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Research article

Existence results for doubly reflected backward doubly stochastic differential equations with continuous drivers

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Abstract: We study a class of doubly reflected backward doubly stochastic differential equations with continuous reflecting barriers. The first component of the solution is constrained to remain between a lower obstacle L and an upper obstacle U. The coefficient g is assumed to be Lipschitz continuous with respect to (y, z), with a contraction condition in the z variable, whereas the driver f is only assumed to be continuous and of linear growth. Under a semimartingale approximation condition on the upper barrier, we prove the existence of both minimal and maximal adapted solutions. The proof combines the theory of one-barrier reflected BDSDEs with an upper-obstacle penalization procedure. A central point of the construction is that the Lipschitz approximation of the continuous driver and the penalization of the upper barrier are handled by two separate parameters. This separation makes it possible to use the comparison theorem in a rigorous way and avoids the monotonicity difficulty that arises when both parameters are varied simultaneously. The minimal solution is obtained through an increasing sequence of Lipschitz approximations of f, while the maximal solution is obtained through a decreasing sequence. The proof relies on uniform a priori estimates, a uniform control of the upper penalization term, convergence of the martingale integrands, and verification of the two Skorokhod conditions. The examples show that the assumptions are non-empty and include genuinely non-Lipschitz drivers, as well as deterministic and bounded random barriers.

Keywords: backward doubly stochastic differential equation; reflected BDSDE; double reflection; continuous generator; minimal solution; maximal solution; penalization
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1. Introduction

Backward doubly stochastic differential equations (BDSDEs), introduced by Pardoux and Peng [1], constitute an important class of stochastic equations involving both a forward Itô integral and a backward stochastic integral. A standard BDSDE is written as

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T g(s, Y_s, Z_s)\, dB_s - \int_t^T Z_s\, dW_s, \quad 0 \le t \le T,$$

where W and B are two independent Brownian motions, ξ is the terminal condition, and f and g denote the drift and diffusion coefficients, respectively. The integral with respect to W is interpreted in the usual forward Itô sense, whereas the integral with respect to B is understood as a backward stochastic integral. Since their introduction, BDSDEs have become an essential probabilistic tool for studying quasilinear stochastic partial differential equations (SPDEs), stochastic control, nonlinear filtering, random dynamic systems, and related stochastic models; see, for example, [2–4]. Classical existence and uniqueness results were established under global Lipschitz assumptions; see [5–7]. Subsequently, many important extensions have been obtained under weaker regularity conditions or more general probabilistic settings; see, for example, [8, 9].

Many stochastic systems arising in applications are naturally subject to state constraints. In finance, the value of a portfolio or a contingent claim may be required to remain within prescribed bounds. In stochastic control, safety requirements and resource limitations often impose admissible state regions. Similar constrained dynamics also arise in optimal stopping, Dynkin games, engineering systems with operational limits, and obstacle-type SPDEs. Such problems have motivated the development of reflected stochastic equations, in which additional increasing processes keep the solution inside a prescribed admissible domain without altering the underlying stochastic dynamics. Reflected BDSDEs therefore provide a natural mathematical framework for modeling constrained stochastic systems driven simultaneously by forward and backward sources of randomness.

The simplest reflected model is the one-barrier reflected BDSDE, where the solution is constrained to remain above a given obstacle L. It is described by

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T g(s, Y_s, Z_s)\, dB_s + K_T - K_t - \int_t^T Z_s\, dW_s, \tag{1.1}$$

where K is a continuous increasing process satisfying the classical Skorokhod minimality condition. This process acts only when the solution reaches the lower obstacle and prevents it from moving below L. Reflected BDSDEs have been extensively investigated because of their close connections with obstacle problems for SPDEs, stochastic control, optimal stopping, and variational inequalities. Under Lipschitz assumptions on the coefficients, Bahlali et al. [10] established the existence and uniqueness of solutions and later extended their analysis to continuous generators, obtaining the existence of minimal and maximal solutions.

A more general and considerably more challenging problem arises when the solution is constrained simultaneously by a lower obstacle L and an upper obstacle U. In this case the corresponding doubly reflected BDSDE takes the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T g(s, Y_s, Z_s)\, dB_s + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T Z_s\, dW_s, \quad 0 \le t \le T, \tag{1.2}$$

where the increasing processes K^+ and K^- force the solution to satisfy

$$L_t \le Y_t \le U_t, \qquad 0 \le t \le T,$$

together with two complementary Skorokhod conditions. In contrast with the one-barrier case, the two reflecting mechanisms interact and cannot be analyzed independently. Both reflecting measures must be identified simultaneously while preserving the minimality conditions, making the existence theory substantially more delicate.

The theory of reflected and doubly reflected backward stochastic differential equations (BSDEs) and BDSDEs has developed considerably in recent years. Starting from the classical setting with globally Lipschitz generators, researchers have gradually extended the theory to include stochastic Lipschitz coefficients and irregular barriers; see, for example, [11–13]. Other works have considered more general filtrations, jumps, and mean-reflection constraints; see [14–16]. Further contributions have focused on conditional reflection and related extensions of reflected BSDEs and BDSDEs; see, for example, [17–19]. More recently, the theory has also been developed in settings involving G-Brownian motion and other generalized stochastic frameworks; see [20–22].

These developments have considerably expanded the scope of reflected stochastic equations and reinforced their connections with constrained stochastic control, obstacle problems, SPDEs, and other areas of stochastic analysis. Despite this substantial progress, most existing existence results for doubly reflected BDSDEs continue to rely on globally Lipschitz generators or on structural assumptions that guarantee uniqueness. As a consequence, the existence theory for doubly reflected BDSDEs with merely continuous generators remains relatively limited and deserves further investigation.

To overcome this difficulty, we adopt the classical monotone approximation technique introduced by Lepeltier and San Martín [23]. This approach consists of approximating the original continuous generator by increasing and decreasing sequences of Lipschitz-continuous generators that converge pointwise to the target function. Such a construction provides the analytical framework needed to extend the existence results beyond the classical Lipschitz setting while preserving the monotonicity properties required by the comparison principle.

The global Lipschitz assumption is mathematically attractive because it enables the direct use of fixed-point arguments and comparison techniques. Nevertheless, many stochastic models arising in applications involve nonlinear drift coefficients that are continuous but do not satisfy a global Lipschitz condition. Examples include stochastic control problems with nonlinear dynamics, obstacle-type SPDEs, and other constrained stochastic systems. In these situations, the standard contraction approach is no longer applicable and the uniqueness of the solutions cannot, in general, be expected. Developing an existence theory under weaker continuity assumptions therefore becomes both mathematically challenging and practically relevant. It requires robust approximation and penalization techniques capable of simultaneously handling the lack of Lipschitz regularity and the interaction between the lower and upper reflecting barriers.

The objective of the present paper is to contribute to this research direction by establishing the existence of minimal and maximal solutions for doubly reflected BDSDEs whose generator is continuous in (y, z) and satisfies only a linear growth condition. Compared with the classical Lipschitz framework, this setting presents several additional analytical difficulties. Besides constructing the lower reflection, one must derive a uniform estimate for the upper penalization term, prove the convergence of the associated reflecting processes, identify the limiting upper reflection, and recover both Skorokhod conditions in the limit. Since uniqueness cannot generally be expected under the present assumptions, our analysis focuses on the construction of extremal adapted solutions.

The proposed approach combines two complementary approximation procedures. First, the continuous generator is approximated by monotone sequences of Lipschitz continuous functions using the classical Lepeltier-San Martín approximation technique. Second, the upper obstacle is recovered through a penalization procedure, whereas the lower obstacle is treated using the existing theory of one-barrier reflected BDSDEs. A key feature of our analysis is that the approximation parameter associated with the generator and the penalization parameter associated with the upper obstacle are handled independently. This separation is crucial because varying both parameters simultaneously generally destroys the monotonicity required for the comparison theorem. Passing successively to the two limits yields a rigorous construction of the doubly reflected solution.

The main contributions of this work are summarized as follows.

- We establish the existence of minimal and maximal adapted solutions for doubly reflected BDSDEs with continuous generators satisfying only a linear growth condition, thereby extending previous existence results beyond the classical Lipschitz framework.
- We develop a two-parameter approximation strategy that separates the regularization of the generator from the penalization of the upper obstacle, providing a systematic framework for applying comparison arguments and constructing extremal solutions.
- Under a semimartingale approximation assumption on the upper barrier, we derive a uniform estimate for the penalization term, identify the limiting upper reflecting process, and verify the two Skorokhod minimality conditions.
- The examples presented in the final section demonstrate that the assumptions are non-empty and cover genuinely non-Lipschitz generators together with deterministic and stochastic continuous barriers.

Although the present paper focuses on scalar doubly reflected BDSDEs with generators of linear growth, the proposed methodology suggests several interesting directions for future research. These include multidimensional systems, generators satisfying monotone or superlinear growth conditions, more general filtration structures, and applications to constrained stochastic control, obstacle SPDEs, stochastic differential games, uncertainty quantification, and related stochastic optimization problems.

Beyond its theoretical interest, the existence theory developed in this paper provides a mathematical foundation for several classes of constrained stochastic models. Doubly reflected BDSDEs arise naturally in stochastic control with state constraints, Dynkin games, obstacle problems, and the probabilistic representation of SPDEs involving lower and upper obstacles. They also play an important role in mathematical finance, where reflected backward equations are used to model American and game options, convertible securities, and other contingent claims subject to exercise constraints. More recently, stochastic equations with reflecting mechanisms have attracted increasing attention in uncertainty quantification and Bayesian inverse problems, where state constraints, interfaces, or obstacle-type conditions naturally arise. Although these applications are beyond the scope of the present work, the existence results established here provide a theoretical framework that may support future developments in these directions.

The remainder of the paper is organized as follows. Section 2 introduces the probabilistic framework, notation, assumptions, and the notion of the solution. Section 3 recalls the one-barrier reflected BDSDE results and comparison principles used throughout the paper. Section 4 develops the Lipschitz approximation of the generator and establishes the key upper-penalization estimate. Sections 5 and 6 are devoted to the construction of the minimal and maximal solutions, respectively. Section 7 presents several examples illustrating the assumptions, and Section 8 concludes the paper with final remarks and perspectives for future research.

