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Hermite–Hadamard-type double-integral inequalities for the operator modulus in Hilbert spaces

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Abstract: Let $(\mathcal{H};\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. Denote by $\mathcal{L}(\mathcal{H})$ the Banach C^*-algebra of bounded linear operators on $\mathcal{H}$. For $\mathcal{A}\in\mathcal{L}(\mathcal{H})$, we define the modulus of $\mathcal{A}$ by $|\mathcal{A}| := (\mathcal{A}^*\mathcal{A})^{1/2}$. We say that the continuous function $S : [\mathfrak{a}_1,\mathfrak{a}_2] \to \mathcal{L}(\mathcal{H})$ is square modulus convex on $[\mathfrak{a}_1,\mathfrak{a}_2]$ if

$$\begin{aligned}&|S((1-\vartheta)\mathfrak{u}+\vartheta\mathfrak{v})|^2\\ &\le (1-\vartheta)|S(\mathfrak{u})|^2+\vartheta|S(\mathfrak{v})|^2\end{aligned}$$

in the operator order of $\mathcal{L}(\mathcal{H})$, for all $\mathfrak{u},\mathfrak{v}\in[\mathfrak{a}_1,\mathfrak{a}_2]$ and $\vartheta\in[0,1]$. In this paper, we have showed, among other things, that

$$\begin{aligned}0&\le 2\min\{\vartheta,1-\vartheta\}\times\left[\frac{1}{\mathfrak{a}_2-\mathfrak{a}_1}\int_a^{\mathfrak{a}_2}|S(s)|^2ds-\frac{1}{(\mathfrak{a}_2-\mathfrak{a}_1)^2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\left|S\left(\frac{s+\sigma}{2}\right)\right|^2dsd\sigma\right]\\ &\le\frac{1}{\mathfrak{a}_2-\mathfrak{a}_1}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}|S(s)|^2ds-\frac{1}{(\mathfrak{a}_2-\mathfrak{a}_1)^2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}|S((1-\vartheta)s+\vartheta\sigma)|^2dsd\sigma\\ &\le 2\max\{\vartheta,1-\vartheta\}\times\left[\frac{1}{\mathfrak{a}_2-\mathfrak{a}_1}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}|S(s)|^2ds-\frac{1}{(\mathfrak{a}_2-\mathfrak{a}_1)^2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\left|S\left(\frac{s+\sigma}{2}\right)\right|^2dsd\sigma\right]\end{aligned}$$

for all $\vartheta\in[0,1]$. Some integral inequalities for symmetric weights were obtained. Applications for exponential functions and modulus quadratic functions were provided as well.

Keywords: Hermite–Hadamard inequality; midpoint inequality; operator valued functions in Hilbert spaces; operator exponential
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1. Introduction

Let $\mathbb{R}$ be the set of real numbers and $\Delta \subseteq \mathbb{R}$ be an interval. A function $\kappa : \Delta \subseteq \mathbb{R} \to \mathbb{R}$ is said to be *convex* in the classical sense if it satisfies the following inequality

$$\kappa\left(t\mathfrak{u} + (1-t)\,\mathfrak{v}\right) \leq tf\left(\mathfrak{u}\right) + (1-t)\,\kappa\left(\mathfrak{v}\right),$$

for all $\mathfrak{u}, \mathfrak{v} \in \Delta$ and $t \in [0,1]$.

Let $\kappa : \Delta \subseteq \mathbb{R} \to \mathbb{R}$ and $\mathfrak{a}_1, \mathfrak{a}_2 \in \Delta$ with $\mathfrak{a}_1 < \mathfrak{a}_2$. Then the inequality

$$\kappa\left(\frac{\mathfrak{a}_1 + \mathfrak{a}_2}{2}\right) \leq \frac{1}{\mathfrak{a}_2 - \mathfrak{a}_1}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \kappa(x)dx \leq \frac{\kappa(\mathfrak{a}_1) + \kappa(\mathfrak{a}_2)}{2} \tag{1.1}$$

holds if κ is convex, and is known in the literature as Hermite–Hadamard inequality, after the name of C. Hermite and J. Hadamard (see [10]). The inequalities in (1.1) hold in the reversed direction if κ is a concave function.

A vast literature related to (1.1) has been produced by a large number of mathematicians [7] since it is considered to be one of the most famous inequalities for convex functions due to its usefulness and many applications in various branches of pure and applied mathematics.

Let V be a vector space over the real or complex number field $\mathbb{K}$ and $\mathfrak{y}_1, \mathfrak{y}_2 \in V,\ \mathfrak{y}_1 \neq \mathfrak{y}_2$. Define the segment

$$[\mathfrak{y}_1, \mathfrak{y}_2] := \{(1-t)\mathfrak{y}_1 + t\mathfrak{y}_2,\ t \in [0,1]\}.$$

We consider the function $\kappa : [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ and the associated function

$$g(\mathfrak{y}_1, \mathfrak{y}_2) : [0,1] \to \mathbb{R},\ g(\mathfrak{y}_1, \mathfrak{y}_2)(t) := \kappa[(1-t)\mathfrak{y}_1 + t\mathfrak{y}_2],\ t \in [0,1].$$

Note that κ is convex on $[\mathfrak{y}_1, \mathfrak{y}_2]$ if and only if $g(\mathfrak{y}_1, \mathfrak{y}_2)$ is convex on $[0,1]$.

For any convex function defined on a segment $[\mathfrak{y}_1, \mathfrak{y}_2] \subset V$, we have the Hermite–Hadamard integral inequality:

$$\kappa\left(\frac{\mathfrak{y}_1 + \mathfrak{y}_2}{2}\right) \leq \int_0^1 \kappa[(1-t)\mathfrak{y}_1 + t\mathfrak{y}_2]dt \leq \frac{\kappa(\mathfrak{y}_1) + \kappa(\mathfrak{y}_2)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite–Hadamard inequality (1.1) for the convex function $g(\mathfrak{y}_1, \mathfrak{y}_2) : [0,1] \to \mathbb{R}$.

In [8], some Hermite–Hadamard-type inequalities were discovered using the fractional operator, which produces some interesting results. The authors A. Munir et al. established a general identity for Caputo-Fabrizio integral operators and the second derivative function. Using this identity new error bounds and estimates for strongly-convex functions were obtained.

Since $\kappa(\mathfrak{y}_1) = \|\mathfrak{y}_1\|^\beta$ ($\mathfrak{y}_1 \in V$ and $1 \le \beta < \infty$) is a convex function, then for any $\mathfrak{y}_1, \mathfrak{y}_2 \in V$, we have the following norm inequality from (1.2) (see [9]):

$$\left\| \frac{\mathfrak{y}_1 + \mathfrak{y}_2}{2} \right\|^\beta \le \int_0^1 \|(1-t)\mathfrak{y}_1 + t\mathfrak{y}_2\|^\beta dt \le \frac{\|\mathfrak{y}_1\|^\beta + \|\mathfrak{y}_2\|^\beta}{2}. \tag{1.3}$$

Denote by $\mathcal{L}(\mathcal{H})$ the Banach C^*-algebra of bounded linear operators on the Hilbert space $\mathcal{H}$. For $A \in \mathcal{L}(\mathcal{H})$, we define the modulus of $\mathcal{A}$ by $|\mathcal{A}| := (\mathcal{A}^*\mathcal{A})^{1/2}$. It is well-known that the modulus of the operators does not satisfy, in general, the triangle inequality $|\mathcal{A}+\mathcal{B}| \le |\mathcal{A}| + |\mathcal{B}|$, so the classical arguments using this inequality can not be used.

We use the following Cauchy–Bunyakowsky–Schwarz discrete operator inequality [6]:

$$\sum_{k=1}^n w_k |z_k|^2 \sum_{k=1}^n w_k |\mathcal{A}_k|^2 \ge \left| \sum_{k=1}^n w_k z_k \mathcal{A}_k \right|^2, \tag{1.4}$$

where $z_k \in \mathbb{C}$, $\mathcal{A}_k \in \mathcal{L}(\mathcal{H})$, $w_k \ge 0$ for $k \in \{1, ..., n\}$, and $\sum_{k=1}^n w_k = 1$.

Like in [6], we consider the following class of operator-valued functions:

Definition 1.1. *We say that the continuous function* $S : [\mathfrak{a}_1, \mathfrak{a}_2] \to \mathcal{L}(\mathcal{H})$ *is square modulus convex (concave) on* $[\mathfrak{a}_1, \mathfrak{a}_2]$ *if*

$$|S((1-t)\mathfrak{u} + t\mathfrak{v})|^2 \le (\ge) (1-t)|S(\mathfrak{u})|^2 + t|S(\mathfrak{v})|^2 \tag{1.5}$$

in the operator order of $\mathcal{L}(\mathcal{H})$, *for all* $\mathfrak{u}, \mathfrak{v} \in [\mathfrak{a}_1, \mathfrak{a}_2]$ *and* $t \in [0, 1]$.

Let $\mathcal{A}, \mathcal{B} \in \mathcal{L}(\mathcal{H})$ and $\mu \in [0,1]$. Then by (1.4), we get

$$\begin{aligned}
|(1-\mu)\mathcal{A} + \mu\mathcal{B}|^2 &= \left|(1-\mu)^{1/2}(1-\mu)^{1/2}\mathcal{A} + \mu^{1/2}\mu^{1/2}\mathcal{B}\right|^2 \\
&\le \left[\left((1-\mu)^{1/2}\right)^2 + \left(\mu^{1/2}\right)^2\right]\left[\left|(1-\mu)^{1/2}\mathcal{A}\right|^2 + \left|\mu^{1/2}\mathcal{B}\right|^2\right] \\
&= (1-\mu+\mu)\left[(1-\mu)|\mathcal{A}|^2 + \mu|\mathcal{B}|^2\right] \\
&= (1-\mu)|\mathcal{A}|^2 + \mu|\mathcal{B}|^2.
\end{aligned}$$

Consider the function $C : [0,1] \to \mathcal{L}(\mathcal{H})$, $C(t) = |(1-t)\mathcal{A} + tB|$. Let $t_1, t_2 \in [0,1]$ and $\mu \in [0,1]$. Then

$$\begin{aligned}
|C((1-\mu)t_1 + \mu t_2)|^2 &= |(1-(1-\mu)t_1 - \mu t_2)\mathcal{A} + ((1-\mu)t_1 + \mu t_2)\mathcal{B}|^2 \\
&= |(1-\mu)((1-t_1)\mathcal{A} + t_1\mathcal{B}) + \mu((1-t_2)\mathcal{A} + t_2\mathcal{B})|^2 \\
&\le (1-\mu)|(1-t_1)\mathcal{A} + t_1\mathcal{B}|^2 + \mu|(1-t_2)\mathcal{A} + t_2\mathcal{B}|^2 \\
&= (1-\mu)|C(t_1)|^2 + \mu|C(t_2)|^2,
\end{aligned}$$

which shows that C is square modulus convex on $[0,1]$.

Assume that κ is nonnegative on Δ and operator convex, namely,

$$\kappa((1-\mu)\mathcal{A} + \mu\mathcal{B}) \le (1-\mu)\kappa(\mathcal{A}) + \mu\kappa(\mathcal{B}),$$

for all $\mu \in [0,1]$ and selfadjoint operators $\mathcal{A}$, $\mathcal{B}$ with spectra in Δ.

For such functions and $\mathcal{A}$, $\mathcal{B}$, we consider

$$\mathcal{D}(t) := [\kappa((1-t)\mathcal{A} + t\mathcal{B})]^{1/2}, t \in [0,1].$$

Then, using a similar proof as above for the modulus function, we conclude that $\mathcal{D}$ is square modulus convex on $[0,1]$.

