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Research article

A flexible moment exponential distribution based on confluent hypergeometric functions: properties, inference, and applications

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Abstract: In this paper, we introduced a novel extension of the moment exponential (ME) distribution using a confluent hypergeometric construction. The proposed model was developed through a slash-type mechanism to provide greater flexibility in modeling kurtosis. We derived the general form of the probability density function and study several of its statistical properties, including moments, skewness, and kurtosis coefficients. Statistical inference was conducted using both the method of moments and maximum likelihood estimation, the latter implemented via the expectation-maximization (EM) algorithm. A simulation study was performed to evaluate the finite-sample performance of the maximum likelihood estimators. Finally, the proposed model was applied to real datasets exhibiting high kurtosis, showing improved fitting performance compared to the classical ME distribution.

Keywords: moment exponential distribution; Kurtosis; maximum likelihood estimation; Slash distribution; EM algorithm
Mathematics Subject Classification: 62E15, 62F12, 62P30

1. Introduction

The slash methodology is based on the ratio of two independent continuous random variables. Its origin can be traced back to the classical slash distribution, which is a symmetric extension of the standard normal distribution. Specifically, a random variable T is said to follow a slash distribution if it can be expressed as

$$T = \frac{Z}{U}, \tag{1.1}$$

where $Z \sim N(0, 1)$, $U \sim \text{Beta}(q, 1)$ with $q > 0$, and Z is independent of U (see Johnson and Kotz [1]). This construction produces heavier tails than the normal distribution, making it suitable for modeling data with extreme observations. Properties, inference, and extensions of this family have been widely studied in Rogers and Tukey [2], Mosteller and Tukey [3], Kadafar [4], Wang and Genton [5].

Dara and Ahmad [6] introduced the ME distribution. A random variable Y is said to follow an ME distribution if its probability density function (PDF) is given by

$$f_Y(y; \beta) = \frac{y}{\beta^2} \exp(-y/\beta), \quad y > 0,\ \beta > 0, \tag{1.2}$$

denoted by $Y \sim ME(\beta)$.

Iriarte et al. [7] extended the ME distribution using the slash methodology by considering $Z \sim ME(\beta)$ and $U \sim \text{Beta}(q, 1)$ in (1), leading to the slash moment exponential (SME) distribution. A random variable T follows an SME distribution if its PDF is given by

$$f_T(t; \beta, q) = q\beta^q t^{-(q+1)} \gamma(q + 2, t/\beta), \tag{1.3}$$

where $t > 0, \beta, q > 0$, and $\gamma(\alpha, v) = \int_0^v u^{\alpha-1} \exp(-u)\, du$ denote the lower incomplete gamma function. This model provides heavier tails than the classical ME distribution.

A key mathematical tool in this work is the confluent hypergeometric function (see Abramowitz and Stegun [8]), defined as

$${}_1F_1(a, b; t) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b - a)} \int_0^1 v^{a-1}(1 - v)^{b-a-1} \exp(tv)\, dv, \quad b > a > 0, \tag{1.4}$$

where $\Gamma(\cdot)$ denotes the gamma function.

The main objective of this paper is to introduce a new extension of the ME distribution based on the slash methodology and incorporating a confluent hypergeometric structure. The proposed model is designed to exhibit heavier tails than the SME distribution, thereby offering enhanced flexibility for modeling highly leptokurtic data and outliers.

The motivation for this work arises from the inherent limitations of standard positive distributions, such as the exponential, Weibull, and gamma, and even the recently introduced ME distribution, when applied to real-world data characterized by extreme kurtosis, high skewness, and atypical observations (outliers). In fields such as reliability engineering, finance, and lifetime analysis, observations in the upper tail are common. Since the baseline ME distribution decays exponentially fast, it systematically underestimates the probability of extreme events, leading to poor fit and unreliable predictive inferences. Although some heavy-tailed variants, such as the SME distribution, have been proposed using the standard slash framework, they often lack the structural flexibility needed to accommodate highly leptokurtic data without distorting the modal body of the distribution.

To address these shortcomings, our proposed confluent hypergeometric moment exponential (CHME) distribution departs from traditional slash-type extensions that rely on an asymmetric Beta$(q, 1)$ mechanism. Instead, our approach employs a symmetric Beta(q, q) mixing distribution in a novel hierarchical scale-mixture formulation. This architectural choice constitutes the core novelty of our work and yields three major contributions:

- It establishes a highly flexible tail-modulating mechanism capable of absorbing exceptionally high kurtosis while preserving the baseline profile.
- It allows us to derive explicit, tractable closed-form expressions for the probability density, cumulative distribution, survival, and hazard functions via the confluent hypergeometric function, thus avoiding unexpressed integrals.
- It enables the implementation of a customized and stable EM algorithm based on its latent structure.

The remainder of the paper is organized as follows. In Section 2, we present the proposed distribution, derive its PDF, and study its main properties, including moments, survival function, and hazard function. In Section 3, we develop statistical inference procedures based on the method of moments, maximum likelihood estimation via the EM algorithm, and Bayesian estimation via Gibbs sampler with metropolis-hastening sub-steps, and conduct a simulation study to evaluate the performance of the estimators, comparing the EM and Bayesian methods. Section 4 illustrates the usefulness of the proposed model by applying it to real datasets exhibiting high kurtosis. Finally, Section 5 concludes the paper.

2. Density function and properties

In this section, we introduce the stochastic representation, PDF, and main properties of the proposed distribution.

2.1. Stochastic representation

The proposed distribution is defined through the following stochastic representation:

$$X = \frac{Y}{2Z}, \tag{2.1}$$

where $Y \sim \mathrm{ME}(\beta)$ and $Z \sim \mathrm{Beta}(q, q)$ are independent random variables, with $\beta > 0$ and $q > 0$. This formulation uses a slash-type mechanism designed to produce heavier tails than the baseline model. Here, the base variable Y governs the core shape, while the latent variable $Z \in (0, 1)$ introduces a random scaling effect. Because Z is bounded between 0 and 1, dividing by it acts as an inflation factor that stretches the right tail of the distribution, with the shape parameter q controlling the intensity of this tail-weighting effect. Hierarchically, this relationship implies that given a fixed manifestation $Z = z$, the model simplifies to $X \mid Z = z \sim \mathrm{ME}\left(\frac{\beta}{2z}\right)$. The resulting unconditional distribution of X is called the *confluent hypergeometric moment exponential* (CHME) distribution, denoted by $X \sim CHME(\beta, q)$.

2.2. Probability density function

The following proposition provides the PDF of the CHME distribution.

Proposition 1. *Let $X \sim CHME(\beta, q)$. Then, the PDF of X is given by*

$$f_X(x;\beta,q) = \frac{2(q+1)x\exp(-2x/\beta)}{\beta^2(2q+1)}\, {}_1F_1\left(q, 2q+2, \frac{2x}{\beta}\right), \tag{2.2}$$

for $x > 0$, $\beta > 0$, and $q > 0$, where ${}_1F_1$ denotes the confluent hypergeometric function.

Proof. Using the stochastic representation in (2.1) and applying the Jacobian transformation with

$$X = \frac{Y}{2Z}, \quad W = Z,$$

we obtain the inverse transformation $Y = 2XW$, $Z = W$, with Jacobian $|J| = 2w$. Thus, the joint PDF of (X, W) is

$$f_{X,W}(x, w) = 2w f_Y(2xw) f_Z(w), \quad 0 < w < 1,\ x > 0.$$

Marginalizing with respect to W yields

$$f_X(x;\beta,q) = \frac{4x}{\beta^2 B(q,q)} \int_0^1 w^{q+1}(1-w)^{q-1}\exp(-2xw/\beta)\, dw. \tag{2.3}$$

To solve this integral in terms of the confluent hypergeometric function, we apply the change of variable $u = 1 - w$ (so $dw = -du$). The integral in (2.3) becomes

$$\begin{aligned}\int_0^1 (1-u)^{q+1}u^{q-1}\exp\left(-\frac{2x}{\beta}(1-u)\right) du &= \exp\left(-\frac{2x}{\beta}\right)\int_0^1 u^{q-1}(1-u)^{(2q+2)-q-1}\exp\left(\frac{2x}{\beta}u\right) du \\ &= \exp\left(-\frac{2x}{\beta}\right) B(q, q+2)\, {}_1F_1\left(q, 2q+2, \frac{2x}{\beta}\right),\end{aligned}$$

where we have used the integral representation of the confluent hypergeometric function given in (1.4). Substituting this result back into the marginal density f_X, we get

$$f_X(x;\beta,q) = \frac{4x\exp(-2x/\beta)}{\beta^2}\frac{B(q,q+2)}{B(q,q)}\, {}_1F_1\left(q, 2q+2, \frac{2x}{\beta}\right).$$

