally,

$$\begin{aligned}(I_n - W_1)A^{\dagger}(I_n - W_2) &= U\begin{bmatrix} A_L{}^D A_L & A_L{}^D B_L \\ O & O \end{bmatrix}\begin{bmatrix} A_L & B_L \\ C_L & E_L \end{bmatrix}^{\dagger}\begin{bmatrix} A_L A_L{}^D & O \\ C_L A_L{}^D & O \end{bmatrix}U^* \\ &= U\begin{bmatrix} A_L{}^D & O \\ O & O \end{bmatrix}\begin{bmatrix} A_L & B_L \\ C_L & E_L \end{bmatrix}\begin{bmatrix} A_L & B_L \\ C_L & E_L \end{bmatrix}^{\dagger}\begin{bmatrix} A_L & B_L \\ C_L & E_L \end{bmatrix}\begin{bmatrix} A_L{}^D & O \\ O & O \end{bmatrix}U^* \\ &= U\begin{bmatrix} A_L{}^D & O \\ O & O \end{bmatrix}\begin{bmatrix} A_L & B_L \\ C_L & E_L \end{bmatrix}\begin{bmatrix} A_L{}^D & O \\ O & O \end{bmatrix}U^* \\ &= U\begin{bmatrix} A_L{}^D & O \\ O & O \end{bmatrix}U^* = A_{(L)}^{(D)}.\end{aligned}$$

This completes the proof. □

Remark 5.2. Under the hypotheses of Theorem 5.1, by the above proof, one can derive

$$A_{(L)}^{(D)} = (I_n - W_1)A^{-}(I_n - W_2),$$

where $A^{-} \in \mathbb{C}^{n\times n}$ satisfies $AA^{-}A = A$.

In the following theorem, we present another representation of $A_{(L)}^{(D)}$ that is derived from Theorem 4.6. The notation $A[\alpha|\beta]$ denotes the submatrix of A determined by the row index set α and column index set β.

Theorem 5.3. *Let $A \in \mathbb{C}^{n\times n}$, $L \leq \mathbb{C}^{n\times n}$, and $\operatorname{rank}(A) = r \geqslant 1$, and let $N = \{1, 2, \ldots, n\}$. If $\alpha, \beta \subseteq N$ are such that $A[\alpha|\beta] \in \mathbb{C}^{r\times r}$ is invertible, then*

$$A_{(L)}^{(D)} = (I_n - W_1)[N|\beta](A[\alpha|\beta])^{-1}(I_n - W_2)[\alpha|N]. \tag{5.1}$$

Proof. Suppose

$$Y_1 = \begin{bmatrix} A & I_n - W_2 \\ I_n - W_1 & A_{(L)}^{(D)} \end{bmatrix} \text{ and } Y_2 = \begin{bmatrix} A[\alpha|\beta] & (I_n - W_2)[\alpha|N] \\ (I_n - W_1)[N|\beta] & A_{(L)}^{(D)} \end{bmatrix}.$$

By Theorem 4.6, we know that $\operatorname{rank}(Y_1) = \operatorname{rank}(A)$. Thus,

$$r = \operatorname{rank}(A[\alpha|\beta]) \leq \operatorname{rank}(Y_2) \leq \operatorname{rank}(Y_1) = \operatorname{rank}(A) = r,$$

which means

$$\operatorname{rank}(Y_2) = \operatorname{rank}(A[\alpha|\beta]) = r. \tag{5.2}$$

Furthermore, by using basic row and column operations of block matrices, we obtain

$$\operatorname{rank}(Y_2) = \operatorname{rank}(A[\alpha|\beta]) + \operatorname{rank}\Big(A_{(L)}^{(D)} - (I_n - W_1)[N|\beta](A[\alpha|\beta])^{-1}(I_n - W_2)[\alpha|N]\Big). \tag{5.3}$$

