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Research article

The relation between α-spectral radius and structure parameters of s-uniform hypergraphs

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Abstract: The relationship between the tensor spectrum and structural parameters of hypergraphs is a hot topic in the field of hypergraph theory in recent years. This work first establishes the relation between the α-spectral radius and diameter for s-uniform hypergraphs of a given order. Second, we obtain the relation between α-spectral radius, connectivity and independence number of 3-uniform hypergraphs with given order. Additionally, the extremal structure achieving the maximum α-spectral radius is characterized for supertrees, specifically through the difference between the maximum degree and its α-spectral radius.

Keywords: s-uniform hypergraphs; α-spectral radius; diameter; independence number; connectivity
Mathematics Subject Classification: 05C50, 05C65

1. Introduction

A hypergraph $F = (S(F), T(F))$ consists of a vertex set $S(F)$ and an edge set $T(F)$, where $T(F)$ is a family of subsets of $S(F)$ (see [1]). For a fixed integer $s \geq 2$, the hypergraph denoted as s-uniform is a hypergraph such that every edge consists of s vertices. Denote by $d_F(v)$ the degree of v in F, which counts the edges containing v. We define the maximum degree $\Delta(F)$ and the minimum degree $\delta(F)$ of F as

$$\Delta(F) = \max\left\{d_F(v)\,\middle|\, v \in V\right\}, \quad \delta(F) = \min\left\{d_F(v)\,\middle|\, v \in V\right\},$$

respectively. A pendant edge at a vertex v is an edge in which every vertex except v has degree one. Given a hypergraph F, another hypergraph F' is a sub-hypergraph where $S(F') \subseteq S(F)$ and $T(F') \subseteq T(F)$. Moreover, $F[A]$ is an induced sub-hypergraph of F if it has vertex set $A \in S(F)$ and edge set $T(F[A]) = \{e \in T(F) : e \subseteq S(F[A])\}$.

A hypergraph F is linear if any two distinct edges share at most one common vertex. Letting $T(F) = \{e_1, e_2, \cdots, e_m\}$, the hypergraph F is *simple*, if $e_i \subseteq e_j$ implies that $i = j$ for any i, j with $1 \le i, j \le m$. In this paper, we only consider simple hypergraphs. In a hypergraph, if two vertices appear together in one edge, then they are called adjacent. Two edges are called adjacent if their intersection is nonempty. The term incident refers to the relationship between a vertex v and an edge e when v lies in e. In the hypergraph F, a path of length l refers to a vertex-edge sequence $v_1, e_1, v_2, e_2, \cdots, v_l, e_l, v_{l+1}$, where the vertices v_t are pairwise different and the edges e_t are pairwise different, where $v_t, v_{t+1} \in e_t$ for $t = 1, \cdots, l$. A path is called a cycle when $l \ge 1$ and $v_1 = v_{l+1}$. A hypergraph F is a hypertree if it is connected (every pair of vertices can be joined by such a path) and contains no cycles. The other undefined terms follow [1] and [2].

Consider an s-uniform hypergraph $F = (S(F), T(F))$ with $|S(F)| = n$. A vertex where degree equals 1 is referred to as a core vertex. A pendant edge $e \in T$ is an edge, and there exists a vertex $v \in e$ where every vertex $w \in e \setminus \{v\}$ has degree one. Let $Q = (v_0, e_1, v_1, \ldots, v_{g-1}, e_g, v_g)$ be a pendant path in F if: (1). vertices $v_1, \ldots, v_{g-1}$ are all of degree 2; (2). vertex v_g is a core vertex; (3). for each $i \in \{1, \ldots, g\}$, all vertices in $e_i \setminus \{v_{i-1}, v_i\}$ are core vertices. Let K_n^s denote the complete linear s-uniform hypergraph on n vertices, i.e., the hypergraph whose edge set consists of all s-subsets of the vertex set with the property that any two distinct edges share no more than one vertex. A complete linear s-uniform hypergraph on t vertices is also called a clique of order t. We denote by $\omega(F)$ the clique number of F, i.e., the maximum possible size of a clique in F. An independent set of F is a vertex set containing no adjacent pair of vertices. The independence number $\alpha(F)$ of F is defined as the maximum possible size of an independent set in F. A vertex set $X \subseteq S(F)$ is a vertex-cut in F whenever $F - X$ is disconnected. If F has a vertex-cut, then its vertex-connectivity $\kappa(F)$ is the minimum size of such a cut. Otherwise, $\kappa(F) = |S(F)| - 1$. We call F s-vertex-connected if $\kappa(F) \ge s$. The *union* $F_1 \cup F_2$ is defined as $F_1 \cup F_2 = (S(F_1) \cup S(F_2), T(F_1) \cup T(F_2))$. The *join* $F_1 + F_2$ of two s-uniform hypergraphs F_1 and F_2 is obtained from $F_1 \cup F_2$ by adding all s-uniform edges that contain at least one vertex from $S(F_1)$ and at least one vertex from $S(F_2)$.

Let $\mathcal{Z} = (a_{i_1 i_2 \cdots i_s})$ be a multidimensional array with indices $1 \le i_j \le n$ $(j = 1, \cdots, s)$ and entries in $\mathbb{C}$. Then, $\mathcal{Z}$ is referred to as an order s dimension n tensor over $\mathbb{C}$.

Definition 1.1 ([3]). *Consider a tensor $\mathcal{Z}$ with order s and dimension n, and take $y = (y_1, \cdots, y_n)^\top \in \mathbb{C}^n$ as a column vector. The ith coordinate of $\mathcal{Z}y^{s-1} \in \mathbb{C}^n$ is given by*

$$(\mathcal{Z}y^{s-1})_i = \sum_{i_2, \cdots, i_s = 1}^{n} a_{i i_2 \cdots i_s} y_{i_2} \cdots y_{i_s}, (i = 1, \cdots, n).$$

Set $y^{[s]} = (y_1^s, \cdots, y_n^s)^\top$. We say that $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{Z}$ if one can find a nonzero $y \in \mathbb{C}^n$ satisfying

$$\mathcal{Z}y^{s-1} = \lambda y^{[s-1]}.$$

Let $\zeta(\mathcal{Z}) = \lambda_1(\mathcal{Z})$ be the spectral radius of $\mathcal{Z}$. That is, the spectral radius is defined as the maximum value of the modulus of all its eigenvalues. The lemma below, which follows from optimization theory, provides an alternative definition.

Lemma 1.2 ([4]). *For any symmetric nonnegative tensor $\mathcal{Z}$ (order s and dimension n) over $\mathbb{C}$, we have*

$$\zeta(\mathcal{Z}) = max\{y^\top \mathcal{Z}y | y \in \mathbb{R}_+^n, ||y||_s^s = 1\},$$

where $\|y\|_s^s = \sum\limits_{i=1}^{n} y_i^s$.

Definition 1.3 ([5]). *Let* $\mathcal{Z}(F)$ *denote the adjacency tensor of an* s*-uniform hypergraph* $F = (S(F), T(F))$ *with* $|S(F)| = n$*, where* $\mathcal{Z}(F)$ *has order* s *and dimension* n*, and where its* $(i_1 \cdots i_s)$*-entry is*

$$\mathcal{Z}_{i_1 \cdots i_s} = \begin{cases} \frac{1}{(s-1)!}, & if\ \{i_1, \cdots, i_s\} \in T(F); \\ 0, & otherwise. \end{cases}$$

Spectral graph theory studies relations between graph properties and matrix spectral. Its extension to hypergraphs, tensor spectral theory, has attracted much attention due to various applications [5, 6]. For more details, please refer to ([7, 8]).