Finally, expanding the Beta functions in terms of Gamma functions gives

$$\frac{B(q,q+2)}{B(q,q)} = \frac{\Gamma(q)\Gamma(q+2)/\Gamma(2q+2)}{\Gamma(q)\Gamma(q)/\Gamma(2q)} = \frac{\Gamma(q+2)\Gamma(2q)}{\Gamma(2q+2)\Gamma(q)} = \frac{q(q+1)\Gamma(q)\Gamma(2q)}{2q(2q+1)\Gamma(2q)\Gamma(q)} = \frac{q+1}{2(2q+1)}.$$

Multiplying this ratio by $\frac{4x}{\beta^2}$ simplifies to the exact leading constant in (2.2), which completes the proof. □

Figure 1 shows the PDF of the CHME distribution by fixing $\beta = 1$ and varying $q = 1, 2, 3, 4$. It can be observed that parameter q indeed gives the distribution higher kurtosis.

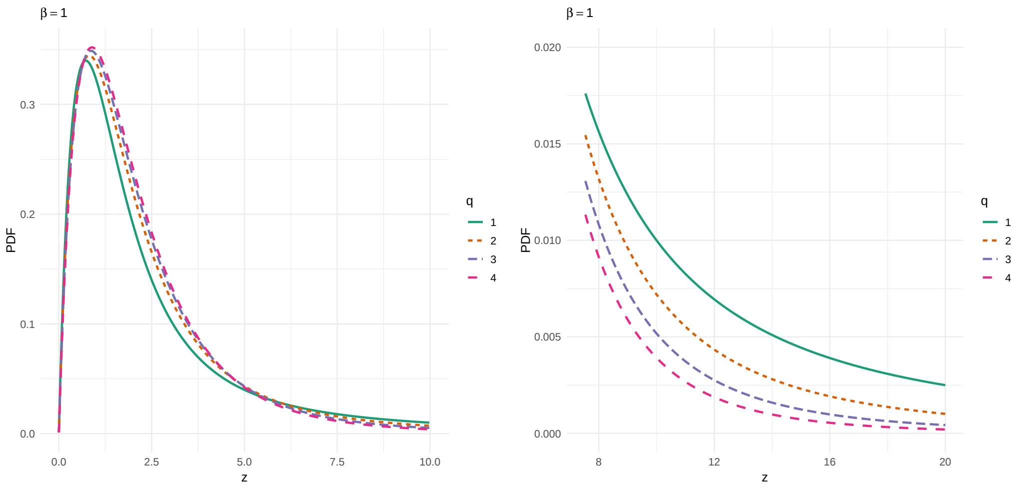

Figure 1. PDF of the CHME distribution, with $\beta = 1$ and varying q.

Asymptotic tail behavior

To rigorously justify how the confluent hypergeometric construction uniquely contributes to tail flexibility, we analyze the asymptotic behavior of $f_X(x; \beta, q)$ as $x \to \infty$. It is well known [8] that the confluent hypergeometric function satisfies the following asymptotic expansion for large values of its argument t:

$$ {}_1F_1(a, b, t) \sim \frac{\Gamma(b)}{\Gamma(a)} e^t t^{a-b}, \quad \text{as } t \to \infty. \tag{2.4} $$

By substituting $a = q$, $b = 2q + 2$, and $t = \frac{2x}{\beta}$ into (2.4), the PDF of the CHME distribution for large x can be expressed as:

$$ \begin{aligned} f_X(x; \beta, q) &\sim \frac{2(q+1)x \exp(-2x/\beta)}{\beta^2(2q+1)} \left[\frac{\Gamma(2q+2)}{\Gamma(q)} \exp\left(\frac{2x}{\beta}\right) \left(\frac{2x}{\beta}\right)^{-q-2} \right] \\ &= \left[\frac{2(q+1)\Gamma(2q+2)}{\beta^2(2q+1)\Gamma(q)} \left(\frac{2}{\beta}\right)^{-q-2} \right] x^{-(q+1)}. \end{aligned} $$

Consequently, as $x \to \infty$, the density function satisfies $f_X(x; \beta, q) \propto x^{-(q+1)}$, which proves a heavy-tailed, power-law decay of Pareto type, where the parameter q directly controls the tail index.

The unique contribution of this construction over traditional heavy-tailed extensions — such as the SME, which uses a Beta$(q, 1)$ factor — lies in the mixing kernel. The symmetric Beta(q, q) density incorporates an additional modulating term $(1 - z)^{q-1}$. When $q < 1$, the mixing distribution becomes U-shaped. This simultaneously places heavy probability mass near zero — acting as a severe scale-inflator that generates extremely heavy tails to capture distant outliers — and near one, which preserves the baseline ME mode intact. This dual action provides a highly adaptive transition from the body to the tail, explaining its superior performance in modeling highly leptokurtic datasets.

2.3. *Alternative representation*

Proposition 2. *If* $X \mid Z = z \sim ME\left(\frac{\beta}{2z}\right)$ *and* $Z \sim Beta(q, q)$*, then* $X \sim CHME(\beta, q)$*.*

Proof. By definition of the hierarchical setup, the conditional distribution of X given $Z = z$ follows a Moment Exponential distribution with scale parameter $\theta = \frac{\beta}{2z}$. Its probability density function is given by:

$$f_{X|Z}(x|z) = \frac{x}{\theta^2} \exp\left(-\frac{x}{\theta}\right) = \frac{4z^2 x}{\beta^2} \exp\left(-\frac{2xz}{\beta}\right), \quad x > 0.$$

Furthermore, since the latent variable Z follows a symmetric Beta distribution, $Z \sim \text{Beta}(q, q)$, its probability density function is expressed as:

$$f_Z(z) = \frac{1}{B(q,q)} z^{q-1}(1-z)^{q-1}, \quad 0 < z < 1.$$

The marginal density of X is obtained by integrating the joint density $f_{X,Z}(x,z) = f_{X|Z}(x|z) f_Z(z)$ over the entire support of Z:

$$\begin{aligned} f_X(x;\beta,q) &= \int_0^1 f_{X|Z}(x|z) f_Z(z)\, dz \\ &= \int_0^1 \left[\frac{4z^2 x}{\beta^2} \exp\left(-\frac{2xz}{\beta}\right)\right]\left[\frac{1}{B(q,q)} z^{q-1}(1-z)^{q-1}\right] dz \\ &= \frac{4x}{\beta^2 B(q,q)} \int_0^1 z^{q+1}(1-z)^{q-1} \exp\left(-\frac{2xz}{\beta}\right) dz. \end{aligned}$$

This integral expression is mathematically identical to the marginalization integral obtained in the proof of Proposition 1, with z serving as the variable of integration instead of w. Consequently, by applying the exact same algebraic reduction, the change of variable $u = 1 - z$, and the confluent hypergeometric integral definition, we arrive at:

$$f_X(x;\beta,q) = \frac{2(q+1)x \exp(-2x/\beta)}{\beta^2(2q+1)}\, {}_1F_1\left(q, 2q+2, \frac{2x}{\beta}\right),$$

which corresponds precisely to the PDF of a $CHME(\beta, q)$ distribution as stated in (2.2). □

2.4. Distribution function

Proposition 3. *The cumulative distribution function (CDF), survival function (SF) and hazard function (HF) of $X \sim CHME(\beta, q)$ are given by*

$$F_X(x;\beta,q) = 1 - \exp(-2x/\beta)\left[{}_1F_1\left(q, 2q, \frac{2x}{\beta}\right) + \frac{x}{\beta}\, {}_1F_1\left(q, 2q+1, \frac{2x}{\beta}\right)\right], \tag{2.5}$$

$$S_X(x;\beta,q) = \exp(-2x/\beta)\left[{}_1F_1\left(q, 2q, \frac{2x}{\beta}\right) + \frac{x}{\beta}\, {}_1F_1\left(q, 2q+1, \frac{2x}{\beta}\right)\right], \tag{2.6}$$

$$h_X(x;\beta,q) = \frac{2(q+1)x\, {}_1F_1\left(q, 2q+2, \frac{2x}{\beta}\right)}{\beta^2(2q+1)\left[{}_1F_1\left(q, 2q, \frac{2x}{\beta}\right) + \frac{x}{\beta}\, {}_1F_1\left(q, 2q+1, \frac{2x}{\beta}\right)\right]}. \tag{2.7}$$