Hence, (5.1) can be obtained from (5.2) and (5.3). □

Example 5.4. Supposing that the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad P_L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to check that $\mathrm{Ind}(P_LAP_L) = 2$. Then,

$$A_{(L)}^{(D)} = \begin{bmatrix} \frac{2}{27} & \frac{1}{9} & \frac{4}{27} & 0 \\ \frac{2}{27} & \frac{1}{9} & \frac{4}{27} & 0 \\ \frac{2}{27} & \frac{1}{9} & \frac{4}{27} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; AA_{(L)}^{(D)} = \begin{bmatrix} \frac{2}{9} & \frac{1}{3} & \frac{4}{9} & 0 \\ \frac{2}{9} & \frac{1}{3} & \frac{4}{9} & 0 \\ \frac{2}{9} & \frac{1}{3} & \frac{4}{9} & 0 \\ \frac{2}{9} & \frac{1}{3} & \frac{4}{9} & 0 \end{bmatrix}, \; A_{(L)}^{(D)}A = \begin{bmatrix} \frac{2}{9} & \frac{1}{3} & \frac{4}{9} & \frac{5}{9} \\ \frac{2}{9} & \frac{1}{3} & \frac{4}{9} & \frac{5}{9} \\ \frac{2}{9} & \frac{1}{3} & \frac{4}{9} & \frac{5}{9} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Taking $\alpha = \{1, 2\}, \beta = \{1, 2\}$, and $N = \{1, 2, 3, 4\}$,

$$A[\alpha|\beta] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad (I_n - W_1)[N|\beta] = \begin{bmatrix} \frac{2}{9} & \frac{1}{3} \\ \frac{2}{9} & \frac{1}{3} \\ \frac{2}{9} & \frac{1}{3} \\ 0 & 0 \end{bmatrix}, \; (I_n - W_2)[\alpha|N] = \begin{bmatrix} \frac{2}{9} & \frac{1}{3} & \frac{4}{9} & 0 \\ \frac{2}{9} & \frac{1}{3} & \frac{4}{9} & 0 \end{bmatrix}.$$

It is obvious that $\mathrm{rank}(A) = \mathrm{rank}(A[\alpha|\beta]) = 2$; that is, $A[\alpha|\beta]$ is invertible. Hence,

$$(I_n - W_1)[N|\beta](A[\alpha|\beta])^{-1}(I_n - W_2)[\alpha|N] = \begin{bmatrix} \frac{2}{27} & \frac{1}{9} & \frac{4}{27} & 0 \\ \frac{2}{27} & \frac{1}{9} & \frac{4}{27} & 0 \\ \frac{2}{27} & \frac{1}{9} & \frac{4}{27} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A_{(L)}^{(D)}.$$

In the next theorem, we use $A\mid_L$ to indicate the restriction of $A \in \mathbb{C}^{n\times n}$ to $L \leq \mathbb{C}^n$; that is,

$$A\mid_L : L \to AL \subseteq \mathbb{C}^{n\times n} \text{ and } A\mid_L x = Ax \quad \text{for } x \in L.$$

Theorem 5.5. *Let $A \in \mathbb{C}^{n\times n}$, and $L \leq \mathbb{C}^n$, $k = \mathrm{Ind}(P_LAP_L)$, $S = \mathcal{R}((P_LAP_L)^k)$, $T = \mathcal{N}((P_LAP_L)^k)$, and $\tilde{A} = (P_LAP_L)^{k+1}\mid_S$. Then, $\tilde{A}$ is invertible, and*

$$A_{(L)}^{(D)} = \tilde{A}^{-1}(P_LAP_L)^k. \tag{5.4}$$

Proof. The restriction map $\tilde{A}$ is trivially surjective from S onto $(P_LAP_L)^{k+1}S$. Therefore, it suffices to verify that $\tilde{A}$ is injective. Indeed, let $x \in S$ satisfy $\tilde{A}x = (P_LAP_L)^{k+1}|_S x = 0$. Then, there exists $y \in \mathbb{C}^n$ such that $x = (P_LAP_L)^k y$. Moreover,

$$(P_LAP_L)^{2k+1}y = \tilde{A}x = 0.$$

Hence, we have $y \in \mathcal{N}((P_LAP_L)^{2k+1}) = T$, which means $x = (P_LAP_L)^k y = 0$.

Letting $X = \tilde{A}^{-1}(P_LAP_L)^k$, we can verify that $X = \tilde{A}^{-1}(P_LAP_L)^k$ satisfies the three equations in (1.2), that is,

$$(P_LAP_L)^{k+1}X = (P_LAP_L)^k, \quad X(P_LAP_L)X = X, \quad \text{and} \quad (P_LAP_L)X = X(P_LAP_L).$$

The verification is given below.