For an s-uniform hypergraph $F = (S(F), T(F))$ with $|S(F)| = n$, its signless Laplacian tensor is $\mathcal{Q}(F) = \mathcal{D}(F) + \mathcal{Z}(F)$, where $\mathcal{D}(F)$ is the order s and dimension n diagonal tensor with $\mathcal{D}_{ii\cdots i} = d_F(i)$ for all $i \in [n]$.

Inspired by Nikiforov [9], Lin and Zhou [10] defined the convex linear combinations $\mathcal{Z}_\alpha(F)$ of $\mathcal{D}(F)$ and $\mathcal{Z}(F)$:

$$\mathcal{Z}_\alpha(F) = \alpha\mathcal{D}(F) + (1-\alpha)\mathcal{Z}(F), (0 \le \alpha < 1).$$

Let the α-spectra of F be the eigenvalue set of $\mathcal{Z}_\alpha(F)$. The α-spectral radius $\zeta_\alpha(F)$ of F is the maximum absolute value of the modulus of all its eigenvalues. For $s \ge 2$, let y be an n-dimensional column vector. Then,

$$y^\top(\mathcal{Z}_\alpha(F)y) = \alpha \sum_{w \in S(F)} d_w y_w^s + (1-\alpha)s \sum_{e \in T(F)} \prod_{v \in e} y_v.$$

Recently, several results exist on the relation between the α-spectral radius and structural parameters for s-uniform hypergraphs. For example, Guo and Zhou [11] gave structure with maximum α-spectrum radius in s-uniform supertrees. Lin and Zhou [12] gave extremal structure with maximum α-spectral radius in hypertrees. Liu and Li [13] determined the unique supertrees with the maximum α-spectral radius among all uniform supertrees with independence number and given degree sequences and matching number,respectively. The extremal structure achieving the maximum α-spectral radius in supertrees was characterized by You et al. [14] using degree sorting. Hou et al. [15] obtained the relation between the α-spectra radius of hypergraphs and line graphs. For unicyclic uniform hypergraphs with prescribed diameter and number of pendant edges, Kang et al. [16] characterized the extremal structure that maximizes the α-spectral radius.

This paper is arranged as follows. First, we give some basic definitions and lemmas in Section 2. Then, in Section 3, by using the method of pendant path and the structure property of hypergraphs with diameter and clique number, we first establish the between the α-spectral radius and diameter in s-uniform hypergraphs with given order. Then, we obtain the between α-spectral radius, connectivity, and independence number of 3-uniform hypergraphs with given order. In Section 4, we also characterized the extremal structure maximizing the α-spectral radius in supertrees via the difference between the maximum degree and the α-spectral radius.

2. Preliminary

A useful method for obtaining an s-uniform hypergraph with larger spectral radius was given by Li et al. [17].

Definition 2.1 ([17]). *Let $F = (S(F), T(F))$ be a hypergraph and $w \in S(F)$, $e_1, e_2, \cdots, e_s \in T(F)$, where $w \notin e_j$, $j = 1, \cdots, s (s \geq 1)$. Set $v_j \in e_j$ and $e'_j = (e_j \setminus \{v_j\}) \cup \{w\}$. Set $F' = (S(F), T(F'))$ as the hypergraph where $T(F') = (T \setminus \{e_1, \cdots, e_s\}) \cup \{e'_1, \cdots, e'_s\}$. So, we move $(e_1, \cdots, e_s)$ from $(v_1, \cdots, v_s)$ to w in F, and we denote this hypergraph by F'.*

Then, Guo et al. obtained an edge-moving operation with larger spectral radius for a hypergraph as follows.

Lemma 2.2 ([11]). *Let F be a s-uniform hypergraph ($s \geq 2$). In F, $w, v_1, \cdots, v_t \in S(F)$ and $e_1, \cdots, e_t \in T(F)$ for $t \geq 1$, where $w \notin e_j$ and $v_j \in e_j$ for $j = 1, \cdots, t$. Set $e'_j = (e_j \backslash \{v_j\}) \cup \{w\}$ for $j = 1, \cdots, t$. If $e'_j \notin T(F)$ for $j = 1, \cdots, t$, set $F' = F - \{e_1, \cdots, e_t\} + \{e'_1, \cdots, e'_t\}$. Set y is the α-positive unit vector of F. If $y_w \geq max\{y_{v_1}, \cdots, y_{v_t}\}$, then $\zeta_\alpha(F') > \zeta_\alpha(F)$.*

Lemma 2.3 ([16]). *Consider two connected s-uniform hypergraphs $F = (S(F), T(F))$ and $F' = (S(F'), T(F'))$. Let F' be a proper sub-hypergraph of F. Then, we obtain $\zeta_\alpha(F') < \zeta_\alpha(F)$.*

In [18], Wang et al. found an s-uniform hypergraph with larger α-spectral radius, for an s-uniform hypergraph with two pendant paths attaching two distinct vertices. Let F be an s-uniform hypergraph. Let $F_{w,v}(g, l)$ be the hypergraph obtained by attaching the path Q_g to $w \in V(F)$ and the path Q_l to $v \in V(F)$, respectively. Similarly, let $F_w(g, l)$ be the hypergraph constructed by attaching both paths Q_g and Q_l to the same vertex $w \in V(F)$.

Lemma 2.4 ([18]). *Let w, v be two vertices with degree more than 1 in hypergraph F. Suppose we have an internal path Q with t length of hypergraph $F_{w,v}(g, l)$ for every $g \geq l \geq 1$. Then, we obtain*

$$\zeta_\alpha(F_{w,v}(g + 1, l - 1)) < \zeta_\alpha(F_{w,v}(g, l)), \quad for \;\; g - l + 1 \geq t \geq 0.$$

In [11], Guo et al. found that an s-uniform hypergraph has larger α-spectral radius, when the s-uniform hypergraph has two pendant paths attached to the same vertex.

Lemma 2.5 ([11]). *Let F be a connected s-uniform ($s \geq 2$) hypergraph, where $|T(F)| \geq 1$ and $w \in S(F)$. For $g \geq l \geq 1$ and $0 \leq \alpha < 1$, we obtain*

$$\zeta_\alpha(F_w(g, l)) > \zeta_\alpha(F_w(g + 1, l - 1)).$$

By Lemma 2.5, the following conclusion is obvious.

Corollary 2.6. *Let F be a connected s-uniform ($s \geq 2$) hypergraph where $|T(F)| \geq 1$ and $w \in S(F)$. For $g \geq l \geq 1$ and $0 \leq \alpha < 1$, we obtain*

$$\zeta_\alpha(F_w(g, l)) > \zeta_\alpha(F_w(g + l - 1, 1)).$$

Two hypergraphs F and F' are *isomorphic* if there exists a bijection $\sigma : S(F) \to S(F')$, where $\{v_1, v_2, \cdots, v_s\} \in T(F)$ iff $\{\sigma(v_1), \sigma(v_2), \cdots, \sigma(v_s)\} \in T(F')$. Such a bijection σ is called an isomorphism between F and F'. If $F = F'$, then an *automorphism* of F denoted by σ. Let $\mathbf{y}$ be a vector focusing on $S(F)$ and σ is a structure-preserving automorphism of F. w is *equivalent* to v in F, written $w \sim v$, provided that $\sigma(w) = v$ for some automorphism σ of F. Let $\mathbf{y}_\sigma$ be the vector where $y_\sigma(w) = y_{\sigma(w)}$ for every $w \in S(F)$.