Proof. Instead of directly integrating the marginal PDF, the CDF can be efficiently derived by utilizing the hierarchical representation of the model and the law of total probability. Conditioning on $Z = z$, we know that $X \mid Z = z \sim \mathrm{ME}\left(\frac{\beta}{2z}\right)$. The conditional CDF of a ME distribution is well-known and given by:

$$F_{X|Z}(x|z) = 1 - \left(1 + \frac{2xz}{\beta}\right)\exp\left(-\frac{2xz}{\beta}\right).$$

Therefore, the marginal CDF of X is obtained by marginalizing out $Z \sim \mathrm{Beta}(q, q)$:

$$\begin{aligned} F_X(x;\beta,q) &= \int_0^1 F_{X|Z}(x|z) f_Z(z)\,dz \\ &= 1 - \int_0^1 \exp\left(-\frac{2xz}{\beta}\right) f_Z(z)\,dz - \frac{2x}{\beta}\int_0^1 z\exp\left(-\frac{2xz}{\beta}\right) f_Z(z)\,dz \\ &= 1 - I_1 - I_2. \end{aligned}$$

To solve I_1, we substitute $f_Z(z)$ and apply the change of variable $u = 1 - z$:

$$\begin{aligned} I_1 &= \frac{1}{B(q,q)}\int_0^1 \exp\left(-\frac{2x}{\beta}(1-u)\right)(1-u)^{q-1}u^{q-1}\,du \\ &= \exp\left(-\frac{2x}{\beta}\right)\frac{1}{B(q,q)}\int_0^1 \exp\left(\frac{2x}{\beta}u\right)u^{q-1}(1-u)^{q-1}\,du \\ &= \exp\left(-\frac{2x}{\beta}\right){}_1F_1\left(q, 2q, \frac{2x}{\beta}\right). \end{aligned}$$

Similarly, for I_2, applying the same change of variable $u = 1 - z$ yields:

$$\begin{aligned} I_2 &= \frac{2x}{\beta B(q,q)}\int_0^1 \exp\left(-\frac{2x}{\beta}(1-u)\right)(1-u)^{q}u^{q-1}\,du \\ &= \frac{2x}{\beta}\exp\left(-\frac{2x}{\beta}\right)\frac{B(q,q+1)}{B(q,q)}\left[\frac{1}{B(q,q+1)}\int_0^1 \exp\left(\frac{2x}{\beta}u\right)u^{q-1}(1-u)^{(2q+1)-q-1}\,du\right] \\ &= \frac{2x}{\beta}\exp\left(-\frac{2x}{\beta}\right)\frac{B(q,q+1)}{B(q,q)}{}_1F_1\left(q, 2q+1, \frac{2x}{\beta}\right). \end{aligned}$$

Expanding the Beta ratio in terms of Gamma functions gives $\frac{B(q,q+1)}{B(q,q)} = \frac{\Gamma(q+1)\Gamma(2q)}{\Gamma(2q+1)\Gamma(q)} = \frac{q}{2q} = \frac{1}{2}$. Substituting this back into I_2, the factor 2 cancels out, resulting in

$$I_2 = \frac{x}{\beta}\exp\left(-\frac{2x}{\beta}\right){}_1F_1\left(q, 2q+1, \frac{2x}{\beta}\right).$$

Combining $1 - I_1 - I_2$ yields the exact expression for F_X in (2.5). The survival function is simply $S_X = 1 - F_X$, and the hazard function follows directly from the definition $h_X = f_X/S_X$, substituting the pdf from Proposition 1 and the newly derived S_X. This completes the proof. □

Figures 2 and 3 show the SF and the HF of the CHME, varying $q = 1, 2, 3, 4$, and fixing $\beta = 1$. It can be noted how the kurtosis, which increases as q decreases, is reflected in the behavior of both SF and HF.

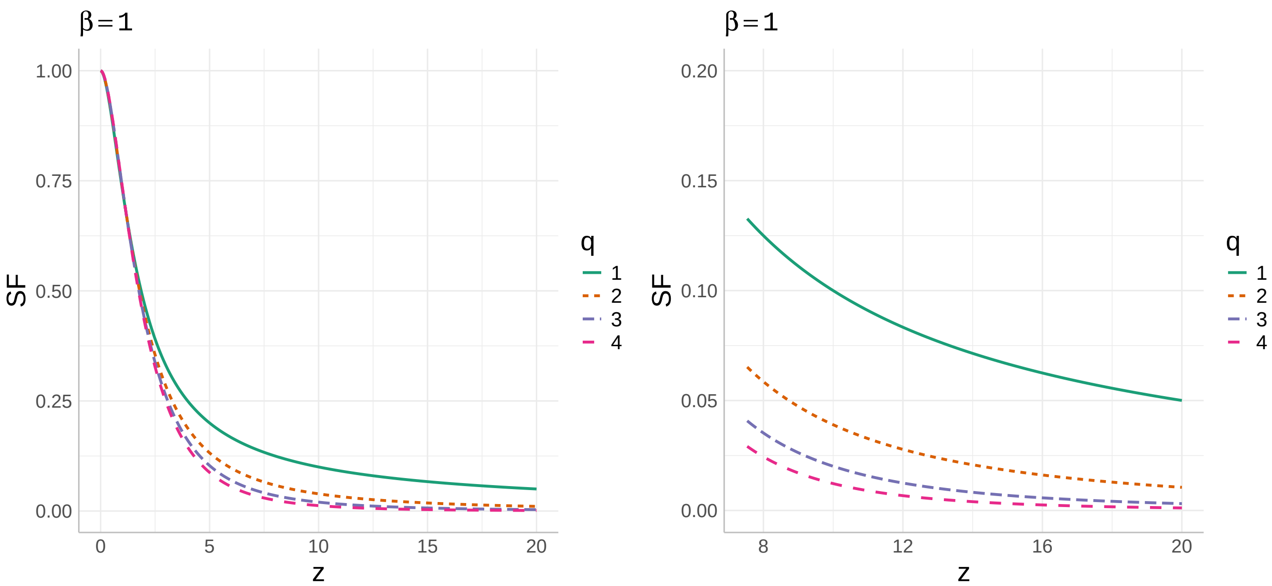

Figure 2. SF of the CHME distribution, with $\beta = 1$ and varying q.

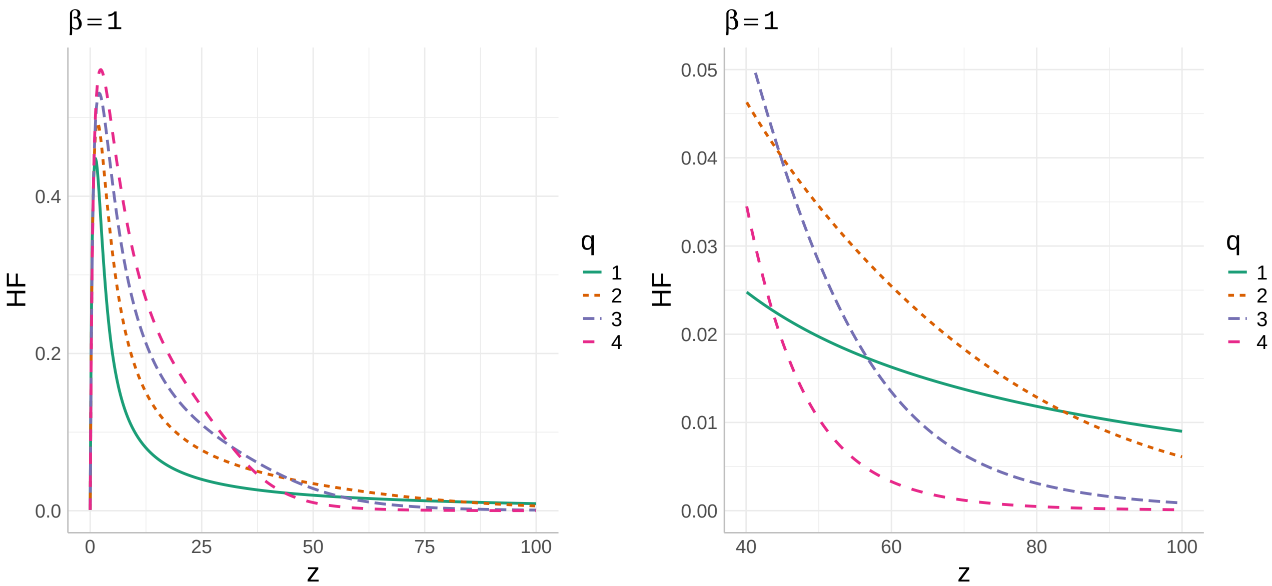

Figure 3. HF of the CHME distribution, with $\beta = 1$ and varying q.

