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Research article

Extensions of entropy-based concomitant models for ordered systems with applications to radiation monitoring data in the Kingdom of Saudi Arabia

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Abstract: In this study, a generating informational function is introduced for concomitants of p-generalized order statistics, providing a unified framework that incorporates several classical entropy-type measures as special cases. The proposed formulation extends existing entropy-based models by combining concomitant structures with generalized order statistics under the Huang–Kotz extension. Since closed-form expressions are generally difficult to obtain, a tractable representation based on the uniform distribution is developed and subsequently applied to several lifetime distributions through suitable transformations. New stochastic ordering properties of the proposed generating informational function are established, and corresponding characterization results are derived for the uniform and exponential distributions. In addition, explicit analytical bounds are obtained to approximate the proposed measure in situations where exact evaluation is analytically intractable. From an inferential perspective, two non-parametric estimators for the generating informational function are proposed, and their consistency properties are investigated. A corrected estimator is further introduced to improve estimation efficiency. The asymptotic behavior of the estimators is numerically illustrated through histogram-based analysis. Numerical studies based on Monte Carlo simulations, bootstrap convergence analysis, and real lifetime datasets demonstrate the effectiveness, stability, and practical applicability of the proposed estimation procedures.

Keywords: generating informational function; generalized order statistics; concomitants; stochastic ordering; non-parametric estimation
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1. Introduction

Information theory provides a fundamental framework for quantifying uncertainty and randomness in probabilistic systems. Since the pioneering work of Claude E. Shannon (Shannon [1]), numerous measures of information have been introduced to characterize different aspects of uncertainty associated with random variables and probability distributions. Among these measures, Shannon entropy has played a central role in statistics, reliability theory, signal processing, and machine learning. Over the years, several alternative and generalized measures of information have been proposed in order to capture different structural properties of distributions, including cumulative forms, residual measures, and fractional generalizations. By establishing the differentiating entropy Z of a randomized continuous variable, the literature created the foundations of theoretical information analysis.

In recent years, considerable attention has been directed toward information measures that act as complements to the classical entropy function. In this context, Lad et al. [2] introduced the concept of the dual counterpart of Shannon entropy, known as extropy. By employing a Maclaurin series representation, the extropy associated with a non–negative random variable Z, having probability density function (PDF) $h_Z(z)$, can be expressed as

$$Ex(Z) = -\frac{1}{2}\int_0^{\infty} h_Z^2(z)\, dz.$$

Within probability and distribution theory, generating functions play a central analytical role. For instance, the statistical moments of a random variable may be obtained through repeated differentiation of its moment generating function evaluated at zero. In information theory, a related concept known as the generating informational function is used to construct different measures that quantify information. This function, originally proposed by Golomb [3], is defined for a non–negative random variable with PDF h_Z as

$$GI_{\xi}(Z) = \int_0^{\infty} h_Z^{\xi}(z)\, dz, \tag{1.1}$$

where $\xi > 0$ and $GI_1(Z) = 1$. In addition,

$$\frac{\partial}{\partial\xi} GI_{\xi}(Z)|_{\xi=1} = \int_0^{\infty} h_Z(z)\ln h_Z(z)\, dz = -ESn(Z), \tag{1.2}$$

where $ESn(Z)$ denotes the Shannon entropy; see [1]. Owing to the importance of the generating informational framework in the development of information measures, this function has recently attracted the attention of several researchers. For example, Kharazmi and Balakrishnan [4] proposed two important divergence measures and demonstrated that many well–known divergence metrics arise as particular cases. Furthermore, the generating informational function has also been examined in the cases of order statistics and record values by Kharazmi and Balakrishnan [5] and Zamani et al. [6].

Motivated by the significance of the term $h_Z^{\xi}(z)$ within information–theoretic analysis, Mohamed and Sakr [7] recently proposed a new measure called the generalized fractional extropy, which is defined as

$$GEx_{\xi}(Z) = \frac{1}{\Gamma(\xi+1)}\int_0^{\infty} h_Z^{\xi}(z)\, dz, \tag{1.3}$$

where $\xi \geq 1$. In their study, Mohamed and Sakr [7] also clarified the omission of the negative sign in this formulation. They argued that removing this sign leads to a more appropriate interpretation of the complementary fractional generalized entropy introduced earlier by Mohamed and Almuqrin [8].

The Farlie–Gumbel–Morgenstern, also referred to as the Morgenstern family, represents a well-known class of bivariate probability distributions. This family was first introduced by Morgenstern [9] in the context of distributions with Cauchy marginals. Such models have proven useful in constructing multivariate distributions when the marginal distributions are specified in advance.

Johnson and Kotz [10] investigated the multivariate extension of this family and presented a comprehensive study of its probabilistic and statistical properties. Later, Huang and Kotz [11] proposed an extension that incorporates an additional parameter in order to strengthen the dependence structure between the component variables.

Over time, several researchers have proposed further generalizations of the Morgenstern family of bivariate distributions. In particular, Huang and Kotz [12] introduced a polynomial-type extension involving a single additional parameter. The cumulative distribution function (CDF) and the corresponding PDF of this model are given, respectively, as follows:

$$H_{Y,Z}(y,z) = H_Y(y)H_Z(z)[1 + \omega(1 - H_Y^{\beta}(y))(1 - H_Z^{\beta}(z))], \tag{1.4}$$

$$h_{Y,Z}(y,z) = h_Y(y)h_Z(z)[1 + \omega((1+\beta)H_Y^{\beta}(y) - 1)((1+\beta)H_Z^{\beta}(z) - 1)], \beta \geq 1, \tag{1.5}$$

with noting that the shape-parameter vector (ω,β)'s allowable range is $\Omega = \{(\omega,\beta) : \frac{-1}{\beta^2} \leq \omega \leq \frac{1}{\beta}, \beta \geq 1\}$, and the correlation coefficient (Corr.) range is $\frac{-3\min(1,\beta^2)}{(\beta+2)^2} \leq Corr. \leq \frac{3\beta}{(\beta+2)^2}$.

Originally, David et al. [13] studied concomitants of order statistics. From some bivariate population with CDF $H_{Y,Z}(y,z)$, let (Y_i, Z_i), $i = 1, 2, ..., m$, be m pairs of independent randomly considered variables. Let $Y_{(l:m)}$ be the lth order statistics, then Z associated with $Y_{(l:m)}$ is called the concomitant of lth order statistics and is denoted by $Z_{[l:m]}$. The PDF and CDF of $Z_{[l:m]}$ are given by:

$$h_{Z,[l:m]}(z) = \int_{-\infty}^{\infty} h_{Z|Y}(z \mid y)\, h_{Y,(l:m)}(y)\, dy, \tag{1.6}$$

$$H_{Z,[l:m]}(y) = \int_{-\infty}^{\infty} H_{Z|Y}(z \mid y)\, h_{Y,(l:m)}(y)\, dy, \tag{1.7}$$

where $h_{Y,(l:m)}(y)$ is the PDF of $Y_{(l:m)}$.

Kamps [14] introduced the framework of generalized order statistics as a unified model for ordered random variables. This formulation includes several well-known models—such as classical order statistics, sequential order statistics, record values, Pfeifer's records, and progressively type-II censored order statistics—as particular cases. Let $m \in \mathbb{N}$, $r \geq 1$, and $p_1, \ldots, p_{m-1} \in \mathbb{R}$. Define

$$P_s = \sum_{j=s}^{m-1} p_j, \qquad 1 \leq s \leq m-1,$$

and assume that the parameters satisfy

$$\alpha_s = r + m - s + P_s \geq 1, \qquad s = 1, 2, \ldots, m-1.$$

Furthermore, denote $\tilde{p} = (p_1, \ldots, p_{m-1}) \in \mathbb{R}^{m-1}$. A number of important classes of ordered random variables arise as special members of the p-generalized order statistics subclass. In particular, when $p_1 = p_2 = \cdots = p_{m-1} = p$, the PDF of the lth p-generalized order statistics, denoted by $Z_{(l,m,p,r)}$, can be expressed as follows (see Kamps [14]):

$$h_{Z,(l,m,p,r)}(z) = \frac{c_{l-1}}{(l-1)!}(1 - H_Z(z))^{\alpha_l - 1} h_Z(z) g_p^{l-1}(H_Z(z)), \tag{1.8}$$

where $1 \leq l \leq m$, $\alpha_i = r + (m-l)(p+1)$, $c_{l-1} = \prod_{j=1}^{l} \alpha_j$, $g_p(z) = h_{Z,p}(z) - h_{Z,p}(0)$, $0 < z < 1$, and

$$h_{Z,p}(z) = \begin{cases} \frac{-(1-z)^{p+1}}{p+1}, & p \neq -1, \\ -\ln(1-z), & p = -1. \end{cases}$$

Throughout this paper, we assume that $\beta \geq 1$, where $\beta \in \mathbb{N}$. Building on the extension proposed by Huang and Kotz, Mohamed [15] derived the PDF and the CDF of the concomitant corresponding to the lth p-generalized order statistic. These functions are given as follows:

$$h_{Z,[l,m,p,r]}(z) = h_Z(z)\left[1 + \omega\Theta_{[l,m,p,r;\beta]}\left(1 - (1+\beta)H_Z^{\beta}(z)\right)\right], \tag{1.9}$$

$$H_{Z,[l,m,p,r]}(z) = H_Z(z)\left[1 + \omega\Theta_{[l,m,p,r;\beta]}\left(1 - H_Z^{\beta}(z)\right)\right], \tag{1.10}$$

with considering that

$$\Theta_{[l,m,p,r;\beta]} = 1 - (1+\beta)\sum_{j=0}^{\beta}\binom{\beta}{j}(-1)^j \prod_{i=1}^{l} \frac{\alpha_i}{\alpha_i + j}, \tag{1.11}$$

and $\alpha_i = r + (m-l)(p+1)$, $1 \leq l \leq m$.

Despite the extensive literature on entropy measures, generalized order statistics, and concomitant models, limited attention has been devoted to the generating informational function for concomitants of p-generalized order statistics. Existing studies on generalized fractional extropy and generating informational functions primarily focus on ordinary random variables, ordered statistics, or coherent systems, without accounting for the dependence structure induced by concomitants under the Huang–Kotz extension. Consequently, the analytical properties established in those works cannot be directly transferred to the present setting. In particular, the interplay among stochastic ordering properties, characterization results, analytical bounds, and non-parametric estimation procedures for the generating informational function of concomitants has not been systematically investigated. The present work fills this gap under the Huang–Kotz framework, subject to the constraints $\xi \geq 1, \beta \geq 1$, and $\omega \neq 0$. The main contributions of this paper can be summarized as follows:

(1) Introducing a generating informational function for concomitants of p-generalized order statistics;

(2) Deriving different analytical expansions and stochastic ordering properties;

(3) Establishing new characterization results and analytical bounds under several distributions;

(4) Proposing non-parametric estimation procedures together with numerical and real-data investigations.

Studying the generating informational function of ordered random variables exists in many references, including Kharazmi and Balakrishnan [16] and Mohamed [17]. Therefore, we aim to discuss the generating informational function of the concomitants of p-generalized order statistics under the Huang–Kotz extension. The rest of this article is organized as follows. In Section 2, different expansions of the generating informational function of the concomitant of p-generalized order statistics are developed, together with applications to some lifetime distributions. In addition, stochastic ordering comparisons and related characterization results are presented. Section 3 contains new analytical bounds and characterization results for the proposed generating informational function. In Section 4, non-parametric estimation procedures are introduced, and their consistency properties together with numerical illustrations are discussed through simulation and real-data applications. Finally, concluding remarks and possible future directions are presented in Section 5.

The notations used in the article are summarized in Table 1.

Table 1. List of main notations used in the paper.

Symbol	Meaning
$Y_{(l:m)}$	lth order statistics
$Z_{[l:m]}$	Concomitant of lth order statistics
$Z_{[l,m,p,r]}$	Concomitant of lth p-generalized order statistics
$h_{U,[l,m,p,r]}(u)$	Uniformly distributed PDF of the concomitant of lth p-generalized order statistics throughout the interval $(0, 1)$
$GI_\xi(Z_{[l,m,p,r]})$	Generating informational function of $Z_{[l,m,p,r]}$
$\leq^{hr}$	Hazard rate order
$\leq^{disp}$	Dispersive order
$\leq^{lir}$	Location-independent riskier order
$\mathfrak{M} = h_Z(.) < \infty$	The mode of the density function h_Z
g	Window size
$\xrightarrow{p}$	Convergence in probability
$SERM$	square error root mean

2. Generating informational function of the concomitant of p-generalized order statistics

In the present section, we explore the characteristics of the generating informational function of the concomitant of p-generalized order statistics. If $Z_{[l,m,p,r]}$ is the concomitant of lth p-generalized order statistics with the PDF defined in (1.9), then, the generating informational function of $Z_{[l,m,p,r]}$ can be considered as

$$
\begin{aligned}
GI_{\xi}(Z_{[l,m,p,r]}) &= \int_0^{\infty}\left(h_{Z,[l,m,p,r]}(z)\right)^{\xi} dz \\
&= \int_0^{\infty}\left(h_Z(z)\left[1+\omega\Theta_{[l,m,p,r;\beta]}\left(1-(1+\beta)H_Z^{\beta}(z)\right)\right]\right)^{\xi} dz \qquad (2.1)\\
&= \int_0^{\infty}\left(\left(1+\omega\Theta_{[l,m,p,r;\beta]}\right)h_{Z,1:1}(z)-\omega\Theta_{[l,m,p,r;\beta]}h_{Z,\beta+1:\beta+1}(z)\right)^{\xi} dz,
\end{aligned}
$$

where $h_{Z,m:m}(z)$ is the PDF of the *mth* order statistic of a random sample of size m of the Z variate (the PDF of the parallel system with lifetime $Z_{m:m} = \{Z_1, \ldots, Z_m\}$).

In the following theorems, we can discuss the expansion of the generating informational function of the concomitant of *lth* p-generalized order statistics according to (2.1).

Theorem 2.1. *Let $Z_{[l,m,p,r]}$ denote the concomitant of the lth p-generalized order statistic with the PDF defined in* (1.9). *Assume that $\xi > 0$,*

$$GI_{\xi}(Z) = \int_0^{\infty}(h_Z(z))^{\xi}\, dz < \infty,$$

and, for the generalized binomial expansion to be valid,

$$\left|\omega\Theta_{[l,m,p,r;\beta]}\left(1-(1+\beta)H_Z^{\beta}(z)\right)\right| < 1, \qquad z > 0.$$

Then the generating informational function of $Z_{[l,m,p,r]}$ admits the representation

$$GI_{\xi}(Z_{[l,m,p,r]}) = \sum_{k=0}^{\infty}\binom{\xi}{k}\left(\omega\Theta_{[l,m,p,r;\beta]}\right)^{k}\int_0^{\infty}(h_Z(z))^{\xi}\left(1-(1+\beta)H_Z^{\beta}(z)\right)^{k} dz. \tag{2.2}$$

Furthermore,

$$GI_{\xi}(Z_{[l,m,p,r]}) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{\xi}{k}\binom{k}{j}(-1)^{j}(1+\beta)^{j}\left(\omega\Theta_{[l,m,p,r;\beta]}\right)^{k}\int_0^{\infty}(h_Z(z))^{\xi}\, H_Z^{\beta j}(z)\, dz. \tag{2.3}$$

Proof. From (2.1),

$$GI_{\xi}(Z_{[l,m,p,r]}) = \int_0^{\infty}(h_Z(z))^{\xi}\left(1+\omega\Theta_{[l,m,p,r;\beta]}\left(1-(1+\beta)H_Z^{\beta}(z)\right)\right)^{\xi} dz.$$

Since

$$\left|\omega\Theta_{[l,m,p,r;\beta]}\left(1-(1+\beta)H_Z^{\beta}(z)\right)\right| < 1,$$

the generalized binomial theorem gives

$$\left(1+\omega\Theta_{[l,m,p,r;\beta]}\left(1-(1+\beta)H_Z^{\beta}(z)\right)\right)^{\xi} = \sum_{k=0}^{\infty}\binom{\xi}{k}\left(\omega\Theta_{[l,m,p,r;\beta]}\right)^{k}\left(1-(1+\beta)H_Z^{\beta}(z)\right)^{k}.$$

Since the above series converges absolutely and

$$\int_0^\infty (h_Z(z))^\xi \; dz < \infty,$$

the interchange between summation and integration is justified by the dominated convergence theorem. Hence,

$$GI_\xi(Z_{[l,m,p,r]}) = \sum_{k=0}^{\infty} \binom{\xi}{k} \left(\omega\Theta_{[l,m,p,r;\beta]}\right)^k \int_0^\infty (h_Z(z))^\xi \left(1 - (1+\beta)H_Z^\beta(z)\right)^k dz,$$

which proves (2.2). Finally, applying the ordinary binomial theorem,

$$\left(1 - (1+\beta)H_Z^\beta(z)\right)^k = \sum_{j=0}^{k} \binom{k}{j} (-1)^j (1+\beta)^j H_Z^{\beta j}(z),$$

and substituting this expression into (2.2) yields (2.3). This completes the proof. □