Lemma 2.7 ([12]). *Let F be a connected hypergraph where the rank is equal to s, and σ an automorphism in F. Let $\mathbf{y}$ be an eigenvector of $\mathcal{Z}_\alpha(F)$, and obtain $\mathbf{y}_\sigma = \mathbf{y}$. If $w \sim v$, then $y_w = y_v$.*

3. The relation between α-spectral radius and structure parameters in s-uniform hypergraphs

We first give two families of hypergraphs, i.e., *hyperbugs* and *kite hypergraphs*.

Definition 3.1. *A hyperbug $B_{g,l,k}$ is an s-uniform hypergraph obtained from the complete s-uniform hypergraph K_g^s by removing the edge $\{w, w_2, \ldots, w_{s-1}, v\}$ and attaching the paths Q_l and Q_k to the vertices w and v, respectively. A hyperbug is called balanced if $|l - k| \leq 1$ (see Figure 1).*

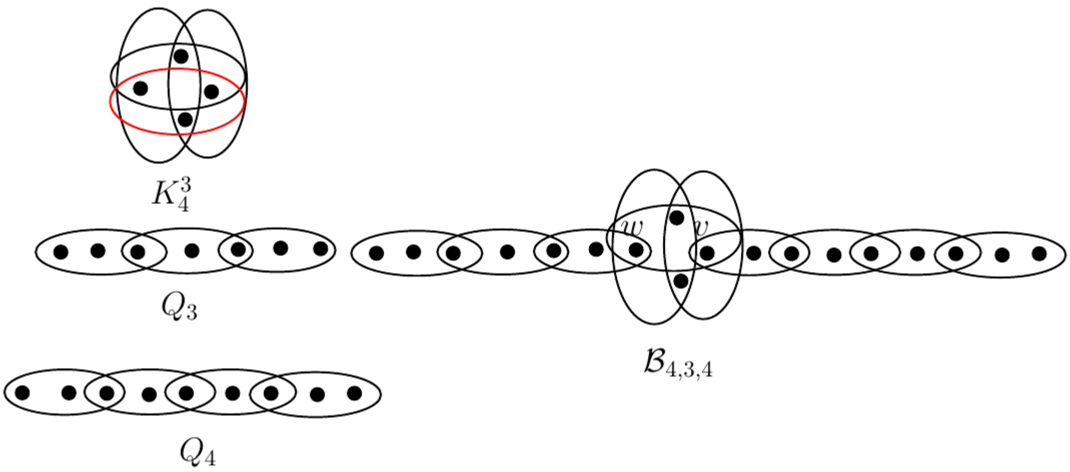

Figure 1. $B_{4,3,4}$.

Definition 3.2. *A kite hypergraph $PK_{g,l}$ is a hypergraph which is s-uniform, obtained by joining a vertex of K_l^s to an end-vertex in the path Q_g (see Figure 2).*

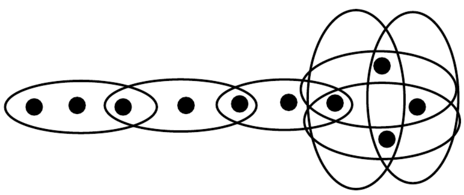

Figure 2. $PK_{3,4}$.

3.1. The relation between the α-spectral radius and diameter in s-uniform hypergraphs with given order

Let F be a connected s-uniform hypergraph that has a path Q as a sub-hypergraph (see Figure 3). We define Q as a *pendant path* where one end of Q is a vertex of degree greater than 1 in F, this vertex is called the *root* of Q.

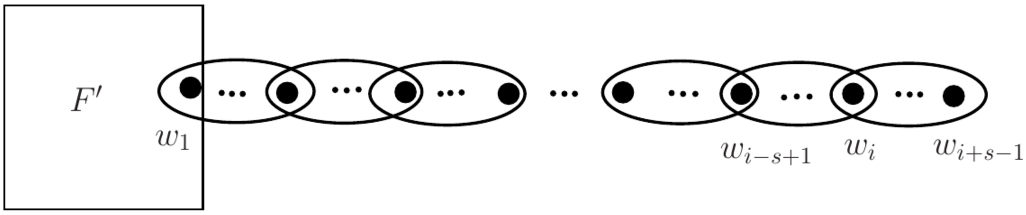

Figure 3. F.

We assume that $\zeta \geq 2$ and $\alpha \neq 1$. In Figure 3, we easily obtain a s-uniform hypergraph that has a pendant path Q, called F, $\zeta_\alpha(F) = \zeta \geq 2$. Then, we can determine the value of an eigenvector to ζ_α for Q.

In fact, let $Q = (w_1, \cdots, w_i, \cdots, w_{i+s-1})$ be a pendant path in F with root w_1. Set $y_1, \cdots, y_i, \cdots, y_{i+s-1} (i = n(s-1)+1, n \in \mathbb{N})$ as the elements of a Perron vector $\zeta_\alpha(F)$ associated with $w_1, \cdots, w_i, \cdots, w_{i+s-1} (i = n(s-1)+1, n \in \mathbb{N})$, respectively. The eigenequation for $\mathcal{Z}_\alpha(F)$ of $y_i (i = n(s-1)+1, n \in \mathbb{N})$ is

$$\zeta y_i^{s-1} = 2\alpha y_i^{s-1} + (1-\alpha) y_{i-s+1} y_{i-s+2} \cdots y_{i-1} + (1-\alpha) y_{i+1} y_{i+2} \cdots y_{i+s-1},$$

which implies that

$$\begin{aligned} y_{i-s+1} y_{i-s+2} \cdots y_{i-1} - \frac{\zeta - 2\alpha}{1-\alpha} y_i^{s-1} + y_{i+1} y_{i+2} \cdots y_{i+s-1} = 0, \\ \frac{y_{i-s+1} y_{i-s+2} \cdots y_{i-1}}{y_{i+1} y_{i+2} \cdots y_{i+s-1}} - \frac{\zeta - 2\alpha}{1-\alpha} \frac{y_i^{s-1}}{y_{i+1} y_{i+2} \cdots y_{i+s-1}} + 1 = 0, \end{aligned}$$

and

$$\frac{y_{i-s+1} y_{i-s+2} \cdots y_{i-1}}{y_i^{s-1}} \cdot \frac{y_i^{s-1}}{y_{i+1} y_{i+2} \cdots y_{i+s-1}} - \frac{\zeta - 2\alpha}{1-\alpha} \frac{y_i^{s-1}}{y_{i+1} y_{i+2} \cdots y_{i+s-1}} + 1 = 0.$$

The above equation is equivalent to the following crucial equation:

$$X^2 - \frac{\zeta - 2\alpha}{1-\alpha} X + 1 = 0. \tag{3.1}$$

We can set γ as the root of (3.1),

$$\gamma := \frac{1}{2} \left(\frac{\zeta - 2\alpha}{1-\alpha} + \sqrt{(\frac{\zeta - 2\alpha}{1-\alpha}^2) - 4} \right), \tag{3.2}$$

and γ is real because $\zeta \geq 2$; so, $\gamma \geq 1$, with strict inequality if $\zeta > 2$.