2.5. Moments

Proposition 4. *Let $X \sim CHME(\beta, q)$. The r-th moment, for $r = 1, 2, \ldots$ and $q > r$, is given by*

$$E(X^r) = \frac{\beta^r \Gamma(r+2)\Gamma(q-r)\Gamma(2q)}{2^r \Gamma(2q-r)\Gamma(q)}. \tag{2.8}$$

Proof. Using the stochastic representation given in Eq (2.1), we can write $X = \frac{Y}{2Z}$, where $Y \sim \mathrm{ME}(\beta)$ and $Z \sim \mathrm{Beta}(q, q)$ are independent random variables. Therefore, the r-th moment of X can be separated as

$$E(X^r) = E\left(\left(\frac{Y}{2Z}\right)^r\right) = \frac{1}{2^r} E(Y^r) E(Z^{-r}).$$

First, we compute the r-th moment of Y. By definition of the ME distribution, its PDF is $f_Y(y) = \frac{y}{\beta^2} \exp(-y/\beta)$ for $y > 0$. Thus,

$$E(Y^r) = \int_0^\infty y^r \frac{y}{\beta^2} \exp(-y/\beta) dy = \frac{1}{\beta^2} \int_0^\infty y^{r+1} \exp(-y/\beta) dy.$$

Applying the change of variable $t = y/\beta$, with $dy = \beta dt$, we obtain:

$$E(Y^r) = \frac{1}{\beta^2} \int_0^\infty (\beta t)^{r+1} \exp(-t) \beta dt = \beta^r \int_0^\infty t^{(r+2)-1} \exp(-t) dt = \beta^r \Gamma(r+2).$$

Next, we compute the negative r-th moment of Z. Since $Z \sim \mathrm{Beta}(q, q)$, its PDF is $f_Z(z) = \frac{1}{B(q,q)} z^{q-1}(1-z)^{q-1}$ for $0 < z < 1$. Thus,

$$E(Z^{-r}) = \int_0^1 z^{-r} \frac{1}{B(q,q)} z^{q-1}(1-z)^{q-1} dz = \frac{1}{B(q,q)} \int_0^1 z^{(q-r)-1}(1-z)^{q-1} dz.$$

This integral converges if and only if $q - r > 0$, which yields the required condition $q > r$. Recognizing the integral as the Beta function $B(q-r, q)$, we have

$$E(Z^{-r}) = \frac{B(q-r,q)}{B(q,q)} = \frac{\frac{\Gamma(q-r)\Gamma(q)}{\Gamma(2q-r)}}{\frac{\Gamma(q)\Gamma(q)}{\Gamma(2q)}} = \frac{\Gamma(q-r)\Gamma(2q)}{\Gamma(2q-r)\Gamma(q)}.$$

Finally, substituting $E(Y^r)$ and $E(Z^{-r})$ back into the initial equation, we obtain

$$E(X^r) = \frac{1}{2^r} \left[\beta^r \Gamma(r+2)\right] \left[\frac{\Gamma(q-r)\Gamma(2q)}{\Gamma(2q-r)\Gamma(q)}\right] = \frac{\beta^r \Gamma(r+2)\Gamma(q-r)\Gamma(2q)}{2^r \Gamma(2q-r)\Gamma(q)},$$

which completes the proof. □

2.6. Mean and variance

Corollary 1. *The mean and variance of $X \sim CHME(\beta, q)$ are given by*

$$E(X) = \frac{\beta \Gamma(q-1)\Gamma(2q)}{\Gamma(2q-1)\Gamma(q)}, \quad q > 1,$$

and

$$Var(X) = \frac{\beta^2 \Gamma(2q)}{\Gamma(q)} \left[\frac{3\Gamma(q-2)}{2\Gamma(2q-2)} - \frac{\Gamma^2(q-1)\Gamma(2q)}{\Gamma^2(2q-1)\Gamma(q)}\right], \quad q > 2.$$

Proof. The mean $E(X)$ follows directly by substituting $r = 1$ into the moment formula (2.8) from Proposition 4. Noting that $\Gamma(1 + 2) = \Gamma(3) = 2! = 2$, the term 2^1 in the denominator cancels out, yielding the stated expression for $E(X)$ valid for $q > 1$.

Similarly, for the second moment we set $r = 2$. Since $\Gamma(2 + 2) = \Gamma(4) = 3! = 6$, we get

$$E(X^2) = \frac{\beta^2(6)\Gamma(q-2)\Gamma(2q)}{4\Gamma(2q-2)\Gamma(q)} = \frac{3\beta^2\Gamma(q-2)\Gamma(2q)}{2\Gamma(2q-2)\Gamma(q)}, \quad \text{for } q > 2.$$

The variance is then computed using the standard relation $\mathrm{Var}(X) = E(X^2) - (E(X))^2$. Factoring out the common terms $\frac{\beta^2\Gamma(2q)}{\Gamma(q)}$ leads directly to the result. □

3. Methods of estimation and simulation

In this section, we develop parameter estimation procedures for the CHME distribution based on the method of moments, maximum likelihood, and Bayesian inference. Additionally, an EM algorithm is proposed to facilitate numerical computation of the maximum likelihood estimators (MLEs).

3.1. Moment estimators

Proposition 5. *Let $X_1, \ldots, X_n$ be a random sample from $CHME(\beta, q)$. The method of moments estimators $\hat{\theta}_M = (\hat{\beta}_M, \hat{q}_M)$ are obtained by solving the system of equations derived by equating the first two theoretical moments with their empirical counterparts.*

From Proposition 4, the first two moments are used to obtain

$$\hat{\beta}_M = \frac{\overline{X}\Gamma(2\hat{q}_M - 1)\Gamma(\hat{q}_M)}{\Gamma(\hat{q}_M - 1)\Gamma(2\hat{q}_M)}, \tag{3.1}$$

$$0 = \frac{\overline{X}^2\Gamma^2(2\hat{q}_M - 1)\Gamma(\hat{q}_M)}{\Gamma^2(\hat{q}_M - 1)\Gamma(2\hat{q}_M)} - \frac{2\overline{X^2}\Gamma(2\hat{q}_M - 2)\Gamma(\hat{q}_M)}{3\Gamma(\hat{q}_M - 2)\Gamma(2\hat{q}_M)}, \tag{3.2}$$

where $\overline{X}$ and $\overline{X^2}$ denote the sample first and second raw moments, respectively. Equation (3.2) must be solved numerically to obtain $\hat{q}_M$, which is then substituted into (3.1) to compute $\hat{\beta}_M$.

Proof. By definition of the method of moments, we equate the first two theoretical moments, $E(X)$ and $E(X^2)$, to the corresponding sample moments, $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$ and $\overline{X^2} = \frac{1}{n}\sum_{i=1}^n X_i^2$. Based on the results from Proposition 4, we form the following system of equations:

$$\frac{\beta\Gamma(q-1)\Gamma(2q)}{\Gamma(2q-1)\Gamma(q)} = \overline{X}, \tag{3.3}$$

$$\frac{3\beta^2\Gamma(q-2)\Gamma(2q)}{2\Gamma(2q-2)\Gamma(q)} = \overline{X^2}. \tag{3.4}$$

From (3.3), we can explicitly isolate the parameter β to obtain the estimator $\hat{\beta}_M$ in terms of $\hat{q}_M$ and $\overline{X}$, which yields exactly Eq (3.1).