Theorem 2.2. *Let $Z_{[l,m,p,r]}$ denote the concomitant of the lth p-generalized order statistic with the PDF defined in* (1.9)*. Assume that $\xi > 0$,*

$$\int_0^\infty (h_Z(z))^\xi \; dz < \infty,$$

and, for the generalized binomial expansion to be valid,

$$\left| \frac{\omega\Theta_{[l,m,p,r;\beta]}\, h_{Z,\beta+1:\beta+1}(z)}{(1+\omega\Theta_{[l,m,p,r;\beta]})\, h_{Z,1:1}(z)} \right| < 1.$$

Then the generating informational function defined in (2.1) *can be expressed as*

$$GI_\xi(Z_{[l,m,p,r]}) = \sum_{k=0}^{\infty} \binom{\xi}{k} (1+\omega\Theta_{[l,m,p,r;\beta]})^{\xi-k} (-\omega\Theta_{[l,m,p,r;\beta]})^k \int_0^\infty h_{Z,1:1}^{\xi-k}(z)\, h_{Z,\beta+1:\beta+1}^{k}(z)\, dz. \tag{2.4}$$

Proof. From the last expression in (2.1),

$$GI_\xi(Z_{[l,m,p,r]}) = \int_0^\infty \left((1+\omega\Theta_{[l,m,p,r;\beta]})h_{Z,1:1}(z) - \omega\Theta_{[l,m,p,r;\beta]}h_{Z,\beta+1:\beta+1}(z)\right)^\xi dz.$$

Let

$$A = (1+\omega\Theta_{[l,m,p,r;\beta]})h_{Z,1:1}(z), \qquad B = \omega\Theta_{[l,m,p,r;\beta]}h_{Z,\beta+1:\beta+1}(z).$$

Then

$$GI_\xi(Z_{[l,m,p,r]}) = \int_0^\infty (A - B)^\xi \, dz.$$

Since

$$\left|\frac{B}{A}\right| = \left| \frac{\omega\Theta_{[l,m,p,r;\beta]}\, h_{Z,\beta+1:\beta+1}(z)}{(1+\omega\Theta_{[l,m,p,r;\beta]})\, h_{Z,1:1}(z)} \right| < 1,$$

the generalized binomial theorem gives

$$(A-B)^{\xi}=\sum_{k=0}^{\infty}\binom{\xi}{k}A^{\xi-k}(-B)^{k}.$$

Hence,

$$(A-B)^{\xi}=\sum_{k=0}^{\infty}\binom{\xi}{k}(1+\omega\Theta_{[l,m,p,r;\beta]})^{\xi-k}(-\omega\Theta_{[l,m,p,r;\beta]})^{k}h_{Z,1:1}^{\xi-k}(z)h_{Z,\beta+1:\beta+1}^{k}(z).$$

Since the above series converges absolutely and

$$\int_{0}^{\infty}(h_Z(z))^{\xi}\,dz<\infty,$$

the interchange between summation and integration is justified by the dominated convergence theorem. Therefore,

$$GI_{\xi}(Z_{[l,m,p,r]})=\sum_{k=0}^{\infty}\binom{\xi}{k}(1+\omega\Theta_{[l,m,p,r;\beta]})^{\xi-k}(-\omega\Theta_{[l,m,p,r;\beta]})^{k}\int_{0}^{\infty}h_{Z,1:1}^{\xi-k}(z)\,h_{Z,\beta+1:\beta+1}^{k}(z)\,dz,$$

which proves (2.4). □

Remark 2.1. *Under the special cases of the p-generalized order statistics, we can see the following:*

(1) If $p=0$ and $r=1$, the p-generalized order statistics turn into order statistics. Then, based on the concomitants of the lth order statistics $Z_{[l:m]}$, Eq (1.11) turns into

$$\begin{aligned}\Theta_{[l,m;\beta]}&=1-(1+\beta)\sum_{j=0}^{\beta}\binom{\beta}{j}(-1)^{j}\left(\prod_{i=1}^{l}\frac{m-i+1}{m-i+1+j}\right)\\&=1-\frac{(1+\beta)Beta(l+\beta,n-l+1)}{Beta(l,n-l+1)},\end{aligned}\tag{2.5}$$

with noting that $Beta(a,b)$ denotes the beta function, defined by $Beta(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, $a,b>0$.

(2) If $p=-1$, and $r=1$, the p-generalized order statistics turn into record values. Then, based on the concomitants of the record value $Z_{[l]}$, Eq (1.11) turns into

$$\Theta_{[l;\beta]}=1-(1+\beta)\sum_{j=0}^{\beta}\binom{\beta}{j}(-1)^{j}\frac{1}{(j+1)^{l}}.\tag{2.6}$$

The following result provides an alternative representation of the generating informational function of the concomitant of the lth p-generalized order statistic through the probability integral transform.

Theorem 2.3. *Let Z be a continuous random variable with CDF H_Z and PDF h_Z. Assume that*

$$\int_{0}^{\infty}h_{Z}^{\xi}(z)\,dz<\infty,$$

for some $\xi \geq 1$*. Define the transformed random variable* $U = H_Z(Z)$*. Then, by the probability integral transform,* $U \sim \text{Uniform}(0,1)$*. Moreover, under the Huang–Kotz family, the density of the concomitant of the lth p-generalized order statistic is*

$$h_{U,[l,m,p,r]}(u) = 1 + \omega\Theta_{[l,m,p,r;\beta]}\left(1-(1+\beta)u^{\beta}\right), \qquad 0<u<1,$$

where $\beta \geq 1$*, and* $\omega \neq 0$*. Consequently,*

$$GI_{\xi}(Z_{[l,m,p,r]}) = \int_0^1 h^{\xi}_{U,[l,m,p,r]}(u)\, h_Z^{\xi-1}\left(H_Z^{-1}(u)\right)\, du. \tag{2.7}$$

Proof. Since H_Z is continuous, the probability integral transform implies that

$$U = H_Z(Z) \sim \text{Uniform}(0,1).$$

Under the Huang–Kotz model, the density of the concomitant corresponding to U is

$$h_{U,[l,m,p,r]}(u) = 1 + \omega\Theta_{[l,m,p,r;\beta]}\left(1-(1+\beta)u^{\beta}\right), \qquad 0<u<1. \tag{2.8}$$

From Eq (1.9),

$$h_{Z,[l,m,p,r]}(z) = h_Z(z)\left[1+\omega\Theta_{[l,m,p,r;\beta]}\left(1-(1+\beta)H_Z^{\beta}(z)\right)\right].$$

Hence,

$$GI_{\xi}(Z_{[l,m,p,r]}) = \int_0^{\infty} h^{\xi}_{Z,[l,m,p,r]}(z)\, dz.$$

Now let $u = H_Z(z)$. Since H_Z is continuously differentiable and strictly increasing,

$$du = h_Z(z)\, dz, \qquad z = H_Z^{-1}(u).$$

Therefore,

$$dz = \frac{du}{h_Z\left(H_Z^{-1}(u)\right)}.$$

Substituting into the above integral gives

$$\begin{aligned} GI_{\xi}(Z_{[l,m,p,r]}) &= \int_0^{\infty} h_Z^{\xi}(z)\left[1+\omega\Theta_{[l,m,p,r;\beta]}\left(1-(1+\beta)H_Z^{\beta}(z)\right)\right]^{\xi} dz \\ &= \int_0^1 h_Z^{\xi-1}\left(H_Z^{-1}(u)\right)\left[1+\omega\Theta_{[l,m,p,r;\beta]}\left(1-(1+\beta)u^{\beta}\right)\right]^{\xi} du \\ &= \int_0^1 h^{\xi}_{U,[l,m,p,r]}(u)\, h_Z^{\xi-1}\left(H_Z^{-1}(u)\right)\, du, \end{aligned}$$

which establishes (2.7). The assumed integrability of h_Z^{ξ} guarantees that the above change of variable is well defined and that the resulting integral is finite. □

We give the following example to illustrate the utility of expression (2.7).

Example 2.1. *According to reliability theory, the l-out-of-m system is one of the basic and widely used system structures. Let* $Z_1, Z_2, \ldots, Z_m$ *denote the independent lifetimes of the m components constituting the system. The system functions as long as at least l components remain operational. Accordingly, the lifetime of an l-out-of-m system is represented by the* $(m-l+1)$*th order statistic* $Z_{m-l+1:m}$. *Assume that the lifetime of each component follows a Weibull distribution with a unit scale parameter and shape parameter* τ. *The CDF of this distribution is*

$$H_Z(z) = 1 - e^{-z^{\tau}},\ z > 0,\ \tau > 0.$$

For this model, it follows that

$$h_Z\left(H_Z^{-1}(u)\right) = \tau(1-u)(-\log(1-u))^{\frac{\tau-1}{\tau}},\ 0 < u < 1.$$

By applying Eq (2.7)*, and assuming* $\xi \geq 1$, $\omega \neq 0$, *and* $\beta \geq 1$, *the generating informational function corresponding to the system lifetime can be expressed as*

$$GI_{\xi}(Z_{[m-l+1:m]}) = \tau^{\xi-1} \int_0^1 h^{\xi}_{U,[m-l+1:m]}(u) \left[(1-u)\,(-\log(1-u))^{\frac{\tau-1}{\tau}}\right]^{\xi-1} du.$$

Because of the analytical complexity of the above integral, obtaining an explicit closed-form expression is generally difficult. Therefore, numerical integration procedures are employed to investigate the behavior of the generating informational function $GI_{\xi}(Z_{[m-l+1:m]})$ *as the shape parameter* τ *varies. The numerical study considers systems corresponding to several different values of m.*

Figure 1 illustrates the behavior of $GI_{\xi}(Z_{[m-l+1:m]})$ as a function of l for several system sizes m and values $\xi = 2, 3, 4$, and 5. In all panels, the measure exhibits a non-monotonic behavior with respect to l. For small values of l, $GI_{\xi}(Z_{[m-l+1:m]})$ decreases as l increases, reaches a minimum at intermediate values of l, and then increases again for larger values of l.

This pattern can be explained through the order-statistic representation of the system lifetime. Since an l-out-of-m system has lifetime $Z_{[m-l+1:m]}$, increasing l decreases the order index $m - l + 1$. For small l, the system lifetime corresponds to higher order statistics, and increasing l shifts the lifetime toward more central order statistics, reducing the variability captured by the informational measure. For sufficiently large values of l, particularly when $l \geq m - l + 1$, the lifetime becomes associated with lower order statistics, corresponding to earlier system failure and stronger sensitivity to component-level variability, which leads to an increase in GI_{ξ}.

The system size m also substantially influences the shape of the curves. Larger systems exhibit smoother changes and a wider range of variation in l, while smaller systems display more rapid transitions. In addition, the location of the minimum tends to shift toward larger values of l as m increases.

A comparison among panels (a)–(d) further shows that increasing ξ changes both the magnitude and sensitivity of the measure. Larger values of ξ generally produce lower values of the index and accentuate the differences between the curves corresponding to different system sizes.

Overall, the numerical results demonstrate that the proposed index effectively captures the interplay between the system configuration (l, m), the order-statistic structure of the system lifetime, and the parameter ξ.

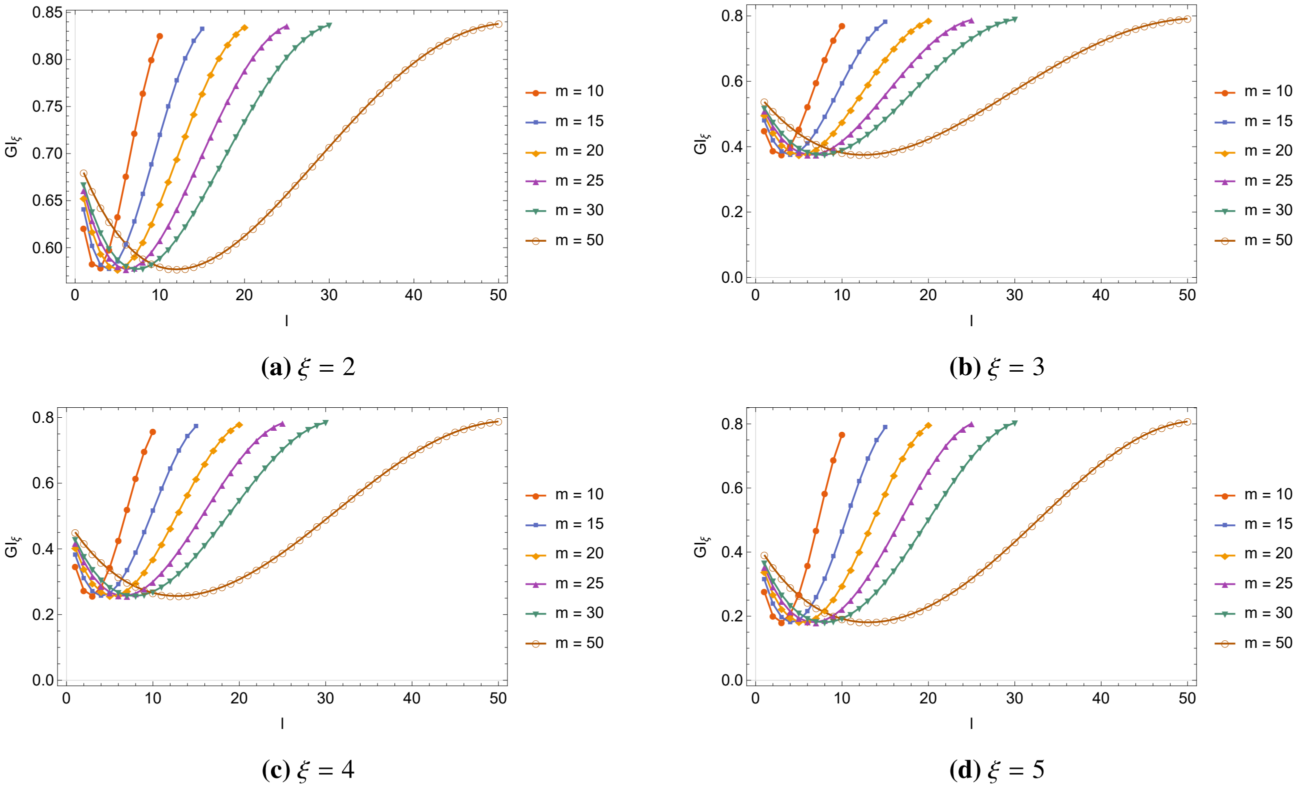


Figure 1. The behavior of $GI_{\xi}(Z_{[m-l+1:m]})$ for the Weibull distribution with $\beta = 2$, $\tau = 2$, and $\omega = 0.5$.

2.1. Stochastic comparison

The generating informational function associated with the concomitant of the lth p–generalized order statistics $Z_{[l,m,p,r]}$ is now investigated under dispersive and location–independent riskier ordering assumptions. Prior to presenting the main results, we briefly recall several concepts that will be required throughout this section.

Let $\mathbb{R}_+ = \{Z : Z \geq 0\}$ denote the class of non–negative absolutely continuous random variables defined on $(0, \infty)$. Additional discussions regarding stochastic orderings and their properties can be found in Shaked and Shanthikumar [18].

Definition 2.1. *Let $Z_1, Z_2 \in \mathbb{R}_+$ be two non–negative random variables with PDFs h_{Z_1} and h_{Z_2}, CDFs H_{Z_1} and H_{Z_2}, survival functions $\bar{H}_{Z_1}$ and $\bar{H}_{Z_2}$, and hazard rate functions*

$$\Psi_{Z_1}(z) = \frac{h_{Z_1}(z)}{\bar{H}_{Z_1}(z)}, \qquad \Psi_{Z_2}(z) = \frac{h_{Z_2}(z)}{\bar{H}_{Z_2}(z)}.$$

Then the following statements hold:

(1) The variable Z_1 is said to exhibit an increasing failure rate (IFR) or a decreasing failure rate (DFR) whenever its hazard rate function $\Psi_{Z_1}(z)$ is monotone increasing or decreasing in z, respectively.

(2) The ordering $Z_1 \leq^{hr} Z_2$ (hazard rate order) holds if

$$\Psi_{Z_1}(z) \geq \Psi_{Z_2}(z),\ z > 0.$$

(3) The variable Z_1 is said to be smaller than Z_2 in the dispersive sense, written $Z_1 \leq^{disp} Z_2$, whenever

$$H_{Z_1}^{-1}(u_1) - H_{Z_1}^{-1}(u_2) \leq H_{Z_2}^{-1}(u_1) - H_{Z_2}^{-1}(u_2), \quad 0 < u_1 \leq u_2 < 1.$$

An equivalent formulation is

$$h_{Z_1}\Big(H_{Z_1}^{-1}(u)\Big) \geq h_{Z_2}\Big(H_{Z_2}^{-1}(u)\Big), \quad 0 < u < 1.$$

(4) The ordering $Z_1 \leq^{lir} Z_2$ (location–independent riskier order) is defined through

$$\int_0^{H_{Z_1}^{-1}(u)} H_{Z_1}(z)\, dz \leq \int_0^{H_{Z_2}^{-1}(u)} H_{Z_2}(z)\, dz, \quad u \in (0,1).$$

The next theorem is obtained by applying representation (2.7).