The following result Theorem 3.3 can be obtained from Eqs (3.1) and (3.2).

Theorem 3.3. *Let F be a linear s-uniform hypergraph with $\zeta := \zeta_\alpha(F) \geq 2$. Let $Q = (w_1, \cdots, w_i, \cdots, w_{i+s-1})$ be a pendant path in F with root w_1. Set $y_1, \cdots, y_i, \cdots, y_{i+s-1}$ as the entries of a Perron eigenvector associated with $\zeta_\alpha(F)$ for $w_1, \cdots, w_i, \cdots, w_{i+s-1}$, where $i = n(s-1)+1, (n \in \mathbb{N})$. If γ is the result in* (3.2)*, then we have*

$$y_{i-s+1} > \gamma y_{i-s+2}, \;\; y_{i-s+1} > \gamma y_i.$$

Proof. Let F be a linear s-uniform hypergraph , and assume that $\zeta \geq 2$. The eigenequation of $\mathcal{Z}_\alpha(F)$ is

$$\begin{aligned}
\zeta y_{i+s-1}^{s-1} &= \alpha y_{i+s-1}^{s-1} + (1-\alpha) y_i y_{i+1} \cdots y_{i+s-2}, \\
\zeta y_{i+s-2}^{s-1} &= \alpha y_{i+s-2}^{s-1} + (1-\alpha) y_i y_{i+1} \cdots y_{i+s-3} y_{i+s-1}, \\
\cdots & \\
\zeta y_{i+1}^{s-1} &= \alpha y_{i+1}^{s-1} + (1-\alpha) y_i y_{i+2} \cdots y_{i+s-1},
\end{aligned}$$

where $y_j (i+1 \leq j \leq i+s-1)$.

Since $\frac{\zeta-2\alpha}{1-\alpha} > \frac{1}{2}\left(\frac{\zeta-2\alpha}{1-\alpha} + \sqrt{(\frac{\zeta-2\alpha}{1-\alpha}^2) - 4}\right) = \gamma$,

$$\begin{aligned}
y_i y_{i+1} \cdots y_{i+s-2} = \frac{\zeta-\alpha}{1-\alpha} y_{i+s-1}^{s-1} > \frac{\zeta-2\alpha}{1-\alpha} y_{i+s-1}^{s-1} &> \gamma y_{i+s-1}^{s-1}, \\
y_i y_{i+1} \cdots y_{i+s-3} y_{i+s-1} &> \gamma y_{i+s-2}^{s-1}, \\
\cdots & \\
y_i y_{i+2} \cdots y_{i+s-1} &> \gamma y_{i+1}^{s-1}.
\end{aligned}$$

Multiplying the $s-1$ inequalities, yields

$$y_i^{s-1} y_{i+1}^{s-2} \cdots y_{i+s-1}^{s-2} > \gamma^{s-1} y_{i+1}^{s-1} y_{i+2}^{s-1} \cdots y_{i+s-1}^{s-1}.$$

So, we have

$$y_i^{s-1} > \gamma^{s-1} y_{i+1} y_{i+2} \cdots y_{i+s-1}. \tag{3.3}$$

By Lemma 2.7, we obtain

$$y_{i+1} = y_{i+2} = \cdots = y_{i+s-2} = y_{i+s-1}.$$

Then,

$$y_i > \gamma y_{i+1}, \tag{3.4}$$

$$y_i > \gamma y_{i+s-1}. \tag{3.5}$$

By the eigenequation of $\mathcal{Z}_\alpha(F)$ for $y_j (i-s+2 \leq j \leq i)$ and the induction assumption,

$$\begin{aligned}
\frac{\zeta-\alpha}{1-\alpha} y_i^{s-1} &= y_{i-s+1} y_{i-s+2} \cdots y_{i-1} + y_{i+1} y_{i+2} \cdots y_{i+s-1}, \\
\frac{\zeta-\alpha}{1-\alpha} y_{i-1}^{s-1} &= y_{i-s+1} y_{i-s+2} \cdots y_{i-2} y_i,
\end{aligned}$$

$$\begin{aligned}
\frac{\zeta-\alpha}{1-\alpha}y_{i-2}^{s-1} &= y_{i-s+1}y_{i-s+2}\cdots y_{i-3}y_{i-1}y_i,\\
&\cdots\\
\frac{\zeta-\alpha}{1-\alpha}y_{i-s+2}^{s-1} &= y_{i-s+1}y_{i-s+3}\cdots y_{i-1}y_i.
\end{aligned}$$

We get

$$\begin{aligned}
\frac{\zeta-2\alpha}{1-\alpha}y_i^{s-1} &\leq (\gamma^{-1})^{s-1}y_i^{s-1}+y_{i-s+1}y_{i-s+2}\cdots y_{i-1}\\
&= \left(\frac{\zeta-2\alpha}{1-\alpha}-\frac{1}{2}\left(\frac{\zeta-2\alpha}{1-\alpha}+\sqrt{(\frac{\zeta-2\alpha}{1-\alpha}^2)-4}\right)\right)^{s-1}y_i^{s-1}\\
&+ y_{i-s+1}y_{i-s+2}\cdots y_{i-1}.
\end{aligned}$$

Then,

$$\begin{aligned}
y_{i-s+1}y_{i-s+2}\cdots y_{i-1} &> \left\{\frac{\zeta-2\alpha}{1-\alpha}-\left[\frac{1}{2}\left(\frac{\zeta-2\alpha}{1-\alpha}-\sqrt{(\frac{\zeta-2\alpha}{1-\alpha}^2)-4}\right)\right]^{s-1}\right\}y_i^{s-1}\\
&> \gamma y_i^{s-1},
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
y_{i-s+1}y_{i-s+2}\cdots y_{i-2}y_i &> \gamma y_{i-1}^{s-1},\\
&\cdots\\
y_{i-s+1}y_{i-s+3}\cdots y_{i-1}y_i &> \gamma y_{i-s+2}^{s-1}.
\end{aligned}$$

Multiplying the $s-1$ inequalities,

$$y_{i-s+1}^{s-1}y_{i-s+2}^{s-2}\cdots y_i^{s-2} > \gamma^{s-1}y_i^{s-1}y_{i-1}^{s-1}\cdots y_{i-s+2}^{s-1}.$$

So, we have

$$y_{i-s+1}^{s-1} > \gamma^{s-1}y_{i-s+2}y_{i-s+3}\cdots y_i. \tag{3.6}$$

We prove that both strict inequalities hold. Suppose, on the contrary, that at least one of them fails. That is, either

$$y_{i-s+1} \leq \gamma y_{i-s+2}$$

or

$$y_{i-s+1} \leq \gamma y_i.$$

We consider the two cases separately.

Case 1: $y_{i-s+1} \leq \gamma y_{i-s+2}$.