To find the equation for $\hat{q}_M$, we square both sides of (3.3) to obtain the following expression for β^2:

$$\beta^2 = \overline{X}^2 \frac{\Gamma^2(2q-1)\Gamma^2(q)}{\Gamma^2(q-1)\Gamma^2(2q)}. \tag{3.5}$$

Similarly, we isolate β^2 from Eq (3.4) as follows:

$$\beta^2 = \overline{X^2} \frac{2\Gamma(2q-2)\Gamma(q)}{3\Gamma(q-2)\Gamma(2q)}. \tag{3.6}$$

Equating (3.5) and (3.6) to eliminate β^2, we get

$$\overline{X}^2 \frac{\Gamma^2(2q-1)\Gamma^2(q)}{\Gamma^2(q-1)\Gamma^2(2q)} = \overline{X^2} \frac{2\Gamma(2q-2)\Gamma(q)}{3\Gamma(q-2)\Gamma(2q)}.$$

Finally, to match the desired root-finding form $g(\hat{q}_M) = 0$, we subtract the right-hand side from the left-hand side. By factoring out the common term $\frac{\Gamma(q)}{\Gamma(2q)}$ from both sides, we obtain

$$\frac{\overline{X}^2 \Gamma^2(2q-1)\Gamma(q)}{\Gamma^2(q-1)\Gamma(2q)} - \frac{2\overline{X^2}\Gamma(2q-2)\Gamma(q)}{3\Gamma(q-2)\Gamma(2q)} = 0,$$

which proves Eq (3.2) and completes the proof. □

3.2. Maximum likelihood estimation

Proposition 6. *Let $X_1, \ldots, X_n$ be a random sample from $CHME(\beta, q)$. The log-likelihood function for $\theta = (\beta, q)$ is given by*

$$\ell(\beta, q) = n \log\left(\frac{2(q+1)}{\beta^2(2q+1)}\right) + \sum_{i=1}^{n} \log(x_i) - \frac{2}{\beta}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \log\left[{}_1F_1\left(q, 2q+2, \frac{2x_i}{\beta}\right)\right]. \tag{3.7}$$

Proof. By definition, the likelihood function for a random sample $X_1, \ldots, X_n$ is the product of the marginal density functions given in Proposition 1

$$L(\beta, q) = \prod_{i=1}^{n} f_X(x_i; \beta, q) = \prod_{i=1}^{n} \left\{ \frac{2(q+1)x_i \exp(-2x_i/\beta)}{\beta^2(2q+1)} \, {}_1F_1\left(q, 2q+2, \frac{2x_i}{\beta}\right)\right\}.$$

Taking the natural logarithm of the likelihood function, $\ell(\beta, q) = \log L(\beta, q)$, and applying the standard properties of logarithms ($\log(ab) = \log a + \log b$ and $\log(\prod a_i) = \sum \log a_i$), we obtain

$$\begin{aligned}
\ell(\beta, q) &= \sum_{i=1}^{n} \log\left(\frac{2(q+1)}{\beta^2(2q+1)}\right) + \sum_{i=1}^{n} \log(x_i) + \sum_{i=1}^{n}\left(-\frac{2x_i}{\beta}\right) + \sum_{i=1}^{n} \log\left[{}_1F_1\left(q, 2q+2, \frac{2x_i}{\beta}\right)\right] \\
&= n \log\left(\frac{2(q+1)}{\beta^2(2q+1)}\right) + \sum_{i=1}^{n} \log(x_i) - \frac{2}{\beta}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \log\left[{}_1F_1\left(q, 2q+2, \frac{2x_i}{\beta}\right)\right].
\end{aligned}$$

This completes the proof. □

The MLEs for β and q are obtained by maximizing the log-likelihood function. The score equations are derived by differentiating $\ell(\beta, q)$ with respect to each parameter and setting the results to zero.

For the scale parameter β, by rewriting the first term as $n\log\left(\frac{2(q+1)}{2q+1}\right) - 2n\log\beta$ and applying the chain rule to the hypergeometric function, we get

$$\begin{aligned}\frac{\partial \ell}{\partial \beta} &= \frac{\partial}{\partial \beta}\left[-2n\log\beta\right] - \frac{\partial}{\partial \beta}\left[\frac{2}{\beta}\sum_{i=1}^{n} x_i\right] + \sum_{i=1}^{n} \frac{\frac{\partial}{\partial \beta}{}_1F_1\left(q, 2q+2, \frac{2x_i}{\beta}\right)}{{}_1F_1\left(q, 2q+2, \frac{2x_i}{\beta}\right)} \\ &= -\frac{2n}{\beta} + \frac{2}{\beta^2}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{C_1(x_i, \beta, q)}{C(x_i, \beta, q)} = 0.\end{aligned}$$

Similarly, for the shape parameter q, we expand the first term as $n\log(q+1) - n\log(2q+1)$ (ignoring terms constant with respect to q). Differentiating this yields

$$\begin{aligned}\frac{\partial \ell}{\partial q} &= n\left(\frac{1}{q+1} - \frac{2}{2q+1}\right) + \sum_{i=1}^{n} \frac{\frac{\partial}{\partial q}{}_1F_1\left(q, 2q+2, \frac{2x_i}{\beta}\right)}{{}_1F_1\left(q, 2q+2, \frac{2x_i}{\beta}\right)} \\ &= n\left(\frac{2q+1-2(q+1)}{(q+1)(2q+1)}\right) + \sum_{i=1}^{n} \frac{C_2(x_i, \beta, q)}{C(x_i, \beta, q)} \\ &= \frac{-n}{(q+1)(2q+1)} + \sum_{i=1}^{n} \frac{C_2(x_i, \beta, q)}{C(x_i, \beta, q)} = 0,\end{aligned}$$

where the functions corresponding to the hypergeometric term and its derivatives are defined as:

$$C(x_i, \beta, q) = {}_1F_1\left(q, 2q+2, \frac{2x_i}{\beta}\right),$$

$$C_1(x_i, \beta, q) = \frac{\partial}{\partial \beta}{}_1F_1\left(q, 2q+2, \frac{2x_i}{\beta}\right), \quad C_2(x_i, \beta, q) = \frac{\partial}{\partial q}{}_1F_1\left(q, 2q+2, \frac{2x_i}{\beta}\right).$$

Since these equations involve derivatives of the confluent hypergeometric function with respect to its parameters and arguments, they do not admit analytical closed-form solutions. Therefore, numerical optimization methods, such as Newton-Raphson or quasi-Newton algorithms (e.g., BFGS), must be used to solve for the MLEs $\hat{\beta}$ and $\hat{q}$.

3.3. EM algorithm for parameter estimation

An alternative estimation approach can be developed using the hierarchical representation of the model

$$\begin{aligned}X_i|Z_i = z_i &\sim \mathrm{ME}\left(\frac{\beta}{2z_i}\right), \\ Z_i &\sim \mathrm{Beta}(q, q).\end{aligned}$$

Based on this formulation, the complete-data log-likelihood function, up to an additive constant, is given by

$$l_c(\theta) = -2n\log\beta - n\log B(q, q) + (q+1)\sum_{i=1}^{n}\log z_i + (q-1)\sum_{i=1}^{n}\log(1-z_i) - \frac{2}{\beta}\sum_{i=1}^{n} z_i x_i. \tag{3.8}$$

(E-step)

The E-step computes the conditional expectation of the complete-data log-likelihood, known as the Q-function, given the observed data $\boldsymbol{X}$ and the current parameter estimates at the k-th iteration, $\theta^{(k)} = (\beta^{(k)}, q^{(k)})$. This function is explicitly expressed as

$$Q(\theta|\theta^{(k)}) = -2n\log\beta - n\log B(q,q) + (q+1)\sum_{i=1}^{n}\hat{u}_i^{(k)} + (q-1)\sum_{i=1}^{n}\hat{v}_i^{(k)} - \frac{2}{\beta}\sum_{i=1}^{n}x_i\hat{w}_i^{(k)}, \tag{3.9}$$

where $\hat{u}_i^{(k)}$, $\hat{v}_i^{(k)}$, and $\hat{w}_i^{(k)}$ denote the conditional expectations defined by

$$\begin{aligned}
\hat{u}_i^{(k)} &= \mathbb{E}[\log Z_i \mid X_i = x_i; \theta^{(k)}],\\
\hat{v}_i^{(k)} &= \mathbb{E}[\log(1-Z_i) \mid X_i = x_i; \theta^{(k)}],\\
\hat{w}_i^{(k)} &= \mathbb{E}[Z_i \mid X_i = x_i; \theta^{(k)}].
\end{aligned}$$

Since the conditional distribution of $Z_i \mid X_i = x_i$ is proportional to $z_i^{q^{(k)}+1}(1-z_i)^{q^{(k)}-1}\exp\left(-\frac{2x_iz_i}{\beta^{(k)}}\right)$, these expectations can be computed numerically utilizing standard quadrature methods.