For every $z \in \mathbb{C}^n$, there exists $\alpha \in \mathbb{C}^n$ such that $(P_LAP_L)^k z = (P_LAP_L)^{k+1}\alpha$. Hence,

$$\begin{aligned}(P_LAP_L)^{k+1}Xz &= (P_LAP_L)^{k+1}\tilde{A}^{-1}(P_LAP_L)^k z = (P_LAP_L)^{k+1}\tilde{A}^{-1}(P_LAP_L)^{k+1}\alpha \\ &= (P_LAP_L)^{k+1}\alpha = (P_LAP_L)^k z.\end{aligned}$$

Consequently, we have $(P_LAP_L)^{k+1}X = (P_LAP_L)^k$. Additionally, because

$$X(P_LAP_L)Xz = \tilde{A}^{-1}(P_LAP_L)^{k+1}\tilde{A}^{-1}(P_LAP_L)^{k+1}\alpha = \tilde{A}^{-1}(P_LAP_L)^{k+1}\alpha = \tilde{A}^{-1}(P_LAP_L)^k z = Xz,$$

we conclude $X(P_LAP_L)X = X$. For any $z \in \mathbb{C}^n$, there exists $\beta \in \mathbb{C}^n$ such that

$$(P_LAP_L)^k z = (P_LAP_L)^{2k+1}\beta.$$

Hence,

$$\begin{aligned}(P_LAP_L)Xz &= P_LAP_L\tilde{A}^{-1}(P_LAP_L)^{2k+1}\beta = P_LAP_L(P_LAP_L)^k\beta = (P_LAP_L)^{k+1}\beta, \\ X(P_LAP_L)z &= \tilde{A}^{-1}(P_LAP_L)^{k+1}z = \tilde{A}^{-1}(P_LAP_L)^{2k+2}\beta = (P_LAP_L)^{k+1}\beta.\end{aligned}$$

This establishes $(P_LAP_L)Xz = X(P_LAP_L)z$, from which we obtain $(P_LAP_L)X = X(P_LAP_L)$. Thus, $X = (P_LAP_L)^D = A^{(D)}_{(L)}$. This completes the proof. □

6. Applications

Recall from the introduction that L and $L^\perp$ are the KVL and KCL subspaces, respectively, corresponding to the cycle and cut spaces of the network. Physically, when the network contains degenerate configurations such as purely capacitive cutsets or inductive loops, the matrix $AP_L + P_{L^\perp}$ becomes singular, and the classical BD inverse ceases to exist. In such circumstances, the BDD inverse remains applicable, and the index $k = \mathrm{Ind}(AP_L + P_{L^\perp})$ provides a quantitative measure of the algebraic degeneracy. The following theorem establishes how the BDD inverse can be employed to solve the resulting constrained linear system.

Theorem 6.1. *Let $A \in \mathbb{C}^{n\times n}$, $L \leq \mathbb{C}^n$, $k = \mathrm{Ind}(AP_L + P_{L^\perp})$, and $\beta \in \mathcal{R}((AP_L + P_{L^\perp})^k)$. Consider the restricted linear equation*

$$A\mathbf{x} + \mathbf{y} = \beta, \quad \mathbf{x} \in L, \ \mathbf{y} \in L^\perp, \tag{6.1}$$

where $\mathbf{x}$ and $\mathbf{y}$ are unknown.

(*a*) *The general solution of Eq* (6.1) *is*

$$\begin{aligned}\mathbf{x} &= A^{(D)}_{(L)}\beta + P_L\eta + (I - A^{(D)}_{(L)})(P_LAP_L)^{k-1}u, \\ \mathbf{y} &= (I_n - AA^{(D)}_{(L)})\beta - AP_L\eta - (I - AA^{(D)}_{(L)})(AP_L)^k u,\end{aligned}$$

where $\eta \in \mathcal{N}(AP_L + P_{L^\perp})$, and $u \in \mathbb{C}^n$ is arbitrary.

(*b*) *If $(\mathbf{x} + \mathbf{y}) \in \mathcal{R}((AP_L + P_{L^\perp})^k)$, then Eq* (6.1) *has a unique solution, and*

$$\mathbf{x} = A^{(D)}_{(L)}\beta \text{ and } \mathbf{y} = (I_n - AA^{(D)}_{(L)})\beta.$$

Proof. (*a*) Suppose that $\mathbf{z} = \mathbf{x} + \mathbf{y}$, where $\mathbf{x} = P_L z$, and $\mathbf{y} = P_{L^\perp} z$. The consistency of (6.1) is equivalent to the consistency of

$$(AP_L + P_{L^\perp})\mathbf{z} = \beta. \tag{6.2}$$

Because $k = \mathrm{Ind}(AP_L + P_{L^\perp})$, and $\beta \in \mathcal{R}((AP_L + P_{L^\perp})^k)$, by [30, Lemma 3.1], we know that (6.2) has solutions, and its general solution is

$$\mathbf{z} = (AP_L + P_{L^\perp})^D\beta + [I - (AP_L + P_{L^\perp})^D(AP_L + P_{L^\perp})]\xi,$$

$$\forall \xi \in \mathcal{R}((AP_L + P_{L^\perp})^{k-1}) + \mathcal{N}((AP_L + P_{L^\perp})).$$