Theorem 2.4. *Let $Z_{1,[l,m,p,r]}$ and $Z_{2,[l,m,p,r]}$ denote two concomitants of the lth p-generalized order statistics whose parent distributions have CDFs H_{Z_1}, H_{Z_2} and PDFs h_{Z_1}, h_{Z_2}, respectively. If $Z_1 \leq^{disp} Z_2$, then for $\xi \geq 1$, $\omega \neq 0$, and $\beta \geq 1$, $GI_\xi(Z_{1,[l,m,p,r]}) \geq GI_\xi(Z_{2,[l,m,p,r]})$.*

Proof. Using representation (2.7), the generating informational function may be expressed as

$$GI_\xi(Z_{1,[l,m,p,r]}) = \int_0^1 h^{\xi}_{U,[l,m,p,r]}(u)\, h^{\xi-1}_{Z_1}\Big(H_{Z_1}^{-1}(u)\Big)\, du.$$

Since $Z_1 \leq^{disp} Z_2$, we have $h_{Z_1}\Big(H_{Z_1}^{-1}(u)\Big) \geq h_{Z_2}\Big(H_{Z_2}^{-1}(u)\Big), \quad 0 < u < 1$. Consequently,

$$GI_\xi(Z_{1,[l,m,p,r]}) \geq \int_0^1 h^{\xi}_{U,[l,m,p,r]}(u)\, h^{\xi-1}_{Z_2}\Big(H_{Z_2}^{-1}(u)\Big)\, du = GI_\xi(Z_{2,[l,m,p,r]}),$$

which establishes the desired inequality. □

Bagai and Kochar [19] showed that if $Z_1 \leq^{hr} Z_2$ and at least one of the variables possesses the DFR property, then the dispersive order $Z_1 \leq^{disp} Z_2$ follows. Combining this observation with Theorem 2.4 leads to the following immediate consequence.

Corollary 2.1. *If $Z_1 \leq^{hr} Z_2$ and either Z_1 or Z_2 has a DFR distribution, then $GI_\xi(Z_{1,[l,m,p,r]}) \geq GI_\xi(Z_{2,[l,m,p,r]})$, for all $\xi \geq 1$, $\omega \neq 0$, and $\beta \geq 1$.*

Another application of representation (2.7) arises in studying the generating informational function corresponding to the concomitant of the lth p-generalized order statistics $Z_{[l,m,p,r]}$ when the component lifetimes are independent but not necessarily identically distributed. The following theorem provides a useful comparison result.

Theorem 2.5. *Assume that the conditions of Theorem 2.4 hold. If $GI_\xi(Z_1) \geq GI_\xi(Z_2)$ and $\inf_{u\in C_1} h_{U,[l,m,p,r]}(u) \geq \sup_{u\in C_2} h_{U,[l,m,p,r]}(u)$, where*

$$C_1 = \Big\{u \in (0,1) : h_{Z_1}\Big(H_{Z_1}^{-1}(u)\Big) > h_{Z_2}\Big(H_{Z_2}^{-1}(u)\Big)\Big\},$$
$$C_2 = \Big\{u \in (0,1) : h_{Z_1}\Big(H_{Z_1}^{-1}(u)\Big) \leq h_{Z_2}\Big(H_{Z_2}^{-1}(u)\Big)\Big\},$$

then, the inequality

$$GI_\xi\Big(Z_{1,[l,m,p,r]}\Big) \geq GI_\xi\Big(Z_{2,[l,m,p,r]}\Big)$$

holds for all $\xi \geq 1$, $\omega \neq 0$, and $\beta \geq 1$.

Proof. From the assumption $GI_\xi(Z_1) \geq GI_\xi(Z_2)$ and representation (1.1), we have

$$GI_\xi(Z_2) - GI_\xi(Z_1) = \int_0^1 \mathfrak{D}(u)\,du \leq 0, \tag{2.9}$$

where

$$\mathfrak{D}(u) = h_{Z_2}^{\xi-1}\Big(H_{Z_2}^{-1}(u)\Big) - h_{Z_1}^{\xi-1}\Big(H_{Z_1}^{-1}(u)\Big).$$

Using representation (2.7), the difference between the two generating informational functions can be written as

$$\begin{aligned}
&GI_\xi\Big(Z_{2,[l,m,p,r]}\Big) - GI_\xi\Big(Z_{1,[l,m,p,r]}\Big)\\
&= \int_0^1 h_{U,[l,m,p,r]}^{\xi}(u)\,\mathfrak{D}(u)\,du\\
&= \int_{C_1} h_{U,[l,m,p,r]}^{\xi}(u)\,\mathfrak{D}(u)\,du + \int_{C_2} h_{U,[l,m,p,r]}^{\xi}(u)\,\mathfrak{D}(u)\,du.
\end{aligned}$$

Because $\mathfrak{D}(u) < 0$ for $u \in C_1$ and $\mathfrak{D}(u) \geq 0$ for $u \in C_2$, we obtain

$$\begin{aligned}
&GI_\xi\Big(Z_{2,[l,m,p,r]}\Big) - GI_\xi\Big(Z_{1,[l,m,p,r]}\Big)\\
&\leq \left(\inf_{u\in C_1} h_{U,[l,m,p,r]}(u)\right)^{\xi} \int_{C_1} \mathfrak{D}(u)\,du + \left(\sup_{u\in C_2} h_{U,[l,m,p,r]}(u)\right)^{\xi} \int_{C_2} \mathfrak{D}(u)\,du.
\end{aligned}$$

By the assumed ordering between the extrema of $h_{U,[l,m,p,r]}(u)$, the above expression is bounded by

$$\left(\sup_{u\in C_2} h_{U,[l,m,p,r]}(u)\right)^{\xi} \int_0^1 \mathfrak{D}(u)\,du.$$

Finally, using (2.9), the last quantity is non–positive. Hence,

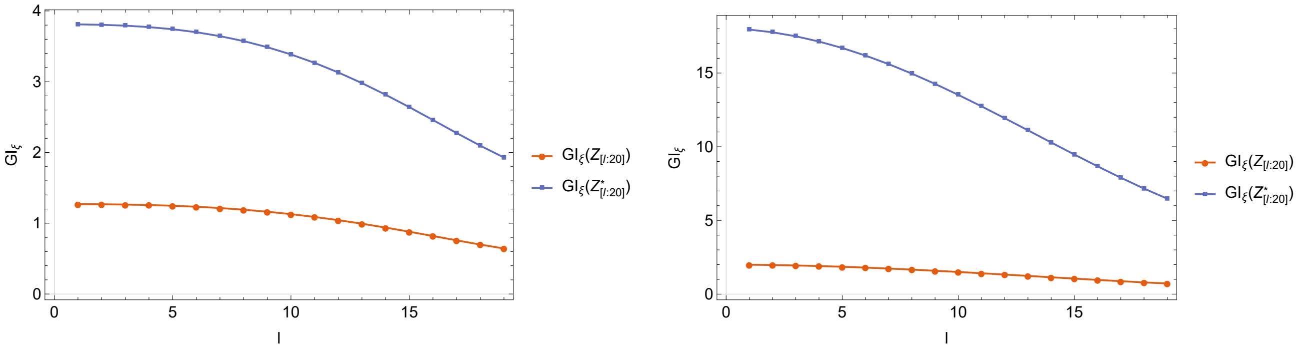
$$GI_\xi\Big(Z_{1,[l,m,p,r]}\Big) \geq GI_\xi\Big(Z_{2,[l,m,p,r]}\Big),$$

which completes the proof. □

The following example illustrates the applicability of Theorem 2.5.

Example 2.2. *Consider two collections of independent and identically distributed component lifetimes* $\{Z_1, \ldots, Z_{20}\}$ *and* $\{Z_1^*, \ldots, Z_{20}^*\}$. *Assume that* Z_i *follows the CDF* $H_{Z_i}(z) = 1 - e^{-2z}, z > 0,\ i = 1, 2, \ldots, 20$, *while* Z_i^* *follows* $H_{Z_i^*}(z) = 1 - e^{-6z}, z > 0,\ i = 1, 2, \ldots, 20$. *For* $\xi = 2$, *it can be verified that* $GI_2(Z) = 1, GI_2(Z^*) = 3$, *which implies that* $GI_2(Z) \leq GI_2(Z^*)$. *Moreover, let* $B_1 = [0, 1)$ *and* $B_2 = \{1\}$. *Then* $\inf_{u\in B_1} h_U(u) = \sup_{u\in B_2} h_U(u) = 0$. *Therefore, applying Theorem 2.5 to the lth order statistic* $Z_{l:20}$ *yields* $GI_2(Z_{[l:20]}) \leq GI_2(Z^*_{[l:20]})$, $l \leq 20$, *and* $\beta \geq 1$; *see Figure 2.*

In the following, we will use the property $Z_1 \leq^{\mathrm{disp}_z} Z_2$ to discuss some further outcomes. Motivated by these developments, we show that the parent distribution of the observations can be uniquely determined through the generating informational function corresponding to the lth p-generalized order statistic.



(a) $\beta = 2$ **(b)** $\beta = 3$

Figure 2. Verification of Example 2.2 for $\beta = 2$ (left) and $\beta = 3$ (right), with $\omega = 0.2$.

Theorem 2.6. *Suppose the assumptions stated in Theorem 2.4 hold. Then the distributions H_{Z_1} and H_{Z_2} belong to the same location family if and only if the ordering $Z_1 \leq^{disp_z} Z_2$ holds and*

$$GI_\xi(Z_{1,[l,m,p,r]}) = GI_\xi(Z_{2,[l,m,p,r]}), \quad \text{for all } l \leq m \text{ with } \beta \geq 1,\ \xi \geq 1,\ \omega \neq 0, \tag{2.10}$$

provided that $h_{U,[l,m,p,r]}(u) > 0$ for every $u \in (0, 1)$.

Proof. The necessity part is immediate. To prove sufficiency, consider the representation of the generating informational function of $Z_{1,[l,m,p,r]}$ given by

$$GI_\xi(Z_{1,[l,m,p,r]}) = \int_0^1 h^\xi_{U,[l,m,p,r]}(u)\, h^{\xi-1}_{Z_1}\left(H^{-1}_{Z_1}(u)\right) du.$$

Similarly, we obtain an analogous expression for $GI_\xi(Z_{2,[l,m,p,r]})$. If equality (2.10) holds for all $l \leq m$ with $\beta \geq 1$, $\xi \geq 1$, and $\omega \neq 0$, then

$$\int_0^1 h^\xi_{U,[l,m,p,r]}(u) \left[h^{\xi-1}_{Z_1}\left(H^{-1}_{Z_1}(u)\right) - h^{\xi-1}_{Z_2}\left(H^{-1}_{Z_2}(u)\right)\right] du = 0. \tag{2.11}$$

Since $\xi \geq 1$, the function $h_{U,[l,m,p,r]}(u)$ is strictly positive on $(0, 1)$. Furthermore, the ordering $Z_1 \leq^{\mathrm{disp}_z} Z_2$ implies

$$h_{Z_1}\left(H^{-1}_{Z_1}(u)\right) \geq h_{Z_2}\left(H^{-1}_{Z_2}(u)\right), \qquad 0 < u < 1.$$

Hence, the integrand in (2.11) is nonnegative on $(0, 1)$, which forces

$$h_{Z_1}\left(H^{-1}_{Z_1}(u)\right) = h_{Z_2}\left(H^{-1}_{Z_2}(u)\right) \quad \text{almost everywhere.}$$

Consequently, the quantile functions differ only by a constant shift, namely

$$H^{-1}_{Z_2}(u) = H^{-1}_{Z_1}(u) + \mathfrak{O}^*$$

for some constant $\mathfrak{O}^*$, which completes the proof. □

Corollary 2.2. *Let $Z_{1,[m,m,p,r]}$ and $Z_{2,[m,m,p,r]}$ denote the maximum generalized order statistics generated from the distributions H_{Z_1} and H_{Z_2}, respectively. Then H_{Z_1} and H_{Z_2} belong to the same location family if and only if $Z_1 \leq^{disp_z} Z_2$, and*

$$GI_\xi(Z_{1,[m,m,p,r]}) = GI_\xi(Z_{2,[m,m,p,r]}), \quad \forall m \geq 1.$$

The following theorem presents a related characterization involving both location and scale transformations.

Theorem 2.7. *Under the assumptions of Theorem 2.6, the distributions H_{Z_1} and H_{Z_2} belong to the same location-scale family if and only if $Z_1 \leq^{disp_z} Z_2$, and*

$$\frac{GI_\xi(Z_{1,[l,m,p,r]})}{GI_\xi(Z_1)} = \frac{GI_\xi(Z_{2,[l,m,p,r]})}{GI_\xi(Z_2)}, \quad \forall\, l \leq m \; with \, \beta \geq 1,\ \xi \geq 1,\ \omega \neq 0. \tag{2.12}$$

Proof. The necessity is straightforward. For sufficiency, using the integral representation of the generating informational function, we obtain

$$\frac{GI_\xi(Z_{1,[l,m,p,r]})}{GI_\xi(Z_1)} = \int_0^1 h^\xi_{U,[l,m,p,r]}(u) \frac{h^{\xi-1}_{Z_1}\Big(H^{-1}_{Z_1}(u)\Big)}{GI_\xi(Z_1)} du. \tag{2.13}$$

Similarly, an analogous expression holds for $Z_{2,[l,m,p,r]}$. Combining (2.12) and (2.13) gives

$$\int_0^1 h^\xi_{U,[l,m,p,r]}(u) \frac{h^{\xi-1}_{Z_1}\Big(H^{-1}_{Z_1}(u)\Big)}{GI_\xi(Z_1)} du = \int_0^1 h^\xi_{U,[l,m,p,r]}(u) \frac{h^{\xi-1}_{Z_2}\Big(H^{-1}_{Z_2}(u)\Big)}{GI_\xi(Z_2)} du. \tag{2.14}$$

Let $\upsilon = \frac{GI_\xi(Z_2)}{GI_\xi(Z_1)}$. Since $Z_1 \leq^{\mathrm{disp}_z} Z_2$, it follows that $GI_\xi(Z_1) \leq GI_\xi(Z_2)$, and therefore $\upsilon \geq 1$. Equation (2.14) can be rewritten as

$$\int_0^1 h^\xi_{U,[l,m,p,r]}(u) \Big[\upsilon\, h^{\xi-1}_{Z_1}\Big(H^{-1}_{Z_1}(u)\Big) - h^{\xi-1}_{Z_2}\Big(H^{-1}_{Z_2}(u)\Big)\Big] du = 0. \tag{2.15}$$

Because of the dispersion ordering, we have

$$\upsilon\, h_{Z_1}\Big(H^{-1}_{Z_1}(u)\Big) \geq h_{Z_2}\Big(H^{-1}_{Z_2}(u)\Big), \quad 0 < u < 1.$$

Hence, the integrand in (2.15) is nonnegative, which implies

$$H^{-1}_{Z_2}(u) = \upsilon\, H^{-1}_{Z_1}(u) + \mathfrak{O}^*,$$

for some constant $\mathfrak{O}^*$. This establishes the result. □

Corollary 2.3. *Under the conditions of Corollary 2.2, the distributions H_{Z_1} and H_{Z_2} belong to the same scale family if and only if $Z_1 \leq^{disp_z} Z_2$, and*

$$\frac{GI_\xi(Z_{1,[m,m,p,r]})}{GI_\xi(Z_1)} = \frac{GI_\xi(Z_{2,[m,m,p,r]})}{GI_\xi(Z_2)}, \quad \forall m \geq 1.$$

3. Bounds and characterization results for the generating informational function

In many practical situations, obtaining explicit expressions for the generating informational function becomes analytically challenging, particularly when the underlying distribution of the components is complicated or when the number of observations is large. In such cases, bounding techniques provide a convenient and efficient alternative for studying the behavior of the measure.

Motivated by this difficulty, we investigate bounds for the generating informational function associated with the concomitant of the lth p-generalized order statistics $Z_{[l,m,p,r]}$. The following theorem provides useful upper and lower bounds under mild regularity conditions.