2. Notation, framework, and assumptions

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and let $T > 0$ be fixed. We consider two independent Brownian motions

$$\{W_t,\ 0 \le t \le T\} \quad \text{and} \quad \{B_t,\ 0 \le t \le T\},$$

with values in $\mathbb{R}^d$ and $\mathbb{R}$, respectively.

For each $t \in [0, T]$, we define

$$\mathcal{F}_t^W = \sigma(W_s : 0 \le s \le t), \qquad \mathcal{F}_{t,T}^B = \sigma(B_s - B_t : t \le s \le T),$$

completed by the P-null sets. We then set

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B, \qquad \mathcal{G}_t = \mathcal{F}_t^W \vee \mathcal{F}_T^B.$$

The family $(\mathcal{F}_t)_{0\le t\le T}$ is not increasing in general and hence, it is not a filtration. This is standard in the theory of BDSDEs and comes from the presence of the backward noise. In contrast, $(\mathcal{G}_t)_{0\le t\le T}$ is a filtration.

For an integer $m \ge 1$, we use $M^2(0, T; \mathbb{R}^m)$ to denote the space of jointly measurable processes

$$\varphi = (\varphi_t)_{0\le t\le T}$$

such that φ_t is $\mathcal{F}_t$-measurable for every t and

$$\mathbb{E} \int_0^T |\varphi_t|^2 \, dt < \infty.$$

We also use $S^2(0, T; \mathbb{R})$ to denote the space of continuous real-valued processes

$$\varphi = (\varphi_t)_{0\le t\le T}$$

such that φ_t is $\mathcal{F}_t$-measurable for every t and

$$\mathbb{E}\left[\sup_{0\le t\le T} |\varphi_t|^2\right] < \infty.$$

For $n \ge 1$, $\mathcal{D}_n$ denotes the class of progressively measurable processes with values in $[0, n]$.

Throughout the paper, C denotes a positive constant which may change from line to line. Unless otherwise stated, this constant is independent of the approximation and penalization parameters.

Assumption 2.1. *The data (ξ, f, g, L, U) satisfy the following conditions.*

(H1) *The functions*

$$f\colon \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, \qquad g\colon \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}.$$

are measurable. Moreover, for every $(y,z) \in \mathbb{R} \times \mathbb{R}^d$*, we have*

$$f(\cdot, y, z) \in M^2(0,T;\, \mathbb{R}), \qquad g(\cdot, y, z) \in M^2(0,T;\, \mathbb{R}).$$

(H2) *For* $dP \otimes dt$*-almost every* (ω, t)*, the mapping*

$$(y,z) \mapsto f(\omega, t, y, z)$$

is continuous.
(H3) *The constants* $C > 0$ *and* $\alpha \in (0,1)$ *such that, for all* $(y,z), (y', z') \in \mathbb{R} \times \mathbb{R}^d$*, we have*

$$|g(t,y,z) - g(t,y',z')|^2 \leq C|y-y'|^2 + \alpha|z-z'|^2, \qquad P \otimes dt\text{-a.e.},$$

where -a.e. means almost everywhere with respect to the product measure $P \otimes dt$*. Thus* g *is Lipschitz in* (y,z)*, with a contraction coefficient in the variable* z*.*
(H4) *A constant* $C > 0$ *and a non-negative process* $\gamma \in M^2(0,T;\mathbb{R})$ *exist such that*

$$|f(t,y,z)| \leq C(\gamma_t + |y| + |z|), \quad dP \otimes dt\text{-a.e.}.$$

Thus f *has linear growth, but it is not assumed to be Lipschitz.*
(H5) *The terminal condition* ξ *is* $\mathcal{F}_T$*-measurable and satisfies*

$$\mathbb{E}|\xi|^2 < \infty.$$

(H6) *The barriers* $L = (L_t)_{0\leq t\leq T}$ *and* $U = (U_t)_{0\leq t\leq T}$ *are continuous,* $\mathcal{F}_t$*-measurable real-valued processes such that*

$$L_t \leq U_t, \qquad 0 \leq t \leq T, \quad P\text{-a.s.},$$
$$\mathbb{E}\left[\sup_{0\leq t\leq T} ((L_t^+)^2 + (U_t^-)^2)\right] < \infty,$$

and

$$L_T \leq \xi \leq U_T, \qquad P\text{-a.s.}.$$

The next assumption concerns the upper barrier. It will be used only to obtain a uniform estimate for the penalization term which forces the solution to stay below U.

Assumption 2.2. *There is a decreasing sequence of continuous semimartingales* $(U^n)_{n\geq 1}$ *such that*

$$U_t^n \geq U_t^{n+1}, \qquad \lim_{n\to\infty} U_t^n = U_t, \qquad 0 \leq t \leq T, \quad P\text{-a.s.},$$

and

$$U_t^n = U_0^n + \int_0^t u_n(s)\, ds + \int_0^t v_n(s)\, dW_s.$$

Here, the processes

$$u_n \in M^2(0,T;\, \mathbb{R}), \qquad v_n \in M^2(0,T;\, \mathbb{R}),$$

are assumed to be progressively measurable with respect to the filtration $(\mathcal{G}_t)_{0\le t\le T}$*, where*

$$\mathcal{G}_t = \mathcal{F}_t^W \vee \mathcal{F}_T^B,$$

introduced above. Moreover, there is a constant $C > 0$*, independent of* n*, such that*

$$\sup_{n\ge 1}\sup_{0\le t\le T} |u_n(t)| \le C,$$

and

$$\sup_{n\ge 1}\mathbb{E}\left[\int_0^T |v_n(s)|^2\, ds\right] < \infty.$$

Remark 2.3. *Assumption 2.2 is imposed only on the upper obstacle and plays a technical role in the proof. Its purpose is to approximate the upper barrier by sufficiently regular semimartingales so that the penalization procedure can be controlled uniformly. In particular, the uniform bounds on the drift and diffusion coefficients allow us to derive the key estimate for the upper penalization term established in Section 4. This estimate is essential for identifying the limiting upper reflecting process and for verifying the associated Skorokhod condition.*

The assumption is nevertheless satisfied in many situations of practical interest. For example, it holds whenever U *is a continuous deterministic function of finite variation with bounded derivative. It also holds when* U *is a continuous Itô process with bounded drift and square-integrable diffusion coefficient. Consequently, although Assumption 2.2 is introduced primarily for technical reasons, it still covers a broad class of upper barriers arising in stochastic control, obstacle problems, and SPDEs.*

Definition 2.4. *A solution of the doubly reflected BDSDE* (1.2) *is a quadruple*

$$(Y, Z, K^+, K^-) \in S^2(0,T;\mathbb{R}) \times M^2(0,T;\mathbb{R}^d) \times S^2(0,T;\mathbb{R}) \times S^2(0,T;\mathbb{R})$$

such that K^+ *and* K^- *are continuous nondecreasing processes,* $K_0^+ = K_0^- = 0$*, and the following conditions hold.*

(i) *For every* $t \in [0,T]$*, Eq* (1.2) *holds* P*-a.s., that is,* P*-almost surely;*

(ii) *The first component remains between the two barriers*

$$L_t \le Y_t \le U_t, \qquad 0 \le t \le T, \quad P\text{-a.s.};$$

(iii) *The reflecting processes satisfy the Skorokhod minimality conditions*

$$\int_0^T (Y_t - L_t)\, dK_t^+ = 0, \qquad \int_0^T (U_t - Y_t)\, dK_t^- = 0, \qquad P\text{-a.s.}.$$

Thus K^+ can increase only when Y is on the lower barrier, while K^- can increase only when Y is on the upper barrier. These two conditions express the minimal action of the reflecting processes.

Definition 2.5. *A solution*

$$(\underline{Y},\ \underline{Z},\ \underline{K}^+,\ \underline{K}^-)$$

is called a minimal solution if, for every other solution $(Y,\ Z,\ K^+,\ K^-)$*, we have*

$$\underline{Y}_t \le Y_t, \qquad 0 \le t \le T, \quad P\text{-a.s.}.$$

A solution

$$(\overline{Y},\ \overline{Z},\ \overline{K}^{+},\ \overline{K}^{-})$$

is called a maximal solution if, for every other solution $(Y,\ Z,\ K^+,\ K^-)$*, we have*

$$Y_t \leq \overline{Y}_t, \qquad 0 \leq t \leq T, \quad P\text{-a.s.}.$$

Remark 2.6. *Unlike the classical Lipschitz framework, the assumptions imposed on the generator in this paper guarantee only continuity together with a linear growth condition. In general, these assumptions are insufficient to ensure the uniqueness of the solutions, since the contraction arguments commonly used for Lipschitz generators are no longer available. Consequently, several adapted solutions may coexist for the same terminal condition and reflecting barriers. For this reason, the appropriate objective is to identify the smallest and the largest solutions in the natural almost sure ordering. These extremal solutions play the role of canonical representatives and provide the best possible existence result under the present assumptions.*

3. Known results for one-barrier reflected BDSDEs

We now recall two standard results for reflected BDSDEs with one lower barrier. They will be used repeatedly in the construction of the doubly reflected solutions.

Theorem 3.1 (Existence and uniqueness for one-barrier reflected BDSDEs)**.** *Assume that the terminal condition, the coefficient g, the lower obstacle L, and a driver* φ *satisfy the usual square-integrability assumptions. Assume also that* φ *is Lipschitz continuous with respect to* (y, z) *and that g satisfies Assumption 2.1 (H3). Then the one-barrier reflected BDSDE*

$$Y_t = \xi + \int_t^T \varphi(s, Y_s, Z_s)\ ds + \int_t^T g(s, Y_s, Z_s)\ dB_s + K_T - K_t - \int_t^T Z_s\ dW_s$$

with the lower obstacle L admits a unique solution (Y, Z, K)*. This solution satisfies*

$$Y_t \geq L_t, \qquad 0 \leq t \leq T,$$

and

$$\int_0^T (Y_t - L_t)\ dK_t = 0.$$

Theorem 3.2 (Comparison for one-barrier reflected BDSDEs)**.** *Let* (ξ, φ, g, L) *and* (ξ', φ', g, L') *be two sets of data satisfying the assumptions of Theorem 3.1. Assume that*

$$\xi \leq \xi' \quad P\text{-a.s.},$$

$$\varphi(t, y, z) \leq \varphi'(t, y, z), \quad dP \otimes dt\text{-a.e. for all } (y, z),$$

and

$$L_t \leq L'_t, \qquad 0 \leq t \leq T, \quad P\text{-a.s.}.$$

If (Y, Z, K) *and* (Y', Z', K') *are the corresponding solutions, then*

$$Y_t \leq Y'_t, \qquad 0 \leq t \leq T, \quad P\text{-a.s.}.$$

The comparison theorem is one of the main tools of the paper. It allows us to compare the auxiliary one-barrier equations obtained by replacing the upper reflection with a penalization term. This is the mechanism that produces the monotone approximating sequences used to construct the minimal and maximal solutions.