The function $\kappa(t) = t^{\iota}$ is operator convex on $(0,\infty)$ if either $1 \le \iota \le 2$ or $-1 \le \iota \le 0$ and is operator concave on $(0,\infty)$ if $0 \le \iota \le 1$. Therefore for $\mathcal{A}, \mathcal{B} > 0$, the function

$$S_{\iota}(t) := ((1-t)\mathcal{A} + t\mathcal{B})^{\iota/2},\ t \in [0,1],$$

is square modulus convex on $[0,1]$ for $1 \le \iota \le 2$ or $-1 \le \iota \le 0$, [6].

We give now an example of a complex-valued function for which the real and imaginary parts are not convex, however the square of the modulus is.

Let $S : [\mathfrak{a}_1, \mathfrak{a}_2] \to \mathbb{C}$ be defined by $S(t) := \mathfrak{u}(t) + \vartheta(t)i$, $t \in [\mathfrak{a}_1, \mathfrak{a}_2]$. Observe that $|S(t)|^2 = \mathfrak{u}^2(t) + \vartheta^2(t)$, $t \in [\mathfrak{a}_1, \mathfrak{a}_2]$. Notice that, if $\mathfrak{u}^2$, ϑ^2 are convex on $[\mathfrak{a}_1, \mathfrak{a}_2]$, then obviously $\mathfrak{u}^2 + \vartheta^2$ is convex on $[\mathfrak{a}_1, \mathfrak{a}_2]$. However, if we take $\mathfrak{u}(t) = t\sin t$, $\vartheta(t) = t\cos t$, $t \in [0, 2\pi]$, then neither $\mathfrak{u}^2$ nor ϑ^2 is convex on $[0,2\pi]$ but $\mathfrak{u}^2 + \vartheta^2$ is still convex on $[0,2\pi]$.

2. Preliminary facts

Following Roberts and Varberg [11], we recall that if $\kappa : \Delta \to \mathbb{R}$ is a convex function, then for any $s_0 \in \mathring{\Delta}$ (the interior of the interval Δ), the limits

$$\kappa'_-(s_0) := \lim_{s \to s_0-} \frac{\kappa(s) - \kappa(s_0)}{s - s_0} \text{ and } \kappa'_+(s_0) := \lim_{s \to s_0+} \frac{\kappa(s) - \kappa(s_0)}{s - s_0}$$

exist and $\kappa'_-(s_0) \le \kappa'_+(s_0)$. The functions κ'_- and κ'_+ are monotonic nondecreasing on $\mathring{\Delta}$ and this property can be extended to the whole interval Δ (see [11]).

From the monotonicity of the lateral derivatives κ'_- and κ'_+, we also have the gradient inequality

$$\kappa'_-(s)(s - \sigma) \ge \kappa(s) - \kappa(\sigma) \ge \kappa'_+(\sigma)(s - \sigma)$$

for any $s, \sigma \in \mathring{\Delta}$.

If $\Delta = [\mathfrak{a}_1, \mathfrak{a}_2]$, then at the end points, we also have the inequalities

$$\kappa(s) - \kappa(\mathfrak{a}_1) \ge \kappa'_+(\mathfrak{a}_1)(s - \mathfrak{a}_1),$$

for any $s \in (\mathfrak{a}_1, \mathfrak{a}_2]$, and

$$\kappa(\sigma) - \kappa(\mathfrak{a}_2) \ge \kappa'_-(\mathfrak{a}_2)(\sigma - \mathfrak{a}_2),$$

for any $\sigma \in [\mathfrak{a}_1, \mathfrak{a}_2)$.

For the operator $G \in \mathcal{L}(\mathcal{H})$, we define the selfadjoint operator

$$\mathrm{Re}(G) = \frac{1}{2}(G^* + G).$$

We have the following fundamental fact providing a characterization of square modulus convex functions [6]:

Proposition 2.1. *Assume that function* $S:[\mathfrak{a}_1,\mathfrak{a}_2]\to\mathcal{L}(\mathcal{H})$ *is continuous on* $[\mathfrak{a}_1,\mathfrak{a}_2]$*. The function* $S:[\mathfrak{a}_1,\mathfrak{a}_2]\to\mathcal{L}(\mathcal{H})$ *is square modulus convex on* $[\mathfrak{a}_1,\mathfrak{a}_2]$ *if and only if for all* $x\in\mathcal{H}\setminus\{0\}$*, the auxiliary function* $\psi_{S,x}:[\mathfrak{a}_1,\mathfrak{a}_2]\to[0,\infty)$, $\psi_{S,x}(\mathfrak{u})=\|S(\mathfrak{u})x\|^2$ *is convex on* $[\mathfrak{a}_1,\mathfrak{a}_2]$*.*

Proof. For the sake of completeness, we give here a simple proof.

The condition (1.5) is equivalent to

$$\left\langle |S((1-t)\mathfrak{u}+t\mathfrak{v})|^2 x,x\right\rangle \le (1-t)\left\langle |S(\mathfrak{u})|^2 x,x\right\rangle + t\left\langle |S(\mathfrak{v})|^2 x,x\right\rangle,$$

namely,

$$\langle [S((1-t)\mathfrak{u}+t\mathfrak{v})]^* S((1-t)\mathfrak{u}+t\mathfrak{v})x,x\rangle \le (1-t)\langle [S(\mathfrak{u})]^* S(\mathfrak{u})x,x\rangle + t\langle [S(\mathfrak{v})]^* S(\vartheta)x,x\rangle$$

or

$$\|S((1-t)\mathfrak{u}+t\mathfrak{v})x\|^2 \le (1-t)\|S(\mathfrak{u})x\|^2 + t\|S(\mathfrak{v})x\|^2,$$

for all $t\in[0,1]$ and $\mathfrak{u},\mathfrak{v}\in[\mathfrak{a}_1,\mathfrak{a}_2]$. □

Observe that, see also [6],

$$\begin{aligned}
\psi'_{\pm S,x}(\mathfrak{u}) &= (\langle S(\mathfrak{u})x, S(\mathfrak{u})x\rangle)'_{\pm} = \left\langle S'_{\pm}(\mathfrak{u})x, S(\mathfrak{u})x\right\rangle + \left\langle S(\mathfrak{u})x, S'_{\pm}(\mathfrak{u})x\right\rangle \\
&= \left\langle (S(\mathfrak{u}))^* S'_{\pm}(\mathfrak{u})x,x\right\rangle + \left\langle (S'_{\pm}(\mathfrak{u}))^* S(\mathfrak{u})x,x\right\rangle \\
&= \left\langle (S(\mathfrak{u}))^* S'_{\pm}(\mathfrak{u})x,x\right\rangle + \left\langle ((S(\mathfrak{u}))^* S'_{\pm}(\mathfrak{u}))^* x,x\right\rangle \\
&= \left\langle 2Re\left((S(\mathfrak{u}))^* S'_{\pm}(\mathfrak{u})\right)x,x\right\rangle
\end{aligned}$$

and

$$\begin{aligned}
\psi'_{+S,x}(\mathfrak{a}_1) &= 2\left\langle Re\left((S(\mathfrak{a}_1))^* S'_{+}(\mathfrak{a}_1)\right)x,x\right\rangle, \\
\psi'_{-S,x}(\mathfrak{a}_2) &= 2\left\langle Re\left((S(\mathfrak{a}_2))^* S'_{-}(\mathfrak{a}_2)\right)x,x\right\rangle,
\end{aligned}$$

for all $x\in\mathcal{H}\setminus\{0\}$.

We have for $t\in(\mathfrak{a}_1,\mathfrak{a}_2)$ and small $h\neq 0$ such that $t+h\in(\mathfrak{a}_1,\mathfrak{a}_2)$,

$$\begin{aligned}
\left\langle \frac{|S(t+h)|^2-|S(t)|^2}{h}x,x\right\rangle &= \frac{1}{h}\left[\left\langle |S(t+h)|^2 x,x\right\rangle - \left\langle |S(t)|^2 x,x\right\rangle\right] \\
&= \frac{1}{h}\left[\|S(t+h)x\|^2 - \|S(t)x\|^2\right],
\end{aligned}$$

for all $x\in\mathcal{H}\setminus\{0\}$.

By taking the lateral limits, we get

$$\begin{aligned}
\lim_{h\to\pm 0}\left\langle \frac{|S(t+h)|^2-|S(t)|^2}{h}x,x\right\rangle &= \lim_{h\to\pm 0}\frac{1}{h}\left[\|S(t+h)x\|^2 - \|S(t)x\|^2\right] \\
&= \left\langle 2Re\left((S(t))^* S'_{\pm}(t)\right)x,x\right\rangle,
\end{aligned}$$

for all $x\in\mathcal{H}\setminus\{0\}$.

Therefore for the function $\psi(t) := |S(t)|^2$, $t \in (\mathfrak{a}_1, \mathfrak{a}_2)$, we have

$$\psi'_{\pm}(t) = 2Re\left((S(t))^* S'_{\pm}(t)\right) \tag{2.1}$$

and

$$\psi'_{+}(\mathfrak{a}_1) = 2Re\left((S(\mathfrak{a}_1))^* S_{+}(\mathfrak{a}_1)\right), \ \psi'_{-}(\mathfrak{a}_2) = 2Re\left((S(\mathfrak{a}_2))^* S'_{-}(\mathfrak{a}_2)\right).$$

We also have the operator gradient inequality

$$\begin{aligned} 2(s-\sigma) Re\left((S(s))^* S'_{-}(s)\right) &\geq |S(s)|^2 - |S(\sigma)|^2 \\ &\geq 2(s-\sigma) Re\left((S(\sigma))^* S'_{+}(\sigma)\right), \end{aligned} \tag{2.2}$$

for any $s, \sigma \in (\mathfrak{a}_1, \mathfrak{a}_2)$.

Moreover, at the end points of the interval, we have the operator inequalities

$$|S(s)|^2 - |S(\mathfrak{a}_1)|^2 \geq 2(s-\mathfrak{a}_1) Re\left((S(\mathfrak{a}_1))^* S'_{+}(\mathfrak{a}_1)\right), \tag{2.3}$$

for any $s \in (\mathfrak{a}_1, \mathfrak{a}_2]$, and

$$2(\mathfrak{a}_2-\sigma) Re\left((S(\mathfrak{a}_2))^* S'_{-}(\mathfrak{a}_2)\right) \geq |S(\mathfrak{a}_2)|^2 - |S(\sigma)|^2, \tag{2.4}$$

for any $\sigma \in [\mathfrak{a}_1, \mathfrak{a}_2)$.

Finally, we notice that for $\mathfrak{a}_1 \leq t_1 < t_2 \leq \mathfrak{a}_2$, we have

$$\begin{aligned} Re\left((S(\mathfrak{a}_1))^* S'_{+}(\mathfrak{a}_1)\right) &\leq Re\left((S(t_1))^* S'_{\pm}(t_1)\right) \\ &\leq Re\left((S(t_2))^* S'_{\pm}(t_2)\right) \leq Re\left((S(\mathfrak{a}_2))^* S'_{-}(\mathfrak{a}_2)\right), \end{aligned} \tag{2.5}$$

in the operator order of $\mathcal{L}(\mathcal{H})$.

The following characterization of strongly differentiable square modulus convex functions may be stated as well:

Proposition 2.2. *Assume that function $S : [\mathfrak{a}_1, \mathfrak{a}_2] \to \mathcal{L}(\mathcal{H})$ is strongly differentiable on $(\mathfrak{a}_1, \mathfrak{a}_2)$. The function $S : [\mathfrak{a}_1, \mathfrak{a}_2] \to \mathcal{L}(\mathcal{H})$ is square modulus convex on $[\mathfrak{a}_1, \mathfrak{a}_2]$ if and only if*

$$Re\left((S(t_1))^* S'(t_1)\right) \leq Re\left((S(t_2))^* S'(t_2)\right), \tag{2.6}$$

for all $\mathfrak{a}_1 \leq t_1 < t_2 \leq \mathfrak{a}_2$.