Since $y_{i+1} = y_{i+2} = \cdots = y_{i+s-2}$, we rewrite Eq (3.6) as

$$y_{i-s+1}^{s-1} > \gamma y_{i-s+2}\cdot\gamma y_{i-s+3}\cdots\gamma y_i = (\gamma y_{i-s+2})^{s-2}\gamma y_i. \tag{3.7}$$

Using the assumption $y_{i-s+1} \leq \gamma y_{i-s+2}$, it follows from (3.7) that

$$y_{i-s+1}^{s-1} > (\gamma y_{i-s+2})^{s-2} \gamma y_i \geq (y_{i-s+1})^{s-2} \gamma y_i.$$

Dividing both sides by the positive quantity y_{i-s+1}^{s-2}, we obtain

$$y_{i-s+1} > \gamma y_i. \tag{3.8}$$

Now, substituting (3.8) back into (3.7) yields

$$y_{i-s+1}^{s-1} > (\gamma y_{i-s+2})^{s-2} \gamma y_i > (\gamma y_{i-s+2})^{s-2} y_{i-s+1}.$$

Dividing by $y_{i-s+1} > 0$, we get

$$y_{i-s+1}^{s-2} > (\gamma y_{i-s+2})^{s-2}.$$

Taking the $(s-2)$-th root on both sides gives

$$y_{i-s+1} > \gamma y_{i-s+2},$$

which contradicts the assumption of Case 1. Thus, Case 1 is impossible.

Case 2: $y_{i-s+1} \leq \gamma y_i$.

Again, from Eq (3.6) we have

$$y_{i-s+1}^{s-1} > (\gamma y_{i-s+2})^{s-2} \gamma y_i. \tag{3.9}$$

Using the assumption $y_{i-s+1} \leq \gamma y_i$, we obtain from (3.9) that

$$y_{i-s+1}^{s-1} > (\gamma y_{i-s+2})^{s-2} y_{i-s+1}.$$

Dividing by $y_{i-s+1} > 0$, we get

$$y_{i-s+1}^{s-2} > (\gamma y_{i-s+2})^{s-2},$$

and hence

$$y_{i-s+1} > \gamma y_{i-s+2}. \tag{3.10}$$

Now, substituting (3.10) into (3.9) gives

$$y_{i-s+1}^{s-1} > (\gamma y_{i-s+2})^{s-2} \gamma y_i > (y_{i-s+1})^{s-2} \gamma y_i.$$

Dividing by $y_{i-s+1}^{s-2} > 0$, we obtain

$$y_{i-s+1} > \gamma y_i,$$

which contradicts the assumption of Case 2. Thus, Case 2 is also impossible.

Since both cases lead to contradictions, the assumption that at least one of the two strict inequalities fails is false. Therefore,

$$y_{i-s+1} > \gamma y_{i-s+2} \quad \text{and} \quad y_{i-s+1} > \gamma y_i.$$

This completes the proof.

□

Lemma 3.4. *If g and $l + r$ are given, then*

$$\zeta(B_{g,l,r}) < \zeta(B_{g,\lfloor \frac{l+r}{2} \rfloor, \lceil \frac{l+r}{2} \rceil}).$$

Proof. By the definition of $B_{g,l,r}$, if $l = r = 1$, then $B_{g,l,r} = K_g^s$, and thus $\zeta(B_{g,l,r}) = \zeta(K_g^s)$. If $l \geq r \geq 2$, there exists an internal path of length 2 in $B_{g,l,r}$. From Lemma 2.4, it follows that when $l - r + 1 \geq 2$,

$$\zeta(B_{g,l,r}) < \zeta(B_{g,l-1,r+1}).$$

Repeating this process, for fixed g and $l + r$, we have

$$\zeta(B_{g,l,r}) < \zeta(B_{g,\lfloor \frac{l+r}{2} \rfloor, \lceil \frac{l+r}{2} \rceil}).$$

□

Theorem 3.5. *Consider the linear s-uniform hypergraph $F = (S(F), T(F))$, $|S(F)| = n$, diam$(F) \geq r$. Setting $r = 1$, we have $\zeta_\alpha(F) = \zeta_\alpha(K_n^s)$. If $r \geq 2$, then*

$$\zeta_\alpha(F) \leq \zeta_\alpha(B_{n-r+2}, \lfloor \frac{r}{2} \rfloor, \lceil \frac{r}{2} \rceil).$$

with equality iff $F = B_{n-r+2}, \lfloor \frac{r}{2} \rfloor, \lceil \frac{r}{2} \rceil$.

Proof. Since the spectral radius $\zeta_\alpha(F)$ is non-decreasing with respect to edge addition, we may assume without loss of generality that F is edge-maximal under the given constraints. The result is correct if $r = 1$, because K_n^s is the only linear s-uniform hypergraph with n vertices and diameter 1. Since $r \geq 2$, let F be a linear s-uniform hypergraph of order n with diam$(F) \geq r$ that maximizes the number of edges. By Lemma 3.4, we next prove that $F = B_{n-r+2,g,r-g}$ for $1 \leq g \leq r - 1$.

Let $\zeta := \zeta_\alpha(F)$. Set w, v are two vertices with distance r in F. Define S_i as the set of vertices shared by edges at distance i from w, where $i = 0, \cdots, r$. Given that F has the greatest possible number of edges, there exists a linear complete hypergraph induced by $S_i \cup S_{i+1}$ for every $i = 0, \cdots, r-1$. We know $|S_0| = 1$; thus, $|S_r| = 1$. Otherwise, we have $|S_r| \geq 2$, and add $S_r \setminus \{v\}$ for all $v \in S_r$. These new edges do not change the distance from w to v; thus, the number of F is not maximal, which produces a contradiction; therefore, $|S_r| = 1$.

Thus, by $\zeta_\alpha(F) \leq \Delta(F)$, we have

$$\Delta(F) \geq \zeta_\alpha(F) > \zeta_\alpha(B_{n-r+2,\lfloor \frac{r}{2} \rfloor, \lceil \frac{r}{2} \rceil}) > \zeta_\alpha(K_{n-r+2}^s - e) > \delta(K_{n-r+2}^s - e) = n - r.$$

Then, $\Delta(F) \geq n - r + 1$. Set w is maximum degree-vertex in F, and set $w \in S_i$. We easily obtain $0 < i < r$, and because

$$d(w) = |S_{i-1}| + |S_i| + |S_{i+1}| - 1,$$

we have

$$\begin{aligned} n - r + 2 &\leq |S_{i-1}| + |S_i| + |S_{i+1}| = n - \sum_{j<i-1} |S_j| - \sum_{j>i+1} |S_j| \\ &\leq n - (r + 1 - 3) = n - r + 2. \end{aligned}$$

Thus, if $j < i-1$ or $j > i+1$, then $|S_i| = 1$; so, $|S_{i-1}| + |S_i| + |S_{i+1}| = n - r + 2$.

If $|S_{i-1}| = |S_{i+1}| = 1$, we obtain $F = B_{n-r+2,i,r-i}$, so Theorem 3.5 is proved in this case. Next, by constructing a hypergraph F' with n vertices and $diam(F') = r$ with $\zeta_\alpha(F') \geq \zeta$, we want to obtain the contradictions for other cases. Let $\mathbf{y} := (y_1, \cdots, y_n)$ be a Perron vector to $\zeta_\alpha(F)$.