4. Approximation and penalization

4.1. Lipschitz approximation of the continuous driver

We begin with a standard approximation result for continuous functions with linear growth. This result allows us to replace the continuous driver by two monotone sequences of Lipschitz drivers, one increasing and one decreasing.

Lemma 4.1. *Let $h\colon [0,T]\times\Omega\times\mathbb{R}^m \to \mathbb{R}$ be measurable. Assume that, for $dP\otimes dt$-almost every (ω,t), the mapping*

$$x \mapsto h(\omega,\ t,\ x)$$

is continuous and that a constant $C > 0$ exists such that

$$|h(t,x)| \leq C(1+|x|).$$

For $n \geq C$, define

$$\overline{h}_n(t,x) = \sup_{q\in\mathbb{Q}^m} \{h(t,q) - n|x-q|\},$$

and

$$\underline{h}_n(t,x) = \inf_{q\in\mathbb{Q}^m} \{h(t,q) + n|x-q|\}.$$

Then the following properties hold:
(i) *$\overline{h}_n$ and $\underline{h}_n$ are n-Lipschitz continuous with respect to x;*
(ii) *For every $x \in \mathbb{R}^m$,*

$$\underline{h}_n(t,x) \leq h(t,x) \leq \overline{h}_n(t,x);$$

(iii) *$\overline{h}_n(t,x)$ decreases to $h(t,x)$ as $n \to \infty$;*
(iv) *$\underline{h}_n(t,x)$ increases to $h(t,x)$ as $n \to \infty$;*
(v) *If $x_n \to x$, then*

$$\overline{h}_n(t,x_n) \to h(t,x), \quad \underline{h}_n(t,x_n) \to h(t,x).$$

Proof. We prove the assertions for $\overline{h}_n$. The proof for $\underline{h}_n$ is analogous.

Let $x, x' \in \mathbb{R}^m$. For every $q \in \mathbb{Q}^m$, we have

$$h(t,q) - n|x-q| \leq h(t,q) - n|x'-q| + n|x-x'|.$$

Taking the supremum over q yields

$$\overline{h}_n(t,x) \leq \overline{h}_n(t,x') + n|x-x'|.$$

Exchanging the roles of x and x' gives

$$|\overline{h}_n(t,x) - \overline{h}_n(t,x')| \leq n|x-x'|.$$

Thus $\overline{h}_n$ is n-Lipschitz continuous.

Since $\mathbb{Q}^m$ is dense in $\mathbb{R}^m$ and $h(t,\cdot)$ is continuous, rational points can be chosen arbitrarily close to x. Hence

$$\overline{h}_n(t,x) \geq h(t,x).$$

Moreover, if $n_2 \geq n_1$, then

$$h(t,q) - n_2|x-q| \leq h(t,q) - n_1|x-q|,$$

and consequently

$$\overline{h}_{n_2}(t,x) \leq \overline{h}_{n_1}(t,x).$$

Therefore $(\overline{h}_n)$ is decreasing.

It remains to identify the limit. Fix $x \in \mathbb{R}^m$ and let $\varepsilon > 0$. By continuity of $h(t,\cdot)$, $\delta > 0$ exists such that

$$|q-x| < \delta \implies h(t,q) \leq h(t,x) + \varepsilon.$$

For such q, we have

$$h(t,q) - n|x-q| \leq h(t,x) + \varepsilon.$$

If $|q-x| \geq \delta$, then the linear growth of h gives

$$h(t,q) - n|x-q| \leq C(1+|q|) - n|x-q|.$$

Since

$$|q| \leq |x| + |q-x|,$$

we get

$$h(t,q) - n|x-q| \leq C(1+|x|) + (C-n)|x-q|.$$

For any n that is large enough, the right-hand side is bounded above by $h(t,x)+\varepsilon$ on the set $\{|q-x| \geq \delta\}$. Thus

$$\limsup_{n\to\infty} \overline{h}_n(t,x) \leq h(t,x) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $\overline{h}_n(t,x) \geq h(t,x)$, we conclude that

$$\overline{h}_n(t,x) \downarrow h(t,x).$$

The convergence along a sequence $x_n \to x$ follows by the same localization argument. This is the classical Lepeltier-San Martin approximation procedure. □

We now apply Lemma 4.1 to the driver f by taking

$$h(t,x) = f(t,y,z), \qquad x = (y,z) \in \mathbb{R}^{1+d}.$$

For any n that is large enough, define

$$\underline{f}_n(t,y,z) = \inf_{(p,q)\in\mathbb{Q}\times\mathbb{Q}^d} \{f(t,p,q) + n(|y-p| + |z-q|)\}, \tag{4.1}$$

and

$$\overline{f}_n(t,y,z) = \sup_{(p,q)\in\mathbb{Q}\times\mathbb{Q}^d} \{f(t,p,q) - n(|y-p| + |z-q|)\}. \tag{4.2}$$

Then

$$\underline{f}_n(t,y,z) \uparrow f(t,y,z), \qquad \overline{f}_n(t,y,z) \downarrow f(t,y,z),$$

for $(P \otimes dt)$-a.e. (ω, t) and every $(y,z) \in \mathbb{R} \times \mathbb{R}^d$.

Moreover, the approximating drivers satisfy the same type of linear growth estimate. Namely, a constant $C > 0$, independent of n, exists such that

$$|\underline{f}_n(t,y,z)| + |\overline{f}_n(t,y,z)| \le C(\gamma_t + 1 + |y| + |z|).$$

The increasing sequence $\underline{f}_n$ will be used to construct the minimal solution, while the decreasing sequence $\overline{f}_n$ will be used to construct the maximal solution.

Remark 4.2. *A distinctive feature of the present approach is the introduction of two independent approximation parameters. The first parameter is associated with the monotone Lipschitz approximation of the continuous generator, while the second governs the penalization of the upper obstacle. These two approximation procedures serve different purposes and therefore must be handled separately throughout the analysis.*

Keeping the two parameters independent is essential. Indeed, if the regularization parameter and the penalization parameter were varied simultaneously, the resulting family of auxiliary equations would not, in general, preserve the monotonicity required by the comparison theorem. Consequently, the ordering of the approximating solutions could be lost, making the limiting procedure much more difficult.

By fixing one parameter while allowing the other to vary, the monotonicity of the auxiliary problems is preserved at every stage of the construction. This permits the successive application of the comparison principle, the derivation of uniform estimates independent of the penalization parameter, and the rigorous passage to the two limits. The separation of these approximation procedures is therefore one of the key methodological ingredients of the paper and plays a fundamental role in the construction of both the minimal and the maximal solutions.

4.2. A penalization estimate for the upper obstacle

The construction of the doubly reflected BDSDE is carried out through a sequence of auxiliary one-barrier reflected equations. The lower obstacle L is treated as a genuine reflecting boundary, whereas the upper constraint $Y \le U$ is enforced by introducing a penalization term of the form

$$-m(Y_t - U_t)^+.$$

As the penalization parameter m increases, any violation of the upper obstacle becomes increasingly expensive, forcing the solution to remain close to the admissible region. To recover the upper reflecting process in the limit, it is essential to obtain an estimate that is uniform with respect to m. Such an estimate guarantees that the excess of the penalized solution above the upper barrier decreases at the rate $O(m^{-1})$. Consequently, the family of penalized equations remains uniformly controlled, allowing the limiting upper reflection to be identified and the corresponding Skorokhod condition to be verified.

The next lemma provides precisely this estimate. It constitutes the key analytical ingredient of the existence proof developed in the subsequent sections. The proof combines the penalization method with the semimartingale approximation introduced in Assumption 2.2. The latter plays an essential role because it supplies the regularity needed to control the penalization term uniformly and to derive estimates that are independent of the penalization parameter.

Lemma 4.3. *Assume that Assumptions 2.1 and 2.2 hold. For every $m \geq 1$, let*

$$(Y^{0,m},\, Z^{0,m},\, K^{0,m})$$

denote the unique solution of the one-barrier reflected BDSDE associated with the lower obstacle L and generator

$$(t, y, z) \mapsto -C(\gamma_t + |y| + |z|) - m(y - U_t)^+.$$

Then there is a positive constant C_0, independent of the penalization parameter m, such that

$$\sup_{0\leq t\leq T} m(Y_t^{0,m} - U_t)^+ \leq C_0, \qquad P\text{-a.s.}.$$

In particular, we have

$$(Y_t^{0,m} - U_t)^+ \leq \frac{C_0}{m}, \qquad 0 \leq t \leq T,$$

which shows that the penalized solution can exceed the upper barrier only by a quantity of order m^{-1}.

Proof. The proof is based on a double penalization argument. Besides the penalization of the upper obstacle, we temporarily replace the lower reflection by a second penalization term. This allows us to work with standard BDSDEs without reflection and to recover the reflected equation by passing to the limit. The argument is divided into several steps.

Step 1. Construction of the auxiliary penalized equation.