The proof follows by Proposition 2.1 and the fact that a scalar-valued differentiable function is convex on $(\mathfrak{a}_1, \mathfrak{a}_2)$ if and only if its derivative is nondecreasing on $(\mathfrak{a}_1, \mathfrak{a}_2)$.

We notice that if $\psi(t) := |S(t)|^2$, $t \in (\mathfrak{a}_1, \mathfrak{a}_2)$, then we have from (2.1)

$$\psi'(t) = 2Re\left((S(t))^* S'(t)\right) = (S(t))^* S'(t) + (S'(t))^* S(t). \tag{2.7}$$

If $S : [\mathfrak{a}_1, \mathfrak{a}_2] \to \mathcal{L}(\mathcal{H})$ is strongly twice differentiable on $(\mathfrak{a}_1, \mathfrak{a}_2)$, then

$$\begin{aligned} \psi''(t) &= \left((S(t))^* S'(t)\right)' + \left((S'(t))^* S(t)\right)' \\ &= \left((S'(t))^* S'(t)\right) + (S(t))^* S''(t) + \left((S''(t))^* S(t)\right) + \left((S'(t))^* S'(t)\right) \\ &= 2|S'(t)|^2 + 2Re\left[(S(t))^* S''(t)\right] = 2\left\{|S'(t)|^2 + Re\left[(S(t))^* S''(t)\right]\right\}, \end{aligned}$$

for all $t \in (\mathfrak{a}_1, \mathfrak{a}_2)$.

We can state then the following result for strongly twice differentiable functions on $(\mathfrak{a}_1, \mathfrak{a}_2)$ as well:

Proposition 2.3. *Assume that function $S : [\mathfrak{a}_1, \mathfrak{a}_2] \to \mathcal{L}(\mathcal{H})$ is strongly twice differentiable on $(\mathfrak{a}_1, \mathfrak{a}_2)$. The function $S : [\mathfrak{a}_1, \mathfrak{a}_2] \to \mathcal{L}(\mathcal{H})$ is square modulus convex on $[\mathfrak{a}_1, \mathfrak{a}_2]$ if and only if*

$$|S'(t)|^2 + Re\left[(S(t))^* S''(t)\right] \geq 0,$$

for all $t \in (\mathfrak{a}_1, \mathfrak{a}_2)$.

We also have the following scalar inequality of interest:

$$\begin{aligned} 0 &\leq 2\min\{t, 1-t\}\left[\frac{\kappa(s)+\kappa(\mathfrak{u})}{2} - \kappa\left(\frac{s+\mathfrak{u}}{2}\right)\right] \\ &\leq (1-t)\kappa(s) + t\kappa(\mathfrak{u}) - \kappa((1-t)s + t\mathfrak{u}) \\ &\leq 2\max\{t, 1-t\}\left[\frac{\kappa(s)+\kappa(\mathfrak{u})}{2} - \kappa\left(\frac{s+\mathfrak{u}}{2}\right)\right], \end{aligned} \tag{2.8}$$

where $\kappa : [\mathfrak{a}_1, \mathfrak{a}_2] \to \mathbb{R}$ is a convex function on $[\mathfrak{a}_1, \mathfrak{a}_2]$ and $t \in [0, 1]$.

The proof follows, for instance, by Corollary 1 from [3] for $n = 2, \beta_1 = 1 - t, \beta_2 = t, t \in [0, 1]$, and $x_1 = s, x_2 = \mathfrak{u}$.

By making use of (2.8) for the $\psi_{S,x} : [\mathfrak{a}_1, \mathfrak{a}_2] \to [0, \infty)$, $\psi_{S,x}(\mathfrak{u}) = \|S(\mathfrak{u})x\|^2$, $x \in \mathcal{H} \setminus \{0\}$, we can state the following result:

Proposition 2.4. *Assume that the continuous function $S : [\mathfrak{a}_1, \mathfrak{a}_2] \to \mathcal{L}(\mathcal{H})$ is square modulus convex on $[\mathfrak{a}_1, \mathfrak{a}_2]$, $s, \sigma \in (\mathfrak{a}_1, \mathfrak{a}_2)$ and $\vartheta \in [0, 1]$, then*

$$\begin{aligned} 0 &\leq 2\min\{\vartheta, 1-\vartheta\}\left[\frac{|S(s)|^2 + |S(\sigma)|^2}{2} - \left|S\left(\frac{s+\sigma}{2}\right)\right|^2\right] \\ &\leq (1-\vartheta)|S(s)|^2 + \vartheta|S(\sigma)|^2 - |S((1-\vartheta)s + \vartheta\sigma)|^2 \\ &\leq 2\max\{\vartheta, 1-\vartheta\}\left[\frac{|S(s)|^2 + |S(\sigma)|^2}{2} - \left|S\left(\frac{s+\sigma}{2}\right)\right|^2\right]. \end{aligned} \tag{2.9}$$

Remark 2.1. *If we take the integral over $\vartheta \in [0, 1]$ in (2.9) and observe that $\int_0^1 \min\{\vartheta, 1-\vartheta\}\, d\vartheta = \frac{1}{4}$, and $\int_0^1 \max\{\vartheta, 1-\vartheta\}\, d\vartheta = \frac{3}{4}$, then we get*

$$\begin{aligned} 0 &\leq \frac{1}{2}\left[\frac{|S(s)|^2 + |S(\sigma)|^2}{2} - \left|S\left(\frac{s+\sigma}{2}\right)\right|^2\right] \\ &\leq \frac{|S(s)|^2 + |S(\sigma)|^2}{2} - \int_0^1 |S((1-\vartheta)s + \vartheta\sigma)|^2\, d\vartheta \\ &\leq \frac{3}{2}\left[\frac{|S(s)|^2 + |S(\sigma)|^2}{2} - \left|S\left(\frac{s+\sigma}{2}\right)\right|^2\right], \end{aligned}$$

for all $s, \sigma \in (\mathfrak{a}_1, \mathfrak{a}_2)$.

We have the following result for general convex functions [5]:

Lemma 2.1. *Let* $\psi : [\mathfrak{a}_1, \mathfrak{a}_2] \subset \mathbb{R} \to \mathbb{R}$ *be a convex function on the interval* $[\mathfrak{a}_1, \mathfrak{a}_2]$*,* $s, \sigma \in (\mathfrak{a}_1, \mathfrak{a}_2)$ *and* $\vartheta \in [0, 1]$ *, then*

$$\begin{aligned} 0 &\le \vartheta (1-\vartheta)(\sigma - s)\left[\psi'_+\left((1-\vartheta) s + \vartheta\sigma\right) - \psi'_-\left((1-\vartheta) s + \vartheta\sigma\right)\right] \\ &\le (1-\vartheta)\psi(s) + \vartheta\psi(\sigma) - \psi\left((1-\vartheta) s + \vartheta\sigma\right) \\ &\le \vartheta (1-\vartheta)(\sigma - s)\left[\psi'_-(\sigma) - \psi'_+(s)\right]. \end{aligned} \tag{2.10}$$

In particular, we have

$$\begin{aligned} 0 &\le \frac{1}{4}(\sigma - s)\left[\psi'_+\left(\frac{s+\sigma}{2}\right) - \psi'_-\left(\frac{s+\sigma}{2}\right)\right] \\ &\le \frac{\psi(s) + \psi(\sigma)}{2} - \psi\left(\frac{s+\sigma}{2}\right) \\ &\le \frac{1}{4}(\sigma - s)\left[\psi'_-(\sigma) - \psi'_+(s)\right]. \end{aligned} \tag{2.11}$$

The constant $\frac{1}{4}$ *is best possible in both inequalities from (2.11).*

Remark 2.2. *If the function* $\psi : [\mathfrak{a}_1, \mathfrak{a}_2] \subset \mathbb{R} \to \mathbb{R}$ *is a differentiable convex function on* $(\mathfrak{a}_1, \mathfrak{a}_2)$ *, then for any* $s, \sigma \in (\mathfrak{a}_1, \mathfrak{a}_2)$ *and* $\vartheta \in [0, 1]$*, we have*

$$\begin{aligned} 0 &\le (1-\vartheta)\psi(s) + \vartheta\psi(\sigma) - \psi\left((1-\vartheta) s + \vartheta\sigma\right) \\ &\le \vartheta (1-\vartheta)(\sigma - s)\left[\psi'(\sigma) - \psi'(s)\right]. \end{aligned} \tag{2.12}$$

The following inequalities in terms of the first derivative may be stated as well:

Proposition 2.5. *Assume that the continuous function* $S : [\mathfrak{a}_1, \mathfrak{a}_2] \to \mathcal{L}(\mathcal{H})$ *is square modulus convex on* $[\mathfrak{a}_1, \mathfrak{a}_2]$ *,* $s, \sigma \in (\mathfrak{a}_1, \mathfrak{a}_2)$ *and* $\vartheta \in [0, 1]$ *, then*

$$\begin{aligned} 0 &\le 2\vartheta (1-\vartheta)(\sigma - s) \times \left[Re\left((S((1-\vartheta) s + \vartheta\sigma))^* S'_+((1-\vartheta) s + \vartheta\sigma)\right)\right. \\ &\quad \left. - Re\left((S((1-\vartheta) s + \vartheta\sigma))^* S'_-((1-\vartheta) s + \vartheta\sigma)\right)\right] \\ &\le (1-\vartheta)|S(s)|^2 + \vartheta|S(\sigma)|^2 - |S((1-\vartheta) s + \vartheta\sigma)|^2 \\ &\le 2\vartheta (1-\vartheta)(\sigma - s)\left[Re\left((S(\sigma))^* S'_-(\sigma)\right) - Re\left((S(s))^* S'_+(s)\right)\right]. \end{aligned} \tag{2.13}$$

If $S : [\mathfrak{a}_1, \mathfrak{a}_2] \to \mathcal{L}(\mathcal{H})$ *is strongly differentiable on* $(\mathfrak{a}_1, \mathfrak{a}_2)$*, then*

$$\begin{aligned} 0 &\le (1-\vartheta)|S(s)|^2 + \vartheta|S(\sigma)|^2 - |S((1-\vartheta) s + \vartheta\sigma)|^2 \\ &\le 2\vartheta (1-\vartheta)(\sigma - s)\left[Re\left((S(\sigma))^* S'(\sigma)\right) - Re\left((S(s))^* S'(s)\right)\right], \end{aligned} \tag{2.14}$$

for $s, \sigma \in (\mathfrak{a}_1, \mathfrak{a}_2)$ *and* $\vartheta \in [0, 1]$ *.*

In particular,

$$\begin{aligned} 0 &\le \frac{|S(s)|^2 + |S(\sigma)|^2}{2} - \left|S\left(\frac{s+\sigma}{2}\right)\right|^2 \\ &\le \frac{1}{2}(\sigma - s)\left[Re\left((S(\sigma))^* S'(\sigma)\right) - Re\left((S(s))^* S'(s)\right)\right], \end{aligned} \tag{2.15}$$

for $s, \sigma \in (a, \mathfrak{a}_2)$ *.*

Proof. Let $x \in \mathcal{H} \setminus \{0\}$, $s, \sigma \in (\mathfrak{a}_1, \mathfrak{a}_2)$, and $\vartheta \in [0,1]$. Then by the inequality (2.10) for the function $\psi_{S,x} : [\mathfrak{a}_1, \mathfrak{a}_2] \to [0,\infty)$, $\psi_{S,x}(\mathfrak{u}) = \|S(\mathfrak{u})x\|^2$, we get

$$\begin{aligned}
&2\vartheta(1-\vartheta)(\sigma-s) \times [\langle Re((S((1-\vartheta)s+\vartheta\sigma))^* S'_+((1-\vartheta)s+\vartheta\sigma))x,x\rangle \\
&\quad - \langle Re((S((1-\vartheta)s+\vartheta\sigma))^* S'_-((1-\vartheta)s+\vartheta\sigma))x,x\rangle] \\
&\le (1-\vartheta)\left\langle |S(s)|^2 x,x\right\rangle + \vartheta\left\langle |S(\sigma)|^2 x,x\right\rangle - \left\langle |S((1-\vartheta)s+\vartheta\sigma)|^2 x,x\right\rangle \\
&\le 2\vartheta(1-\vartheta)(\sigma-s) \times [\langle Re((S(\sigma))^* S'_-(\sigma))x,x\rangle - \langle Re((S(s))^* S'_+(s))x,x\rangle],
\end{aligned}$$

which is equivalent to (2.13). □

Remark 2.3. *If we integrate the inequality (2.14) over $\vartheta \in [0,1]$, then we obtain*

$$\begin{aligned}
0 &\le \frac{|S(s)|^2 + |S(\sigma)|^2}{2} - \int_0^1 |S((1-\vartheta)s+\vartheta\sigma)|^2 d\vartheta \\
&\le \frac{1}{3}(\sigma-s)[Re((S(\sigma))^* S'(\sigma)) - Re((S(s))^* S'(s))],
\end{aligned}$$

for all $s, \sigma \in (\mathfrak{a}_1, \mathfrak{a}_2)$.