We first focus on the case $|S_{i-1}| = 1$ and $|S_{i+1}| \geq 2$. When $|S_i| = 1$, the proof is corrected. We set $|S_i| \geq 2$. Set $S_{i-1} = \{a\}, S_{i+2} = \{b\}$, and $y_b \geq y_a$. Seclecting a vertex $w \in S_i$, removing the edge e_{wa}, and adding the edge e_{wb}, we obtain F' as the resulting hypergraph. By symmetry and Lemma 2.7, $y_{w'} = y_w$ for every $w' \in S_i$; we obtain that

$$\begin{aligned}
0 \geq \zeta_\alpha(F') - \zeta &\geq y^\top \mathcal{Z}_\alpha(F') y - y^\top \mathcal{Z}_\alpha(F) y \\
&= \alpha(y^\top D(F') y - y^\top D(F) y) + (1-\alpha)(y^\top \mathcal{Z}(F') y - y^\top \mathcal{Z}(F) y) \\
&= \alpha(y_{e_{wb}}^2 - y_{e_{wa}}^2) + 2(1-\alpha)s(y_{e_{wb}} - y_{e_{wa}}) \\
&= (y_{e_{wb}} - y_{e_{wa}})(2s(1-\alpha) + \alpha(y_{e_{wb}} + y_{e_{wa}})) \\
&\geq 0,
\end{aligned}$$

this implies that $\zeta_\alpha(F') = \zeta_\alpha(F)$ and $\mathbf{y}$ is an eigenvector associated with $\zeta_\alpha(F)$. Because a neighbor domain of a in F' is a proper subset of the neighbor domain of a for F, the eigenequations for $\zeta_\alpha(F')$ and $\zeta_\alpha(F)$ produce a contradiction.

For the cases $|S_{i-1}| \geq 2$ and $|S_{i+1}| = 1$, the proof is same. Next, we focus only on $|S_{i-1}| \geq 2$ and $|S_{i+1}| \geq 2$.

Set $S_{i-2} = \{a\}, c \in S_{i-1}, d \in S_{i+2}$, and $S_{i+2} = \{b\}$. First, we prove that

$$y_c \geq y_a. \tag{3.11}$$

Since $i \geq 3$, and $S_{i-3} = \{z\}$, by Theorem 3.3, we obtain $y_z < y_a$. Thus, letting $l := |S_{i-1}|$, by the eigenequation of the vertex a, we obtain

$$\zeta y_a^{s-1} = \alpha(l+1) y_a^{s-1} + (1-\alpha)(y_{e_{az\setminus\{a\}}} + l x_{e_{ac\setminus\{a\}}}).$$

By $y_z < y_a$ and Theorem 3.3, we obtain

$$\zeta y_a^{s-1} < \alpha(l+1) y_a^{s-1} + (1-\alpha)(y_a^{s-1} + l y_c^{s-1}),$$

so

$$\left(\frac{y_c}{y_a}\right)^{s-1} > \frac{\zeta - 1 - \alpha l}{(1-\alpha) l}.$$

Because

$$\zeta - 1 \geq n - r - 1 = |S_{i-1}| + |S_i| + |S_{i+1}| - 3 \geq |S_{i-1}| = l,$$

inequality (3.11) is proved. Using symmetry, we obtain $y_d > y_b$. By symmetry, we obtain that $y_b > y_a$, so $y_a < y_b < y_d$. For a vertex $w \in S_{i-1}$, removing the edges e_{wa}, and adding the edges e_{wd} for every $d \in S_{i+1}$, F' is the resulting hypergraph. In other words, F' is obtained by moving the vertex w from S_{i-1} into S_i. This proves that

$$0 \geq \zeta_\alpha(F') - \zeta \geq y^\top \mathcal{Z}_\alpha(F') y - y^\top \mathcal{Z}_\alpha(F) y$$

$$
\begin{aligned}
&\geq \alpha(|S_{i+1}|y_{e_{wd}}^2 - y_{e_{wa}}^2) + 2(1-\alpha)s(|S_{i+1}|y_{e_{wd}} - y_{e_{wa}}) \\
&> \alpha(y_{e_{wd}}^2 - y_{e_{wa}}^2) + 2(1-\alpha)s(y_{e_{wd}} - y_{e_{wa}}) \\
&> 0.
\end{aligned}
$$

Thus, we have a contradiction, proving Theorem 3.5. □

3.2. The relation between the α-spectral radius, the connectivity, and the independence number of 3-uniform hypergraphs with given order

Let n, f, and c be three integers. We denote by $\mathcal{F}_{n,f,c}$ the families of hypergraphs of n vertices with connectivity f and independence number c. We first give the next crucial result.

Lemma 3.6. *Let $F = (S(F), T(F))$ be a connected 3-uniform hypergraph. Let $\{v_1, v_2, \ldots, v_{2s}\} \subseteq N_F(v)$ such that $vv_{2i-1}v_{2i} \in T(F)$ for $1 \leq i \leq s$, and let $\{v_{2s+1}, v_{2s+2}, \ldots, v_{2(s+l)}\} \subseteq S(F) - N_F(v)$. Suppose $y = (y_1, y_2, \cdots, y_n)^\top$ is the positive unit vector of $\zeta_\alpha(F)$, where y_i corresponds to the vertex $v_i (1 \leq i \leq n)$. Let F^* be the hypergraph obtained from F by removing the edges $\{vv_{2i-1}v_{2i} \mid 1 \leq i \leq s\}$ and adding the edges $\{vv_{2i-1}v_{2i} \mid s+1 \leq i \leq s+l\}$. If $y_j \leq y_i$ for $j \leq i$, then $\zeta_\alpha(F) \leq \zeta_\alpha(F^*)$. Furthermore, if $y_j < y_i$ for $j < i$, then $\zeta_\alpha(F) < \zeta_\alpha(F^*)$.*

Proof. By the structure of the hypergraph, F^* is obtained from F by removing the edges $\{vv_{2i-1}v_{2i} \mid 1 \leq i \leq s\}$ and adding the edges $\{vv_{2i-1}v_{2i} \mid s+1 \leq i \leq s+l\}$. Hence, $d_F(v) = d_{F^*}(v)$.

$$
\begin{aligned}
y^\top(\mathcal{Z}_\alpha(F^*) - \mathcal{Z}_\alpha(F))y &= \sum_{i=2k+1}^{2k+2l} \alpha y_i^3 - \sum_{i=1}^{2k} \alpha y_i^3 + (1-\alpha)[-y_v \sum_{i=1}^{s} y_{2i-1}y_{2i} \\
&\quad + y_v \sum_{i=s+1}^{s+l} y_{2i-1}y_{2i} - y_v \sum_{i=1}^{s} y_{2i-1}y_{2i} + y_v \sum_{i=s+1}^{s+l} y_{2i-1}y_{2i}] \\
&= \sum_{i=2k+1}^{2k+2l} \alpha y_i^3 - \sum_{i=1}^{2k} \alpha y_i^3 + 2(1-\alpha)y_v(\sum_{i=s+1}^{s+l} y_{2i-1}y_{2i} - \sum_{i=1}^{s} y_{2i-1}y_{2i}),
\end{aligned}
$$

If $y_j \leq y_i$ for $j \leq i$, then $\sum\limits_{i=2k+1}^{2k+2l} \alpha y_i^3 \geq \sum\limits_{i=1}^{2k} \alpha y_i^3$ and $\sum\limits_{i=s+1}^{s+l} y_{2i-1}y_{2i} \geq \sum\limits_{i=1}^{s} y_{2i-1}y_{2i}$. Furthermore,

$$
y^\top(\mathcal{Z}_\alpha(F^*) - \mathcal{Z}_\alpha(F))y \geq 0.
$$

We get

$$
\zeta_\alpha(F^*) = \max_{\|y\|^2=1} y^\top \mathcal{Z}_\alpha(F^*)y \geq y^\top \mathcal{Z}_\alpha(F^*)y > y^\top \mathcal{Z}_\alpha(F)y = \zeta_\alpha(F).
$$