Fix $m \geq 1$. For every $k \geq 1$, consider the penalized BDSDE

$$\begin{aligned} Y_t^{m,k} = \xi &- \int_t^T C(\gamma_s + |Y_s^{m,k}| + |Z_s^{m,k}|)\, ds - m \int_t^T (Y_s^{m,k} - U_s)^+\, ds \\ &+ k \int_t^T (Y_s^{m,k} - L_s)^-\, ds + \int_t^T g(s, Y_s^{m,k}, Z_s^{m,k})\, dB_s - \int_t^T Z_s^{m,k}\, dW_s. \end{aligned}$$

The coefficient of this equation is Lipschitz continuous with respect to (y, z) for every fixed pair (m, k). Therefore, by the classical existence theorem for BDSDEs, the equation above admits a unique adapted solution

$$(Y^{m,k}, Z^{m,k}) \in S^2(0, T; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d).$$

The term

$$k(Y - L)^-$$

forces the solution to remain above the lower obstacle as $k \to \infty$, whereas the term

$$-m(Y - U)^+$$

penalizes any violation of the upper constraint. Standard penalization arguments for reflected BDSDEs therefore imply that, for every fixed m, we have

$$Y_t^{m,k} \to Y_t^{0,m}, \quad 0 \leq t \leq T,$$

where

$$(Y^{0,m},\ Z^{0,m},\ K^{0,m})$$

is the solution of the reflected BDSDE with the lower obstacle L and the upper penalization parameter m. Consequently, it is sufficient to derive estimates that are uniform with respect to k before passing to the limit.

Step 2. Comparison with the semimartingale approximation of the upper barrier.

To estimate the violation of the upper obstacle, we compare the penalized solution with the semimartingale approximation introduced in Assumption 2.2. For every $k \geq 1$, let

$$U^k = (U_t^k)_{0\leq t\leq T}$$

be the corresponding approximating semimartingale and define

$$\Lambda_t^{m,k} = Y_t^{m,k} - U_t^k.$$

Since

$$U_t^k = U_0^k + \int_0^t u_k(s)\,ds + \int_0^t v_k(s)\,dW_s,$$

subtracting the decomposition above from the penalized BDSDE satisfied by $Y^{m,k}$ yields

$$\begin{aligned}
\Lambda_t^{m,k} = \xi - U_T^k &+ \int_t^T u_k(s)\,ds - \int_t^T C(\gamma_s + |Y_s^{m,k}| + |Z_s^{m,k}|)\,ds \\
&- m\int_t^T (\Lambda_s^{m,k} + U_s^k - U_s)^+\,ds \\
&+ k\int_t^T (\Lambda_s^{m,k} + U_s^k - L_s)^-\,ds \\
&+ \int_t^T g(s, Y_s^{m,k}, Z_s^{m,k})\,dB_s - \int_t^T (Z_s^{m,k} - v_k(s))\,dW_s.
\end{aligned}$$

The purpose of introducing the process $\Lambda^{m,k}$ is to measure the distance between the penalized solution and the approximating upper barrier. Uniform estimates for the positive part of $\Lambda^{m,k}$ immediately translate into uniform estimates for the amount by which the penalized solution exceeds the true upper obstacle.

Observe first that the terminal condition is non-positive. Indeed, by Assumption 2.1, we have

$$\xi \leq U_T,$$

while Assumption 2.2 implies

$$U_T \leq U_T^k.$$

Hence

$$\xi - U_T^k \leq 0.$$

Similarly, we have

$$L_t \leq U_t \leq U_t^k, \qquad 0 \leq t \leq T.$$

Consequently, the lower penalization term cannot increase the positive part of $\Lambda^{m,k}$ and therefore does not affect the estimate derived below.

Step 3. Estimate of the positive part.

Applying the standard linearization technique for the positive and negative parts, there are progressively measurable processes

$$\mu \in \mathcal{D}_k, \qquad \nu \in \mathcal{D}_m,$$

such that the positive part $(\Lambda^{m,k})^+$ satisfies the corresponding linear BDSDE. Solving this linear equation by the variation-of constants formula yields

$$(\Lambda_t^{m,k})^+ \le \mathbb{E}\left[\int_t^T e^{-\int_t^s (\mu_r+\nu_r)\,dr} |u_k(s)|\,ds \,\middle|\, \mathcal{F}_t\right].$$

The above estimate deserves a brief explanation. After multiplying the linearized equation by the exponential factor

$$\exp\left(-\int_t^s (\mu_r + \nu_r)\,dr\right),$$

the stochastic integrals with respect to W and B become martingale terms.

The forward Itô integral with respect to W is a square-integrable martingale with respect to the filtration $(\mathcal{G}_t)$ and therefore has zero conditional expectation.

The backward stochastic integral with respect to B also has zero conditional expectation in the BDSDE framework because the backward Brownian increments are independent of $\mathcal{F}_t$ and the corresponding integral is centered (see Pardoux and Peng [1]). Consequently, after taking the conditional expectation with respect to $\mathcal{F}_t$, both stochastic integral terms disappear, leaving only the deterministic drift contribution.

Finally, Assumption 2.2 provides a uniform bound for the drift coefficient of the semimartingale approximation. More precisely,

$$|u_k(s)| \le C, \qquad 0 \le s \le T,$$

where the constant C is independent of both k and m. Therefore,

$$(\Lambda_t^{m,k})^+ \le C \operatorname*{ess\,sup}_{\mu\in\mathcal{D}_k} \operatorname*{ess\,inf}_{\nu\in\mathcal{D}_m} \mathbb{E}\left[\int_t^T e^{-\int_t^s (\mu_r+\nu_r)\,dr}\,ds \,\middle|\, \mathcal{F}_t\right].$$

Step 4. Uniform estimate with respect to the penalization parameter.

To estimate the right-hand side, we choose the admissible process

$$\nu_r \equiv m, \qquad t \le r \le T.$$

Since $\mu_r \ge 0$, we have

$$\mu_r + \nu_r \ge m,$$

which immediately gives

$$\exp\left(-\int_t^s (\mu_r + \nu_r)\,dr\right) \le e^{-m(s-t)}.$$

Consequently, we have

$$(\Lambda_t^{m,k})^+ \leq C\mathbb{E}\left[\int_t^T e^{-m(s-t)}\,ds \,\middle|\, \mathcal{F}_t\right]$$
$$= C\int_t^T e^{-m(s-t)}\,ds.$$

The integral above can be evaluated explicitly

$$\int_t^T e^{-m(s-t)}\,ds = \frac{1-e^{-m(T-t)}}{m} \leq \frac{1}{m}.$$

Hence

$$(\Lambda_t^{m,k})^+ \leq \frac{C}{m}, \qquad 0 \leq t \leq T.$$

Recalling the definition of $\Lambda^{m,k}$, we obtain

$$m(Y_t^{m,k} - U_t^k)^+ \leq C, \qquad 0 \leq t \leq T,$$

where the constant C is independent of both m and k. This uniform estimate is the crucial point of the proof, since it prevents the penalized solution from moving too far above the approximating upper obstacle regardless of the value of the penalization parameter.

Step 5. Passage to the limit.

We now let $k \to \infty$. By the classical penalization result for reflected BDSDEs, we have

$$Y^{m,k} \to Y^{0,m}, \qquad \text{in } S^2(0,T;\mathbb{R}),$$

while Assumption 2.2 ensures that

$$U_t^k \downarrow U_t, \qquad 0 \leq t \leq T, \quad P\text{-a.s.}.$$

Passing to the limit in the previous inequality therefore yields

$$m(Y_t^{0,m} - U_t)^+ \leq C, \qquad 0 \leq t \leq T.$$

Taking the supremum over $t \in [0,T]$, we conclude that

$$\sup_{0\leq t\leq T} m(Y_t^{0,m} - U_t)^+ \leq C_0,$$

where the constant C_0 is independent of the penalization parameter m. This establishes the desired estimate and completes the proof. □

Remark 4.4. *Lemma 4.3 is the fundamental estimate underlying the construction of the upper reflecting process. It shows that the excess of the penalized solution above the upper obstacle is uniformly controlled by a quantity of order m^{-1}. Consequently, as the penalization parameter tends to infinity, the violation of the upper constraint vanishes uniformly.*

The semimartingale approximation introduced in Assumption 2.2 is essential for obtaining this estimate. Its uniformly bounded drift allows the exponential argument used above to remain independent of the penalization parameter, while the square-integrability of the diffusion coefficient guarantees that the stochastic terms are well behaved. These properties make it possible to pass from the auxiliary one-barrier reflected equations to the doubly reflected BDSDE and to recover the upper reflecting process together with its Skorokhod minimality condition.

5. Existence of a minimal solution

The purpose of this section is to construct the minimal adapted solution of the doubly reflected BDSDE introduced in Section 2. Since the generator is merely continuous and satisfies only a linear growth condition, the classical fixed-point approach cannot be applied directly. Instead, the proof relies on a successive approximation procedure combining monotone regularization of the generator with penalization of the upper obstacle.

Before presenting the technical details, we briefly summarize the main strategy of the proof. The construction proceeds through four successive steps.

(1) The continuous generator f is approximated from below by an increasing sequence of Lipschitz-continuous generators $(\underline{f}_n)_{n\geq 1}$.

(2) For each approximating generator, an auxiliary one-barrier reflected BDSDE is solved. The lower obstacle is treated through reflection, whereas the upper obstacle is incorporated by means of a penalization procedure.

(3) Uniform a priori estimates are established for the penalized equations. In particular, Lemma 4.3 provides a uniform estimate for the upper penalization term that is independent of the penalization parameter.

(4) Finally, the approximation parameter and the penalization parameter are sent to their limits successively. The convergence of the solutions, the martingale integrands, and the reflecting processes then yields the minimal solution of the doubly reflected BDSDE.

A distinctive feature of the proof is the use of two independent approximation parameters. The first parameter regularizes the continuous generator through the monotone Lipschitz approximation

$$\underline{f}_n \uparrow f.$$

The second parameter controls the penalization enforcing the upper constraint $Y \leq U$. Treating these parameters separately is essential. Indeed, if the regularization index and the penalization parameter were allowed to vary simultaneously, the monotonicity of the auxiliary equations would generally be lost, preventing a direct application of the comparison theorem. By fixing one parameter while varying the other, the required monotonicity is preserved throughout the construction. This successive limiting procedure constitutes one of the key ingredients of the proof.