The following bounds in terms of the second derivative holds, see for instance [4] for a detailed proof:

Lemma 2.2. *Let $\kappa : [\mathfrak{a}_1, \mathfrak{a}_2] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $(\mathfrak{a}_1, \mathfrak{a}_2)$. If there exists the constants d, D such that*

$$d \le \kappa''(t) \le D \text{ for any } t \in (\mathfrak{a}_1, \mathfrak{a}_2), \tag{2.16}$$

then

$$\begin{aligned}
\frac{1}{2}\vartheta(1-\vartheta)d(\sigma-s)^2 &\le (1-\vartheta)\kappa(s) + \vartheta\kappa(\sigma) - \kappa((1-\vartheta)s+\vartheta\sigma) \\
&\le \frac{1}{2}\vartheta(1-\vartheta)D(\sigma-s)^2,
\end{aligned} \tag{2.17}$$

for any $s, \sigma \in (\mathfrak{a}_1, \mathfrak{a}_2)$ and $\vartheta \in [0,1]$.

In particular, we have

$$\frac{1}{8}(\sigma-s)^2 d \le \frac{\kappa(s)+\kappa(\sigma)}{2} - \kappa\left(\frac{s+\sigma}{2}\right) \le \frac{1}{8}(\sigma-s)^2 D, \tag{2.18}$$

for any $s, \sigma \in (\mathfrak{a}_1, \mathfrak{a}_2)$.

The constant $\frac{1}{8}$ is best possible in both inequalities in (2.18).

Indeed if we take $\kappa(t) = t^2$, then $\kappa''(t) = 2$ and we obtain in all terms of (2.18) the same quantity $\frac{1}{4}(\sigma-s)^2$.

For strongly twice differentiable square modulus convex functions, we have the following upper and lower bounds:

Proposition 2.6. *Assume that $S : [\mathfrak{a}_1, \mathfrak{a}_2] \to \mathcal{L}(\mathcal{H})$ is strongly twice differentiable on $(\mathfrak{a}_1, \mathfrak{a}_2)$ and there exist $C, \mathcal{D} \in \mathcal{L}(\mathcal{H})$ with $C, \mathcal{D} \geq 0$ and*

$$(0 \leq) C \leq |S'(t)|^2 + Re\left[(S(t))^* S''(t)\right] \leq \mathcal{D} \text{ for all } t \in (\mathfrak{a}_1, \mathfrak{a}_2). \tag{2.19}$$

Then for any $s, \sigma \in [\mathfrak{a}_1, \mathfrak{a}_2]$ and $\vartheta \in [0, 1]$, we have

$$\begin{aligned} \vartheta(1-\vartheta)(\sigma - s)^2 C &\leq (1-\vartheta)|S(s)|^2 + \vartheta|S(\sigma)|^2 - |S((1-\vartheta)s + \vartheta\sigma)|^2 \\ &\leq \vartheta(1-\vartheta)(\sigma - s)^2 \mathcal{D}. \end{aligned} \tag{2.20}$$

In particular,

$$\begin{aligned} \frac{1}{4}(\sigma - s)^2 C &\leq \frac{|S(s)|^2 + |S(\sigma)|^2}{2} - \left|S\left(\frac{s+\sigma}{2}\right)\right|^2 \\ &\leq \frac{1}{4}(\sigma - s)^2 \mathcal{D}. \end{aligned} \tag{2.21}$$

The proof follows by Lemma 2.2 for the function $\psi_{S,x} : [\mathfrak{a}_1, \mathfrak{a}_2] \to [0, \infty)$, $\psi_{S,x}(\mathfrak{u}) = \|S(\mathfrak{u})x\|^2$, $x \in \mathcal{H} \setminus \{0\}$, and by observing that

$$\psi''_{S,x}(t) = 2\left\langle \left\{ |S'(t)|^2 + Re\left[(S(t))^* S''(t)\right] \right\} x, x \right\rangle,$$

for $t \in (\mathfrak{a}_1, \mathfrak{a}_2)$.

Corollary 2.7. *With the assumptions of Proposition 2.6, we have the trapezoid-type inequality*

$$\begin{aligned} \frac{1}{6}(\sigma - s)^2 C &\leq \frac{|S(s)|^2 + |S(\sigma)|^2}{2} - \int_0^1 |S((1-\vartheta)s + \vartheta\sigma)|^2 d\vartheta \\ &\leq \frac{1}{6}(\sigma - s)^2 \mathcal{D} \end{aligned} \tag{2.22}$$

and the midpoint-type inequality

$$\begin{aligned} \frac{1}{12}(\sigma - s)^2 C &\leq \int_0^1 |S((1-\vartheta)s + \vartheta\sigma)|^2 d\vartheta - \left|S\left(\frac{s+\sigma}{2}\right)\right|^2 \\ &\leq \frac{1}{12}(\sigma - s)^2 \mathcal{D} \end{aligned} \tag{2.23}$$

for any $s, \sigma \in [\mathfrak{a}_1, \mathfrak{a}_2]$.

Proof. The inequality (2.22) follows by (2.20) on integrating over $\vartheta \in [0, 1]$.

From (2.21), we get

$$\begin{aligned} &\frac{1}{4}((1-\vartheta)s + \vartheta\sigma - (1-\vartheta)\sigma - \vartheta s)^2 C \\ &\leq \frac{|S((1-\vartheta)s + \vartheta\sigma)|^2 + |S((1-\vartheta)\sigma + \vartheta s)|^2}{2} - \left|S\left(\frac{s+\sigma}{2}\right)\right|^2 \\ &\leq \frac{1}{4}((1-\vartheta)s + \vartheta\sigma - (1-\vartheta)\sigma - \vartheta s)^2 \mathcal{D}, \end{aligned} \tag{2.24}$$

namely,

$$\frac{1}{4}(1-2\vartheta)^2(\sigma-s)^2\mathcal{C} \le \frac{|S((1-\vartheta)s+\vartheta\sigma)|^2+|S((1-\vartheta)\sigma+\vartheta s)|^2}{2} - \left|S\left(\frac{s+\sigma}{2}\right)\right|^2 \le \frac{1}{4}(1-2\vartheta)^2(\sigma-s)^2\mathcal{D}. \tag{2.25}$$

Taking the integral over $\vartheta \in [0,1]$ and observing that

$$\int_0^1 (1-2\vartheta)^2 d\vartheta = \frac{1}{3}$$

and

$$\int_0^1 |S((1-\vartheta)\sigma+\vartheta s)|^2 d\vartheta = \int_0^1 |S((1-\vartheta)s+\vartheta\sigma)|^2 d\vartheta,$$

then by (2.25), we derive (2.23). □

3. Double-integral inequalities

As outlined in the seminal paper [2], the consideration of double integrals can provide more insights on improving some known results. In this spirit, we can state the following double-integral inequalities for operator square modulus convex functions:

Theorem 3.1. *Assume that the continuous function $S : [\mathfrak{a}_1, \mathfrak{a}_2] \to \mathcal{L}(\mathcal{H})$ is square modulus convex on $[\mathfrak{a}_1, \mathfrak{a}_2]$. Then for all $\vartheta \in [0,1]$,*

$$\begin{aligned} 0 &\le 2\min\{\vartheta, 1-\vartheta\} \times \left[\frac{1}{\mathfrak{a}_2-\mathfrak{a}_1}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S(s)|^2 ds - \frac{1}{(\mathfrak{a}_2-\mathfrak{a}_1)^2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \left|S\left(\frac{s+\sigma}{2}\right)\right|^2 dsd\sigma\right] \\ &\le \frac{1}{\mathfrak{a}_2-\mathfrak{a}_1}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S(s)|^2 ds - \frac{1}{(\mathfrak{a}_2-\mathfrak{a}_1)^2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S((1-\vartheta)s+\vartheta\sigma)|^2 dsd\sigma \\ &\le 2\max\{\vartheta, 1-\vartheta\} \times \left[\frac{1}{\mathfrak{a}_2-\mathfrak{a}_1}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S(s)|^2 ds - \frac{1}{(\mathfrak{a}_2-\mathfrak{a}_1)^2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \left|S\left(\frac{s+\sigma}{2}\right)\right|^2 dsd\sigma\right]. \end{aligned} \tag{3.1}$$

Proof. If we take the double integral in (2.9), then we get

$$\begin{aligned} &2\min\{\vartheta, 1-\vartheta\} \times \left[\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \left(\frac{|S(s)|^2+|S(\sigma)|^2}{2}\right) dsd\sigma - \int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \left|S\left(\frac{s+\sigma}{2}\right)\right|^2 dsd\sigma\right] \\ &\quad \le (1-\vartheta)\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S(s)|^2 dsd\sigma + \vartheta\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S(\sigma)|^2 dsd\sigma \\ &\quad\quad - \int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S((1-\vartheta)s+\vartheta\sigma)|^2 dsd\sigma \\ &\quad \le 2\max\{\vartheta, 1-\vartheta\} \\ &\quad\quad \times \left[\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \left(\frac{|S(s)|^2+|S(\sigma)|^2}{2}\right) dsd\sigma - \int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \left|S\left(\frac{s+\sigma}{2}\right)\right|^2 dsd\sigma\right], \end{aligned} \tag{3.2}$$

for all $\vartheta \in [0,1]$.