Theorem 3.1. *Let $Z_{[l,m,p,r]}$ denote the lth p-generalized order statistic generated from m independent and identically distributed random variables having PDF h_Z.*

(i) Suppose that the density h_Z possesses a finite mode $\mathfrak{M} = h_Z(q) < \infty$, where $q = \sup\{z : h_Z(z) \leq \mathfrak{M}\}$, denotes the modal point of the density function. Then, for $\beta \geq 1$, $\omega \neq 0$, and $\xi \geq 1$, the following bound holds:

$$GI_\xi(Z_{[l,m,p,r]}) \leq \mathfrak{M}^{\xi-1}\, GI_\xi(U_{[l,m,p,r]}),$$

where

$$GI_\xi(U_{[l,m,p,r]}) = \int_0^1 h^{\xi}_{U,[l,m,p,r]}(u)\, du = \int_0^1 \Big(1 + \omega\Theta_{[l,m,p,r;\beta]}\Big(1 - (1+\beta)u^{\beta}\Big)\Big)^{\xi}\, du. \tag{3.1}$$

(ii) For $\beta \geq 1$, $\omega \neq 0$, and $\xi \geq 1$, the generating informational function satisfies

$$S_1^{\xi} GI_\xi(Z) \leq GI_\xi(Z_{[l,m,p,r]}) \leq S_2^{\xi} GI_\xi(Z),$$

where

$$S_1 = \inf_{u\in(0,1)} h_{U,[l,m,p,r]}(u), \qquad S_2 = \sup_{u\in(0,1)} h_{U,[l,m,p,r]}(u).$$

Proof. The proof of part (i) follows directly from the boundedness of the density function and, therefore, is omitted for brevity. We focus on establishing part (ii). Since

$$h_{U,[l,m,p,r]}(u) \geq \inf_{u\in(0,1)} h_{U,[l,m,p,r]}(u) = S_1,$$

it follows from representation (2.7) that

$$\begin{aligned} GI_\xi(Z_{[l,m,p,r]}) &= \int_0^1 \Big[h_{U,[l,m,p,r]}(u)\Big]^{\xi}\, h_Z^{\xi-1}\Big(H_Z^{-1}(u)\Big)\, du \\ &\geq S_1^{\xi} \int_0^1 h_Z^{\xi-1}\Big(H_Z^{-1}(u)\Big)\, du \\ &= S_1^{\xi} GI_\xi(Z). \end{aligned}$$

The upper bound can be obtained in a completely analogous manner by using $h_{U,[l,m,p,r]}(u) \leq S_2$. This completes the proof. □

Part (i) of Theorem 3.1 provides an upper bound for the generating informational function of the lth p-generalized order statistic in terms of the modal value of the parent distribution. The bound depends on the generating informational function associated with the uniform representation of the statistic.

Part (ii) shows that, under suitable conditions, the generating informational function of $Z_{[l,m,p,r]}$ lies between two scaled versions of the generating informational function of the underlying distribution.

An alternative formulation of the generating informational function for a non-negative absolutely continually random variable Z with CDF $H_Z(z)$ and inverted hazard rate function $\delta(z) = \frac{h_Z(z)}{H_Z(z)}$ can be found as

$$\begin{aligned} GI_\xi(Z) &= \frac{1}{\xi}\int_0^{\infty} \delta^{\xi-1}(z)\, h_{\xi;\xi}(z)\, dz \\ &= \frac{1}{\xi}\,\mathbb{E}_{\xi;\xi}\Big[\delta^{\xi-1}(Z)\Big], \end{aligned} \tag{3.2}$$

where $\mathbb{E}_{\xi;\xi}$ denotes the expectation with respect to the PDF

$$h_{\xi;\xi}(z) = \xi\, h_Z(z)\, H_Z^{\xi-1}(z). \tag{3.3}$$

The density in (3.3) corresponds to the distribution of the maximum of ξ independent and identically distributed random variables having common density h_Z.

Theorem 3.2. *According to the PDF and CDF of the concomitant of the lth p-generalized order statistics given in* (1.9) *and* (1.10)*, respectively, the corresponding reversed hazard rate function is*

$$\delta_{[l,m,p,r;\beta]}(z) = \frac{h_{Z,[l,m,p,r]}(z)}{H_{Z,[l,m,p,r]}(z)} = \delta_Z(z)\, \delta^*_{[l,m,p,r;\beta]}(z),$$

where $\delta_Z(z) = \frac{h_Z(z)}{H_Z(z)}$, *and*

$$\delta^*_{[l,m,p,r;\beta]}(z) = \frac{1 + \omega\Theta_{[l,m,p,r;\beta]}\left(1 - (1+\beta)H_Z^{\beta}(z)\right)}{1 + \omega\Theta_{[l,m,p,r;\beta]}\left(1 - H_Z^{\beta}(z)\right)}, \qquad 0 < H_Z(z) < 1.$$

Suppose that $\omega\Theta_{[l,m,p,r;\beta]} > 0$, $1 - \omega\Theta_{[l,m,p,r;\beta]} > 0$, $\xi \geq 1$, $\omega \neq 0$, *and* $\beta \geq 1$. *Then* $\delta^*_{[l,m,p,r;\beta]}(u)$ *is strictly decreasing on* $(0,1)$ *and satisfies*

$$1 - \beta A \leq \delta^*_{[l,m,p,r;\beta]}(u) \leq 1, \qquad A = \omega\Theta_{[l,m,p,r;\beta]}.$$

Consequently, $(1-\beta A)\delta_Z(z) \leq \delta_{[l,m,p,r;\beta]}(z) \leq \delta_Z(z)$, *and*

$$\frac{(1-\beta A)^{\xi-1}}{\xi}\, \mathbb{E}^{([l,m,p,r])}_{\xi;\xi}\left[\delta_Z^{\xi-1}(Z)\right] \leq GI_{\xi}(Z_{[l,m,p,r]}) \leq \frac{1}{\xi}\, \mathbb{E}^{([l,m,p,r])}_{\xi;\xi}\left[\delta_Z^{\xi-1}(Z)\right],$$

where $\mathbb{E}^{([l,m,p,r])}_{\xi;\xi}$ *denotes expectation with respect to*

$$h^{([l,m,p,r])}_{\xi;\xi}(z) = \xi h_{Z,[l,m,p,r]}(z) H^{\xi-1}_{Z,[l,m,p,r]}(z).$$

Proof. Let $A = \omega\Theta_{[l,m,p,r;\beta]}$. Then

$$\delta^*_{[l,m,p,r;\beta]}(u) = \frac{1 + A - A(1+\beta)u^{\beta}}{1 + A - Au^{\beta}}, \qquad 0 < u < 1.$$

Differentiating with respect to u yields

$$\frac{d}{du}\delta^*_{[l,m,p,r;\beta]}(u) = -\frac{A\beta^2(1+A)u^{\beta-1}}{(1 + A - Au^{\beta})^2}.$$

Since $A > 0, \beta \geq 1, u^{\beta-1} > 0$, and $(1 + A - Au^{\beta})^2 > 0$, we have

$$\frac{d}{du}\delta^*_{[l,m,p,r;\beta]}(u) < 0,$$

showing that $\delta^*_{[l,m,p,r;\beta]}(u)$ is strictly decreasing on $(0,1)$. Moreover,

$$\delta^*_{[l,m,p,r;\beta]}(0) = 1, \qquad \delta^*_{[l,m,p,r;\beta]}(1) = 1 - \beta A,$$

and hence

$$1-\beta A \le \delta^{*}_{[l,m,p,r;\beta]}(u) \le 1.$$

Since

$$\delta_{[l,m,p,r;\beta]}(z) = \delta_Z(z)\delta^{*}_{[l,m,p,r;\beta]}(z),$$

it follows that

$$(1-\beta A)\delta_Z(z) \le \delta_{[l,m,p,r;\beta]}(z) \le \delta_Z(z).$$

Raising both sides to the power $\xi-1$ (recall that $\xi \ge 1$) gives

$$(1-\beta A)^{\xi-1}\delta_Z^{\xi-1}(z) \le \delta^{\xi-1}_{[l,m,p,r;\beta]}(z) \le \delta_Z^{\xi-1}(z).$$

Finally, applying the representation in Eq (3.2) with respect to the concomitant distribution,

$$GI_\xi(Z_{[l,m,p,r]}) = \frac{1}{\xi}\mathbb{E}^{([l,m,p,r])}_{\xi;\xi}\left[\delta^{\xi-1}_{[l,m,p,r;\beta]}(Z)\right],$$

yields

$$\frac{(1-\beta A)^{\xi-1}}{\xi}\,\mathbb{E}^{([l,m,p,r])}_{\xi;\xi}\left[\delta_Z^{\xi-1}(Z)\right] \le GI_\xi(Z_{[l,m,p,r]}) \le \frac{1}{\xi}\,\mathbb{E}^{([l,m,p,r])}_{\xi;\xi}\left[\delta_Z^{\xi-1}(Z)\right],$$

which completes the proof. □

Remark 3.1. *It should be noted that Theorem 3.2 establishes bounds for $GI_\xi(Z_{[l,m,p,r]})$ by exploiting the monotonicity of $\delta^{*}_{[l,m,p,r;\beta]}(u)$ with respect to u. Consequently, the theorem provides bounds with respect to the transformed variable u, but it does not imply that $GI_\xi(Z_{[l,m,p,r]})$ is monotonic in the parameter l, as illustrated in Figure 1. The bounds illustrated in Figure 1 are consistent with the revised theorem. For example, when $m = 20$ and $l = 11$, we obtain*

$$\omega\Theta_{[l,m,p,r;\beta]} = 0.142857 > 0,$$

and

$$1-\omega\Theta_{[l,m,p,r;\beta]} = 0.857143 > 0,$$

so that the assumptions of Theorem 3.2 are satisfied and the derived bounds are applicable.

From a practical viewpoint, these conditions identify parameter configurations under which the concomitant model preserves the required monotonicity of the reversed hazard rate, thereby guaranteeing the validity of the theoretical bounds. When these conditions are not satisfied, the monotonicity argument no longer holds, and hence the bounds should not be interpreted as theoretically guaranteed. Such cases reflect parameter settings outside the scope of the theorem rather than a contradiction of the proposed results.

3.1. Characterization findings for the generating informational function

In this subsection, we study several characterization properties associated with the generating informational function of the lth p-generalized order statistic. Measures related to extropy and entropy have recently been employed to characterize symmetric continuous distributions; see Gupta and Chaudhary [20] and Alomani et al. [21]. These works demonstrate that such information measures often yield identical values for upper and lower order statistics when the underlying distribution is symmetric. Next, we present a characterization result formulated through the generating informational function. To this end, we consider the concomitant of the lth p-generalized order statistics $Z_{[l,m,p,r]}$. Before proceeding to the main theorem, we recall a useful lemma derived from the Müntz–Szász theorem. This result, discussed for example in Kamps [22], is frequently employed in moment-type characterization problems.

Lemma 3.1. *Let $\varsigma(y)$ be an integrable function defined on the finite bounded interval (a_1, a_2). Assume that $\int_{a_1}^{a_2} z^{m_i}\varsigma(z)\,dz = 0,\ i \geq 1$. Then $\varsigma(z) = 0$ for almost every $z \in (a_1, a_2)$, provided that $\{m_i\}_{i\geq 1}$ is an increasing sequence of positive integers satisfying $\sum_{i=1}^{\infty} \frac{1}{m_i} = \infty$.*

This classical statement implies that the set $\{z^{m_1}, z^{m_2}, \ldots\}$ forms a complete system on the interval (a_1, a_2). A related extension involving functions of the form $\{V^{m_i}(z), m_i \geq 1\}$ was obtained by Hwang and Lin [23] under additional regularity assumptions on $V(z)$.

Theorem 3.3. *Let $Z_{1,[l,m,p,r]}$ and $Z_{2,[l,m,p,r]}$ denote two concomitants of the lth p-generalized order statistics generated from independent and identically distributed components having CDFs H_{Z_1} and H_{Z_2} with densities h_{Z_1} and h_{Z_2}, respectively. Then H_{Z_1} and H_{Z_2} belong to the same distributional family differing only by a location parameter if and only if $GI_\xi\Big(Z_{1,[l,m,p,r]}\Big) = GI_\xi\Big(Z_{2,[l,m,p,r]}\Big)$, $\xi \geq 1$, $\beta \geq 1$, and $\omega \neq 0$.*

Proof. **Necessity.** Suppose that H_{Z_1} and H_{Z_2} are identical up to a location shift. Then there exists a constant $e_1 \in \mathbb{R}$ such that $H_{Z_2}(z) = H_{Z_1}(z - e_1)$. Consequently, $h_{Z_2}(z) = h_{Z_1}(z - e_1)$. Therefore,

$$\begin{aligned}
GI_\xi\Big(Z_{2,[l,m,p,r]}\Big) &= \int_{e_1}^{\infty} h^{\xi}_{Z_2,[l,m,p,r]}(z)\,dz \\
&= \int_{e_1}^{\infty} h^{\xi}_{Z_1,[l,m,p,r]}(z - e_1)\,dz \\
&= \int_{0}^{\infty} h^{\xi}_{Z_1,[l,m,p,r]}(x)\,dx \\
&= GI_\xi\Big(Z_{1,[l,m,p,r]}\Big),
\end{aligned}$$

where $x = z - a_1$.

Sufficiency. For the concomitant of p-generalized order statistics, we have

$$\begin{aligned}
h_{U,[l,m,p,r]}(u) &= 1 + \omega\Theta_{[l,m,p,r;\beta]}\left(1 - (1+\beta)u^{\beta}\right) \\
&= u^{\beta}\left[(1 + \omega\Theta_{[l,m,p,r;\beta]})u^{-\beta} - (1+\beta)\omega\Theta_{[l,m,p,r;\beta]}\right], 0 < u < 1.
\end{aligned} \tag{3.4}$$

Using the assumption $GI_\xi\Big(Z_{1,[l,m,p,r]}\Big) = GI_\xi\Big(Z_{2,[l,m,p,r]}\Big)$, it follows that

$$\int_0^1 h^{\xi}_{U,[l,m,p,r]}(u)\Big[h^{\xi-1}_{Z_1}\Big(H^{-1}_{Z_1}(u)\Big) - h^{\xi-1}_{Z_2}\Big(H^{-1}_{Z_2}(u)\Big)\Big]du = 0.$$

This can be rewritten as

$$\int_0^1 u^{\xi\beta}\,\varrho_{[l,m,p,r;\beta]}(u)\left[h^{\xi-1}_{Z_1}\Big(H^{-1}_{Z_1}(u)\Big) - h^{\xi-1}_{Z_2}\Big(H^{-1}_{Z_2}(u)\Big)\right]du = 0,$$

where

$$\varrho_{[l,m,p,r;\beta]}(u) = \left[(1+\omega\Theta_{[l,m,p,r;\beta]})u^{-\beta} - (1+\beta)\omega\Theta_{[l,m,p,r;\beta]}\right], 0 < u < 1. \tag{3.5}$$

Lemma 3.1 is applied to the function

$$\varsigma(u) = \varrho_{[l,m,p,r;\beta]}(u)\left[h_{Z_1}^{\xi-1}\left(H_{Z_1}^{-1}(u)\right) - h_{Z_2}^{\xi-1}\left(H_{Z_2}^{-1}(u)\right)\right],$$

and utilizing the complete sequence $\{u^{\xi\beta}, \xi\beta \leq 1\}$, we determine (i.e., $\varsigma(u) = 0$)

$$h_{Z_1}\left(H_{Z_1}^{-1}(u)\right) = h_{Z_2}\left(H_{Z_2}^{-1}(u)\right), \quad \text{a.e. } u \in (0,1),$$

where a.e. stands for almost everywhere. Hence,

$$H_{Z_2}^{-1}(u) = H_{Z_1}^{-1}(u) + \mathfrak{O}^*,$$

for some constant $\mathfrak{O}^*$, which proves that the distributions differ only by a location parameter. □

Corollary 3.1. *Under the assumptions of Theorem 3.3, the distributions H_{Z_1} and H_{Z_2} are of the same type differing by a location parameter if and only if $GI_\xi\left(Z_{1,[m,m,p,r]}\right) = GI_\xi\left(Z_{2,[m,m,p,r]}\right)$, $\xi \geq 1$, $\beta \geq 1$, and $\omega \neq 0$.*

The following theorem describes an additional expansion.