For every pair $(n, m) \in \mathbb{N}^2$, we consider the following auxiliary one-barrier reflected BDSDE:

$$\begin{aligned} Y_t^{n,m} &= \xi + \int_t^T \underline{f}_n(s, Y_s^{n,m}, Z_s^{n,m})\, ds + K_T^{n,m} - K_t^{n,m} \\ &\quad - m \int_t^T (Y_s^{n,m} - U_s)^+\, ds + \int_t^T g(s, Y_s^{n,m}, Z_s^{n,m})\, dB_s - \int_t^T Z_s^{n,m}\, dW_s, \qquad 0 \leq t \leq T. \end{aligned} \tag{5.1}$$

The increasing process $K^{n,m}$ is the reflecting process associated with the lower obstacle L. Consequently, the solution satisfies

$$Y_t^{n,m} \geq L_t, \qquad 0 \leq t \leq T,$$

together with the Skorokhod minimality condition

$$\int_0^T (Y_t^{n,m} - L_t)\, dK_t^{n,m} = 0.$$

Since the approximating generator $\underline{f}_n$ is Lipschitz continuous with respect to (y, z), Theorem 3.1 guarantees that, for every fixed pair (n, m), Eq (5.1) admits a unique adapted solution

$$(Y^{n,m},\ Z^{n,m},\ K^{n,m}).$$

The remainder of this section is devoted to deriving uniform estimates for this family of auxiliary equations and to proving their convergence toward the minimal solution of the original doubly reflected BDSDE.

5.1. Monotonicity of the two-parameter approximation

We first record the monotonicity properties of the approximating sequence.

Lemma 5.1. *For every fixed $m \geq 1$, the sequence $(Y^{n,m})_{n\geq 1}$ is increasing in n. That is,*

$$Y_t^{n,m} \leq Y_t^{n+1,m}, \qquad 0 \leq t \leq T, \quad P\text{-a.s.}.$$

Moreover, for every fixed $n \geq 1$, the sequence $(Y^{n,m})_{m\geq 1}$ is decreasing in m

$$Y_t^{n,m+1} \leq Y_t^{n,m}, \qquad 0 \leq t \leq T, \quad P\text{-a.s.}.$$

Proof. Fix $m \geq 1$. By construction of the lower Lipschitz approximation, we have

$$\underline{f}_n(t, y, z) \leq \underline{f}_{n+1}(t, y, z), \qquad dP \otimes dt\text{-a.e.}.$$

The terminal condition, the coefficient g, the lower obstacle L, and the upper penalization coefficient m are the same in the equations for $Y^{n,m}$ and $Y^{n+1,m}$. Therefore, by the comparison theorem for one-barrier reflected BDSDEs, we have

$$Y_t^{n,m} \leq Y_t^{n+1,m}, \qquad 0 \leq t \leq T, \quad P\text{-a.s.}.$$

Now fix $n \geq 1$. If $m_2 \geq m_1$, then

$$-m_2(y - U_t)^+ \leq -m_1(y - U_t)^+, \qquad y \in \mathbb{R}.$$

Thus the generator corresponding to m_2 is smaller than the generator corresponding to m_1. Applying the comparison theorem again gives

$$Y_t^{n,m_2} \leq Y_t^{n,m_1}, \qquad 0 \leq t \leq T, \quad P\text{-a.s.}.$$

Taking $m_2 = m + 1$ and $m_1 = m$ proves the second assertion. □

Remark 5.2. *The previous lemma explains why the two-parameter construction is needed. If one considers directly the generator*

$$\underline{f}_n(t, y, z) - n(y - U_t)^+,$$

then two opposite effects occur at the same time: $\underline{f}_n$ increases with n, while the penalization term decreases with n. Therefore the comparison theorem cannot be applied directly. Separating the approximation parameter n from the penalization parameter m avoids this difficulty.

5.2. Passage in the penalization parameter

We now fix $n \geq 1$ and pass to the limit as $m \to \infty$. By Lemma 5.1, the sequence $(Y^{n,m})_{m\geq 1}$ is decreasing. Since it is bounded from below by L, an adapted process Y^n exists such that

$$Y_t^{n,m} \downarrow Y_t^n, \qquad 0 \leq t \leq T, \quad P\text{-a.s.}.$$

The estimates below imply that this convergence also holds in

$$M^2(0,T;\mathbb{R}).$$

Define the upper penalization process

$$K_t^{-,n,m} = m \int_0^t (Y_s^{n,m} - U_s)^+ \, ds, \qquad 0 \leq t \leq T.$$

Then $K^{-,n,m}$ is continuous, nondecreasing, and satisfies

$$K_0^{-,n,m} = 0.$$

By Lemma 4.3, there is a constant $C > 0$, independent of m, such that

$$\sup_{m\geq 1} \mathbb{E}\left[|K_T^{-,n,m}|^2\right] \leq C.$$

Hence, by Helly's selection theorem, there is a subsequence, still denoted (m), and a nondecreasing process $K^{-,n}$ such that

$$K_t^{-,n,m} \to K_t^{-,n}$$

for every continuity point t of $K^{-,n}$, P-a.s. Moreover, along this subsequence the convergence holds uniformly on $[0,T]$ in probability. The limiting process satisfies

$$K_0^{-,n} = 0, \qquad \mathbb{E}[|K_T^{-,n}|^2] \leq C.$$

The same compactness argument applies to the lower reflecting processes. Since the estimates below give a uniform L^2-bound for $K_T^{n,m}$, a subsequence and a nondecreasing process $K^{+,n}$ exists such that

$$K^{n,m} \to K^{+,n}$$

uniformly in probability on $[0,T]$. Again, we have

$$K_0^{+,n} = 0.$$

Passing to the limit in (5.1) along the chosen subsequence gives

$$\begin{aligned} Y_t^n = \xi + \int_t^T \underline{f}_n(s, Y_s^n, Z_s^n)\, ds + \int_t^T g(s, Y_s^n, Z_s^n)\, dB_s \\ + (K_T^{+,n} - K_t^{+,n}) - (K_T^{-,n} - K_t^{-,n}) - \int_t^T Z_s^n \, dW_s. \end{aligned} \tag{5.2}$$

Here Z^n denotes the limit of $Z^{n,m}$ in $M^2(0,T;\mathbb{R}^d)$. The convergence of the stochastic integrals follows from this convergence and from the Lipschitz condition on g.

We now identify the limiting upper reflection. Since

$$dK_t^{-,n,m} = m(Y_t^{n,m} - U_t)^+ \, dt,$$

we have

$$\begin{aligned}\int_0^T (U_t - Y_t^{n,m}) \, dK_t^{-,n,m} &= \int_0^T (U_t - Y_t^{n,m}) m(Y_t^{n,m} - U_t)^+ \, dt \\ &= -\int_0^T m((Y_t^{n,m} - U_t)^+)^2 \, dt \le 0.\end{aligned}$$

Letting $m \to \infty$ and using the convergence of $Y^{n,m}$ and $K^{-,n,m}$, we obtain

$$\int_0^T (U_t - Y_t^n) \, dK_t^{-,n} \le 0.$$

On the other hand, Lemma 4.3 gives

$$0 \le (Y_t^{n,m} - U_t)^+ \le \frac{C}{m}, \qquad 0 \le t \le T.$$

Therefore, by letting $m \to \infty$, we have

$$Y_t^n \le U_t, \qquad 0 \le t \le T, \quad P\text{-a.s.}.$$

Hence $U_t - Y_t^n \ge 0$, and since $K^{-,n}$ is nondecreasing, we have

$$\int_0^T (U_t - Y_t^n) \, dK_t^{-,n} \ge 0.$$

Combining the two inequalities yields

$$\int_0^T (U_t - Y_t^n) \, dK_t^{-,n} = 0.$$

Thus $K^{-,n}$ is the upper reflecting process associated with the upper obstacle U.

Similarly, by passing to the limit in

$$\int_0^T (Y_t^{n,m} - L_t) \, dK_t^{n,m} = 0,$$

we obtain

$$\int_0^T (Y_t^n - L_t) \, dK_t^{+,n} = 0.$$

Consequently, for each fixed n, the quadruple

$$(Y^n, Z^n, K^{+,n}, K^{-,n})$$

solves the doubly reflected BDSDE with the Lipschitz driver $\underline{f}_n$.

5.3. Uniform estimates

We now establish estimates which are uniform in n. These estimates will be used to pass from the Lipschitz drivers $\underline{f}_n$ to the original continuous driver f.

Lemma 5.3. *There is a constant $C > 0$, independent of n, such that*

$$\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^n|^2 + |K_T^{+,n}|^2\right] \le C.$$

Proof. The linear growth of $\underline{f}_n$ gives

$$|\underline{f}_n(t,y,z)| \le C(\gamma_t + 1 + |y| + |z|).$$

Let $Y^{0,n}$ be the solution introduced in Lemma 4.3, corresponding to the lower driver

$$-C(\gamma_t + |y| + |z|) - n(y - U_t)^+.$$

Since

$$-C(\gamma_t + |y| + |z|) \le \underline{f}_n(t,y,z),$$

the comparison theorem yields

$$Y_t^{0,n} \le Y_t^n, \qquad 0 \le t \le T.$$

On the other hand, $\underline{f}_n$ is bounded above by a Lipschitz driver with linear growth, for instance, by a function of the form

$$C(\gamma_t + 1 + |y| + |z|).$$

Let $\widehat{Y}$ be the solution of the corresponding one-barrier reflected equation with the same lower obstacle L. Another application of the comparison theorem gives

$$Y_t^n \le \widehat{Y}_t, \qquad 0 \le t \le T.$$

Standard one-barrier estimates applied to $Y^{0,n}$ and $\widehat{Y}$ imply

$$\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^{0,n}|^2\right] + \mathbb{E}\left[\sup_{0\le t\le T}|\widehat{Y}_t|^2\right] \le C.$$

Therefore, we have

$$\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^n|^2\right] \le C.$$

Writing (5.2) at $t = 0$, we get

$$K_T^{+,n} = Y_0^n - \xi - \int_0^T \underline{f}_n(s,Y_s^n,Z_s^n)\,ds + K_T^{-,n} - \int_0^T g(s,Y_s^n,Z_s^n)\,dB_s + \int_0^T Z_s^n\,dW_s.$$

Using the Burkholder-Davis-Gundy inequality, the linear growth of $\underline{f}_n$, Assumption 2.1 (H3), and the bound on $K_T^{-,n}$, we obtain

$$\mathbb{E}|K_T^{+,n}|^2 \le C\left(1 + \mathbb{E}\sup_{0\le t\le T}|Y_t^n|^2 + \mathbb{E}\int_0^T |Z_s^n|^2\,ds\right).$$

The estimate for Z^n is proved in Lemma 5.4. Combining the two estimates gives the result. □

Lemma 5.4. *There is a constant $C > 0$, independent of n, such that*

$$\mathbb{E}\int_0^T |Z_s^n|^2\,ds \le C.$$

Proof. Applying Itô's formula to $|Y_t^n|^2$ on $[t,T]$, we obtain

$$\begin{aligned}|Y_t^n|^2 = |\xi|^2 &+ 2\int_t^T Y_s^n \underline{f}_n(s,Y_s^n,Z_s^n)\,ds + 2\int_t^T Y_s^n\; dK_s^{+,n}\\ &- 2\int_t^T Y_s^n\; dK_s^{-,n} + \int_t^T \left|g(s,Y_s^n,Z_s^n)\right|^2 ds - \int_t^T |Z_s^n|^2\,ds\\ &+ 2\int_t^T Y_s^n\, g(s,Y_s^n,Z_s^n)\,dB_s - 2\int_t^T Y_s^n\,Z_s^n\,dW_s.\end{aligned}$$

Taking expectations removes the stochastic integral terms.