Observe that

$$\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S(s)|^2 \, dsd\sigma = \int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S(\sigma)|^2 \, dsd\sigma = (\mathfrak{a}_2 - \mathfrak{a}_1)\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S(s)|^2 \, ds$$

and by (3.2), we get

$$\begin{aligned}
&2\min\{\vartheta, 1-\vartheta\}\left[(\mathfrak{a}_2 - \mathfrak{a}_1)\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S(s)|^2 \, ds - \int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{a}^{\mathfrak{a}_2} \left|S\left(\frac{s+\sigma}{2}\right)\right|^2 dsd\sigma\right] \\
&\leq (\mathfrak{a}_2 - \mathfrak{a}_1)\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S(s)|^2 \, ds - \int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S((1-\vartheta)s + \vartheta\sigma)|^2 \, dsd\sigma \\
&\leq 2\max\{\vartheta, 1-\vartheta\}\left[(\mathfrak{a}_2 - \mathfrak{a}_1)\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S(s)|^2 \, ds - \int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \left|S\left(\frac{s+\sigma}{2}\right)\right|^2 dsd\sigma\right],
\end{aligned}$$

which is equivalent to (3.1). □

For functions that are strongly differentiable, we have the following upper bounds:

Theorem 3.2. *Assume that* $S : [\mathfrak{a}_1, \mathfrak{a}_2] \to \mathcal{L}(\mathcal{H})$ *is square modulus convex on* $[\mathfrak{a}_1, \mathfrak{a}_2]$ *and strongly differentiable on* $(\mathfrak{a}_1, \mathfrak{a}_2)$, *then for all* $\vartheta \in [0,1]$,

$$\begin{aligned}
0 &\leq \frac{1}{\mathfrak{a}_2 - \mathfrak{a}_1}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S(s)|^2 \, ds - \frac{1}{(\mathfrak{a}_2 - \mathfrak{a}_1)^2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S((1-\vartheta)s + \vartheta\sigma)|^2 \, dsd\sigma \\
&\leq 4\vartheta(1-\vartheta)\frac{1}{\mathfrak{a}_2 - \mathfrak{a}_1}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\left(s - \frac{\mathfrak{a}_1 + \mathfrak{a}_2}{2}\right) Re\left((S(s))^* S'(s)\right) ds. \qquad (3.3)
\end{aligned}$$

In particular,

$$\begin{aligned}
0 &\leq \frac{1}{\mathfrak{a}_2 - \mathfrak{a}_1}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S(s)|^2 \, ds - \frac{1}{(\mathfrak{a}_2 - \mathfrak{a}_1)^2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \left|S\left(\frac{s+\sigma}{2}\right)\right|^2 dsd\sigma \qquad (3.4) \\
&\leq \frac{1}{\mathfrak{a}_2 - \mathfrak{a}_1}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\left(s - \frac{\mathfrak{a}_1 + \mathfrak{a}_2}{2}\right) Re\left((S(s))^* S'(s)\right) ds.
\end{aligned}$$

Proof. If we take the double integral in (2.14), then we get

$$\begin{aligned}
&(1-\vartheta)\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S(s)|^2 \, dsd\sigma + \vartheta\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S(\sigma)|^2 \, dsd\sigma \\
&- \int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S((1-\vartheta)s + \vartheta\sigma)|^2 \, dsd\sigma \\
&\leq 2\vartheta(1-\vartheta)\times\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} (\sigma - s)\left[Re\left((S(\sigma))^* S'(\sigma)\right) - Re\left((S(s))^* S'(s)\right)\right] dsd\sigma, \qquad (3.5)
\end{aligned}$$

for all $\vartheta \in [0,1]$.

Observe that

$$\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} (\sigma - s)\left[Re\left((S(\sigma))^* S'(\sigma)\right) - Re\left((S(s))^* S'(s)\right)\right] dsd\sigma$$

$$
\begin{aligned}
&= \int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\left[\sigma Re\left((S(\sigma))^* S'(\sigma)\right) + sRe\left((S(s))^* S'(s)\right)\right] ds d\sigma \\
&\quad - \int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\left[sRe\left((S(\sigma))^* S'(\sigma)\right) + \sigma Re\left((S(s))^* S'(s)\right)\right] ds d\sigma \\
&= 2(\mathfrak{a}_2 - \mathfrak{a}_1)\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} sRe\left((S(s))^* S'(s)\right) ds - 2\frac{\mathfrak{a}_2^2 - \mathfrak{a}_1^2}{2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} Re\left((S(s))^* S'(s)\right) ds \\
&= 2(\mathfrak{a}_2 - \mathfrak{a}_1)\left[\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} sRe\left((S(s))^* S'(s)\right) ds - \frac{\mathfrak{a}_1 + \mathfrak{a}_2}{2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} Re\left((S(s))^* S'(s)\right) ds\right],
\end{aligned}
$$

and by (3.5), we get

$$
\begin{aligned}
&(\mathfrak{a}_2 - \mathfrak{a}_1)\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S(s)|^2 ds - \int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S((1-\vartheta)s + \vartheta\sigma)|^2 ds d\sigma \\
&\leq 4\vartheta(1-\vartheta)(\mathfrak{a}_2 - \mathfrak{a}_1)\left[\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\left(s - \frac{\mathfrak{a}_1 + \mathfrak{a}_2}{2}\right) Re\left((S(s))^* S'(s)\right) ds\right],
\end{aligned}
$$

which is equivalent to (3.3). □

For functions that are strongly twice differentiable, we have some simple lower and upper bounds:

Theorem 3.3. *Assume that* $S : [\mathfrak{a}_1, \mathfrak{a}_2] \to \mathcal{L}(\mathcal{H})$ *is strongly twice differentiable on* $(\mathfrak{a}_1, \mathfrak{a}_2)$ *and there exist* $C, \mathcal{D} \in \mathcal{L}(\mathcal{H})$ *with* $C, \mathcal{D} \geq 0$ *and the condition (2.19) is satisfied, then for all* $\vartheta \in [0, 1]$,

$$
\begin{aligned}
\frac{1}{6}\vartheta(1-\vartheta)(\mathfrak{a}_2 - \mathfrak{a}_1)^2 C &\leq \frac{1}{\mathfrak{a}_2 - \mathfrak{a}_1}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S(s)|^2 ds - \frac{1}{(\mathfrak{a}_2 - \mathfrak{a}_1)^2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S((1-\vartheta)s + \vartheta\sigma)|^2 ds d\sigma \\
&\leq \frac{1}{6}\vartheta(1-\vartheta)(\mathfrak{a}_2 - \mathfrak{a}_1)^2 \mathcal{D}. \qquad (3.6)
\end{aligned}
$$

In particular,

$$
\begin{aligned}
\frac{1}{24}(\mathfrak{a}_2 - \mathfrak{a}_1)^2 C &\leq \frac{1}{\mathfrak{a}_2 - \mathfrak{a}_1}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S(s)|^2 ds - \frac{1}{(\mathfrak{a}_2 - \mathfrak{a}_1)^2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \left|S\left(\frac{s+\sigma}{2}\right)\right|^2 ds d\sigma \\
&\leq \frac{1}{24}(\mathfrak{a}_2 - \mathfrak{a}_1)^2 \mathcal{D}. \qquad (3.7)
\end{aligned}
$$

Proof. By taking the double integral in (2.20), we get

$$
\begin{aligned}
&\vartheta(1-\vartheta)\left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} (\sigma - s)^2 ds d\sigma\right) C \\
&\leq (1-\vartheta)\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S(s)|^2 ds d\sigma + \vartheta\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S(\sigma)|^2 ds d\sigma \\
&\quad - \int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} |S((1-\vartheta)s + \vartheta\sigma)|^2 ds d\sigma \\
&\leq \vartheta(1-\vartheta)\left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} (\sigma - s)^2 ds d\sigma\right)\mathcal{D}. \qquad (3.8)
\end{aligned}
$$

Since
$$\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}(\sigma-s)^2\,dsd\sigma=\frac{1}{6}(\mathfrak{a}_2-\mathfrak{a}_1)^4,$$
hence by (3.8), we get (3.6).

The inequality (3.7) follows from (3.6) by taking $\vartheta = 1/2$. □

4. Some weighted integral inequalities

In this section, we provide some weighted integral inequalities for square modulus convex functions and give some examples for simple symmetrical weights. We note that the adoption of weighted integrals is of interest, with prospective applications in quantum information theory or numerical analysis, fields where operator inequalities enjoy extensive applications. However the details are not provided in this paper.

Theorem 4.1. *Assume that the continuous function $S:[\mathfrak{a}_1,\mathfrak{a}_2]\to\mathcal{L}(\mathcal{H})$ is square modulus convex on $[\mathfrak{a}_1,\mathfrak{a}_2]$, $s,t\in(\mathfrak{a}_1,\mathfrak{a}_2)$ and $\beta:[0,1]\to[0,\infty)$, integrable and symmetrical on $[0,1]$, i.e., $\beta(1-\vartheta)=\beta(\vartheta)$ for $\vartheta\in[0,1]$, then*

$$\begin{aligned}
&\frac{4\int_0^{1/2}\sigma\beta(\sigma)\,d\sigma}{\int_0^1\beta(\vartheta)\,d\vartheta}\left[\frac{|S(s)|^2+|S(t)|^2}{2}-\left|S\left(\frac{s+t}{2}\right)\right|^2\right]\\
&\le\frac{\int_0^1\vartheta\beta(\vartheta)}{\int_0^1\beta(\vartheta)\,d\vartheta}\left(|S(s)|^2+|S(t)|^2\right)-\frac{1}{\int_0^1\beta(\vartheta)\,d\vartheta}\int_0^1\beta(\vartheta)\,|S((1-\vartheta)s+\vartheta t)|^2\,d\vartheta\\
&\le\frac{4\int_{1/2}^1\vartheta\beta(\vartheta)\,d\vartheta}{\int_0^1\beta(\vartheta)\,d\vartheta}\left[\frac{|S(s)|^2+|S(t)|^2}{2}-\left|S\left(\frac{s+t}{2}\right)\right|^2\right].
\end{aligned}\tag{4.1}$$

Moreover, if $w:[\mathfrak{a}_1,\mathfrak{a}_2]\to[0,\infty)$ is integrable, then also

$$\begin{aligned}
&\frac{4\int_0^{1/2}\sigma\beta(\sigma)\,d\sigma}{\int_0^1\beta(\vartheta)\,d\vartheta}\times\left[\frac{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}w(t)\,|S(t)|^2\,dt}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}w(t)\,dt}-\frac{1}{\left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}w(t)\,dt\right)^2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}w(s)\,w(t)\left|S\left(\frac{s+t}{2}\right)\right|^2dsdt\right]\\
&\le\frac{2\int_0^1\vartheta\beta(\vartheta)}{\int_0^1\beta(\vartheta)\,d\vartheta}\frac{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}w(t)\,|S(t)|^2\,dt}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}w(t)\,dt}-\frac{1}{\int_0^1\beta(\vartheta)\,d\vartheta\left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}w(t)\,dt\right)^2}\\
&\quad\times\int_0^1\beta(\vartheta)\left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}w(s)\,w(t)\,|S((1-\vartheta)s+\vartheta t)|^2\,dsdt\right)d\vartheta\\
&\le\frac{4\int_{1/2}^1\vartheta\beta(\vartheta)\,d\vartheta}{\int_0^1\beta(\vartheta)\,d\vartheta}\times\left[\frac{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}w(t)\,|S(t)|^2\,dt}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}w(t)\,dt}-\frac{1}{\left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}w(t)\,dt\right)^2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}w(s)\,w(t)\left|S\left(\frac{s+t}{2}\right)\right|^2dsdt\right].
\end{aligned}\tag{4.2}$$

Proof. We multiply inequality (2.9) by $\beta(\vartheta)\ge0$ and integrate over $\vartheta\in[0,1]$ to get

$$2\int_0^1\min\{\vartheta,1-\vartheta\}\beta(\vartheta)\,d\vartheta\left[\frac{|S(s)|^2+|S(t)|^2}{2}-\left|S\left(\frac{s+t}{2}\right)\right|^2\right]\tag{4.3}$$

$$\leq \int_0^1 (1-\vartheta)\beta(\vartheta)\,d\vartheta\,|S(s)|^2 + \int_0^1 \vartheta\beta(\vartheta)\,d\vartheta\,|S(\sigma)|^2 - \int_0^1 \beta(\vartheta)\,d\vartheta\,|S((1-\vartheta)s+\vartheta t)|^2$$
$$\leq 2\int_0^1 \max\{\vartheta, 1-\vartheta\}\beta(\vartheta)\,d\vartheta \left[\frac{|S(s)|^2+|S(t)|^2}{2} - \left|S\left(\frac{s+t}{2}\right)\right|^2\right].$$

Since $\beta(1-\vartheta) = \beta(\vartheta)$ for $\vartheta \in [0,1]$, then we get

$$\int_0^1 (1-\vartheta)\beta(\vartheta)\,d\vartheta = \int_0^1 (1-\vartheta)\beta(1-\vartheta)\,d\vartheta = \int_0^1 \sigma\beta(\sigma)\,d\sigma,$$
$$\int_0^1 \min\{\vartheta, 1-\vartheta\}\beta(\vartheta)\,d\vartheta = \int_0^{1/2} \vartheta\beta(\vartheta)\,d\vartheta + \int_{1/2}^1 (1-\vartheta)\beta(\vartheta)\,d\vartheta$$
$$= \int_0^{1/2} \vartheta\beta(\vartheta)\,d\vartheta + \int_{1/2}^1 (1-\vartheta)\beta(1-\vartheta)\,d\vartheta$$
$$= \int_0^{1/2} \vartheta\beta(\vartheta)\,d\vartheta + \int_0^{1/2} \sigma\beta(\sigma)\,d\sigma = 2\int_0^{1/2} \sigma\beta(\sigma)\,d\sigma$$

and, similarly,

$$\int_0^1 \max\{\vartheta, 1-\vartheta\}\beta(\vartheta)\,d\vartheta = \int_0^{1/2} (1-\vartheta)\beta(\vartheta)\,d\vartheta + \int_{1/2}^1 \vartheta\beta(\vartheta)\,d\vartheta$$
$$= 2\int_{1/2}^1 \vartheta\beta(\vartheta)\,d\vartheta.$$

By (4.3), we then get

$$4\int_0^{1/2} \sigma\beta(\sigma)\,d\sigma \left[\frac{|S(s)|^2+|S(t)|^2}{2} - \left|S\left(\frac{s+t}{2}\right)\right|^2\right]$$
$$\leq \int_0^1 \vartheta\beta(\vartheta)\left(|S(s)|^2 + d\vartheta\,|S(\sigma)|^2\right) - \int_0^1 \beta(\vartheta)\,|S((1-\vartheta)s+\vartheta t)|^2\,d\vartheta$$
$$\leq 4\int_{1/2}^1 \vartheta\beta(\vartheta)\,d\vartheta \left[\frac{|S(s)|^2+|S(t)|^2}{2} - \left|S\left(\frac{s+t}{2}\right)\right|^2\right],$$

which is equivalent to (4.1).