(M-step)

In the M-step, the Q-function is maximized with respect to θ to find the updated parameter estimates $\theta^{(k+1)}$. By taking the partial derivatives of $Q(\theta|\theta^{(k)})$ with respect to β and q, and setting them equal to zero, we obtain the specific update equations.

The update equation for the scale parameter β yields the following closed-form expression:

$$\hat{\beta}^{(k+1)} = \frac{1}{n}\sum_{i=1}^{n}x_i\hat{w}_i^{(k)}, \tag{3.10}$$

whereas the update for the shape parameter q is obtained by solving the following non-linear equation:

$$2\left[\psi\left(\hat{q}^{(k+1)}\right) - \psi\left(2\hat{q}^{(k+1)}\right)\right] = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{u}_i^{(k)} + \hat{v}_i^{(k)}\right), \tag{3.11}$$

where $\psi(\cdot) = \Gamma'(\cdot)/\Gamma(\cdot)$ represents the digamma function. The E and M steps are iterated until some convergence criterion is achieved. For example, we propose

$$\max(|\beta^{(k)} - \beta|, |q^{(k)} - q|) < 1e-5.$$

3.4. Bayesian inference

Thanks to the development of the EM algorithm, some results can be used to perform Bayesian inference to offer alternative parameter estimates. First, the following Bayesian model can be defined:

$$X_i \mid \beta, q \sim CHME(\beta, q), \quad \beta \sim IG(a_\beta, b_\beta), \quad q \sim Ga(a_q, b_q), \quad i = 1, \ldots, n, \tag{3.12}$$

where $IG(a,b)$ denotes the inverse Gamma (IG) distribution with shape (a) and scale (b) parameters, and $Ga(a,b)$ denotes the Gamma distribution with shape (a) and rate (b) parameters. The IG distribution is chosen over β to achieve conjugacy, and the Gamma distribution over q to offer prior flexibility. Given that the parameters are considered random, their estimates can be obtained by studying the behavior of their posterior distribution $f(\beta \mid q, z, \boldsymbol{x})$ and $f(q \mid \beta, z, \boldsymbol{x})$. However, obtaining those expressions can be difficult. Therefore, a Gibbs sampler can be used to simulate values from their posterior distributions, but using their conditional posterior distributions instead, which are easier to compute. Afterward, an approximation of the posterior mean can be obtained as an estimator.

Proposition 7. *Under model* (3.12)*, the following conditional posterior distributions are obtained:*

$$\begin{aligned}
\beta \mid q, z, \boldsymbol{x} &\sim IG\left(2n + a_\beta,\ b_\beta + 2\sum_{i=1}^{n} z_i x_i\right), \\
f(q \mid \beta, z, \boldsymbol{x}) &\propto (B(q,q))^{-n} q^{a_q - 1} \exp(-b_q q) \exp\left(q \sum_{i=1}^{n} \log(z_i(1-z_i))\right), \quad q \in (0, \infty), \\
f(z_i \mid \beta, q, \boldsymbol{x}) &\propto z_i^{q+1}(1-z_i)^{q-1} \exp\left(\frac{-2 z_i x_i}{\beta}\right), \quad z_i \in (0,1), \quad i = 1, \ldots, n.
\end{aligned}$$

Proof. Using the hierarchical representation given in Proposition 2, model (3.12) is equivalent to the augmentation

$$X_i \mid Z_i = z_i \sim ME\left(\frac{\beta}{2z_i}\right), \quad Z_i \sim \text{Beta}(q,q), \quad \beta \sim IG(a_\beta, b_\beta), \quad q \sim Ga(a_q, b_q), \quad i = 1, \ldots, n.$$

Hence, the following joint posterior distribution is obtained:

$$\pi(\beta, q, z, \mid \boldsymbol{x}) \propto \beta^{-(a_\beta + 2n + 1)} \exp\left(\frac{b_\beta + 2\sum_{i=1}^{n} z_i x_i}{\beta}\right) q^{a_q - 1} \exp(-b_q q)(B(q,q))^{-n} \prod_{i=1}^{n} z_i^{q+1}(1-z_i)^{q-1},$$

leading to the presented results. □

As the posterior distribution kernels for q and z_i do not belong to known distributions, a Gibbs sampler with Metropolis-Hastings sub-steps will be computed. It is worth noting that $f(q \mid \beta, z, \boldsymbol{x})$ and $f(z_i \mid \beta, q, \boldsymbol{x})$ are ensured to be measures because model (3.12) has proper priors.

3.5. Simulation study

Following a strategy similar to that proposed by Iriarte et al. [7] for the SME distribution, random samples from the CHME distribution can be generated using the stochastic representation described in Section 2. The procedure is summarized in Algorithm 1.

Algorithm 1 Simulating from the CHME distribution.

1: Generate $R \sim U(0,1)$.
2: Compute $Y = -\beta\left(W_{-1}\left(\frac{R-1}{e}\right) + 1\right)$. ▷ $W_{-1}(\cdot)$ is used to invert the CDF of the ME distribution.
3: Generate $Z \sim \text{Beta}(q,q)$.
4: Compute $X = \dfrac{Y}{2Z}$.

Table 1. Recovery parameter simulation study for the CHME distribution, under the EM method.

Parameters			n = 50				n = 100				n = 200			
β	q	Estimator	Bias	RMSE	SE	CP	Bias	RMSE	SE	CP	Bias	RMSE	SE	CP
3	1	β	-0.071	0.494	0.492	0.959	-0.029	0.353	0.345	0.946	-0.011	0.242	0.243	0.953
		q	-0.175	0.810	0.425	0.960	-0.058	0.245	0.219	0.960	-0.023	0.155	0.146	0.953
	3	β	-0.063	0.419	0.415	0.957	-0.047	0.299	0.293	0.948	-0.028	0.210	0.207	0.950
		q	-3.627	9.603	11.565	0.939	-1.137	3.192	2.676	0.955	-0.396	1.256	1.043	0.968
	5	β	-0.010	0.385	0.386	0.939	-0.033	0.264	0.279	0.963	-0.011	0.201	0.197	0.949
		q	-8.651	18.395	31.201	0.918	-3.791	9.341	11.051	0.942	-1.359	3.977	3.854	0.931
5	1	β	-0.099	0.809	0.814	0.950	-0.040	0.572	0.573	0.956	-0.039	0.426	0.405	0.936
		q	-0.186	0.614	0.401	0.974	-0.065	0.237	0.221	0.957	-0.031	0.170	0.148	0.944
	3	β	-0.110	0.718	0.695	0.939	-0.056	0.506	0.489	0.941	-0.026	0.345	0.344	0.946
		q	-3.495	9.851	11.121	0.930	-1.129	3.704	2.831	0.937	-0.378	1.409	1.062	0.947
	5	β	-0.041	0.625	0.646	0.958	-0.029	0.468	0.462	0.951	-0.025	0.323	0.329	0.949
		q	-8.727	17.796	31.517	0.917	-3.809	9.162	10.997	0.934	-1.927	4.941	4.633	0.946
10	1	β	-0.227	1.660	1.639	0.960	-0.153	1.189	1.152	0.947	-0.088	0.827	0.812	0.941
		q	-0.202	1.200	0.488	0.970	-0.074	0.256	0.224	0.960	-0.033	0.159	0.148	0.949
	3	β	-0.160	1.340	1.375	0.961	-0.088	0.960	0.975	0.963	-0.040	0.703	0.686	0.935
		q	-3.670	10.269	11.427	0.939	-1.035	3.434	2.644	0.930	-0.417	1.510	1.093	0.955
	5	β	-0.124	1.286	1.305	0.949	-0.035	0.939	0.924	0.946	-0.072	0.666	0.660	0.939
		q	-7.389	16.127	26.939	0.928	-3.768	9.039	10.907	0.921	-1.501	4.293	4.071	0.945
20	1	β	-0.550	3.362	3.288	0.952	-0.165	2.365	2.286	0.941	-0.181	1.626	1.624	0.948
		q	-0.178	0.540	0.385	0.962	-0.072	0.280	0.224	0.972	-0.033	0.162	0.148	0.943
	3	β	-0.368	2.687	2.759	0.949	-0.108	1.973	1.940	0.932	-0.057	1.358	1.369	0.951
		q	-3.778	10.262	11.725	0.945	-1.009	3.288	2.572	0.951	-0.438	1.748	1.123	0.953
	5	β	-0.170	2.481	2.585	0.940	-0.121	1.840	1.853	0.950	-0.124	1.293	1.319	0.953
		q	-7.981	16.549	28.411	0.928	-3.731	8.803	10.721	0.928	-1.589	4.513	4.141	0.946

Table 2. Recovery parameter simulation study for the CHME distribution, under the Bayesian method.