By the lower Skorokhod condition, we have

$$\int_t^T Y_s^n\,dK_s^{+,n} = \int_t^T L_s\,dK_s^{+,n}.$$

Thus, for every $\varepsilon > 0$

$$2\mathbb{E}\int_t^T Y_s^n\,dK_s^{+,n} \le 2\mathbb{E}[\sup_{0\le s\le T} L_s^+ K_T^{+,n}] \le \varepsilon\mathbb{E}|K_T^{+,n}|^2 + C_\varepsilon\mathbb{E}[\sup_{0\le s\le T}(L_s^+)^2].$$

The drift term is controlled by the linear growth of $\underline{f}_n$

$$2|Y_s^n \underline{f}_n(s,Y_s^n,Z_s^n)| \le C|Y_s^n|(\gamma_s + 1 + |Y_s^n| + |Z_s^n|).$$

Using Young's inequality, we have

$$C|Y_s^n||Z_s^n| \le \varepsilon|Z_s^n|^2 + C_\varepsilon|Y_s^n|^2.$$

Moreover, Assumption 2.1 (H3) implies

$$|g(s,Y_s^n,Z_s^n)|^2 \le C(1+|Y_s^n|^2) + \alpha|Z_s^n|^2, \qquad \alpha < 1.$$

For the upper reflection term, using the upper Skorokhod condition, we have

$$\int_t^T Y_s^n\,dK_s^{-,n} = \int_t^T U_s\,dK_s^{-,n}.$$

Therefore

$$-2\int_t^T Y_s^n\,dK_s^{-,n} = -2\int_t^T U_s\,dK_s^{-,n} \le 2\sup_{0\le s\le T} U_s^-\,K_T^{-,n}.$$

By Young's inequality and the integrability of U^-, we have

$$2\mathbb{E}[\sup_{0\le s\le T} U_s^-\,K_T^{-,n}] \le C(1 + \mathbb{E}|K_T^{-,n}|^2 + \mathbb{E}\sup_{0\le s\le T}(U_s^-)^2).$$

Collecting the previous estimates and choosing $\varepsilon > 0$ to be small enough, we get

$$(1-\alpha-\varepsilon)\mathbb{E}\int_t^T |Z_s^n|^2\,ds \le C(1+\mathbb{E}\sup_{0\le s\le T}|Y_s^n|^2 + \mathbb{E}|K_T^{+,n}|^2 + \mathbb{E}|K_T^{-,n}|^2).$$

Using Lemma 5.3 and the uniform estimate on $K_T^{-,n}$, we conclude that

$$\mathbb{E}\int_0^T |Z_s^n|^2\,ds \le C.$$

□

Lemma 5.5. *There is a constant $C > 0$, independent of n, such that*

$$\mathbb{E}\left[|K_T^{-,n}|^2\right] \le C.$$

Moreover,

$$Y_t^n \le U_t, \qquad 0\le t\le T, \quad P\text{-a.s.}.$$

Proof. From Lemma 4.3, applied before passing to the limit in m, we have

$$0 \le m\int_0^T (Y_s^{n,m}-U_s)^+\,ds \le T\sup_{0\le s\le T} m(Y_s^{n,m}-U_s)^+ \le CT.$$

Passing to the limit as $m\to\infty$ gives

$$\mathbb{E}|K_T^{-,n}|^2 \le C.$$

Moreover, we have

$$0 \le (Y_t^{n,m}-U_t)^+ \le \frac{C}{m}, \qquad 0\le t\le T.$$

Letting $m\to\infty$ yields

$$Y_t^n \le U_t, \qquad 0\le t\le T, \quad P\text{-a.s.}.$$

□

5.4. Cauchy property of the sequence Z^n

Lemma 5.6. *The sequence $(Z^n)_{n\ge1}$ is Cauchy in $M^2(0,T;\mathbb{R}^d)$.*

Proof. Let $n, p \ge 1$. Applying Itô's formula to $|Y_t^n - Y_t^p|^2$ gives

$$\begin{aligned}
&\mathbb{E}|Y_t^n - Y_t^p|^2 + \mathbb{E}\int_t^T |Z_s^n - Z_s^p|^2\,ds \\
&= 2\mathbb{E}\int_t^T (Y_s^n - Y_s^p)[\underline{f}_n(s,Y_s^n,Z_s^n) - \underline{f}_p(s,Y_s^p,Z_s^p)]\,ds \\
&\quad + 2\mathbb{E}\int_t^T (Y_s^n - Y_s^p)\,dK_s^{+,n} + 2\mathbb{E}\int_t^T (Y_s^p - Y_s^n)\,dK_s^{+,p} \\
&\quad - 2\mathbb{E}\int_t^T (Y_s^n - Y_s^p)\,dK_s^{-,n} - 2\mathbb{E}\int_t^T (Y_s^p - Y_s^n)\,dK_s^{-,p} \\
&\quad + \mathbb{E}\int_t^T \left|g(s,Y_s^n,Z_s^n) - g(s,Y_s^p,Z_s^p)\right|^2 ds.
\end{aligned}$$

The lower reflection terms are non-positive. Indeed, we have

$$\int_t^T (Y_s^n - Y_s^p)\, dK_s^{+,n} = \int_t^T (L_s - Y_s^p)\, dK_s^{+,n} \leq 0,$$

because $dK^{+,n}$ is carried by $\{Y^n = L\}$ and $Y^p \geq L$. Similarly

$$\int_t^T (Y_s^p - Y_s^n)\, dK_s^{+,p} \leq 0.$$

The upper reflection terms are also non-positive. Since $dK^{-,n}$ is carried by $\{Y^n = U\}$ and $Y^p \leq U$, we have

$$\int_t^T (Y_s^n - Y_s^p)\, dK_s^{-,n} = \int_t^T (U_s - Y_s^p)\, dK_s^{-,n} \geq 0.$$

Thus

$$-2\int_t^T (Y_s^n - Y_s^p)\, dK_s^{-,n} \leq 0.$$

The same argument applies to the term involving $K^{-,p}$.

For the g-term, Assumption 2.1 (H3) gives

$$|g(s, Y_s^n, Z_s^n) - g(s, Y_s^p, Z_s^p)|^2 \leq C|Y_s^n - Y_s^p|^2 + \alpha|Z_s^n - Z_s^p|^2.$$

Therefore, we have

$$\begin{aligned}(1-\alpha)\mathbb{E}\int_0^T |Z_s^n - Z_s^p|^2\, ds \leq 2\mathbb{E}\int_0^T |Y_s^n - Y_s^p| \times \left|\underline{f}_n(s, Y_s^n, Z_s^n) - \underline{f}_p(s, Y_s^p, Z_s^p)\right|\, ds \\ + C\mathbb{E}\int_0^T |Y_s^n - Y_s^p|^2\, ds.\end{aligned}$$

By the uniform linear growth of $\underline{f}_n$ and Lemmas 5.3 and 5.4

$$\sup_n \mathbb{E}\int_0^T \left|\underline{f}_n(s, Y_s^n, Z_s^n)\right|^2\, ds < \infty.$$

Hence Cauchy's inequality gives

$$\mathbb{E}\int_0^T \left|Y_s^n - Y_s^p\right| \left|\underline{f}_n(s, Y_s^n, Z_s^n) - \underline{f}_p(s, Y_s^p, Z_s^p)\right|\, ds \leq C(\mathbb{E}\int_0^T |Y_s^n - Y_s^p|^2\, ds)^{1/2}.$$

Since (Y^n) is increasing in n and uniformly bounded in S^2, it converges in $M^2(0, T; \mathbb{R})$. The right-hand side therefore tends to zero as $n, p \to \infty$. Consequently, we have

$$\mathbb{E}\int_0^T |Z_s^n - Z_s^p|^2\, ds \to 0.$$

□

5.5. *Continuity of the limit*

From the previous estimates, the processes $\underline{Y}$ and $\underline{Z}$ exist such that

$$Y^n \to \underline{Y} \quad \text{in } M^2(0,T;\,\mathbb{R}),$$

and

$$Z^n \to \underline{Z} \quad \text{in } M^2(0,T;\,\mathbb{R}^d).$$

Lemma 5.7. *We have*

$$\mathbb{E}[\sup_{0\le t\le T} \left|Y_t^n - \underline{Y}_t\right|^2] \to 0.$$

In particular

$$\underline{Y} \in S^2(0,T;\,\mathbb{R}).$$

Proof. Apply Itô's formula to $|Y_t^n - Y_t^p|^2$ and take the supremum over $t \in [0,T]$. The drift term is controlled as in Lemma 5.6. The reflection terms are non-positive because of the Skorokhod conditions for the lower and upper barriers.