Now, if we multiply (4.1) by $w(s)\,w(t) \geq 0$ and take the double integral, then we get

$$\frac{4\int_0^{1/2}\sigma\beta(\sigma)\,d\sigma}{\int_0^1 \beta(\vartheta)\,d\vartheta} \times \left[\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,w(t)\left(\frac{|S(s)|^2+|S(t)|^2}{2}\right)dsdt - \int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,w(t)\left|S\left(\frac{s+t}{2}\right)\right|^2 dsdt\right]$$
$$\leq \frac{\int_0^1 \vartheta\beta(\vartheta)}{\int_0^1 \beta(\vartheta)\,d\vartheta}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,w(t)\left(|S(s)|^2 + |S(t)|^2\right)dsdt$$
$$-\frac{1}{\int_0^1 \beta(\vartheta)\,d\vartheta}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\left(\int_0^1 \beta(\vartheta)\,w(s)\,w(t)\,|S((1-\vartheta)s+\vartheta t)|^2\,d\vartheta\right)dsdt$$

$$\leq \frac{4\int_{1/2}^{1}\vartheta\beta(\vartheta)\,d\vartheta}{\int_{0}^{1}\beta(\vartheta)\,d\vartheta}$$
$$\times\left[\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,w(t)\left(\frac{|S(s)|^2+|S(t)|^2}{2}\right)dsdt-\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,w(t)\left|S\left(\frac{s+t}{2}\right)\right|^2 dsdt\right]. \tag{4.4}$$

Since

$$\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,w(t)\left(\frac{|S(s)|^2+|S(t)|^2}{2}\right)dsdt=\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,dt\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,|S(t)|^2\,dt$$

and by Fubini's theorem,

$$\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\left(\int_0^1 \beta(\vartheta)\,w(s)\,w(t)\,|S((1-\vartheta)s+\vartheta t)|^2\,d\vartheta\right)dsdt$$
$$=\int_0^1\beta(\vartheta)\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,w(t)\,|S((1-\vartheta)s+\vartheta t)|^2\,dsdtd\vartheta.$$

Then by (4.4), by division with $\left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,dt\right)^2$, we get (4.2). □

Remark 4.1. *If we take, for instance,* $\beta(\vartheta)=\left|\vartheta-\frac{1}{2}\right|$, $\vartheta\in[0,1]$*, then from (4.1), we get*

$$\frac{1}{3}\left[\frac{|S(s)|^2+|S(t)|^2}{2}-\left|S\left(\frac{s+t}{2}\right)\right|^2\right]\leq\frac{1}{2}\left(|S(s)|^2+|S(t)|^2\right)-4\int_0^1\left|\vartheta-\frac{1}{2}\right||S((1-\vartheta)s+\vartheta t)|^2\,d\vartheta$$
$$\leq\frac{5}{3}\left[\frac{|S(s)|^2+|S(t)|^2}{2}-\left|S\left(\frac{s+t}{2}\right)\right|^2\right],$$

while from (4.2),

$$\frac{1}{3}\left[\frac{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,|S(t)|^2\,dt}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,dt}-\frac{1}{\left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,dt\right)^2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,w(t)\left|S\left(\frac{s+t}{2}\right)\right|^2 dsdt\right]$$
$$\leq\frac{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,|S(t)|^2\,dt}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,dt}-\frac{4}{\left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,dt\right)^2}\int_0^1\left|\vartheta-\frac{1}{2}\right|\left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,w(t)\,|S((1-\vartheta)s+\vartheta t)|^2\,dsdt\right)d\vartheta$$
$$\leq\frac{5}{3}\left[\frac{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,|S(t)|^2\,dt}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,dt}-\frac{1}{\left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,dt\right)^2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,w(t)\left|S\left(\frac{s+t}{2}\right)\right|^2 dsdt\right].$$

In the case of twice differentiable functions, the bounds are simpler, as shown by the next result:

Theorem 4.2. *Assume that* $S:[\mathfrak{a}_1,\mathfrak{a}_2]\to\mathcal{L}(\mathcal{H})$ *is strongly twice differentiable on* $(\mathfrak{a}_1,\mathfrak{a}_2)$ *and there exist* $C,\mathcal{D}\in\mathcal{L}(\mathcal{H})$ *with* $C,\mathcal{D}\geq 0$ *and the condition (2.19) is satisfied,* $s,t\in(\mathfrak{a}_1,\mathfrak{a}_2)$ *and* $\beta:[0,1]\to[0,\infty)$*, integrable and symmetrical on* $[0,1]$*, then we trapezoid type inequalities*

$$\frac{1}{\int_0^1\beta(\vartheta)\,d\vartheta}\int_0^1\vartheta(1-\vartheta)\beta(\vartheta)\,d\vartheta\,(t-s)^2\,C$$

$$\leq \frac{1}{\int_0^1 \beta(\vartheta)\,d\vartheta}\int_0^1 \vartheta\beta(\vartheta)\,d\vartheta\left(|S\,(s)|^2+|S\,(t)|^2\right)$$
$$-\frac{1}{\int_0^1 \beta(\vartheta)\,d\vartheta}\int_0^1 \beta(\vartheta)\,|S\,((1-\vartheta)\,s+\vartheta t)|^2\,d\vartheta$$
$$\leq \frac{1}{\int_0^1 \beta(\vartheta)\,d\vartheta}\int_0^1 \vartheta\,(1-\vartheta)\beta(\vartheta)\,d\vartheta\,(t-s)^2\,\mathcal{D}. \tag{4.5}$$

Moreover, if $w:[\mathfrak{a}_1,\mathfrak{a}_2]\to[0,\infty)$ *is integrable, then also*

$$\frac{2}{\int_0^1 \beta(\vartheta)\,d\vartheta}\int_0^1 \vartheta\,(1-\vartheta)\beta(\vartheta)\,d\vartheta\times\left[\frac{1}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,ds}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,t^2dt-\left(\frac{1}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,ds}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,tdt\right)^2\right]C$$
$$\leq \frac{2\int_0^1 \vartheta\beta(\vartheta)}{\int_0^1 \beta(\vartheta)\,d\vartheta}\frac{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,|S\,(t)|^2\,dt}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,dt}-\frac{1}{\int_0^1 \beta(\vartheta)\,d\vartheta\left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,dt\right)^2}$$
$$\times\int_0^1 \beta(\vartheta)\left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,w(t)\,|S\,((1-\vartheta)\,s+\vartheta t)|^2\,dsdt\right)d\vartheta$$
$$\leq \frac{2}{\int_0^1 \beta(\vartheta)\,d\vartheta}\int_0^1 \vartheta\,(1-\vartheta)\beta(\vartheta)\,d\vartheta\times\left[\frac{1}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,ds}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,t^2dt-\left(\frac{1}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,ds}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,tdt\right)^2\right]C. \tag{4.6}$$

Also, we have the midpoint-type inequalities

$$\frac{1}{\int_0^1 \beta(\vartheta)\,d\vartheta}\int_0^1 \left(\vartheta-\frac{1}{2}\right)^2\beta(\vartheta)\,d\vartheta\,(t-s)^2\,C$$
$$\leq \frac{1}{\int_0^1 \beta(\vartheta)\,d\vartheta}\int_0^1 \beta(\vartheta)\,|S\,((1-\vartheta)\,s+\vartheta t)|^2\,d\vartheta-\left|S\left(\frac{s+t}{2}\right)\right|^2$$
$$\leq \frac{1}{\int_0^1 \beta(\vartheta)\,d\vartheta}\int_0^1 \left(\vartheta-\frac{1}{2}\right)^2\beta(\vartheta)\,d\vartheta\,(t-s)^2\,\mathcal{D} \tag{4.7}$$

and

$$\frac{2}{\int_0^1 \beta(\vartheta)\,d\vartheta}\int_0^1 \left(\vartheta-\frac{1}{2}\right)^2\beta(\vartheta)\,d\vartheta\times\left[\frac{1}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,ds}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,t^2dt-\left(\frac{1}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,ds}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,tdt\right)^2\right]C$$
$$\leq \frac{1}{\int_0^1 \beta(\vartheta)\,d\vartheta\left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,dt\right)^2}\times\int_0^1 \beta(\vartheta)\left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,w(t)\,|S\,((1-\vartheta)\,s+\vartheta t)|^2\,dsdt\right)d\vartheta$$
$$-\frac{1}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,ds}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,w(t)\left|S\left(\frac{s+t}{2}\right)\right|^2 dsdt$$

$$
\begin{aligned}
&\leq \frac{2}{\int_0^1 \beta(\vartheta)\, d\vartheta} \int_0^1 \left(\vartheta - \frac{1}{2}\right)^2 \beta(\vartheta)\, d\vartheta \\
&\quad \times \left[\frac{1}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\, ds} \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\, t^2 dt - \left(\frac{1}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\, ds} \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\, t dt \right)^2 \right] \mathcal{D}. \qquad (4.8)
\end{aligned}
$$

Proof. If we multiply (2.20) by $\beta(\vartheta) \geq 0$, integrate, and use the symmetry of $\beta(\cdot)$, then we get

$$
\begin{aligned}
&\int_0^1 \vartheta(1-\vartheta)\beta(\vartheta)\, d\vartheta\, (t-s)^2\, C \\
&\leq \int_0^1 \vartheta\beta(\vartheta)\, d\vartheta \left(|S(s)|^2 + |S(t)|^2 \right) - \int_0^1 \beta(\vartheta)\, |S((1-\vartheta)s + \vartheta t)|^2\, d\vartheta \\
&\leq \int_0^1 \vartheta(1-\vartheta)\beta(\vartheta)\, d\vartheta\, (t-s)^2\, \mathcal{D},
\end{aligned}
$$

which by division with $\int_0^1 \beta(\vartheta)\, d\vartheta > 0$ gives (4.5).