Parameters			n = 50				n = 100				n = 200			
β	q	Estimator	Bias	RMSE	SE	CP	Bias	RMSE	SE	CP	Bias	RMSE	SE	CP
3	1	β	0.080	0.509	0.497	0.949	0.032	0.342	0.345	0.954	0.020	0.246	0.243	0.941
		q	0.145	0.390	0.342	0.959	0.063	0.226	0.216	0.962	0.035	0.152	0.147	0.951
	3	β	-0.080	0.378	0.396	0.945	-0.035	0.272	0.283	0.954	-0.023	0.199	0.201	0.951
		q	-0.430	0.737	0.982	0.915	-0.203	0.669	0.860	0.916	-0.053	0.608	0.703	0.932
	5	β	-0.124	0.369	0.373	0.929	-0.108	0.257	0.262	0.944	-0.069	0.189	0.186	0.936
		q	-1.987	2.056	1.172	0.634	-1.470	1.595	1.147	0.750	-0.961	1.197	1.088	0.811
5	1	β	0.108	0.821	0.826	0.952	0.001	0.559	0.571	0.951	0.013	0.404	0.404	0.951
		q	0.137	0.405	0.338	0.961	0.053	0.227	0.214	0.971	0.028	0.150	0.146	0.965
	3	β	-0.113	0.627	0.665	0.938	-0.041	0.484	0.473	0.946	-0.037	0.320	0.335	0.956
		q	-0.463	0.724	0.970	0.906	-0.209	0.668	0.855	0.915	-0.049	0.618	0.702	0.918
	5	β	-0.221	0.611	0.622	0.929	-0.158	0.441	0.438	0.925	-0.114	0.326	0.311	0.926
		q	-2.013	2.081	1.164	0.622	-1.485	1.608	1.140	0.756	-0.977	1.206	1.081	0.829
10	1	β	0.156	1.622	1.646	0.954	0.059	1.156	1.147	0.956	0.055	0.829	0.809	0.945
		q	0.126	0.380	0.332	0.966	0.081	0.260	0.222	0.961	0.034	0.155	0.146	0.959
	3	β	-0.303	1.248	1.316	0.941	-0.105	0.895	0.943	0.961	-0.062	0.673	0.671	0.938
		q	-0.427	0.705	0.982	0.919	-0.190	0.649	0.866	0.924	-0.055	0.599	0.700	0.940
	5	β	-0.446	1.262	1.239	0.916	-0.336	0.898	0.874	0.924	-0.240	0.652	0.621	0.929
		q	-1.968	2.038	1.180	0.648	-1.471	1.598	1.145	0.756	-0.972	1.213	1.085	0.823
20	1	β	0.183	3.416	3.276	0.936	0.159	2.302	2.308	0.950	0.162	1.657	1.627	0.949
		q	0.139	0.404	0.338	0.963	0.073	0.246	0.219	0.972	0.029	0.153	0.146	0.963
	3	β	-0.548	2.593	2.643	0.939	-0.221	1.872	1.886	0.949	-0.142	1.292	1.340	0.959
		q	-0.429	0.726	0.983	0.910	-0.183	0.649	0.868	0.930	-0.032	0.594	0.709	0.951
	5	β	-0.962	2.520	2.480	0.912	-0.679	1.769	1.749	0.926	-0.495	1.266	1.245	0.934
		q	-2.010	2.077	1.165	0.641	-1.467	1.587	1.152	0.763	-1.040	1.245	1.059	0.802

Table 3. Simulation study comparing AIC and BIC values for the CHME, SME, and ME distributions.

Parameters			$n = 50$		$n = 100$		$n = 200$	
β	q	Distribution	AIC	BIC	AIC	BIC	AIC	BIC
3	1	CHME	345.57	349.39	689.99	695.2	1380.04	1386.63
		SME	345.58	349.41	690	695.21	1380.04	1386.64
		ME	471.08	473	978.07	980.68	2031.71	2035.01
	3	CHME	295.21	299.04	589.71	594.92	1178.79	1185.39
		SME	295.32	299.15	589.95	595.16	1179.21	1185.8
		ME	301.78	303.69	605.01	607.62	1212.63	1215.93
5	1	CHME	398.2	402.02	793.84	799.05	1586.22	1592.82
		SME	398.21	402.04	793.84	799.05	1586.22	1592.82
		ME	526.04	527.95	1083.89	1086.5	2254.25	2257.55
	3	CHME	347.01	350.84	691.99	697.2	1382.41	1389.01
		SME	347.14	350.97	692.22	697.43	1382.78	1389.38
		ME	353.38	355.29	706.41	709.01	1415.52	1418.82
10	1	CHME	467.05	470.88	931.8	937.01	1862.31	1868.91
		SME	467.06	470.89	931.79	937	1862.27	1868.86
		ME	598.55	600.46	1224.96	1227.56	2557.61	2560.91
	3	CHME	415.15	418.97	831.24	836.46	1660.9	1667.5
		SME	415.27	419.09	831.47	836.68	1661.35	1667.94
		ME	421.71	423.62	846.95	849.55	1694.38	1697.68
20	1	CHME	536.2	540.03	1071.43	1076.64	2139.96	2146.55
		SME	536.19	540.01	1071.43	1076.64	2139.87	2146.47
		ME	665.45	667.36	1385.02	1387.63	2831.66	2834.95
	3	CHME	486.32	490.15	969.79	975	1936.22	1942.82
		SME	486.46	490.28	970.01	975.22	1936.61	1943.21
		ME	493.43	495.34	985.27	987.88	1968.36	1971.66

It follows that $X \sim \text{CHME}(\beta, q)$, where $W_{-1}(\cdot)$ denotes the negative branch of the Lambert W function (see Corless et al. [9]). As noted in Step 2, this function is explicitly utilized to invert the CDF of the ME distribution, thereby ensuring an exact inversion method and full reproducibility for readers attempting to simulate data from this model. To assess the performance of the proposed estimators, a Monte Carlo simulation study with 1,000 replications was conducted under both the EM and Bayesian estimation methods. The objective was to evaluate the model's ability to recover the true parameters β and q. A total of 12 scenarios were considered, combining different parameter values and sample sizes $n = 50, 100$, and 200. The EM algorithm used $\beta^{(0)} = 0.2$ and $q^{(0)} = 2$ as initial values of (β, q), while the hyperparameters of the Bayesian method were fixed at $a_\beta = b_\beta = a_q = b_q = 1$. For each replication of the EM method, we computed the bias, root mean square error (RMSE), standard error (SE), and coverage probability (CP) based on asymptotic normal 95% confidence intervals. The Monte Carlo averages of these measures are reported in Table 1. To compare these results with the Bayesian estimation method, the same measures were also presented in Table 2, but using the posterior mean and posterior standard deviation approximation as the parameter estimator and their SE of each experiment, respectively. The selected parameter combinations allow us to analyze the effect of increasing both β and q on the estimation accuracy. In addition, a second simulation study with 1,000 replications was conducted to compare the performance of the CHME model with the SME and ME distributions. For each generated dataset, the Akaike information criterion (AIC) and Bayesian information criterion (BIC) were computed (see Akaike [10], Schwarz [11]). The Monte Carlo averages are presented in Table 3 for the same parameter configurations and sample sizes.

The results indicate that the performance of the CHME model improves as the sample size increases, as evidenced by decreasing bias, RMSE, and SE and coverage probabilities that approach the nominal 0.95 level. This behavior remains stable even for relatively large values of β. It is also worth noting that when β and/or q are larger, the Bayesian method yields better results, particularly for smaller sample sizes. However, the Bayesian method tends to have a worse behavior regarding the CP, which is no surprise given that the approach is built to cover that frequentist interval. Nevertheless, both alternatives delivered a good behavior in terms of reducing bias, RMSE, and SE.