Let $\xi = ((AP_L + P_{L^\perp})^{k-1})u + \eta$, for any $u \in \mathbb{C}^n$ and $\eta \in \mathcal{N}((AP_L + P_{L^\perp}))$. We get

$$\begin{aligned}
\mathbf{x} = P_L z &= A^{(D)}_{(L)}\beta + [P_L - P_L(AP_L + P_{L^\perp})^D(AP_L + P_{L^\perp})]\xi \\
&= A^{(D)}_{(L)}\beta + [P_L - A^{(D)}_{(L)}P_L(AP_L + P_{L^\perp})][(AP_L + P_{L^\perp})^{k-1}u + \eta] \\
&= A^{(D)}_{(L)}\beta + P_L\eta + [P_L - A^{(D)}_{(L)}P_LAP_L](AP_L + P_{L^\perp})^{k-1}u \\
&= A^{(D)}_{(L)}\beta + P_L\eta + (P_LAP_L)^{k-1}u - A^{(D)}_{(L)}(P_LAP_L)^k u \\
&= A^{(D)}_{(L)}\beta + P_L\eta + (I - A^{(D)}_{(L)}A)(P_LAP_L)^{k-1}u,
\end{aligned}$$

$$\begin{aligned}
\mathbf{y} = \beta - Ax &= (I_n - AA^{(D)}_{(L)})\beta - AP_L\eta - (A - AA^{(D)}_{(L)}A)(P_LAP_L)^{k-1}u \\
&= (I_n - AA^{(D)}_{(L)})\beta - AP_L\eta - (I - AA^{(D)}_{(L)})(AP_L)^k u.
\end{aligned}$$

(*b*) Let $\mathbf{z} = \mathbf{x} + \mathbf{y} \in \mathcal{R}((AP_L + P_{L^\perp})^k)$ be written as $z = (AP_L + P_{L^\perp})^k z_1$ for some $z_1 \in \mathbb{C}^n$. Then, $(AP_L + P_{L^\perp})^{k+1}z_1 = \beta$, and

$$\begin{aligned}
\mathbf{z} &= (AP_L + P_{L^\perp})^k z_1 \\
&= (AP_L + P_{L^\perp})^D(AP_L + P_{L^\perp})^{k+1}z_1 \\
&= (AP_L + P_{L^\perp})^D\beta.
\end{aligned}$$

Therefore, $\mathbf{z} = (AP_L + P_{L^\perp})^D\beta$ is the unique solution of (6.2). In consequence, the unique solution to (6.1) is

$$\mathbf{x} = P_L z = A^{(D)}_{(L)}\beta \quad \text{and} \quad \mathbf{y} = \beta - Ax = (I_n - AA^{(D)}_{(L)})\beta.$$

□

Example 6.2. To validate the numerical behavior of the BDD inverse when computing the unique solution to the Eq (6.1) subject to the constraint

$$(\mathbf{x} + \mathbf{y}) \in \mathcal{R}((AP_L + P_{L^\perp})^k),$$

we consider four cases corresponding to $k = \mathrm{Ind}(AP_L + P_{L^\perp}) = 0, 1, 2, 3$. In each case, we choose a consistent vector $\beta \in \mathcal{R}((AP_L + P_{L^\perp})^k)$ and compute the relative residual

$$\mathrm{RelRes} = \frac{\|A\mathbf{x} + \mathbf{y} - \beta\|_F}{\|\beta\|_F}, \quad \mathbf{x} = A^{(D)}_{(L)}\beta,\ \mathbf{y} = (I_n - AA^{(D)}_{(L)})\beta,$$

where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix.

Case I ($k=0$). Let

$$A=\begin{bmatrix}2 & 1\\ 0 & 0\end{bmatrix},\ P_L=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}.$$

Then, $\mathrm{Ind}(AP_L+P_{L^\perp})=0$, and

$$A_{(L)}^{(D)}=P_L(AP_L+P_{L^\perp})^D=P_L(AP_L+P_{L^\perp})^{-1}=\begin{bmatrix}1/2 & 0\\ 0 & 0\end{bmatrix}.$$

Taking $\beta=[1,1]^T\in\mathbb{C}^2$, we obtain

$$\mathbf{x}=A_{(L)}^{(D)}\beta=\begin{bmatrix}0.5\\ 0\end{bmatrix},\ \mathbf{y}=(I_2-AA_{(L)}^{(D)})\beta=\begin{bmatrix}0\\ 1\end{bmatrix},$$

and hence, $\mathrm{RelRes}=1.11\times10^{-16}$.