Further, if we multiply (4.5) with $w(s)\, w(t) \geq 0$ and take the double integral on $[\mathfrak{a}_1, \mathfrak{a}_2]$, then we get

$$
\begin{aligned}
&\frac{1}{\int_0^1 \beta(\vartheta)\, d\vartheta} \int_0^1 \vartheta(1-\vartheta)\beta(\vartheta)\, d\vartheta \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\, w(t)\, (t-s)^2\, ds dt C \\
&\leq \frac{1}{\int_0^1 \beta(\vartheta)\, d\vartheta} \int_0^1 \vartheta\beta(\vartheta)\, d\vartheta \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\, w(t) \left(|S(s)|^2 + |S(t)|^2 \right) ds dt \\
&\quad - \frac{1}{\int_0^1 \beta(\vartheta)\, d\vartheta} \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\, w(t) \left(\int_0^1 \beta(\vartheta)\, |S((1-\vartheta)s + \vartheta t)|^2\, d\vartheta \right) ds dt \\
&\leq \frac{1}{\int_0^1 \beta(\vartheta)\, d\vartheta} \int_0^1 \vartheta(1-\vartheta)\beta(\vartheta)\, d\vartheta \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\, w(t)\, (t-s)^2\, ds dt \mathcal{D}. \qquad (4.9)
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\, w(t)\, (t-s)^2\, ds dt \\
&= \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\, w(t) \left(t^2 - 2ts + s^2 \right) ds dt \\
&= \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\, w(t)\, t^2 ds dt - 2 \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\, w(t)\, ts ds dt + \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\, w(t)\, s^2 ds dt \\
&= 2 \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\, ds \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\, t^2 dt - 2 \left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\, t dt \right)^2,
\end{aligned}
$$

then by (4.9) and dividing with $\left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\, dt \right)^2$, we get (4.6).

Further, by (2.21), we have, like in the proof of Corollary 2.7, that

$$\left(\frac{1}{2}-\vartheta\right)^2 (t-s)^2 C \le \frac{|S((1-\vartheta)s+\vartheta t)|^2 + |S((1-\vartheta)t+\vartheta s)|^2}{2} - \left|S\left(\frac{s+t}{2}\right)\right|^2$$
$$\le \left(\frac{1}{2}-\vartheta\right)^2 (t-s)^2 \mathcal{D},$$

for $s, t \in (\mathfrak{a}_1, \mathfrak{a}_2)$ and $\vartheta \in [0,1]$.

If we multiply by $\beta(\vartheta) \ge 0$ and integrate, then we get

$$\begin{aligned}
&\int_0^1 \left(\vartheta-\frac{1}{2}\right)^2 \beta(\vartheta)\, d\vartheta\, (t-s)^2 C \\
&\le \frac{1}{2}\int_0^1 \beta(\vartheta)\, |S((1-\vartheta)s+\vartheta t)|^2 d\vartheta + \frac{1}{2}\int_0^1 |S((1-\vartheta)t+\vartheta s)|^2 \beta(\vartheta)\, d\vartheta - \int_0^1 \beta(\vartheta)\, d\vartheta \left|S\left(\frac{s+t}{2}\right)\right|^2 \\
&\le \int_0^1 \left(\vartheta-\frac{1}{2}\right)^2 \beta(\vartheta)\, d\vartheta\, (t-s)^2 \mathcal{D}.
\end{aligned} \tag{4.10}$$

Since $\beta(1-\vartheta) = \beta(\vartheta)$, $\vartheta \in [0,1]$, then

$$\begin{aligned}
\int_0^1 \beta(\vartheta)\, |S((1-\vartheta)s+\vartheta t)|^2 d\vartheta &= \int_0^1 \beta(1-\vartheta)\, |S((1-\vartheta)s+\vartheta t)|^2 d\vartheta \\
&= \int_0^1 \beta(\lambda)\, |S(\lambda s+(1-\lambda)t)|^2 d\lambda
\end{aligned}$$

and by (4.10), we derive (4.7).

Finally, if we multiply (4.7) with $w(s)\, w(t) \ge 0$ and take the double integral on $[\mathfrak{a}_1, \mathfrak{a}_2]$, then we get (4.8). □

Remark 4.2. *If we take the symmetric function $\beta(\vartheta) = \vartheta(1-\vartheta)$, $t \in [0,1]$, then by (4.5), we get that*

$$\frac{1}{5}(t-s)^2 C \le \frac{1}{2}\left(|S(s)|^2 + |S(t)|^2\right) - 6\int_0^1 \vartheta(1-\vartheta)\, |S((1-\vartheta)s+\vartheta t)|^2 d\vartheta$$
$$\le \frac{1}{5}(t-s)^2 \mathcal{D}$$

and by (4.6),

$$\begin{aligned}
&\frac{2}{5}\left[\frac{1}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\, ds}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\, t^2 dt - \left(\frac{1}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\, ds}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\, t dt\right)^2\right] C \\
&\le \frac{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\, |S(t)|^2 dt}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\, dt} - \frac{6}{\left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\, dt\right)^2}\int_0^1 \vartheta(1-\vartheta)\left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\, w(t)\, |S((1-\vartheta)s+\vartheta t)|^2 ds dt\right) d\vartheta \\
&\le \frac{2}{5}\left[\frac{1}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\, ds}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\, t^2 dt - \left(\frac{1}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\, ds}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\, t dt\right)^2\right] C.
\end{aligned}$$

Also, if we take $\beta(\vartheta) = \left|\vartheta - \frac{1}{2}\right|$ in (4.7), then we get

$$\begin{aligned}\frac{1}{8}(t-s)^2 C &\le 4\int_0^1 \left|\vartheta - \frac{1}{2}\right| |S((1-\vartheta)s + \vartheta t)|^2 d\vartheta - \left|S\left(\frac{s+t}{2}\right)\right|^2 \\ &\le \frac{1}{8}(t-s)^2 \mathcal{D}\end{aligned}$$

and by (4.8), that

$$\begin{aligned}&\frac{1}{4}\left[\frac{1}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,ds}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,t^2 dt - \left(\frac{1}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,ds}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,t dt\right)^2\right] C \\ &\le \frac{4}{\left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,dt\right)^2}\int_0^1 \left|\vartheta - \frac{1}{2}\right| \left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,w(t)\,|S((1-\vartheta)s + \vartheta t)|^2 dsdt\right) d\vartheta \\ &\quad - \frac{1}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,ds}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,w(t)\left|S\left(\frac{s+t}{2}\right)\right|^2 dsdt \\ &\le \frac{1}{4}\left[\frac{1}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,ds}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,t^2 dt - \left(\frac{1}{\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(s)\,ds}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} w(t)\,t dt\right)^2\right] \mathcal{D}.\end{aligned}$$

5. Some examples

In this section, let $\mathcal{A}$ be a normal operator in $\mathcal{L}(\mathcal{H})$, i.e., $\mathcal{A}^*\mathcal{A} = \mathcal{A}\mathcal{A}^*$. Consider the function $\psi_{\mathcal{A}}(t) := \left|e^{t\mathcal{A}}\right|^2$, with $t \in \mathbb{R}$. Observe that, since $\mathcal{A}$ is normal, then

$$\psi_{\mathcal{A}}(t) := \left|e^{t\mathcal{A}}\right|^2 = \left(e^{t\mathcal{A}}\right)^* e^{t\mathcal{A}} = e^{t\mathcal{A}^*}e^{t\mathcal{A}} = e^{t(\mathcal{A}^*+\mathcal{A})} = e^{2tRe(\mathcal{A})},$$

for all $t \in \mathbb{R}$.

Observe that, by utilising the results from [1], where the strong differentiability is proved, we have that

$$\psi'_{\mathcal{A}}(t) = 2Re(\mathcal{A})\,e^{2t(\mathcal{A})},\ t \in \mathbb{R},$$

and

$$\psi''_{\mathcal{A}}(t) = 4\,[Re(\mathcal{A})]^2\,e^{2tRe(\mathcal{A})},\ t \in \mathbb{R}.$$

For a selfadjoint operator P, by utilising the continuous functional calculus for selfadjoint operators, we have that

$$P^2 e^{tP} \ge 0 \text{ for all } t \in \mathbb{R},$$

which shows that, for $P = 2\,[Re(\mathcal{A})]$,

$$\psi''_{\mathcal{A}}(t) = 4\,[Re(\mathcal{A})]^2\,e^{2tRe(\mathcal{A})},\ t \in \mathbb{R}.$$

We conclude that the function $S(t) := e^{t\mathcal{A}}$ is square modulus convex on any subinterval $[\mathfrak{a}_1, \mathfrak{a}_2]$ of $\mathbb{R}$.

If we use Proposition 2.4 for $S(t) := e^{t\mathcal{A}}$, then for $s, \sigma \in \mathbb{R}$ and $\vartheta \in [0,1]$, we have the following operator inequality:

$$\begin{aligned}
0 &\le 2\min\{\vartheta, 1-\vartheta\}\left[\frac{e^{2sRe(\mathcal{A})}+e^{2\sigma Re(\mathcal{A})}}{2} - e^{(s+\sigma)Re(\mathcal{A})}\right] \\
&\le (1-\vartheta)\, e^{2sRe(\mathcal{A})} + \vartheta e^{2\sigma Re(\mathcal{A})} - e^{2((1-\vartheta)s+\vartheta\sigma)Re(\mathcal{A})} \\
&\le 2\max\{\vartheta, 1-\vartheta\}\left[\frac{e^{2sRe(\mathcal{A})}+e^{2\sigma Re(\mathcal{A})}}{2} - e^{(s+\sigma)Re(\mathcal{A})}\right].
\end{aligned}$$

Also, from the inequality (2.14) for $S(t) := e^{t\mathcal{A}}$, we also get

$$\begin{aligned}
0 &\le (1-\vartheta)\, e^{2sRe(\mathcal{A})} + \vartheta e^{2\sigma Re(\mathcal{A})} - e^{2((1-\vartheta)s+\vartheta\sigma)Re(\mathcal{A})} \\
&\le 4\vartheta(1-\vartheta)(\sigma - s)\left[Re(\mathcal{A})\, e^{2\sigma Re(\mathcal{A})} - Re(\mathcal{A})\, e^{2sRe(\mathcal{A})}\right],
\end{aligned}$$

for $s, \sigma \in \mathbb{R}$ and $\vartheta \in [0,1]$. If we take in this inequality $\vartheta = 1/2$, then we obtain

$$\begin{aligned}
0 &\le \frac{e^{2sRe(\mathcal{A})}+e^{2\sigma Re(\mathcal{A})}}{2} - e^{(s+\sigma)Re(\mathcal{A})} \\
&\le (\sigma - s)\left[Re(\mathcal{A})\, e^{2\sigma(\mathcal{A})} - Re(\mathcal{A})\, e^{2sRe(\mathcal{A})}\right],
\end{aligned}$$

for $s, \sigma \in \mathbb{R}$.

Also, if $0 < m \le Re(\mathcal{A}) \le M$, then for $t \in [\mathfrak{a}_1, \mathfrak{a}_2] \subset (0, \infty)$,

$$4m^2 e^{2m\mathfrak{a}_1} \le 4m^2 e^{2mt} \le \psi''_{\mathcal{A}}(t) = 4\left[Re(\mathcal{A})\right]^2 e^{2tRe(\mathcal{A})} \le 4M^2 e^{2tM} \le 4M^2 e^{2\mathfrak{a}_2 M}.$$

If we make use of the inequality (2.20), then we get

$$\begin{aligned}
&(0 \le)\, 4m^2 e^{2m\mathfrak{a}_1}\vartheta(1-\vartheta)(\sigma - s)^2 \\
&\le (1-\vartheta)\, e^{2sRe(\mathcal{A})} + \vartheta e^{2\sigma Re(\mathcal{A})} - e^{2((1-\vartheta)s+\vartheta\sigma)Re(\mathcal{A})} \\
&\le 4M^2 e^{2\mathfrak{a}_2 M}\vartheta(1-\vartheta)(\sigma - s),
\end{aligned}$$

for $s, \sigma \in [\mathfrak{a}_1, \mathfrak{a}_2]$. In particular, we have

$$\begin{aligned}
(0 \le)\, m^2 e^{2m\mathfrak{a}_1}(\sigma - s)^2 &\le \frac{e^{2sRe(\mathcal{A})}+e^{2\sigma Re(\mathcal{A})}}{2} - e^{(s+\sigma)Re(\mathcal{A})} \\
&\le M^2 e^{2\mathfrak{a}_2 M}(\sigma - s)^2.
\end{aligned}$$

If G is an invertible operator in $\mathcal{L}(\mathcal{H})$, then it is known, see, for instance, [1], that

$$\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \exp(tG)\, dt = G^{-1}\left[\exp(\mathfrak{a}_2 G) - \exp(\mathfrak{a}_1 G)\right] \tag{5.1}$$

for any $\mathfrak{a}_1 < \mathfrak{a}_2$.