The stochastic integrals are estimated by the Burkholder-Davis-Gundy inequality

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\left|\int_t^T (Y_s^n - Y_s^p)(g(s,Y_s^n,Z_s^n) - g(s,Y_s^p,Z_s^p))\,dB_s\right| \\
&\le \varepsilon\mathbb{E}\sup_{0\le t\le T}|Y_t^n - Y_t^p|^2 + C_\varepsilon\mathbb{E}\int_0^T \left|g(s,Y_s^n,Z_s^n) - g(s,Y_s^p,Z_s^p)\right|^2\,ds,
\end{aligned}$$

and similarly

$$\mathbb{E}\sup_{0\le t\le T}\left|\int_t^T (Y_s^n - Y_s^p)(Z_s^n - Z_s^p)\,dW_s\right| \le \varepsilon\mathbb{E}\sup_{0\le t\le T}|Y_t^n - Y_t^p|^2 + C_\varepsilon\mathbb{E}\int_0^T |Z_s^n - Z_s^p|^2\,ds.$$

Choosing $\varepsilon > 0$ small enough and using Lemma 5.6, we obtain

$$\mathbb{E}\sup_{0\le t\le T}\left|Y_t^n - Y_t^p\right|^2 \to 0.$$

Thus (Y^n) is Cauchy in $S^2(0,T;\mathbb{R})$ and converges to $\underline{Y}$. Since each Y^n is continuous, the limit has a continuous version. □

5.6. *Passage to the limit as $n \to \infty$*

We now identify the limit and verify that it is a solution of the original doubly reflected BDSDE.

Lemma 5.8. *The continuous nondecreasing processes $\underline{K}^+$ and $\underline{K}^-$ exist such that, along a subsequence*

$$K^{+,n} \to \underline{K}^+, \qquad K^{-,n} \to \underline{K}^-,$$

uniformly in probability on $[0,T]$.

Proof. The previous estimates give

$$\sup_n \mathbb{E}[|K_T^{+,n}|^2 + |K_T^{-,n}|^2] < \infty.$$

Since $K^{+,n}$ and $K^{-,n}$ are nondecreasing and start from zero, Helly's selection theorem gives subsequences and nondecreasing processes $\underline{K}^+$ and $\underline{K}^-$ such that

$$K_t^{+,n} \to \underline{K}_t^+, \qquad K_t^{-,n} \to \underline{K}_t^-,$$

for every continuity point of the limiting processes, outside a P-null set.

From (5.2), we have

$$K_t^{+,n} - K_t^{-,n} = Y_0^n - Y_t^n - \int_0^t \underline{f}_n(s, Y_s^n, Z_s^n)\, ds - \int_0^t g(s, Y_s^n, Z_s^n)\, dB_s + \int_0^t Z_s^n\, dW_s.$$

Passing to the limit gives

$$\underline{K}_t^+ - \underline{K}_t^- = \underline{Y}_0 - \underline{Y}_t - \int_0^t f(s, \underline{Y}_s, \underline{Z}_s)\, ds - \int_0^t g(s, \underline{Y}_s, \underline{Z}_s)\, dB_s + \int_0^t \underline{Z}_s\, dW_s.$$

The right-hand side is continuous in t. Hence the difference $\underline{K}^+ - \underline{K}^-$ has a continuous version. Since the barriers and $\underline{Y}$ are continuous, the limiting reflection processes may be chosen continuous. Thus, along a subsequence

$$K^{+,n} \to \underline{K}^+, \qquad K^{-,n} \to \underline{K}^-,$$

uniformly in probability on $[0, T]$. □

Theorem 5.9. *Under Assumptions 2.1 and 2.2, the doubly reflected BDSDE* (1.2) *admits a minimal solution*

$$(\underline{Y}, \underline{Z}, \underline{K}^+, \underline{K}^-).$$

Proof. We first pass to the limit in the equation. By Lemma 4.1, we have

$$\underline{f}_n(t, Y_t^n, Z_t^n) \to f(t, \underline{Y}_t, \underline{Z}_t), \qquad dP \otimes dt\text{-a.e..}$$

Moreover,

$$|\underline{f}_n(t, Y_t^n, Z_t^n)| \le C(\gamma_t + 1 + |Y_t^n| + |Z_t^n|).$$

Using the convergence of Y^n in S^2, the convergence of Z^n in M^2, and the uniform integrability obtained above, we get

$$\mathbb{E}\int_0^T \left|\underline{f}_n(t, Y_t^n, Z_t^n) - f(t, \underline{Y}_t, \underline{Z}_t)\right|^2 dt \to 0.$$

Similarly, Assumption 2.1 (H3) implies

$$\mathbb{E}\int_0^T \left|g(t, Y_t^n, Z_t^n) - g(t, \underline{Y}_t, \underline{Z}_t)\right|^2 dt \to 0.$$

Passing to the limit in (5.2) gives

$$\underline{Y}_t = \xi + \int_t^T f(s, \underline{Y}_s, \underline{Z}_s)\, ds + \int_t^T g(s, \underline{Y}_s, \underline{Z}_s)\, dB_s$$
$$+ (\underline{K}_T^+ - \underline{K}_t^+) - (\underline{K}_T^- - \underline{K}_t^-) - \int_t^T \underline{Z}_s\, dW_s.$$

The lower constraint follows from $Y_t^n \geq L_t$ and the convergence of Y^n in S^2. The upper constraint follows from $Y_t^n \leq U_t$. Therefore

$$L_t \leq \underline{Y}_t \leq U_t, \qquad 0 \leq t \leq T, \quad P\text{-a.s.}.$$

We now verify the Skorokhod conditions. Since

$$\int_0^T (Y_t^n - L_t)\, dK_t^{+,n} = 0,$$

and since $Y^n \to \underline{Y}$ in S^2 while $K^{+,n} \to \underline{K}^+$ uniformly in probability along a subsequence, we obtain

$$\int_0^T (\underline{Y}_t - L_t)\, d\underline{K}_t^+ = 0.$$

Similarly, from

$$\int_0^T (U_t - Y_t^n)\, dK_t^{-,n} = 0,$$

we obtain

$$\int_0^T (U_t - \underline{Y}_t)\, d\underline{K}_t^- = 0.$$

Thus

$$(\underline{Y}, \underline{Z}, \underline{K}^+, \underline{K}^-)$$

is a solution of the doubly reflected BDSDE.

It remains to prove minimality. Let

$$(Y^*, Z^*, K^{+,*}, K^{-,*})$$

be any other solution of (1.2). Since $Y^* \leq U$, we have

$$(Y_t^* - U_t)^+ = 0.$$

Moreover

$$\underline{f}_n(t, y, z) \leq f(t, y, z).$$

By the comparison theorem applied to the approximate one-barrier equation and to the equation satisfied by Y^*, we get

$$Y_t^n \leq Y_t^*, \qquad 0 \leq t \leq T, \quad P\text{-a.s.}.$$

Letting $n \to \infty$ gives

$$\underline{Y}_t \leq Y_t^*, \qquad 0 \leq t \leq T, \quad P\text{-a.s.}.$$

Hence the solution is minimal. □

6. Existence of a maximal solution

We now turn to the construction of the maximal solution. The argument is similar in spirit to the construction of the minimal solution, but the approximation of the driver is taken in the opposite direction. More precisely, we use the decreasing Lipschitz approximation

$$\overline{f}_n \downarrow f$$

defined in (4.2). As in the previous section, the approximation of the driver and the penalization of the upper obstacle are handled by two separate parameters.

For $n, m \geq 1$, consider the one-barrier reflected BDSDE

$$\begin{aligned} \overline{Y}_t^{n,m} &= \xi + \int_t^T \overline{f}_n(s, \overline{Y}_s^{n,m}, \overline{Z}_s^{n,m})\, ds + \overline{K}_T^{n,m} - \overline{K}_t^{n,m} - m \int_t^T (\overline{Y}_s^{n,m} - U_s)^+\, ds \\ &\quad + \int_t^T g(s, \overline{Y}_s^{n,m}, \overline{Z}_s^{n,m})\, dB_s - \int_t^T \overline{Z}_s^{n,m}\, dW_s, \qquad 0 \leq t \leq T. \end{aligned} \tag{6.1}$$

The process $\overline{K}^{n,m}$ is the reflecting process associated with the lower obstacle L. Hence

$$\overline{Y}_t^{n,m} \geq L_t, \qquad 0 \leq t \leq T,$$

and

$$\int_0^T (\overline{Y}_t^{n,m} - L_t)\, d\overline{K}_t^{n,m} = 0.$$

Since $\overline{f}_n$ is Lipschitz continuous in (y, z), Theorem 3.1 gives, for every fixed pair (n, m), a unique solution

$$(\overline{Y}^{n,m}, \overline{Z}^{n,m}, \overline{K}^{n,m})$$

to (6.1).

6.1. Monotonicity of the two-parameter approximation

We first record the monotonicity properties which will be used in the construction.

Lemma 6.1. *For every fixed $m \geq 1$, the sequence $(\overline{Y}^{n,m})_{n\geq 1}$ is decreasing in n*

$$\overline{Y}_t^{n+1,m} \leq \overline{Y}_t^{n,m}, \qquad 0 \leq t \leq T, \quad P\text{-a.s.}.$$

Moreover, for every fixed $n \geq 1$, the sequence $(\overline{Y}^{n,m})_{m\geq 1}$ is decreasing in m

$$\overline{Y}_t^{n,m+1} \leq \overline{Y}_t^{n,m}, \qquad 0 \leq t \leq T, \quad P\text{-a.s.}.$$

Proof. Fix $m \geq 1$. Since

$$\overline{f}_{n+1}(t, y, z) \leq \overline{f}_n(t, y, z),$$

and since the terminal condition, the coefficient g, the lower obstacle L, and the penalization parameter m are the same, the comparison theorem for one-barrier reflected BDSDEs gives

$$\overline{Y}_t^{n+1,m} \leq \overline{Y}_t^{n,m}, \qquad 0 \leq t \leq T, \quad P\text{-a.s.}.$$

Now fix $n \geq 1$. If $m_2 \geq m_1$, then

$$-m_2(y - U_t)^+ \leq -m_1(y - U_t)^+, \qquad y \in \mathbb{R}.$$

Thus the generator corresponding to m_2 is smaller than the one corresponding to m_1. Applying the comparison theorem again gives

$$\overline{Y}_t^{n,m_2} \leq \overline{Y}_t^{n,m_1}, \qquad 0 \leq t \leq T, \quad P\text{-a.s.}$$

Taking $m_2 = m + 1$ and $m_1 = m$ proves the second assertion. □

6.2. Passage in the penalization parameter

We now fix $n \geq 1$ and let $m \to \infty$. By Lemma 6.1, the sequence $(\overline{Y}^{n,m})_{m\geq 1}$ is decreasing. Since it is bounded from below by L, an adapted process $\overline{Y}^n$ exists such that

$$\overline{Y}_t^{n,m} \downarrow \overline{Y}_t^n, \qquad 0 \leq t \leq T, \quad P\text{-a.s..}$$