Theorem 3.4. *Under the assumptions of Theorem 3.3, the distributions H_{Z_1} and H_{Z_2} belong to the same family but differ by both location and scale parameters if and only if*

$$\frac{GI_\xi\left(Z_{1,[l,m,p,r]}\right)}{GI_\xi(Z_1)} = \frac{GI_\xi\left(Z_{2,[l,m,p,r]}\right)}{GI_\xi(Z_2)}, \ \xi \geq 1,\ \omega \neq 0, \beta \geq 1.$$

Proof. The necessity follows immediately. For sufficiency, using representation (1.1), we obtain

$$\frac{GI_\xi\left(Z_{1,[l,m,p,r]}\right)}{GI_\xi(Z_1)} = \int_0^1 h^{\xi}_{U,[l,m,p,r]}(u)\frac{h_{Z_1}^{\xi-1}\left(H_{Z_1}^{-1}(u)\right)}{GI_\xi(Z_1)}\,du.$$

A similar expression holds for Z_2. If the ratio equality holds, then

$$\int_0^1 h^{\xi}_{U,[l,m,p,r]}(u)\left(\delta h_{Z_1}^{\xi-1}\left(H_{Z_1}^{-1}(u)\right) - h_{Z_2}^{\xi-1}\left(H_{Z_2}^{-1}(u)\right)\right)du = 0,$$

where

$$\delta = \frac{GI_\xi(Z_2)}{GI_\xi(Z_1)} = \frac{\int_0^1 h_{Z_2}^{\xi-1}(H_{Z_2}^{-1}(u))\,du}{\int_0^1 h_{Z_1}^{\xi-1}(H_{Z_1}^{-1}(u))\,du}.$$

Therefore, we can write

$$\int_0^1 u^{\xi\beta}\,\varrho_{[l,m,p,r;\beta]}(u)\left(\delta h_{Z_1}^{\xi-1}\left(H_{Z_1}^{-1}(u)\right) - h_{Z_2}^{\xi-1}\left(H_{Z_2}^{-1}(u)\right)\right)du = 0.$$

Applying Lemma 3.1 again yields the desired conclusion. □

Corollary 3.2. *Under the assumptions of Theorem 3.3, the distributions H_{Z_1} and H_{Z_2} are of the same type differing by a location parameter if and only if*

$$\frac{GI_{\xi}\Big(Z_{1,[m,m,p,r]}\Big)}{GI_{\xi}(Z_1)} = \frac{GI_{\xi}\Big(Z_{2,[m,m,p,r]}\Big)}{GI_{\xi}(Z_2)}, \, \xi \geq 1, \, \omega \neq 0, \, \beta \geq 1.$$

The next investigations show that the uniform and exponential distributions may be characterized using the generating informational function of the concomitant of the lth p-generalized order statistics $Z_{[l,m,p,r]}$.

Theorem 3.5. *Let Z be a random variable supported on $(0,1)$ with density h_Z. Then Z follows the uniform distribution on $(0,1)$ if and only if*

$$GI_{\xi}\Big(Z_{[l,m,p,r]}\Big) = GI_{\xi}(Z)\, GI_{\xi}\Big(U_{[l,m,p,r]}\Big), \, \xi \geq 1, \, \omega \neq 0, \, \beta \geq 1.$$

Proof. First, assume that Z has the standard uniform distribution. Then $h_Z(z) = 1$ for $0 < z < 1$, which implies $h_Z(H_Z^{-1}(u)) = 1$, $0 < u < 1$. From Theorem 2.3, it follows that

$$GI_{\xi}\Big(Z_{[l,m,p,r]}\Big) = GI_{\xi}\Big(U_{[l,m,p,r]}\Big) = \int_0^1 h^{\xi}_{U,[l,m,p,r]}(u)\, du. \tag{3.6}$$

Since $GI_{\xi}(Z) = 1$, the necessity follows. Conversely, assume that

$$GI_{\xi}\Big(Z_{[l,m,p,r]}\Big) = GI_{\xi}(Z)\, GI_{\xi}\Big(U_{[l,m,p,r]}\Big).$$

Then

$$\int_0^1 h^{\xi}_{U,[l,m,p,r]}(u) \left(h_Z^{\xi-1}(H_Z^{-1}(u)) - GI_{\xi}(Z)\right) du = 0.$$

Or equivalently,

$$\int_0^1 u^{\xi\beta}\, \varrho_{[l,m,p,r;\beta]}(u) \left(h_Z^{\xi-1}(H_Z^{-1}(u)) - GI_{\xi}(Z)\right) du = 0,$$

where $\varrho_{[l,m,p,r;\beta]}(u)$ is defined in (3.5). Applying Lemma 3.1 and in the same manner as in Theorem 3.3 yields

$$h_Z^{\xi-1}(H_Z^{-1}(u)) = GI_{\xi}(Z), \quad \text{a.e. } u \in (0,1).$$

Hence,

$$\frac{dH_Z^{-1}(u)}{du} = \frac{1}{h_Z(H_Z^{-1}(u))} = \frac{1}{GI_{\xi}^{\frac{1}{\xi-1}}(Z)}.$$

Integrating gives

$$H_Z^{-1}(u) = \frac{u}{GI_{\xi}^{\frac{1}{\xi-1}}(Z)} + \mathfrak{O}^*,$$

where $\mathfrak{O}^*$ is a constant that is arbitrary. Since $\lim_{u\to 0} H_Z^{-1}(u) = 0$, we obtain $\mathfrak{O}^* = 0$. Using $H_Z^{-1}(1) = 1$ yields $GI_{\xi}(Z) = 1$, and therefore $H_Z(z) = z$, which proves that Z follows the uniform distribution on $(0,1)$. $\square$

Theorem 3.6. *Let Z be a random variable that follows the density h_Z. Then Z follows the exponential distribution with the parameter η (i.e., $H_Z(z) = 1 - e^{-\eta z}$, $z \geq 0$) if and only if*

$$GI_{\xi}\Big(Z_{[l,m,p,r]}\Big) = (\xi GI_{\xi}(Z))A_1^{\xi} \sum_{k=0}^{\infty} \binom{\xi}{k} \left(-\frac{A_2}{A_1}\right)^k B(\beta k + 1, \xi), \tag{3.7}$$

where $A_1 = 1 + \omega\Theta_{[l,m,p,r;\beta]}$, $A_2 = (1+\beta)\omega\Theta_{[l,m,p,r;\beta]}$, $\left|\frac{A_2}{A_1}u^{\beta}\right| < 1$, $\xi \geq 1$, $\beta \geq 1$, and $\omega \neq 0$.

Proof. Consider a random variable Z that follows an exponential distribution. By applying (1.1), the generating informational function of Z is obtained as $GI_{\xi}(Z) = \frac{\eta^{\xi-1}}{\xi}$. Moreover, since $h_Z\left(H_Z^{-1}(u)\right) = \eta(1-u)$, substituting this relation into Eq (2.7) leads to

$$\begin{aligned} GI_{\xi}\Big(Z_{[l,m,p,r]}\Big) &= \int_0^1 h^{\xi}_{U,[l,m,p,r]}(u)\, h_Z^{\xi-1}\left(H_Z^{-1}(u)\right) du \\ &= \eta^{\xi-1} \int_0^1 h^{\xi}_{U,[l,m,p,r]}(u)\, (1-u)^{\xi-1}\, du \\ &= (\xi GI_{\xi}(Z)) \int_0^1 \left(A_1 - A_2 u^{\beta}\right)^{\xi} (1-u)^{\xi-1}\, du \\ &= (\xi GI_{\xi}(Z))A_1^{\xi} \sum_{k=0}^{\infty} \binom{\xi}{k} \left(-\frac{A_2}{A_1}\right)^k B(\beta k + 1, \xi), \end{aligned} \tag{3.8}$$

where the last result is obtained by factoring A_1^{ξ}, then using the generalized binomial expansion with the condition $\left|\frac{A_2}{A_1}u^{\beta}\right| < 1$. The necessity is then deduced. We presume that Eq (3.7) stands for a fixed value of l in order to prove sufficiency. Using a consequence in Eq (3.8) and the proof of Theorem 3.3, we derive the following relation:

$$\int_0^1 h^{\xi}_{U,[l,m,p,r]}(u)\, h_Z^{\xi-1}\left(H_Z^{-1}(u)\right) du = (\xi GI_{\xi}(Z)) \int_0^1 h^{\xi}_{U,[l,m,p,r]}(u)\, (1-u)^{\xi-1}\, du.$$

It is comparable to

$$\int_0^1 h^{\xi}_{U,[l,m,p,r]}(u) \left[h_Z^{\xi-1}\left(H_Z^{-1}(u)\right) - (\xi GI_{\xi}(Z))(1-u)^{\xi-1}\right] du.$$

From (2.8), it holds that

$$\int_0^1 u^{\xi\beta}\, \varrho_{[l,m,p,r;\beta]}(u) \left[h_Z^{\xi-1}\left(H_Z^{-1}(u)\right) - (\xi GI_{\xi}(Z))(1-u)^{\xi-1}\right] du = 0,$$

where $\varrho_{[l,m,p,r;\beta]}(u)$ is defined in (3.5). Applying Lemma 3.1 and in the same manner as in Theorem 3.3 yields

$$h_Z\left(H_Z^{-1}(u)\right) = (\xi GI_{\xi}(Z))^{\frac{1}{\xi-1}}(1-u), \text{ a.e. } u \in (0,1).$$

Hence,

$$\frac{dH_Z^{-1}(u)}{du} = \frac{1}{h_Z(H_Z^{-1}(u))} = \frac{1}{(\xi GI_{\xi}(Z))^{\frac{1}{\xi-1}}(1-u)}.$$

Integrating gives

$$H_Z^{-1}(u) = \frac{-\log(1-u)}{(\xi GI_\xi(Z))^{\frac{1}{\xi-1}}} + \mathfrak{D}^*,$$

where $\mathfrak{D}^*$ is a constant that is arbitrary. Since $\lim_{u\to 0} H_Z^{-1}(u) = 0$, we obtain $\mathfrak{D}^* = 0$. As a result, we get $H_Z^{-1}(u) = \frac{-\log(1-u)}{(\xi GI_\xi(Z))^{\frac{1}{\xi-1}}}$, $u > 0$. This suggests that the CDF is $H_Z(u) = 1 - \exp\left\{-(\xi GI_\xi(Z))^{\frac{1}{\xi-1}} u\right\}$, $u > 0$. As a result, Z has a scale parameter and an exponential distribution with $(\xi GI_\xi(Z))^{\frac{1}{\xi-1}}$. The theorem is so established. □

4. Non-parametric estimation of the generating informational function

This section introduces a non-parametric approach for estimating the generating informational function associated with the concomitant of the lth p-generalized order statistics $Z_{[l,m,p,r]}$. Let $Z_1, Z_2, \ldots, Z_M$ be a random sample of size M drawn from an absolutely continuous distribution supported on $(0, \infty)$, and denote the corresponding order statistics by

$$Z_{1:M} \leq Z_{2:M} \leq \cdots \leq Z_{M:M}.$$

These observations are assumed to be independent of the underlying system components and are used solely to construct an empirical approximation of the distribution function. This approximation will then be utilized to estimate the generating informational function. Using the identity

$$\frac{dH_Z^{-1}(u)}{du} = \frac{1}{h_Z(H_Z^{-1}(u))},\ 0 < u < 1,$$

we can deduce that the derivative of the quantile function can be approximated by the finite difference scheme

$$\frac{dH_Z^{-1}(u)}{du} \approx \frac{M(X_{l+g:M} - Z_{l-g:M})}{2g},\ l = g+1, \ldots, M-g.$$

Recall the generating informational function corresponding to $Z_{[l,m,p,r]}$ written in (2.7). Therefore, to estimate $GI_\xi(Z_{[l,m,p,r]})$, we employ the difference-based technique proposed by Vasicek [24], which provides an approximation to the derivative of the quantile function. This method relies on the empirical distribution function and leads to the estimator

$$\begin{aligned}\widehat{GI}_\xi(Z_{[l,m,p,r]}) &= \frac{1}{M}\sum_{j=1}^{M}\left(h_{U,[l,m,p,r]}\Big(\frac{j}{M+1}\Big)\right)^{\xi}\left(\frac{2g}{M(Z_{j+g:M} - Z_{j-g:M})}\right)^{\xi-1} \\ &= \frac{1}{M}\sum_{j=1}^{M}\left(1 + \omega\Theta_{[l,m,p,r;\beta]}\Big(1 - (1+\beta)\Big(\frac{j}{M+1}\Big)^{\beta}\Big)\right)^{\xi} \times \left(\frac{2g}{M(Z_{j+g:M} - Z_{j-g:M})}\right)^{\xi-1}.\end{aligned} \tag{4.1}$$

The window size is denoted by g, a positive integer that satisfies $1 \leq g < M/2$. For boundary indices, we adopt the convention that $Z_{j-g:M} = Z_{1:M}$ when $j - g \leq 1$ and $Z_{j+g:M} = Z_{M:M}$ when $j + g \geq M$.

Following the modification introduced by Ebrahimi et al. [25], an improved version of the estimator can be constructed as

$$
\begin{aligned}
\widehat{GI}^{*}_{\xi}(Z_{[l,m,p,r]}) &= \frac{1}{M}\sum_{j=1}^{M}\left(h_{U,[l,m,p,r]}\left(\frac{j}{M+1}\right)\right)^{\xi}\left(\frac{\lambda_j g}{M(Z_{j+g:M}-Z_{j-g:M})}\right)^{\xi-1} \\
&= \frac{1}{M}\sum_{j=1}^{M}\left(1+\omega\Theta_{[l,m,p,r;\beta]}\left(1-(1+\beta)\left(\frac{j}{M+1}\right)^{\beta}\right)\right)^{\xi}\times\left(\frac{\lambda_j g}{M(Z_{j+g:M}-Z_{j-g:M})}\right)^{\xi-1},
\end{aligned}
\tag{4.2}
$$

where the weights λ_j are defined by

$$
\lambda_j=\begin{cases} 1+\dfrac{j-1}{g}, & 1\le j\le g,\\ 2, & g+1\le j\le M-g,\\ 1+\dfrac{M-j}{g}, & M-g+1\le j\le M.\end{cases}
\tag{4.3}
$$

An essential first step in assessing estimators for parametric performs is establishing estimator consistency. The consistent quality of the estimators specified in Eqs (4.1) and (4.2) is confirmed by the following proposition. The method used in the proof is similar to that of Vasicek [24]'s Theorem 1, which established a commonly used method for demonstrating consistency in entropy-structured statistics. Park [26] and Xiong et al. [27] have also used this technique to verify the validity of their respective test statistics.

Proposition 4.1. *Let $Z_1, Z_2, \ldots, Z_M$ be a random sample of size M drawn from a population with density h_Z and CDF H_Z. Assume that the underlying distribution has finite variance. Then, based on* (4.1) *and* (4.2)*, the following hold:*

(1) $\widehat{GI}_{\xi}(Z_{[l,m,p,r]}) \xrightarrow{p} GI_{\xi}(Z_{[l,m,p,r]})$*;*

(2) $\widehat{GI}^{*}_{\xi}(Z_{[l,m,p,r]}) \xrightarrow{p} GI_{\xi}(Z_{[l,m,p,r]})$*,*

as $M\to\infty$, $g\to\infty$, and $\frac{g}{M}\to 0$, where $\xrightarrow{p}$ denotes convergence in probability, for all $\xi\ge 1$, $\omega\ne 0$, and $\beta\ge 1$.