This implies that for invertible $Re(\mathcal{A})$, we have

$$\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \exp(2tRe(\mathcal{A}))\, dt = \frac{1}{2}(Re(\mathcal{A}))^{-1}\left[\exp(2\mathfrak{a}_2 Re(\mathcal{A})) - \exp(2\mathfrak{a}_1 Re(\mathcal{A}))\right].$$

By Corollary 2.2, we then obtain

$$\begin{aligned}
0 &\leq \frac{2}{3}(\mathfrak{a}_2-\mathfrak{a}_1)^2 m^2 e^{2m\mathfrak{a}_1} \\
&\leq \frac{e^{2\mathfrak{a}_1 Re(\mathcal{A})}+e^{2\mathfrak{a}_2 Re(\mathcal{A})}}{2} - \frac{1}{2}(Re(\mathcal{A}))^{-1}\left[\frac{\exp(2\mathfrak{a}_2 Re(\mathcal{A})) - \exp(2\mathfrak{a}_1 Re(\mathcal{A}))}{\mathfrak{a}_2-\mathfrak{a}_1}\right] \\
&\leq \frac{2}{3}(\mathfrak{a}_2-\mathfrak{a}_1)^2 M^2 e^{2\mathfrak{a}_2 M}
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq \frac{1}{3}(\mathfrak{a}_2-\mathfrak{a}_1)^2 m^2 e^{2m\mathfrak{a}_1} \\
&\leq \frac{1}{2}(Re(\mathcal{A}))^{-1}\left[\frac{\exp(2\mathfrak{a}_2 Re(\mathcal{A})) - \exp(2\mathfrak{a}_1 Re(\mathcal{A}))}{\mathfrak{a}_2-\mathfrak{a}_1}\right] - e^{(\mathfrak{a}_2+\mathfrak{a}_1)Re(\mathcal{A})} \\
&\leq \frac{1}{3}(\mathfrak{a}_2-\mathfrak{a}_1)^2 M^2 e^{2\mathfrak{a}_2 M}.
\end{aligned}$$

Moreover, we have for $\vartheta \in (0,1)$ that

$$\begin{aligned}
&\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \exp(2((1-\vartheta)s+\vartheta\sigma)Re(\mathcal{A}))\,dsd\sigma \\
&= \int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \exp(2(1-\vartheta)sRe(\mathcal{A}))\exp(2\vartheta\sigma Re(\mathcal{A}))\,dsd\sigma \\
&= \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \exp(2(1-\vartheta)sRe(\mathcal{A}))\,ds \int_{a}^{\mathfrak{a}_2} \exp(2\vartheta\sigma Re(\mathcal{A}))\,d\sigma \\
&= \frac{1}{2(1-\vartheta)}(Re(\mathcal{A}))^{-1}\left[\exp(2\mathfrak{a}_2(1-\vartheta)Re(\mathcal{A})) - \exp(2\mathfrak{a}_1(1-\vartheta)Re(\mathcal{A}))\right] \\
&\quad\times \frac{1}{2\vartheta}(Re(\mathcal{A}))^{-1}\left[\exp(2\mathfrak{a}_2\vartheta Re(\mathcal{A})) - \exp(2\mathfrak{a}_1\vartheta(\mathcal{A}))\right]
\end{aligned}$$

and, for $\vartheta = 1/2$,

$$\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \exp((s+\sigma)Re(\mathcal{A}))\,dsd\sigma = (Re(\mathcal{A}))^{-2}\left[\exp(\mathfrak{a}_2 Re(\mathcal{A})) - \exp(\mathfrak{a}_1 Re(\mathcal{A}))\right]^2.$$

If $0 < m \leq Re(\mathcal{A}) \leq M$, for $t \in [\mathfrak{a}_1, \mathfrak{a}_2] \subset (0,\infty)$, then by Theorem 3.3, we get

$$\begin{aligned}
0 &\leq \frac{2}{3}\vartheta(1-\vartheta)(\mathfrak{a}_2-\mathfrak{a}_1)^2 m^2 e^{2m\mathfrak{a}_1} \\
&\leq \frac{1}{2}(Re(\mathcal{A}))^{-1}\left[\frac{\exp(2\mathfrak{a}_2 Re(\mathcal{A})) - \exp(2\mathfrak{a}_1 Re(\mathcal{A}))}{\mathfrak{a}_2-\mathfrak{a}_1}\right] \\
&\quad - \frac{1}{2(1-\vartheta)}(Re(\mathcal{A}))^{-1}\left[\frac{\exp(2\mathfrak{a}_2(1-\vartheta)Re(\mathcal{A})) - \exp(2\mathfrak{a}_1(1-\vartheta)Re(\mathcal{A}))}{\mathfrak{a}_2-\mathfrak{a}_1}\right] \\
&\quad \times \frac{1}{2\vartheta}(Re(\mathcal{A}))^{-1}\left[\frac{\exp(2\mathfrak{a}_2\vartheta Re(\mathcal{A})) - \exp(2\mathfrak{a}_1\vartheta Re(\mathcal{A}))}{\mathfrak{a}_2-\mathfrak{a}_1}\right]
\end{aligned}$$

$$\leq \frac{2}{3}\vartheta(1-\vartheta)(\mathfrak{a}_2-\mathfrak{a}_1)^2 M^2 e^{2\mathfrak{a}_2 M}.$$

In particular, for $\vartheta = 1/2$, we get

$$\begin{aligned}
0 &\leq \frac{1}{6}(\mathfrak{a}_2-\mathfrak{a}_1)^2 m^2 e^{2m\mathfrak{a}_1} \\
&\leq \frac{1}{2}(Re(\mathcal{A}))^{-1}\left[\frac{\exp(2\mathfrak{a}_2 Re(\mathcal{A}))-\exp(2\mathfrak{a}_1 Re(\mathcal{A}))}{\mathfrak{a}_2-\mathfrak{a}_1}\right] \\
&\quad -(Re(\mathcal{A}))^{-2}\left[\frac{\exp(\mathfrak{a}_2 Re(\mathcal{A}))-\exp(\mathfrak{a}_1 Re(\mathcal{A}))}{\mathfrak{a}_2-\mathfrak{a}_1}\right]^2 \\
&\leq \frac{1}{6}(\mathfrak{a}_2-\mathfrak{a}_1)^2 M^2 e^{2\mathfrak{a}_2 M}.
\end{aligned}$$

For distinct operators $\mathcal{A}, \mathcal{B} \in \mathcal{L}(\mathcal{H})$, we consider the function $S(t) = (1-t)\mathcal{A} + t\mathcal{B}$ and $\psi_{\mathcal{A},\mathcal{B}} : [0,1] \to \mathcal{L}(\mathcal{H})$ defined by $\psi_{\mathcal{A},\mathcal{B}}(t) = |S(t)|^2 = |(1-t)\mathcal{A} + t\mathcal{B}|^2$. Observe that

$$\begin{aligned}
\psi_{\mathcal{A},\mathcal{B}}(t) &= (1-t)|\mathcal{A}|^2 + t(1-t)(\mathcal{A}^*\mathcal{B} + \mathcal{B}^*\mathcal{A}) + t^2|\mathcal{B}|^2, \\
\psi'_{\mathcal{A},\mathcal{B}}(t) &= -2(1-t)|\mathcal{A}|^2 + (1-2t)(\mathcal{A}^*\mathcal{B} + \mathcal{B}^*\mathcal{A}) + 2t|\mathcal{B}|^2 \\
&= 2t\left[|\mathcal{A}|^2 - (\mathcal{A}^*\mathcal{B} + \mathcal{B}^*\mathcal{A}) + |\mathcal{B}|^2\right] + \mathcal{A}^*\mathcal{B} + \mathcal{B}^*\mathcal{A} - 2|\mathcal{A}|^2 \\
&= 2t|\mathcal{A}-\mathcal{B}|^2 + \mathcal{A}^*\mathcal{B} + \mathcal{B}^*\mathcal{A} - |\mathcal{A}|^2 - |\mathcal{B}|^2 + |\mathcal{B}|^2 - |\mathcal{A}|^2 \\
&= 2t|\mathcal{A}-\mathcal{B}|^2 - |\mathcal{A}-\mathcal{B}|^2 + |\mathcal{B}|^2 - |\mathcal{A}|^2 \\
&= (2t-1)|\mathcal{A}-\mathcal{B}|^2 + |\mathcal{B}|^2 - |\mathcal{A}|^2
\end{aligned}$$

and

$$\psi''_{\mathcal{A},\mathcal{B}}(t) = 2|\mathcal{A}-\mathcal{B}|^2,$$

for all $t \in [0,1]$.

Assume that there exist $m, M > 0$ such that $0 < m \leq |\mathcal{A}-\mathcal{B}| \leq M$ in the operator order. Then $2m^2 \leq \psi''_{\mathcal{A},\mathcal{B}}(t) \leq 2M^2$. By using Theorem 4.2 for $s = 0$ and $t = 1$, we get from (4.5) that

$$\begin{aligned}
&\frac{2m^2}{\int_0^1 \beta(\vartheta)\,d\vartheta}\int_0^1 \vartheta(1-\vartheta)\beta(\vartheta)\,d\vartheta \\
&\leq \frac{1}{\int_0^1 \beta(\vartheta)\,d\vartheta}\int_0^1 \vartheta\beta(\vartheta)\,d\vartheta\left(|\mathcal{A}|^2 + |\mathcal{B}|^2\right) - \frac{1}{\int_0^1 \beta(\vartheta)\,d\vartheta}\int_0^1 \beta(\vartheta)|(1-\vartheta)\mathcal{A} + \vartheta\mathcal{B}|^2\,d\vartheta \\
&\leq \frac{2M^2}{\int_0^1 \beta(\vartheta)\,d\vartheta}\int_0^1 \vartheta(1-\vartheta)\beta(\vartheta)\,d\vartheta
\end{aligned}$$

and from (4.7) that

$$\frac{2m^2}{\int_0^1 \beta(\vartheta)\,d\vartheta}\int_0^1 \left(\vartheta - \frac{1}{2}\right)^2 \beta(\vartheta)\,d\vartheta$$

$$\leq \frac{1}{\int_0^1 \beta(\vartheta)\, d\vartheta} \int_0^1 \beta(\vartheta) \left|(1-\vartheta)\mathcal{A} + \vartheta\mathcal{B}\right|^2 d\vartheta - \left|\frac{\mathcal{A}+\mathcal{B}}{2}\right|^2$$
$$\leq \frac{2M^2}{\int_0^1 \beta(\vartheta)\, d\vartheta} \int_0^1 \left(\vartheta - \frac{1}{2}\right)^2 \beta(\vartheta)\, d\vartheta.$$

6. Conclusions

In this paper, we established some Hermite–Hadamard-type inequalities for operator square modulus convex functions on an interval of real numbers. Some integral inequalities for symmetric weights are obtained. Applications for exponential functions and modulus quadratic functions were provided as well.

Author contributions

Silvestru Sever Dragomir, Ghada N. AlNemer, and Ahad Hamoud AlOtaibi: Investigation, writing – original draft, writing – review & editing. The authors declare that they have contributed equally to this paper. All authors have read and approved the final version of the manuscript for publication.

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The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Professor Silvestru Sever Dragomir is the Guest Editor for AIMS Mathematics and was not involved in the editorial review and the decision to publish this article. The authors declare that they have no competing interests.

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