In terms of model comparison, the CHME distribution consistently yields lower AIC and BIC values than its competitors. This suggests that the proposed model is flexible enough to capture its own data-generating mechanism and provides a better overall fit than the SME and ME distributions.

4. Application

In this section, we illustrate the practical usefulness of the proposby applying it to a real dataset and comparing its performance with that of its performance with the SME and ME models. The dataset corresponds to the stress-rupture life (in hours) of Kevlar 49/epoxy strands subjected to a constant pressure at 90% of the ultimate stress level (see Andrews and Herzberg [12]). A total of $n = 101$ complete failure times were recorded, with no censoring. This dataset is available in the `R` package `DataSetsUni` under the variable `data_Kevlar`.

Table 4 presents a descriptive summary of the data, including the sample size, mean, variance, skewness (b_1), and kurtosis (b_2). The high kurtosis value ($b_2 = 16.709$) indicates the presence of heavy tails and extreme observations, making this dataset particularly suitable for evaluating models designed to capture atypical behavior.

Table 4. Descriptive summary of the stress-rupture life application, showing the sample size, mean, variance, and asymmetry and kurtosis coefficients.

n	$\bar{z}$	s^2	b_1	b_2
101	1.025	1.253	3.002	16.709

Table 5 reports the maximum likelihood estimates for the CHME, SME, and ME distributions, along with their standard errors (in parentheses), log-likelihood values, and model selection criteria (AIC and BIC).

Table 5. Estimation results for the CHME, SME, and ME distributions: parameter estimates (standard deviation in parentheses), log-likelihood, AIC and BIC.

	CHME	SME	ME
β	**0.38 (0.04)**	0.25 (0.05)	0.51 (0.04)
q	**2.21 (0.71)**	1.81 (0.49)	- - - (- - -)
Log-likelihood	**-121.06**	-122.2	-132.46
AIC	**246.12**	248.4	266.91
BIC	**251.35**	253.63	269.53

The results show that the CHME model achieves the highest log-likelihood and the lowest AIC and BIC values among the competing models, indicating a superior overall fit.

Figure 4 displays the fitted pdf and cdf for the three models. The CHME distribution provides a closer fit to the empirical behavior of the data, particularly in the tail region.

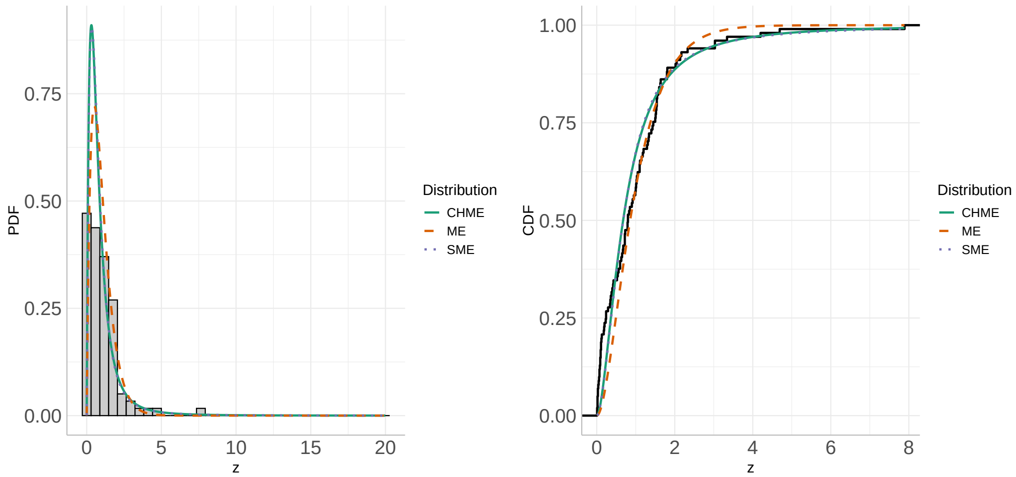

Figure 4. Fitted PDF (left) and CDF (right) of the CHME, SME, and ME distributions to the data.

Figure 5 presents the corresponding QQ plots. It can be observed that the CHME model better aligns with the empirical quantiles, especially in the upper tail, where extreme observations are present. This suggests that the proposed model captures the tail behavior more accurately than the SME and ME distributions.

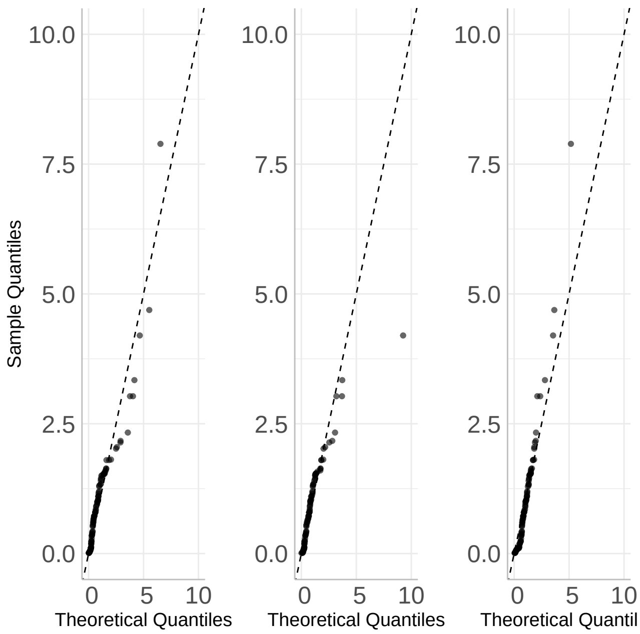

Figure 5. QQ plot for the CHME, SME, and ME distributions.

Overall, these results demonstrate that the CHME distribution provides a more flexible and accurate modeling framework for highly skewed and leptokurtic data, outperforming existing alternatives in both goodness-of-fit measures and graphical diagnostics.

5. Conclusions

In this paper, we introduced a new extension of the ME distribution, namely the confluent hypergeometric moment exponential distribution. The proposed model is constructed through a slash-type mechanism combined with a confluent hypergeometric structure, which provides additional flexibility for modeling skewed and heavy-tailed data.

Several theoretical properties of the CHME distribution were derived, including its PDF, CDF, moments, and shape characteristics. The proposed model admits a convenient stochastic representation, which facilitates both theoretical analysis and simulation.

From an inferential perspective, parameter estimation was performed using the method of moments, maximum likelihood, and Bayesian inference. Due to the complexity of the likelihood function, a latent variable representation was used to develop an EM algorithm and Bayesian estimators, providing a practical and efficient approach to parameter estimation.

A comprehensive Monte Carlo simulation study was conducted to evaluate the performance of the proposed estimators. The results indicate that the estimators are consistent and exhibit good finite-sample behavior, with decreasing bias, RMSE, and standard errors as the sample size increases, and coverage probabilities approaching the nominal level. In general, both MLEs and the Bayesian method are good alternatives. Additionally, model comparison results based on AIC and BIC show that the CHME distribution consistently outperforms the classical ME distribution and is competitive with the SME model.

The practical usefulness of the proposed distribution was illustrated through an application to stress-rupture life data of Kevlar 49/epoxy strands. The results demonstrated that the CHME model provides a better fit than competing models, particularly in capturing extreme observations and tail behavior, as confirmed by both information criteria and graphical diagnostics, such as QQ plots.

Overall, the CHME distribution constitutes a flexible and robust alternative for modeling positive data with high skewness and kurtosis. The additional flexibility introduced by the confluent hypergeometric structure allows for improved modeling of heavy-tailed phenomena, making the proposed model particularly suitable for applications involving atypical observations.

Future research may include the extension of the CHME model to regression frameworks, censored data settings, and multivariate contexts, as well as the exploration of alternative estimation procedures and computational strategies.

Author contributions

Nekeva M. Olmos: Conceptualization of this study, Methodology, Writing—original draft preparation. Wilson E. Caimanque: Conceptualization, Investigation. Yolanda M. Gómez: Methodology, Conceptualization¸ Writing—original draft preparation, Writing—review and editing. Francisco Segovia: Software¸ Investigation. Osvaldo Venegas: Formal analysis, Validation, Writing—review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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