Hence, $\zeta_\alpha(F) < \zeta_\alpha(F^*)$. The lemma is proved. □

Theorem 3.7. *Let n, f, and c be three integers. Among all the 3-uniform connected hypergraphs of n vertices with connectivity f and independence number c, only the hypergraph $K_f^3 + (K_1 \cup (K_{n-f-c}^3 + (c-1)K_1))$ achieves the maximal α-spectral radius.*

Proof. Let F^* be the 3-uniform hypergraph with the maximal α-spectral radius with $S(F^*) = \{v_1, v_2, \cdots, v_n\}$ in $\mathcal{F}_{n,f,c}$. Denote the positive unit vector of $A(F^*)$ by $[y] = (y_{v_1}, y_{v_2}, \cdots, y_{v_n})$, y_{v_i} matching the vertex $v_i (i = 1, 2, \cdots, n)$. Since F^* is f-connected and $\boldsymbol{\alpha}(F^*) = c$, then $n = |S(F^*)| \geq f + c$.

Let $S = \{v_{i_1}, \cdots, v_{i_c}\}$ be an independent set of F^* and $T = \{v_{j_1}, \cdots, v_{j_f}\}$ be a cut set. Let $F^* - T = C_1 \cup \cdots \cup C_l$, where $l \geq 2$. Without loss of generality, suppose that $|S(C_1)| \geq |S(C_2)| \geq \cdots |S(C_l)|$. Let $a_1 = |S(C_1) \cap S|$, $a_2 = |S(C_1) - S|$, $b_1 = |T \cap S|$, and $b_2 = |T - S|$.

Claim 1. *$C_1 = K^3_{a_2} + a_1 K_1$, $F^*[T] = K^3_{b_2} + b_1 K_1$, and $F[S(C_1) \cup T] = K^3_{a_2+b_2} + (a_1 + b_1)K_1$.*

Let $F^* - T = C_1 \cup \cdots \cup C_l$, where $l \geq 2$, be the decomposition into connected components. Without loss of generality, assume that $|S(C_1)| \geq |S(C_2)| \geq \cdots \geq |S(C_l)|$. If C_1 were a component consisting of an isolated vertex, then every C_i would be a single-vertex component. This contradicts the extremal choice of F^*. Therefore, C_1 contains at least one edge. Hence, C_1 is a connected 3-uniform sub-hypergraph, so $|S(C_1)| \geq 3$. Now we show that $C_1 = K^3_{a_2} + a_1 K_1$. Let $u, v, w \in S(C_1) \setminus S$ and $t \in S(C_1) \cap S$. Suppose that $uvw \notin F^*$. Since S remains an independent set of $F^* + uvw$, we have $\boldsymbol{\alpha}(F^*) = \boldsymbol{\alpha}(F^* + uvw)$. It is easy to see that $\kappa(F^*) = \kappa(F^* + uvw)$. Hence, $F^* + uvw \in \mathcal{F}_{n,f,c}$. By Lemma 2.3, $\zeta(F^* + uvw) > \zeta(F^*)$, contradicting the choice of F^*. Therefore, $uvw \in C_1 - S$. By similar reasoning, we show that $F^*[T] = K^3_{b_2} + b_1 K_1$. Let $u', v', w' \in S(T) \setminus S$ and $t' \in S(T) \cap S$. Suppose that $u'v'w' \notin F^*$. T remains a cut set of $F^* + u'v'w'$, and hence it is also a cut set of F^*. It is easy to see that $\kappa(F^*) = \kappa(F^* + u'v'w')$. Hence, $F^* + u'v'w' \in \mathcal{F}_{n,f,c}$. By Lemma 2.3, $\zeta(F^* + u'v'w') > \zeta(F^*)$, contradicting the choice of F^*. Therefore, $u'v'w' \in T - S$. This complete the proof of Claim 1.

If $n = f + c$, it is easy to see that $F^* - S$ is a cut set. With a similar proof to Claim 1, we have $F^* = K^3_f + cK_1$. Now we can assume that $n \geq f + c + 1$.

Claim 2. *C_i is an isolated vertex for $i = 2, \cdots, l$, and $l = 2$.*

Otherwise, suppose that $|S(C_2)| \geq 2$. Then, $S(C_2) - S \neq \emptyset$. Let $u, v \in S(C_2) - S$. If $S(C_3) \neq \emptyset$, letting $w \in S(C_3)$, then $F^* + uvw$ is an f-connected hypergraph with independence number c and $\zeta(F^* + uvw) > \zeta(F^*)$, a contradiction. Hence, $F - T$ contains exactly two components, hence $l = 2$. Let $h_1 = |S(C_2) \cap S|$ and $h_2 = |S(C_2) - S|$. For every vertex $x, y \in S(C_2) - S$ and all vertices $p, z \in S(C_2)$, with similar proof in Claim 1, we have $xyp, ypz \in T(C_2)$. Since S is an independent set, then $C_2 = K_{h_2} + h_1 K_1$. Let $w \in S(C_1) - S$. If $y_w \geq y_u$, let

$$F' = F^* - \{xyu | x, y \in N_{C_2}(u)\} + \{xyw\},$$

by Lemma 2.2, $\zeta(F') > \zeta(F^*)$. By the definition, F' is f-connected, and $\boldsymbol{\alpha}(F') = c$, a contradiction. Else, if $y_w \leq y_u$, we can obtain a contradiction with similar discussion. Thus, Claim 2 is proved.

By Claim 2, we assume $S(C_2) = v_1$.

Claim 3. $v_1 \in S$.

Otherwise, suppose that $v_1 \notin S$. Since $\kappa(F^*) = f$, then $N_{F^*}(v_1) = |T|$. So, we have $N_{F^*}(v_1) \cap S \neq \emptyset$, i.e., $S \cap T \neq \emptyset$. Let $w_1 \in S \cap T$ and $w'_1 \in T - S, w'_1 \in N_{F^*}(w_1)$. First, we consider $|S \cap T| \geq 2$. By Claim 1 and $n \geq f + c + 1$, we have $S(C_1) - S \neq \emptyset$. Let $w_2, w'_2 \in S(C_1) - S$. If $y_{w_2} \geq y_{w_1}$ and $y_{w'_2} \geq y_{w'_1}$, then $F' = F^* - w_1 w'_1 v_1 + w_2 w'_2 v_1$. Observe that $(T - w_1 - w'_1) \cup \{w_2, w'_2\}$ is a cut set and S is still a maximum independent set of F'. Hence, $\kappa(F') = f$ and $\boldsymbol{\alpha}(F') = c$. So, $F' \in \mathcal{F}_{n,f,c}$, a contradiction since $\zeta_\alpha(F') > \zeta_\alpha(F^*)$ by Lemma 3.6. Else, if $y_{w_2} \leq y_{w_1}$ and $y_{w_2'} \leq y_{w_1'}$, let

$$F'' = F^* - \{xw_1w_1'|y \in S - w_1\} + \{xw_2w_2'|y \in S - w_1\}.$$

Then, T is still a cut set and $(S - w_1) \cup w_2$ is independent in F'' with the greatest possible number of vertices. Hence, $F'' \in \mathcal{F}_{n,f,c}$. By Lemma 3.6, $\zeta_\alpha(F'') > \zeta(F_\alpha^*)$. Now we can assume that $S \cap T = w_1$. Let $F_1 = F^* + \{xyw_1|x, y \in S(F^*) - N_{F^*}(w_1)\}$. Clearly, T is still a cut set of F_1 and $(S - w_1) \cup v_1$ is an independent set. Hence, $F_1 \in \mathcal{F}_{n,f,c}$. By Lemma 2.2, we obtain $\zeta(F_1) > \zeta(F^*)$, contradicting to the choice of F^*. Thus, Claim 3 is proved.