Case II ($k=1$). Let

$$A=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\qquad P_L=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Then, $\mathrm{Ind}(AP_L+P_{L^\perp})=1$, and

$$A_{(L)}^{(D)}=P_L(AP_L+P_{L^\perp})^D=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Taking $\beta=[1,0,1]^T\in\mathcal{R}(AP_L+P_{L^\perp})$, we have

$$\mathbf{x}=A_{(L)}^{(D)}\beta=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix},\ \mathbf{y}=(I_3-AA_{(L)}^{(D)})\beta=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix},$$

and hence, $\mathrm{RelRes}=1.05\times10^{-16}$.

Case III ($k=2$). Let

$$A=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},\ P_L=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Then, $\mathrm{Ind}(AP_L+P_{L^\perp})=2$, and

$$A_{(L)}^{(D)}=P_L(AP_L+P_{L^\perp})^D=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Taking $\beta = [1, 0, 0, 1]^T \in \mathcal{R}((AP_L + P_{L^\perp})^2)$, we have

$$\mathbf{x} = A_{(L)}^{(D)}\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y} = (I_4 - AA_{(L)}^{(D)})\beta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

and hence, RelRes $= 1.33 \times 10^{-16}$.

Case IV ($k = 3$). Let

$$A = \begin{bmatrix} 0 & 1+1i & 0+1i & 0 & 0 & 1 \\ 0 & 1+1i & 1+1i & 0+1i & 1 & 1+1i \\ 0 & 0+1i & 0+1i & 1+1i & 0+1i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1+1i & 1 & 1+1i & 1+1i & 0+1i & 0+1i \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad P_L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, $k := \operatorname{Ind}(AP_L + P_{L^\perp}) = 3$ and

$$A_{(L)}^{(D)} = P_L(AP_L + P_{L^\perp})^D = \begin{bmatrix} 0 & 0.056-0.192i & 0.056-0.192i & 0.056-0.192i & 0 & 0 \\ 0 & 0.04-0.28i & 0.04-0.28i & 0.04-0.28i & 0 & 0 \\ 0 & 0.16-0.12i & 0.16-0.12i & 0.16-0.12i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Taking

$$\beta = \begin{bmatrix} -15 & -21+3i & -12-9i & 0 & -18+21i & 1 \end{bmatrix}^T \in \mathcal{R}((AP_L + P_{L^\perp})^3),$$

we have

$$\mathbf{x} = A_{(L)}^{(D)}\beta = \begin{bmatrix} -3+6i \\ -3+9i \\ -6+3i \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = (I_6 - AA_{(L)}^{(D)})\beta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3+12i \\ 1 \end{bmatrix},$$

and hence, RelRes $= 4.83 \times 10^{-15}$.

The relative residuals in Table 1 are all of the order of machine precision, confirming the numerical stability and accuracy of the BDD inverse for solving consistent restricted linear equations regardless of the index of $AP_L + P_{L^\perp}$.

Table 1. Relative residuals for indices $k = 0, 1, 2, 3$.

Case	$k = \mathrm{Ind}(AP_L + P_{L^\perp})$	Relative residual
I	0	1.11×10^{-16}
II	1	1.05×10^{-16}
III	2	1.33×10^{-16}
IV	3	4.83×10^{-15}

7. Conclusions

In this work, we introduce the BDD inverse that extends the BD inverse and the Bott–Duffin group inverse. The properties, characterizations, and representations of the BDD inverse are presented. Moreover, its applications in solving the restricted linear equation (6.1) are discussed.

Based on the above results, future research directions are outlined as follows:

- Perturbation analysis and perturbation bounds for the BDD inverse;
- Iterative computation of the BDD inverse;
- Investigation of the BDD inverse for dual matrices and its applications, as dual matrices constitute a hot topic in recent research.

Author contributions

Jing Zhou: Writing—original draft, methodology, conceptualization, formal analysis; Tan Mei: Formal analysis, validation, investigation; Jiale Gao: Writing—review and editing, validation; Kezheng Zuo: Supervision, conceptualization, writing—review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

No potential conflict of interest was reported by the authors.

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