Proof. To determine whether the estimator $\widehat{GI}_{\xi}(Z_{[l,m,p,r]})$ is consistent, the proof follows similar arguments to those in Noughabi and Arghami [28]. As $g\to\infty$ and $M\to\infty$ with $\frac{g}{M}\to 0$, the difference-based estimator satisfies

$$
\frac{2g}{M(Z_{l+g:M}-Z_{l-g:M})}=\frac{H_M\left(Z_{l+g:M}\right)-H_M\left(Z_{l-g:M}\right)}{Z_{l+g:M}-Z_{l-g:M}},
$$

where H_M denotes the empirical distribution function. By the Glivenko–Cantelli theorem, $H_M(z)\to H_Z(z)$ uniformly almost surely. Hence,

$$
\frac{H_M\left(Z_{l+g:M}\right)-H_M\left(Z_{l-g:M}\right)}{Z_{l+g:M}-Z_{l-g:M}}\simeq\frac{H_Z\left(Z_{l+g:M}\right)-H_Z\left(Z_{l-g:M}\right)}{Z_{l+g:M}-Z_{l-g:M}}.
$$

Applying the mean value theorem, we obtain

$$
\frac{H_Z\left(Z_{l+g:M}\right)-H_Z\left(Z_{l-g:M}\right)}{Z_{l+g:M}-Z_{l-g:M}}\simeq\frac{h_Z\left(Z_{l+g:M}\right)+h_Z\left(Z_{l-g:M}\right)}{2}\simeq h_Z(Z_{l:M}).
$$

Moreover, since $H_M(Z_{l:M}) = \frac{l}{M+1}$, we can write

$$\widehat{GI}_\xi(Z_{[l,m,p,r]}) = \frac{1}{M}\sum_{l=1}^{M}\left(\frac{2g}{M(Z_{l+g:M} - Z_{l-g:M})}\right)^{\xi-1}\left(h_{U,[l,m,p,r]}\left(\frac{l}{M+1}\right)\right)^{\xi}.$$

Using the above approximations,

$$\widehat{GI}_\xi(Z_{[l,m,p,r]}) \simeq \frac{1}{M}\sum_{l=1}^{M} h_Z^{\xi-1}(Z_{l:M})\left(h_{U,[l,m,p,r]}(H_M(Z_{l:M}))\right)^{\xi}.$$

Since the approximation depends on the empirical distribution function's almost sure convergence. (i.e., $H_M(Z_{l:M}) \xrightarrow{a.s.} H_Z(Z_{l:M})$), we further obtain

$$\simeq \frac{1}{M}\sum_{l=1}^{M} h_Z^{\xi-1}(Z_{l:M})\left(h_{U,[l,m,p,r]}(H_Z(Z_{l:M}))\right)^{\xi}.$$

By the invariance of order statistics,

$$= \frac{1}{M}\sum_{l=1}^{M} h_Z^{\xi-1}(Z_l)\left(h_{U,[l,m,p,r]}(H_Z(Z_l))\right)^{\xi}.$$

Applying the strong law of large numbers yields

$$\frac{1}{M}\sum_{l=1}^{M} h_Z^{\xi-1}(Z_l)\left(h_{U,[l,m,p,r]}(H_Z(Z_l))\right)^{\xi} \xrightarrow{a.s.} E\left[h_Z^{\xi-1}(Z)\left(h_{U,[l,m,p,r]}(H_Z(Z))\right)^{\xi}\right].$$

The expectation can be expressed as

$$= \int_0^\infty h_Z^{\xi}(z)\left(h_{U,[l,m,p,r]}(H_Z(z))\right)^{\xi} dz = GI_\xi(Z_{[l,m,p,r]}).$$

Thus,

$$\widehat{GI}_\xi(Z_{[l,m,p,r]}) \xrightarrow{a.s.} GI_\xi(Z_{[l,m,p,r]}),$$

which implies convergence in probability. The consistency of $\widehat{GI}^*_\xi(Z_{[l,m,p,r]})$ follows by analogous arguments, completing the proof. □

Remark 4.1. *The above argument provides heuristic support for the consistency of the proposed estimator. A rigorous proof would require establishing the asymptotic behavior of the spacing estimator under suitable regularity conditions for the concomitant model, which is left for future investigation.*

Next, we investigate the asymptotic behavior of the proposed estimators via Monte Carlo simulation. The main objective is to examine whether the standardized forms

$$\frac{\widehat{GI}_\xi(Z_{[l,m,p,r]}) - \mathbb{E}[\widehat{GI}_\xi(Z_{[l,m,p,r]})]}{\sqrt{\mathrm{Var}(\widehat{GI}_\xi(Z_{[l,m,p,r]}))}},$$

and

$$\frac{\widehat{GI}^*_\xi(Z_{[l,m,p,r]}) - \mathbb{E}[\widehat{GI}^*_\xi(Z_{[l,m,p,r]})]}{\sqrt{\mathrm{Var}(\widehat{GI}^*_\xi(Z_{[l,m,p,r]}))}},$$

converge in distribution to the standard normal law. To this end, we generate 1000 independent samples from the exponential distribution with a unit rate, each of size $M = 100$, and set $\xi = 2$. Based on the lth order statistic $Z_{l:m}$, the estimators defined in (4.1) and (4.2) are then computed for each sample. The resulting histograms, displayed in Figure 3, indicate that both estimators are approximately normally distributed for moderate sample sizes, supporting their asymptotic normality. However, we find that the second estimator is significantly closer to the normal distribution than the first estimator.

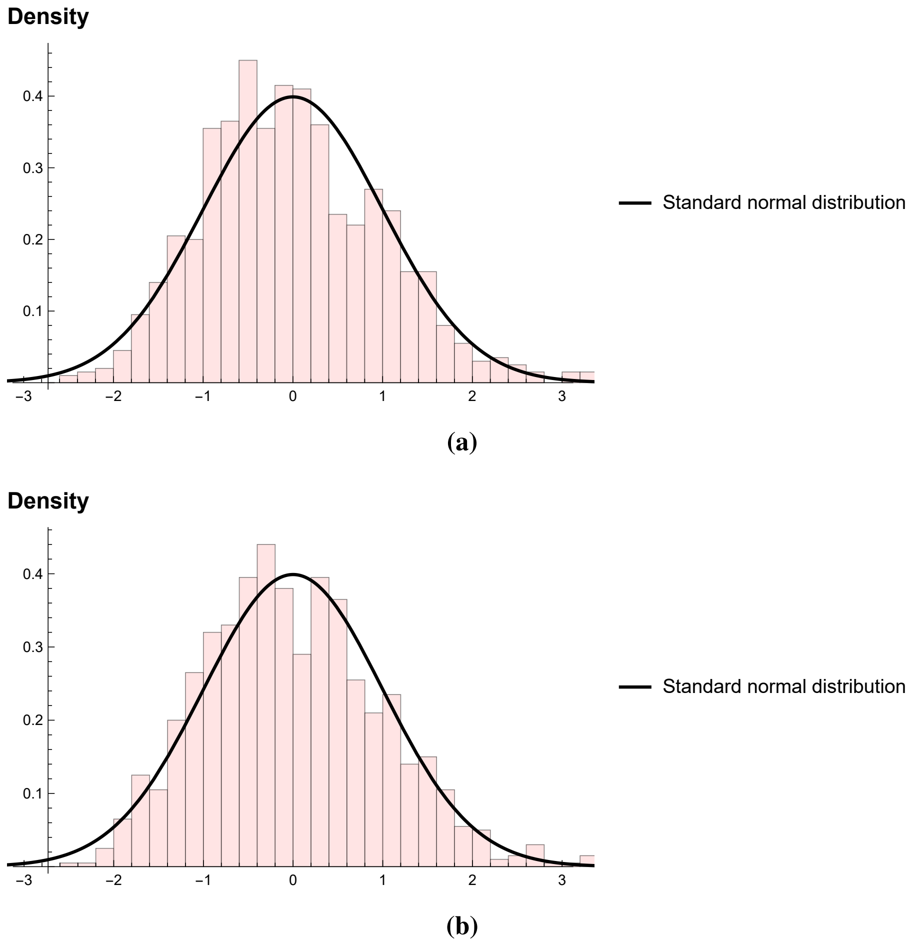


Figure 3. Asymptotic behavior of $\widehat{GI}_2(Z_{[40:50]})$ (up) and $\widehat{GI}^*_2(Z_{[40:50]})$ (down), with $M = 100$, $g = 10, \beta = 2$, and $\omega = 0.5$.

Remark 4.2. *The histograms suggest that the sampling distributions of the proposed estimators become increasingly symmetric and approximately bell-shaped as the sample size increases. Although these empirical findings are consistent with an approximately normal sampling distribution, they should not be interpreted as a proof of asymptotic normality. Establishing a formal central limit theorem for the proposed estimators remains an open problem.*

The estimator $\widehat{GI}_{\xi}(Z_{[l,m,p,r]})$'s square error root mean (SERM) is not invariant under scale changes, but it is invariant across location adjustments of the underlying random variable Z. The points of view of Ebrahimi et al. [25] can be modified to establish this property.

Theorem 4.1. *Let $Z_1, Z_2, \ldots, Z_M$ be a random sample of size M drawn from a population with CDF H_Z and PDF h_Z. Consider the affine transformation $X_j = aZ_j + b$, $a > 0$, $b \in \mathbb{R}$. Let $\widehat{GI}_{\xi}(Z_{[l,m,p,r]})$ and $\widehat{GI}_{\xi}(X_{[l,m,p,r]})$ denote the estimators of $GI_{\xi}(Z_{[l,m,p,r]})$, and $GI_{\xi}(X_{[l,m,p,r]})$ constructed from the samples $\{Z_j\}$ and $\{X_j\}$, respectively, $j = 1, 2, \ldots, M$. Then, for all $\xi \geq 1$, $\omega \neq 0$, and $\beta \geq 1$, the following properties hold:*

(i) $E\Big(\widehat{GI}_{\xi}(X_{[l,m,p,r]})\Big) = \frac{1}{a^{(\xi-1)}} E\Big(\widehat{GI}_{\xi}(Z_{[l,m,p,r]})\Big)$,

(ii) $\mathrm{Var}\Big(\widehat{GI}_{\xi}(X_{[l,m,p,r]})\Big) = \frac{1}{a^{(2\xi-2)}} \mathrm{Var}\Big(\widehat{GI}_{\xi}(Z_{[l,m,p,r]})\Big)$,

(iii) $\mathrm{SERM}\Big(\widehat{GI}_{\xi}(X_{[l,m,p,r]})\Big) = \frac{1}{a^{(\xi-1)}} \mathrm{SERM}\Big(\widehat{GI}_{\xi}(Z_{[l,m,p,r]})\Big)$.

Proof. From the estimator definition in (4.1), we write

$$\begin{aligned}
\widehat{GI}_{\xi}(X_{[l,m,p,r]}) &= \frac{1}{M} \sum_{j=1}^{M} \left(1 + \omega \Theta_{[l,m,p,r;\beta]} \left(1 - (1+\beta)\left(\frac{j}{M+1}\right)^{\beta}\right)\right)^{\xi} \left(\frac{2g}{M(X_{j+g:M} - X_{j-g:M})}\right)^{\xi-1} \\
&= \frac{1}{M} \sum_{j=1}^{M} \left(1 + \omega \Theta_{[l,m,p,r;\beta]} \left(1 - (1+\beta)\left(\frac{j}{M+1}\right)^{\beta}\right)\right)^{\xi} \left(\frac{2g}{M(a\,Z_{j+g:M} - a\,Z_{j-g:M})}\right)^{\xi-1} \\
&= \frac{1}{a^{(\xi-1)}} \widehat{GI}_{\xi}(Z_{[l,m,p,r]}).
\end{aligned}$$

The stated results for the expectation, variance, and SERM follow directly from the standard scaling properties of these measures. □

We obtain the same results in Theorem 4.1 for the estimator $\widehat{GI}^{*}_{\xi}(Z_{[l,m,p,r]})$ given in (4.2).

4.1. Simulation results

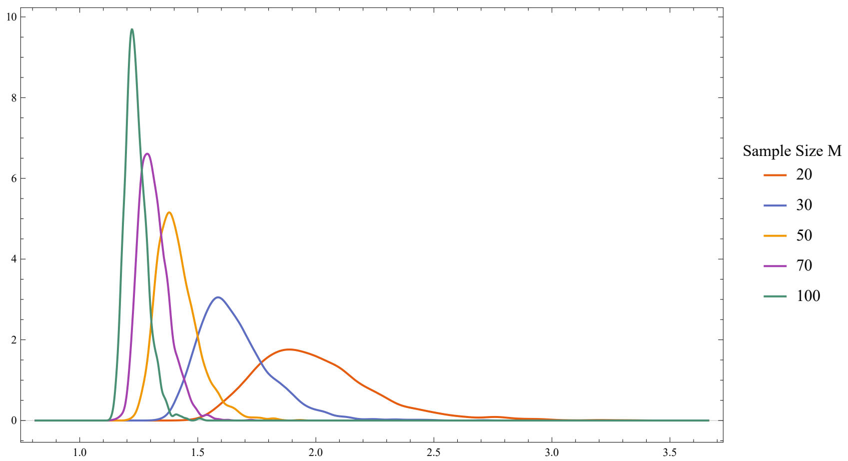
Equations (4.1) and (4.2) were used to calculate the average bias and SERM of the estimators for the standard exponential distribution in order to assess their performance. The computations were performed for sample sizes $M = 50, 100, 150$ with various choices of g and ξ. The results of 1000 simulation runs are reported in Tables 2 and 3, where the theoretical value $GI_{\xi}(Z_{[30,50,0,1]})$ was computed using Eq (2.1) with $\beta = 2$ and $\omega = 0.5$. We observe that the choice of window size g and the parameter ξ has a significant impact on the estimation results. The following observations can be made:

(1) The bias, SERM, and standard deviation (SD) of the generating informational function estimators for the system $Z_{[30,50,0,1]}$ decrease as the sample size M increases. This indicates that the estimators become more accurate and stable with larger sample sizes.

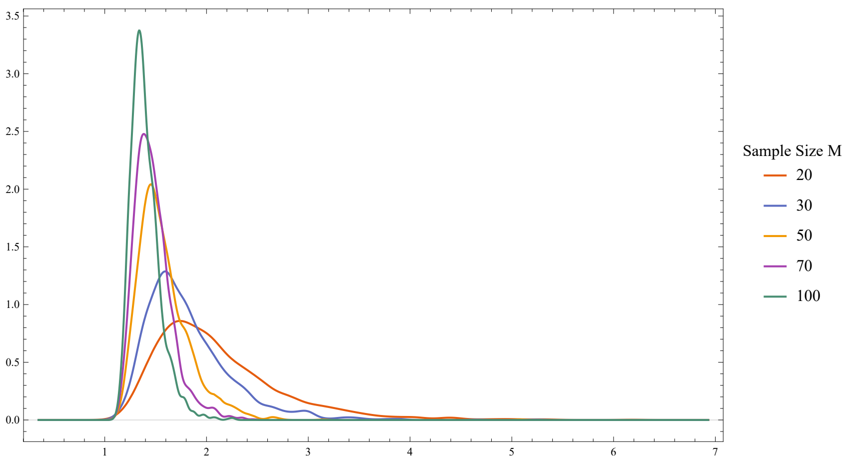
(2) For increasing ξ, and fixed g, the bias, SERM, and SD of the generating informational function estimators for the system $Z_{[30,50,0,1]}$ decrease.

(3) A comparison between the two estimation methods shows that the second estimator with the correction factor consistently outperforms the first one in terms of lower bias, SERM, and SD, demonstrating its superior efficiency.

Furthermore, assuming a standard uniform distribution, Figure 4 illustrates the estimated density functions of the generating informational function estimators $\widehat{GI}_{\xi}(Z_{[30,50,0,1]})$ and $\widehat{GI}^{*}_{\xi}(Z_{[30,50,0,1]})$ for $M = 20, 30, 50, 70$, and 100, with $\xi = 3$, $g = 10$, $\beta = 2$, and $\omega = 0.5$. It is observed that, as the sample size M increases, the distributions of both estimators become progressively more concentrated around the true value, indicating consistency and a reduction in variability. Moreover, the estimator $\widehat{GI}^{*}_{\xi}$, which incorporates the correction factor, exhibits a more pronounced concentration and reduced dispersion compared to $\widehat{GI}_{\xi}$. This suggests that the corrected estimator provides improved finite-sample performance and greater stability.



(a)



(b)

Figure 4. Estimated PDFs of the proposed estimators $\widehat{GI}_{\xi}(Z_{[30,50,0,1]})$ (left) and $\widehat{GI}^{*}_{\xi}(Z_{[30,50,0,1]})$ (right) for different sample sizes M following a standard uniform distribution.

Table 2. The bias, SERM, and SD findings of the estimator $\widehat{GI}_{\xi}(Z_{[30,50,0,1]})$ with traditional exponential distribution component lifetimes for different values of g and ξ with $\beta = 2$ and $\omega = 0.5$.