Since $\kappa(F^*) = f$, then $T = N_{F^*}(v_1)$. By Claim 3, we have $S \cap T = \emptyset$. By Claim 1 and 2, F^* can be obtained from $K_f^3 + (K_{n-f-c}^3 + (c-1)K_1)$ by adding a new vertex and connecting the vertex to vertices of $K_f^3 + (K_{n-f-c}^3 + (c-1)K_1)$. This completes the proof. □

4. The difference between maximum degree and the extremal α-spectral property

We review the definition and properties of supertrees.

Definition 4.1 ([17]). *A supertree is defined as a connected hypergraph that contains no cycles.*

Proposition 4.2 ([17]). *Every connected acyclic hypergraph F is linear. Given an s-uniform supertree F with n vertices, its number of edges is* $\frac{n-1}{s-1}$.

Based on the supertrees, Hu et al. gave the definition of hyperstar in [19].

Definition 4.3 ([19]). *Set* $F = (S, T)$ *is a s-uniform hypergraph of order n, and size m. If the vertex set S can be classified as* $m = 1$ *partitions* $S_0 \cup S_1 \cup \cdots \cup S_m$. *where* $|S_0| = 1$, $|S_1| = \cdots = |S_m| = s - 1$, *and* $T = \{S_0 \cup S_i | i \in [m]\}$, *then we denote by* $S_n^s = F$ *is a hyperstar.*

For a hypergraph F of maximum degree Δ and $0 \leq \alpha < 1, \zeta_\alpha(F) \leq \Delta$, the equality holds iff F has a regular sub-hypergraph of degree Δ. Set $\gamma_\alpha(F) = \Delta - \zeta_\alpha(F)$. We let $\gamma_\alpha(F)$ be a measure of irregularity in the hypergraph F. The following theorem gives the upper bound of $\gamma_\alpha(F)$ for the 3-uniform supertrees.

Theorem 4.4. *Let* T_n^3 *be a 3-uniform supertree with* $n \geq 2$ *vertices. If* $0 \leq \alpha < 1$, *we obtain*

$$\gamma_\alpha(T_n^3) \leq n - 1 + \frac{5 - 2n}{3}\alpha,$$

where the equality holds iff $T_n^3 \cong S_n^3$.

Proof. Let Δ be the maximum degree of T_n^3. We obtain a subgraph $S_{2\Delta+1}^3$ in T_n^3.

Calculation of $\zeta_\alpha(S_{2\Delta+1}^3)$**:**

By the eigenequation of $\mathcal{Z}_\alpha(S_n^s)$, we obtain that

$$\begin{cases} \zeta y_1^{s-1} = \frac{n-1}{s-1}\alpha y_1^{s-1} + \frac{n-1}{s-1}(1-\alpha)y_2^{s-1} & ① \\ \zeta y_2 = \alpha y_2 + (1-\alpha)y_1 & ② \end{cases}$$

$$② \Longrightarrow (\zeta - \alpha)y_2 = (1-\alpha)y_1 \Longrightarrow y_2 = \frac{1-\alpha}{\zeta-\alpha}y_1, y_1 = \frac{\zeta-\alpha}{1-\alpha}y_2.$$

$$①\Longrightarrow (\zeta - \tfrac{n-1}{s-1}\alpha)y_1^{s-1} = \tfrac{n-1}{s-1}(1-\alpha)y_2^{s-1}.$$

So, we have

$$\zeta - \frac{n-1}{s-1}\alpha = \frac{n-1}{s-1}\frac{(1-\alpha)^s}{(\zeta-\alpha)^{s-1}}.$$

Then,

$$(s-1)\zeta(\zeta-\alpha)^{s-1} - (n-1)\alpha(\zeta-\alpha)^{s-1} - (n-1)(1-\alpha)^s = 0.$$

Set $\zeta_\alpha(S^{\,s}_{2\Delta+1})$ as the maximum real root of $(s-1)\zeta(\zeta-\alpha)^{s-1} - \Delta\alpha(\zeta-\alpha)^{s-1} - \Delta(1-\alpha)^s = 0$.

For the equation

$$(s-1)\zeta(\zeta-\alpha)^{s-1} - \Delta\alpha(\zeta-\alpha)^{s-1} - \Delta(1-\alpha)^s = 0,$$

let $\zeta - \alpha = t$. Then $\zeta = t + \alpha$, and thus

$$(s-1)(t+\alpha)t^{s-1} - \Delta\alpha t^{s-1} - \Delta(1-\alpha)^s = 0.$$

That is,

$$(s-1)t^s + (s-1-\Delta)\alpha t^{s-1} - \Delta(1-\alpha)^s = 0.$$

Letting $A = \frac{(s-1-\Delta)}{s-1}\alpha$, and $B = \frac{-\Delta(1-\alpha)^s}{s-1}$, we have another equation:

$$t^s + At^{s-1} + B = 0.$$

When $s = 3$, by the solution method of the cubic equation in one variable, solving the above equation gives its largest real root $-\frac{2}{3}A = -\frac{1}{3}\alpha(2-\Delta)$. Thus,

$$\zeta_\alpha(S^{\,3}_{2\Delta+1}) = \frac{1}{3}\alpha(2\Delta-1).$$

Using Lemma 2.3,

$$\zeta_\alpha(T_n^3) \geq \zeta_\alpha(S^{\,3}_{2\Delta+1}),$$

and the equality holds iff $T_n^3 \cong S^{\,3}_{2\Delta+1}$, where T_n^3 is a connected hypergraph.

By calculation, we have

$$\zeta_\alpha(S^{\,3}_{2\Delta+1}) = \frac{1}{3}\alpha(2\Delta-1).$$

Thus,

$$\gamma_\alpha(T_n^3) \leq f(\Delta),$$

where

$$f(t) = t - \frac{1}{3}\alpha[2(t-1)-1] = (1-\frac{2}{3}\alpha)t + \alpha.$$

Observe that

$$f'(t) = 1 - \frac{2}{3}\alpha > 0.$$

Since $0 \leq \alpha < 1$, and $f(t)$ is strictly increasing, then

$$\gamma_\alpha(T_n^3) \leq f(\Delta) \leq f(n-1) = n - 1 + \frac{5-2n}{3}\alpha,$$

and the equality holds iff $\Delta = n-1$ and $\zeta_\alpha(T_n^3) = \zeta_\alpha(S^3_{2\Delta+1})$, or, equivalently, $T_n^3 \cong S_n^3$. □

Author contributions

Qiannan Niu: Conceptualization, Formal analysis, Writing–original draft; Lei Zhang: Supervision, Writing–review & editing; Haizhen Ren: Supervision, Validation.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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