g	ξ	$M = 50$			$M = 100$			$M = 150$		
		Bias	SERM	SD	Bias	SERM	SD	Bias	SERM	SD
	2	0.303679	0.417678	0.286908	0.240428	0.268763	0.120177	0.224938	0.248548	0.105785
2	3	2.247761	20.964705	20.854288	1.009380	2.171864	1.924018	0.844591	1.696011	1.471490
	4	1208.362762	29597.536205	29587.656857	32.880886	398.489701	397.329536	14.121118	149.957756	149.366103
	2	0.241985	0.283881	0.148503	0.175401	0.197860	0.091605	0.158993	0.175998	0.075511
3	3	0.800002	2.172387	2.020729	0.521635	0.871800	0.698871	0.429005	0.594041	0.411106
	4	11.032093	110.293963	109.795748	5.323754	55.332113	55.102965	1.642567	6.006834	5.780783
	2	0.229648	0.269757	0.141600	0.156823	0.179283	0.086929	0.133894	0.149883	0.067393
4	3	0.547773	0.768350	0.539070	0.367342	0.490492	0.325188	0.317235	0.399054	0.242207
	4	1.979678	5.144661	4.750893	1.324803	4.123072	3.906390	0.697880	3.521418	1.545445
	2	0.231525	0.264086	0.127097	0.154906	0.176709	0.085073	0.122940	0.137815	0.062310
5	3	0.533537	0.726773	0.493742	0.352783	0.466574	0.305497	0.269416	0.328414	0.187900
	4	1.699990	5.581988	5.319485	0.848654	1.876567	1.674543	0.651967	1.449676	1.295445
	2	0.241693	0.272744	0.126451	0.156594	0.176629	0.081748	0.122973	0.137613	0.061798
6	3	0.549226	0.770305	0.540382	0.330541	0.432636	0.279274	0.260084	0.322991	0.191615
	4	1.539328	3.082138	2.671550	0.740059	1.207861	0.955067	0.568712	1.051150	0.884457
	2	0.261189	0.290002	0.126083	0.161849	0.182341	0.084024	0.123890	0.138224	0.061326
7	3	0.562161	0.751482	0.498948	0.323966	0.420439	0.268117	0.250649	0.316329	0.193070
	4	1.498610	3.840824	3.538165	0.720775	1.352126	1.144569	0.531812	0.862267	0.679074
	2	0.281136	0.311906	0.135152	0.163736	0.181115	0.077454	0.128001	0.142307	0.062216
8	3	0.579125	0.772932	0.512153	0.316921	0.393296	0.233018	0.242744	0.292274	0.162868
	4	1.413731	3.268595	2.948520	0.671390	1.018798	0.766662	0.506573	0.818484	0.643206
	2	0.301990	0.329243	0.131225	0.175040	0.192533	0.080227	0.132695	0.145924	0.060740
9	3	0.593244	0.723039	0.413542	0.323167	0.390666	0.219616	0.247879	0.293528	0.157288
	4	1.354531	2.188355	1.719622	0.741222	1.301293	1.070092	0.475846	0.677428	0.482402
	2	0.319571	0.348158	0.138230	0.186795	0.203985	0.081999	0.137237	0.149592	0.059559
10	3	0.650036	0.780035	0.431386	0.345274	0.416334	0.232752	0.249387	0.290745	0.149535
	4	1.358943	2.160659	1.680638	0.658654	1.009079	0.764853	0.466257	0.627834	0.420663
	2	0.340877	0.366949	0.135914	0.193662	0.211335	0.084645	0.143492	0.155332	0.059511
11	3	0.680155	0.831291	0.478186	0.356725	0.414095	0.210396	0.251989	0.293334	0.150230
	4	1.430477	2.122909	1.569376	0.711534	1.236222	1.011428	0.461980	0.621370	0.415749
	2	0.359474	0.383588	0.133924	0.205262	0.221205	0.082497	0.147459	0.158903	0.059243
12	3	0.718680	0.846041	0.446637	0.366399	0.427344	0.220053	0.265765	0.307703	0.155157
	4	1.463814	1.979502	1.333211	0.689121	0.895574	0.572269	0.486444	0.712865	0.521364
	2	0.377739	0.401156	0.135122	0.211285	0.226440	0.081488	0.155232	0.166510	0.060269
13	3	0.736128	0.848100	0.421388	0.393180	0.455796	0.230678	0.274738	0.316814	0.157846
	4	1.563609	2.121666	1.434800	0.723238	0.912484	0.556653	0.487306	0.630980	0.401036
	2	0.401283	0.424083	0.137249	0.222138	0.235661	0.078721	0.162551	0.174320	0.062995
14	3	0.792477	0.893314	0.412504	0.403653	0.461774	0.224387	0.280166	0.321080	0.156920
	4	1.574473	2.062371	1.332736	0.724954	0.926838	0.577757	0.493913	0.633142	0.396328
	2	0.413030	0.434193	0.133969	0.233209	0.246748	0.080650	0.167180	0.177663	0.060154
15	3	0.835522	0.950088	0.452521	0.417987	0.481895	0.239931	0.300003	0.336944	0.153470
	4	1.696730	2.169047	1.351924	0.803803	1.052032	0.679067	0.518590	0.650913	0.393583
	2	0.442101	0.464967	0.144090	0.243342	0.257597	0.084547	0.174875	0.185012	0.060430
16	3	0.863678	0.992733	0.489712	0.441913	0.494251	0.221464	0.305562	0.343971	0.158027
	4	1.727413	2.177051	1.325652	0.798962	0.996550	0.595925	0.554139	0.711689	0.446799
	2	0.463532	0.483984	0.139277	0.254576	0.267474	0.082098	0.184105	0.193856	0.060740
17	3	0.899851	1.016750	0.473575	0.455971	0.511495	0.231885	0.319469	0.359917	0.165853
	4	1.835836	2.303906	1.392705	0.838219	1.077909	0.678038	0.580944	0.744592	0.465982
	2	0.483483	0.502866	0.138338	0.267775	0.280729	0.084333	0.192380	0.202613	0.063607
18	3	0.958570	1.067253	0.469459	0.483810	0.539254	0.238286	0.324158	0.362620	0.162609
	4	1.912306	2.390369	1.434925	0.871470	1.092422	0.659059	0.579044	0.752809	0.481311
	2	0.500194	0.521421	0.147333	0.277863	0.290106	0.083431	0.199927	0.209219	0.061690
19	3	0.994412	1.090091	0.446813	0.494980	0.545135	0.228514	0.340930	0.375447	0.157327
	4	1.996387	2.486668	1.483293	0.895060	1.127709	0.686342	0.587445	0.743937	0.456683
	2	0.511272	0.530780	0.142649	0.288624	0.301088	0.085779	0.201924	0.210952	0.061082
20	3	1.076606	1.176614	0.474936	0.508494	0.558611	0.231373	0.361789	0.400574	0.172041
	4	2.144100	2.622371	1.510611	0.936843	1.122274	0.618229	0.623498	0.743341	0.404931

Table 3. The bias, SERM, and SD findings of the estimator $\widehat{GI}^*_{\xi}(Z_{[30,50,0,1]})$ with traditional exponential distribution component lifetimes for different values of g and ξ with $\beta = 2$ and $\omega = 0.5$.

g	ξ	$M = 50$			$M = 100$			$M = 150$		
		Bias	SERM	SD	Bias	SERM	SD	Bias	SERM	SD
2	2	0.250276	0.316977	0.194614	0.216794	0.241573	0.106627	0.207797	0.226095	0.0891467
	3	1.07716	7.49645	7.42237	0.686105	1.00027	0.728242	0.649682	0.859788	0.563442
	4	153.033	3699.92	3698.61	5.99371	50.5292	50.1975	3.95743	21.7784	21.4265
3	2	0.19403	0.233885	0.130659	0.152277	0.173376	0.0829332	0.142868	0.157894	0.067259
	3	0.463309	0.881668	0.750497	0.368735	0.48351	0.312912	0.329899	0.396525	0.220107
	4	2.17374	14.687	14.5326	1.38365	7.94118	7.82363	0.901979	3.92207	3.81886
4	2	0.174504	0.211332	0.119264	0.130153	0.152362	0.0792496	0.116789	0.133004	0.0636746
	3	0.327844	0.444423	0.300201	0.252615	0.325523	0.205411	0.23891	0.285518	0.156419
	4	0.698521	1.2543	1.04232	0.547302	0.915577	0.734358	0.456678	0.764258	0.632845
5	2	0.169993	0.203189	0.111357	0.125096	0.146663	0.0765952	0.103588	0.119095	0.0587933
	3	0.303143	0.411007	0.277683	0.231608	0.289444	0.173683	0.194165	0.235522	0.133372
	4	0.550446	1.15038	1.01065	0.382251	0.600821	0.463772	0.342566	0.533248	0.408863
6	2	0.170705	0.202338	0.108685	0.122616	0.143681	0.0749338	0.101581	0.117126	0.0583359
	3	0.29336	0.406076	0.28092	0.207176	0.265859	0.166695	0.177891	0.219866	0.129279
	4	0.506151	0.821854	0.647822	0.329984	0.466799	0.330335	0.284526	0.39799	0.27842
7	2	0.181059	0.210622	0.107661	0.124479	0.146015	0.0763619	0.0996242	0.114667	0.0568053
	3	0.294229	0.406401	0.280481	0.195276	0.253573	0.161841	0.165091	0.205279	0.122062
	4	0.502508	1.0217	0.890027	0.298709	0.449469	0.336018	0.258425	0.363429	0.255661
8	2	0.191273	0.222372	0.113476	0.123049	0.141835	0.0705765	0.101426	0.116604	0.0575554
	3	0.294957	0.38765	0.251667	0.18465	0.237894	0.150069	0.155434	0.191691	0.112242
	4	0.456506	0.857627	0.726398	0.273639	0.391877	0.280656	0.229987	0.347373	0.260465
9	2	0.202219	0.230692	0.111079	0.129159	0.14765	0.0715807	0.102938	0.117497	0.0566791
	3	0.290728	0.366535	0.223327	0.181241	0.228738	0.139614	0.155982	0.189768	0.108135
	4	0.429094	0.663314	0.506083	0.281426	0.438281	0.336159	0.219822	0.298675	0.202302
10	2	0.208442	0.238306	0.115563	0.136225	0.154515	0.0729586	0.104494	0.117739	0.0542814
	3	0.318589	0.398503	0.239506	0.191886	0.242932	0.149056	0.150657	0.18304	0.104005
	4	0.421531	0.648565	0.493145	0.250531	0.359751	0.258305	0.202485	0.270084	0.178822
11	2	0.219618	0.246848	0.112759	0.139309	0.158612	0.0758727	0.108507	0.121459	0.054603
	3	0.321382	0.409926	0.254594	0.195356	0.238972	0.137705	0.148918	0.1832	0.106757
	4	0.441369	0.639067	0.462401	0.264727	0.40048	0.300657	0.194128	0.256757	0.168128
12	2	0.227528	0.252806	0.110245	0.145726	0.163105	0.0732979	0.109524	0.122093	0.0539817
	3	0.335872	0.413088	0.240603	0.195441	0.240039	0.139431	0.154055	0.18735	0.106669
	4	0.463325	0.625633	0.420623	0.25452	0.340233	0.225897	0.197508	0.264266	0.175664
13	2	0.234757	0.259091	0.109678	0.147933	0.164311	0.0715491	0.114551	0.126955	0.0547612
	3	0.338008	0.408303	0.229161	0.207731	0.252959	0.14442	0.157054	0.192181	0.110815
	4	0.470728	0.649565	0.447829	0.263468	0.345658	0.223862	0.197758	0.262734	0.173063
14	2	0.246762	0.27043	0.110693	0.153977	0.168664	0.0688726	0.118742	0.131591	0.0567429
	3	0.354342	0.416293	0.218607	0.20791	0.250632	0.140033	0.156959	0.189501	0.106234
	4	0.464952	0.625172	0.418131	0.252887	0.335725	0.220925	0.192872	0.253216	0.164151
15	2	0.247269	0.269321	0.106785	0.160142	0.175073	0.0707804	0.120836	0.132453	0.0542717
	3	0.372329	0.444067	0.242127	0.215858	0.262046	0.148646	0.16593	0.195635	0.103688
	4	0.500959	0.666708	0.440151	0.27563	0.363873	0.237672	0.197289	0.253174	0.158743
16	2	0.261763	0.285695	0.114521	0.165149	0.180619	0.0731741	0.12557	0.136834	0.0543919
	3	0.368337	0.440537	0.241783	0.223969	0.261421	0.134896	0.166233	0.197541	0.106773
	4	0.494098	0.637799	0.403507	0.277696	0.355245	0.221659	0.207878	0.26848	0.169991
17	2	0.271251	0.292479	0.109447	0.17138	0.185294	0.0704834	0.13146	0.142413	0.0547985
	3	0.381363	0.44926	0.237599	0.227691	0.266989	0.139497	0.170488	0.202482	0.109291
	4	0.505237	0.650363	0.409726	0.27967	0.359429	0.225887	0.215887	0.275593	0.171388
18	2	0.277955	0.297957	0.107381	0.178614	0.192762	0.0725225	0.135646	0.147039	0.0567805
	3	0.400686	0.466531	0.239078	0.241825	0.280798	0.142789	0.172816	0.203014	0.106587
	4	0.518733	0.662334	0.412034	0.290115	0.375469	0.238468	0.211156	0.272814	0.172831
19	2	0.2813	0.303065	0.112833	0.183162	0.196379	0.0708613	0.140812	0.151172	0.0550267
	3	0.404814	0.46197	0.222693	0.241424	0.276802	0.13547	0.179639	0.207387	0.103682
	4	0.53372	0.684195	0.428309	0.295969	0.386695	0.248993	0.214239	0.277031	0.175723
20	2	0.281077	0.300973	0.107665	0.189332	0.202724	0.0724964	0.140544	0.150709	0.0544377
	3	0.439294	0.501597	0.242238	0.24737	0.283544	0.138652	0.188461	0.217326	0.108281
	4	0.56571	0.708171	0.426215	0.307325	0.38067	0.224747	0.220815	0.272354	0.159509

4.2. Real data application

This subsection looks at a real-world dataset to show the environmental gamma radiation dose rate as determined by the national network for continuously environment radiation monitoring and early warning in the Kingdom of Saudi Arabia. The measurements, which show the normal variation in the radiation from the environmental background concentrations at the Turaib during April 2, 2024, are displayed in (nSv/h), which yields 144 observations. The Saudi Open Data Portal provides public access to the dataset at: https://open.data.gov.sa/en/datasets/view/960fb600-e757-446a-8283-1caec0b132ff/resources.

To assess the independence assumption, the radiation measurements were analyzed as a temporal sequence. Figure 5 presents the time series, sample autocorrelation function (ACF), and partial autocorrelation function (PACF). The ACF and PACF exhibit no significant autocorrelations, and the Ljung–Box test yields $X^2 = 17.83$ with 20 degrees of freedom ($p-value = 0.5986$). Therefore, the null hypothesis of no serial correlation cannot be rejected, indicating that the observations may reasonably be regarded as independent for the subsequent analysis.

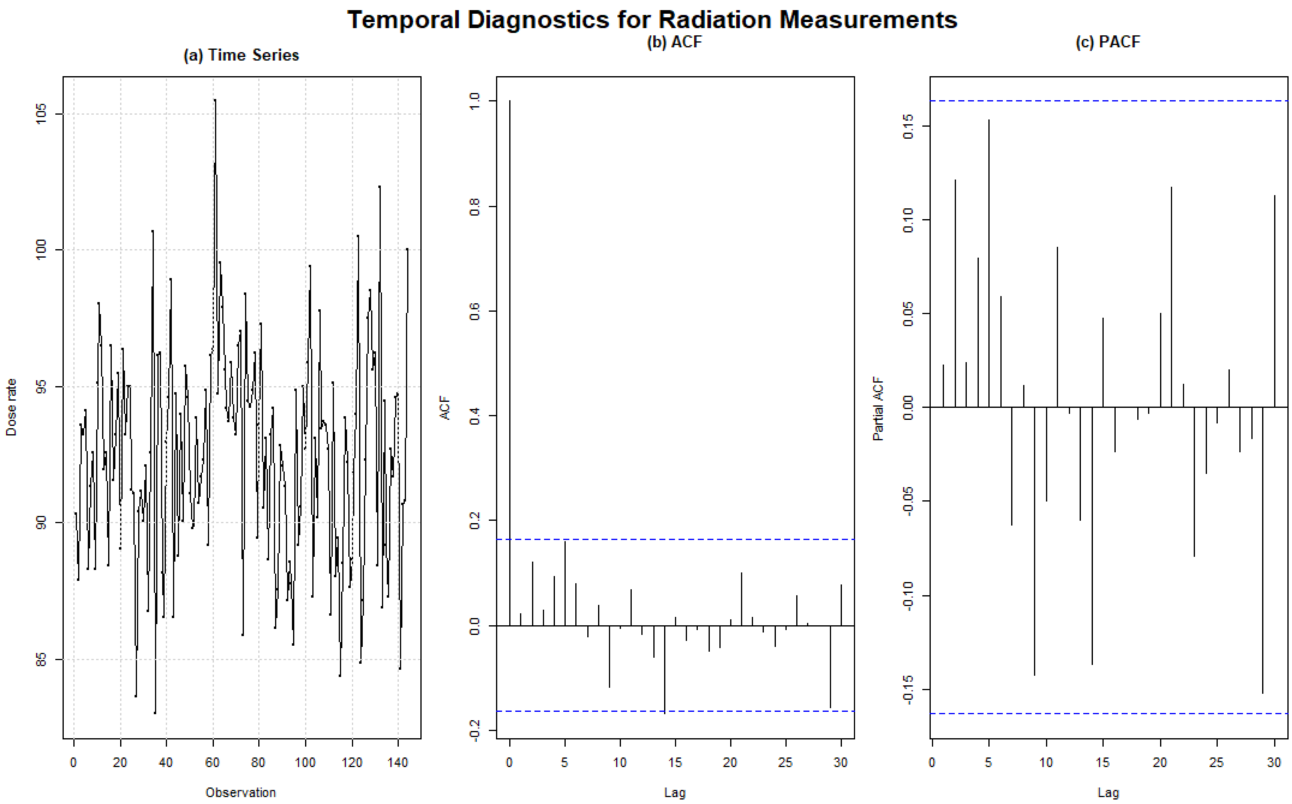


Figure 5. Temporal diagnostics for the environmental gamma radiation dose-rate measurements. Panel (a) displays the observations in their recorded order, while Panels (b) and (c) present the ACF and PACF, respectively.

The observed dataset was fitted using a gamma distribution, whose PDF is given by

$$h_Z(z) = \frac{z^{\alpha-1}\exp(-z/\theta)}{\Gamma(\alpha)\theta^\alpha}, \ \alpha, \theta > 0, \ z \geq 0.$$

The unknown parameters were estimated via the maximum likelihood method. The resulting estimates are $\alpha = 530.8017$ for the shape parameter and $\theta = 0.1741854$ for the scale parameter. To examine how well the proposed model describes the data, two standard goodness-of-fit tests were applied:

(1) Kolmogorov–Smirnov statistic = 0.059145 with p-value = 0.695;

(2) Anderson–Darling statistic = 0.42317 with p-value = 0.8252.

The relatively large p-values obtained from both tests suggest that the gamma distribution adequately captures the behavior of the sample data. Moreover, the plots in Figure 6 indicate an overall good agreement between the gamma model and the observed data, with minor deviations in the tails.

The gamma distribution was adopted for the subsequent analysis because it provided an adequate fit to the observed radiation data according to the Kolmogorov–Smirnov and Anderson–Darling goodness-of-fit tests. The purpose of this application is to demonstrate the implementation of the proposed methodology rather than to perform a comprehensive comparison among competing lifetime distributions. Nevertheless, other positive-support models, such as the Weibull and lognormal distributions, may also be considered in future studies.

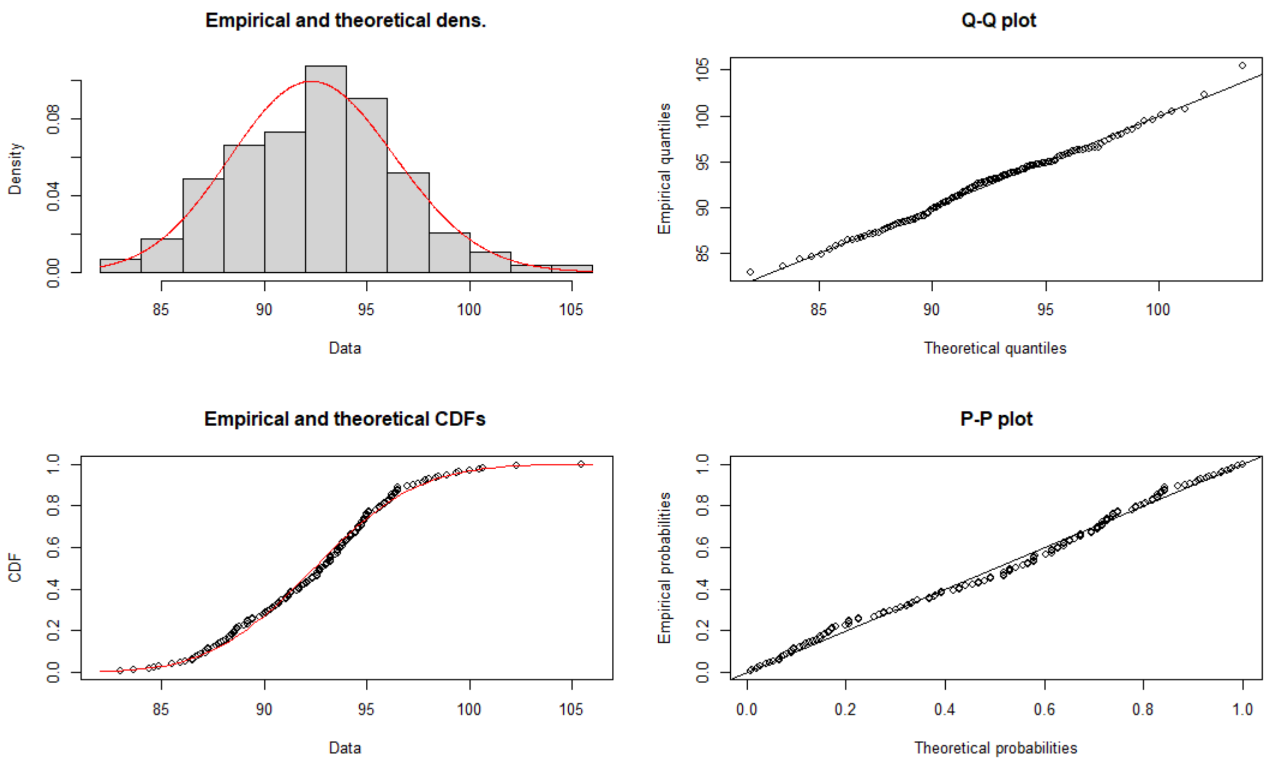


Figure 6. Diagnostic plots for the fitted gamma distribution: Empirical and fitted density (top left), Q–Q plot (top right), empirical and theoretical CDFs (bottom left), and P–P plot (bottom right).

Figure 7 presents the deviation from the fitted parametric benchmark and the proposed estimators $\widehat{GI}_{\xi}(Z_{[30,50,0,1]})$ and $\widehat{GI}^{*}_{\xi}(Z_{[30,50,0,1]})$, denoted by DFPB, for the real dataset under different values of g and ξ. The results show that the DFPB is strongly affected by both parameters. For all considered values of ξ, the DFPB decreases rapidly for small values of g and reaches values close to zero around moderate values of g. As ξ increases from 3 to 5, the DFPB becomes substantially smaller, indicating an improvement in estimation accuracy.

A comparison of the two estimators reveals that the modified estimator $\widehat{GI}^*_\xi(Z_{[30,50,0,1]})$ generally yields lower DFPB than $\widehat{GI}_\xi(Z_{[30,50,0,1]})$, particularly for larger values of g when $\xi = 3$. Although both estimators display similar overall trends, the modified estimator maintains a more stable DFPB pattern and remains closer to zero across a wider range of parameter values. These findings suggest that the DFPB-corrected estimator provides a modest but noticeable improvement in estimation accuracy while preserving the overall robustness of the original estimator.

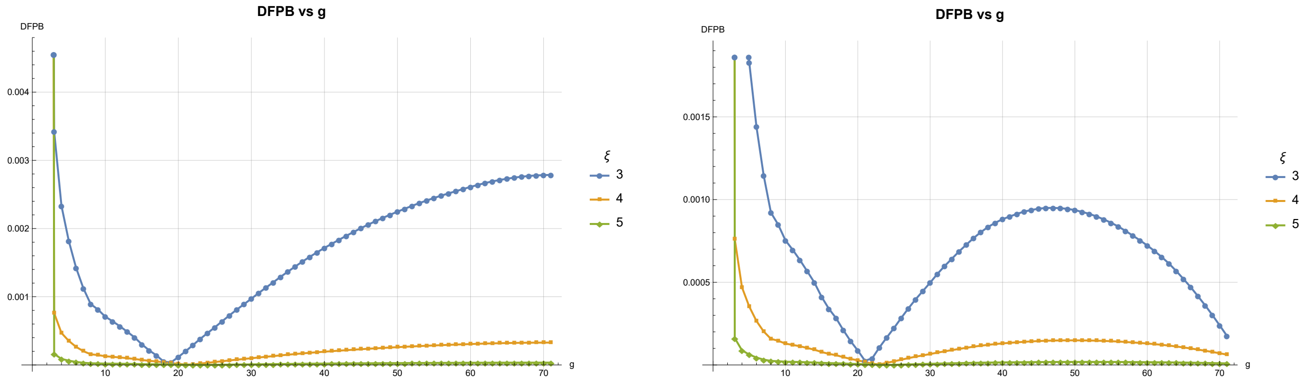


Figure 7. The DFPB of the theoretical value and the proposed estimators $\widehat{GI}_\xi(Z_{[30,50,0,1]})$ (left) and $\widehat{GI}^*_\xi(Z_{[30,50,0,1]})$ (right) for the normal variation in the radiation from the environmental background concentrations.

To evaluate the finite-sample performance and stability of the proposed estimators $\widehat{GI}_\xi(Z_{[30,50,0,1]})$ and $\widehat{GI}^*_\xi(Z_{[30,50,0,1]})$, a bootstrap-based convergence analysis is conducted. Specifically, a resampling scheme with $B = 500$ bootstrap replications is implemented for different combinations of the window size g and the tuning parameter ξ. For each replication, the estimator is recalculated, and its running mean is tracked to assess convergence behavior. In addition, key bootstrap statistics, including the mean, SD, and percentile-based confidence intervals (CI), are computed. Figures 8 and 9 illustrate the evolution of the running means across iterations, while the corresponding numerical results are summarized in Table 4. This combined graphical and numerical analysis provides insight into the stability, variability, and sensitivity of the estimator under different parameter settings.

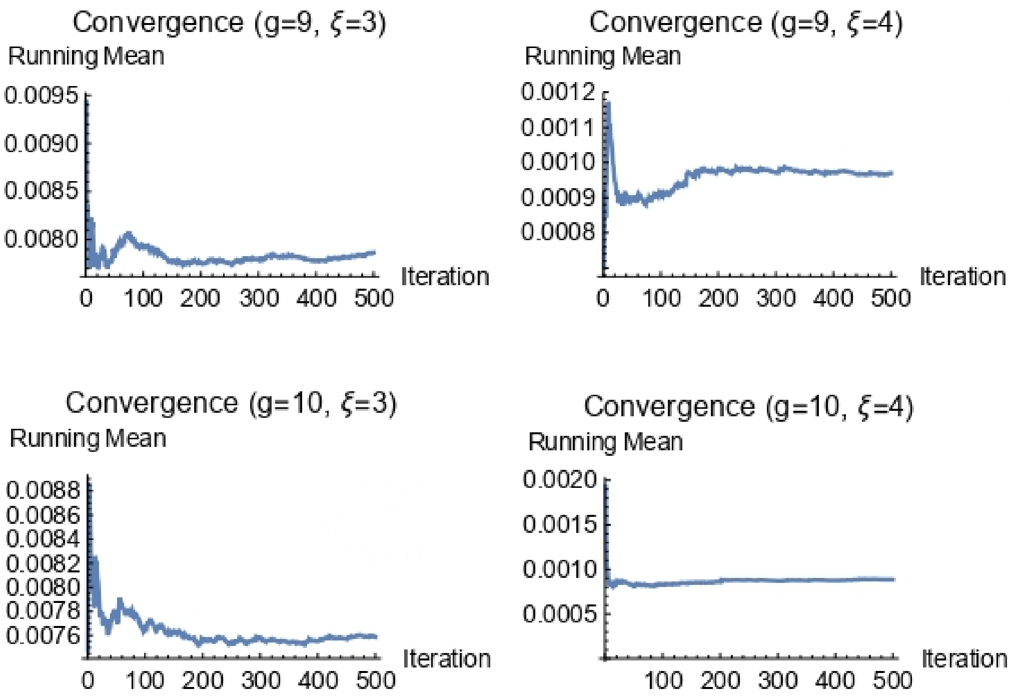


Figure 8. Bootstrap convergence of the estimator $\widehat{GI}_{\xi}(Z_{[30,50,0,1]})$ using $B = 500$ replications of the normal variation in the radiation from the environment background concentrations.

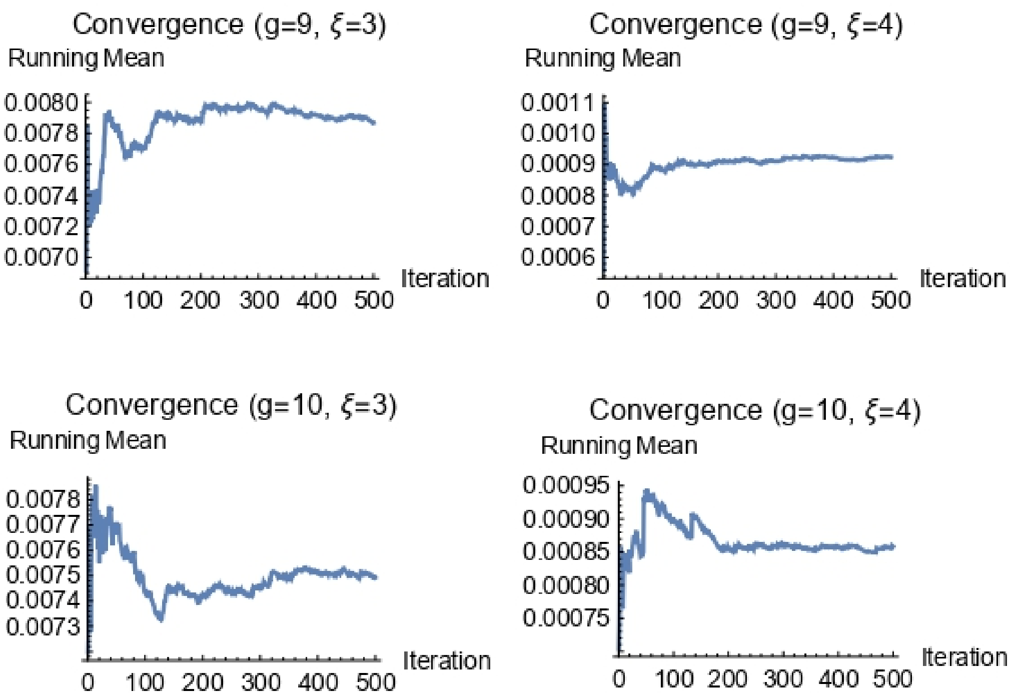


Figure 9. Bootstrap convergence of the estimator $\widehat{GI}^{*}_{\xi}(Z_{[30,50,0,1]})$ using $B = 500$ replications of the normal variation in the radiation from the environmental background concentrations.

Table 4. Bootstrap results for $\widehat{GI}_\xi(Z_{[30,50,0,1]})$ and $\widehat{GI}^*_\xi(Z_{[30,50,0,1]})$ for different values of g and ξ. The table reports the bootstrap mean, SD, and the 95% CI obtained from the bootstrap samples.

g	ξ	$\widehat{GI}_\xi(Z_{[30,50,0,1]})$				$\widehat{GI}^*_\xi(Z_{[30,50,0,1]})$			
		Mean	SD	CI Lower	CI Upper	Mean	SD	CI Lower	CI Upper
9	3	0.00781215	0.00163255	0.00556887	0.011709	0.00690941	0.00145572	0.00486465	0.0106866
9	4	0.000941256	0.000493098	0.000443288	0.00197795	0.000810005	0.000434604	0.000412884	0.00185275
10	3	0.00760114	0.0015417	0.00534299	0.0116567	0.00657412	0.00127584	0.00476481	0.00971742
10	4	0.000891988	0.000366593	0.000464934	0.00190604	0.000723652	0.000298462	0.000398113	0.00152884

The results demonstrate that the proposed estimators exhibit stable and consistent behavior under the bootstrap framework. As shown in Figures 8 and 9, the running means converge rapidly and stabilize well before the maximum number of iterations, indicating strong numerical reliability. The bootstrap summaries in Table 4 further confirm this observation, as the estimated means remain close across different configurations, while the standard deviations are relatively small, leading to reasonably tight CI.

Moreover, the findings reveal that the estimators are sensitive to the choice of the parameter ξ, with moderate values of ξ producing smaller estimates and reduced variability. On the other hand, increasing the window size g slightly improves stability by reducing dispersion without significantly altering the central tendency. Overall, moderate parameter choices (e.g., $g = 9$ or 10) provide a good balance between accuracy and variability. These results confirm that the estimator is robust, convergent, and suitable for practical applications involving real data.

5. Conclusions

This study investigated the generating informational function of the concomitants of p-generalized order statistics under the extension proposed by Huang and Kotz, subject to the constraints $\xi \geq 1$, $\omega \neq 0$, and $\beta \geq 1$. Several expansions of the proposed model were developed and analyzed from different perspectives. Due to the analytical complexity of the model, obtaining closed-form expressions is generally difficult. Therefore, the formulation based on the uniform distribution was adopted, providing a flexible and tractable framework for analysis. In addition, applications involving lifetime distributions were presented through the generating informational function of concomitants of ordinary order statistics. Furthermore, stochastic ordering comparisons were established, accompanied by illustrative examples for the concomitants of ordinary order statistics. Based on these comparisons, several characterization results were derived using stochastic ordering techniques. To address the difficulty of deriving exact entropy expressions, especially for large values of m or in the presence of heterogeneous feature behavior, a set of informative bounds was obtained. These approximations not only enhance the theoretical understanding of the generating informational function in complex system structures but also facilitate practical computation. Moreover, important general characterization results were established for both uniform and exponential distributions, highlighting the significance and applicability of the proposed model. From an application standpoint, two non-parametric estimation procedures were introduced, and their consistency and asymptotic normality were investigated. Numerical results demonstrated the performance of both estimators, indicating that the estimator incorporating a correction factor is more efficient. Finally, a bootstrap convergence analysis of the estimators was conducted, emphasizing the impact of selecting appropriate values for the window size g and the parameter ξ.

Finally, several promising directions for future research remain. These include extending the proposed methodology to heavy-tailed and other lifetime distributions, comparing the generating informational function with other information measures such as Shannon and Rényi entropies, and investigating additional applications in reliability and environmental monitoring. Another important direction is to conduct a comprehensive comparison between the proposed non-parametric estimators and recent distribution estimation and reliability assessment methodologies, including the generative data–physics fusion framework of Song et al. [29], to further assess their relative strengths, computational efficiency, and practical applicability.

Author contributions

Mohamed Said Mohamed and Hanan H. Sakr: Conceptualization, formal analysis, investigation, methodology, software, visualization, resources, writing—original draft, validation, and writing—review & editing. They have read and approved the final version of the article for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors report no conflicts of interest in connection with this research.

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