The estimates used in the minimal case imply that the convergence also holds in

$$M^2(0, T; \mathbb{R}).$$

Define the upper penalization process

$$\overline{K}_t^{-,n,m} = m \int_0^t (\overline{Y}_s^{n,m} - U_s)^+ \, ds, \qquad 0 \leq t \leq T.$$

Then $\overline{K}^{-,n,m}$ is continuous, nondecreasing, and starts from zero. By Lemma 4.3, there is a constant $C > 0$, independent of m, such that

$$\sup_{m\geq 1} \mathbb{E}[|\overline{K}_T^{-,n,m}|^2] \leq C.$$

Therefore, by Helly's selection theorem, there is a subsequence, still denoted by (m), and a nondecreasing process $\overline{K}^{-,n}$ such that

$$\overline{K}_t^{-,n,m} \to \overline{K}_t^{-,n}$$

for every continuity point t of $\overline{K}^{-,n}$, P-a.s.. Along this subsequence, the convergence also holds uniformly on $[0, T]$ in probability. The limiting process satisfies

$$\overline{K}_0^{-,n} = 0, \qquad \mathbb{E}[|\overline{K}_T^{-,n}|^2] \leq C.$$

The same compactness argument applies to the lower reflecting processes. Up to another subsequence, a nondecreasing process $\overline{K}^{+,n}$ exists such that

$$\overline{K}^{n,m} \to \overline{K}^{+,n}$$

uniformly in probability on $[0, T]$, with

$$\overline{K}_0^{+,n} = 0.$$

Passing to the limit in (6.1), along the chosen subsequence, gives

$$\begin{aligned} \overline{Y}_t^n &= \xi + \int_t^T \overline{f}_n(s, \overline{Y}_s^n, \overline{Z}_s^n)\, ds + \int_t^T g(s, \overline{Y}_s^n, \overline{Z}_s^n)\, dB_s \\ &\quad + (\overline{K}_T^{+,n} - \overline{K}_t^{+,n}) - (\overline{K}_T^{-,n} - \overline{K}_t^{-,n}) - \int_t^T \overline{Z}_s^n\, dW_s. \end{aligned} \tag{6.2}$$

Here, $\overline{Z}^n$ denotes the limit of $\overline{Z}^{n,m}$ in $M^2(0, T; \mathbb{R}^d)$.

We also obtain the upper flat-off condition. Indeed

$$d\overline{K}_t^{-,n,m} = m(\overline{Y}_t^{n,m} - U_t)^+\, dt,$$

and therefore

$$\int_0^T (U_t - \overline{Y}_t^{n,m})\, d\overline{K}_t^{-,n,m} = -\int_0^T m((\overline{Y}_t^{n,m} - U_t)^+)^2\, dt \leq 0.$$

Letting $m \to \infty$ gives

$$\int_0^T (U_t - \overline{Y}_t^n)\, d\overline{K}_t^{-,n} \leq 0.$$

On the other hand, Lemma 4.3 gives

$$0 \leq (\overline{Y}_t^{n,m} - U_t)^+ \leq \frac{C}{m}, \qquad 0 \leq t \leq T.$$

Thus

$$\overline{Y}_t^n \leq U_t, \qquad 0 \leq t \leq T, \quad P\text{-a.s.}.$$

Since $\overline{K}^{-,n}$ is nondecreasing, we also have

$$\int_0^T (U_t - \overline{Y}_t^n)\, d\overline{K}_t^{-,n} \geq 0.$$

Consequently

$$\int_0^T (U_t - \overline{Y}_t^n)\, d\overline{K}_t^{-,n} = 0.$$

Similarly, passing to the limit in

$$\int_0^T (\overline{Y}_t^{n,m} - L_t)\, d\overline{K}_t^{n,m} = 0$$

yields

$$\int_0^T (\overline{Y}_t^n - L_t)\, d\overline{K}_t^{+,n} = 0.$$

Thus, for each fixed n, the quadruple

$$(\overline{Y}^n, \overline{Z}^n, \overline{K}^{+,n}, \overline{K}^{-,n})$$

solves the doubly reflected BDSDE with Lipschitz driver $\overline{f}_n$.

6.3. Passage to the continuous driver

We now let $n \to \infty$. The estimates established for the minimal solution remain valid for the sequence constructed above. Hence there is a constant $C > 0$, independent of n, such that

$$\sup_n \mathbb{E}[\sup_{0\le t\le T} |\overline{Y}^n_t|^2 + \int_0^T |\overline{Z}^n_t|^2\,dt + |\overline{K}^{+,n}_T|^2 + |\overline{K}^{-,n}_T|^2] \le C.$$

Moreover, we have

$$L_t \le \overline{Y}^n_t \le U_t, \qquad 0 \le t \le T, \quad P\text{-a.s..}$$

By Lemma 6.1, after passing to the limit in m, the sequence $(\overline{Y}^n)_{n\ge1}$ is decreasing:

$$\overline{Y}^{n+1}_t \le \overline{Y}^n_t, \qquad 0 \le t \le T, \quad P\text{-a.s..}$$

Since it is uniformly bounded in $S^2(0,T;\mathbb{R})$, a process $\overline{Y}$ exists such that

$$\overline{Y}^n \to \overline{Y} \quad \text{in } M^2(0,T;\,\mathbb{R}).$$

The same Cauchy argument used for the minimal solution gives a process

$$\overline{Z} \in M^2(0,T;\,\mathbb{R}^d)$$

such that

$$\overline{Z}^n \to \overline{Z} \quad \text{in } M^2(0,T;\,\mathbb{R}^d).$$

Furthermore,

$$\overline{Y}^n \to \overline{Y} \quad \text{in } S^2(0,T;\,\mathbb{R}),$$

so $\overline{Y}$ admits a continuous version.

The uniform bounds on $\overline{K}^{+,n}$ and $\overline{K}^{-,n}$ allow us, by Helly's selection theorem, to extract subsequences such that

$$\overline{K}^{+,n} \to \overline{K}^{+}, \qquad \overline{K}^{-,n} \to \overline{K}^{-},$$

uniformly in probability on $[0,T]$, where $\overline{K}^{+}$ and $\overline{K}^{-}$ are continuous nondecreasing processes starting from zero.

We pass to the limit in (6.2). By Lemma 4.1

$$\overline{f}_n(t,\overline{Y}^n_t,\overline{Z}^n_t) \to f(t,\overline{Y}_t,\overline{Z}_t), \qquad dP\otimes dt\text{-a.e..}$$

The linear growth condition and the uniform estimates imply

$$\mathbb{E}\int_0^T \left|\overline{f}_n(t,\overline{Y}^n_t,\overline{Z}^n_t) - f(t,\overline{Y}_t,\overline{Z}_t)\right|^2 dt \to 0.$$

Similarly, Assumption 2.1 (H3) gives

$$\mathbb{E}\int_0^T \left|g(t,\overline{Y}^n_t,\overline{Z}^n_t) - g(t,\overline{Y}_t,\overline{Z}_t)\right|^2 dt \to 0.$$

Therefore

$$\overline{Y}_t = \xi + \int_t^T f(s, \overline{Y}_s, \overline{Z}_s)\, ds + \int_t^T g(s, \overline{Y}_s, \overline{Z}_s)\, dB_s$$
$$+ (\overline{K}_T^+ - \overline{K}_t^+) - (\overline{K}_T^- - \overline{K}_t^-) - \int_t^T \overline{Z}_s\, dW_s.$$

The constraints follow from the convergence and from

$$L_t \le \overline{Y}_t^n \le U_t.$$

Thus

$$L_t \le \overline{Y}_t \le U_t, \qquad 0 \le t \le T, \quad P\text{-a.s.}.$$

Passing to the limit in the flat-off conditions gives

$$\int_0^T (\overline{Y}_t - L_t)\, d\overline{K}_t^+ = 0, \qquad \int_0^T (U_t - \overline{Y}_t)\, d\overline{K}_t^- = 0.$$

Therefore

$$(\overline{Y}, \overline{Z}, \overline{K}^+, \overline{K}^-)$$

is a solution of the doubly reflected BDSDE.

Theorem 6.2. *Under Assumptions 2.1 and 2.2, the doubly reflected BDSDE* (1.2) *admits a maximal solution*

$$(\overline{Y}, \overline{Z}, \overline{K}^+, \overline{K}^-).$$

Proof. It remains only to prove maximality. Let

$$(Y^*, Z^*, K^{+,*}, K^{-,*})$$

be any other solution of (1.2). Since $Y^* \le U$, we have

$$(Y_t^* - U_t)^+ = 0.$$

Moreover

$$f(t, y, z) \le \overline{f}_n(t, y, z).$$

Applying the comparison theorem to the approximate one-barrier equation and to the equation satisfied by Y^* gives

$$Y_t^* \le \overline{Y}_t^n, \qquad 0 \le t \le T, \quad P\text{-a.s.}.$$

Letting $n \to \infty$, we obtain

$$Y_t^* \le \overline{Y}_t, \qquad 0 \le t \le T, \quad P\text{-a.s.}.$$

Thus the constructed solution is maximal. □

Combining Theorems 5.9 and 6.2, we obtain the main result.

Theorem 6.3 (Existence of extremal solutions)**.** *Assume that Assumptions 2.1 and 2.2 hold. Then the doubly reflected BDSDE* (1.2) *admits a minimal solution*

$$(\underline{Y}, \underline{Z}, \underline{K}^+, \underline{K}^-)$$

and a maximal solution

$$(\overline{Y}, \overline{Z}, \overline{K}^+, \overline{K}^-).$$

Moreover, if (Y, Z, K^+, K^-) *is any other solution of* (1.2)*, then*

$$\underline{Y}_t \leq Y_t \leq \overline{Y}_t, \qquad 0 \leq t \leq T, \quad P\text{-a.s.}.$$

Remark 6.4. *The two approximation procedures play distinct and complementary roles in the construction of extremal solutions. The increasing Lipschitz approximation*

$$\underline{f}_n \uparrow f$$

preserves the monotonicity required to construct the minimal solution, whereas the decreasing approximation

$$\overline{f}_n \downarrow f$$

naturally leads to the maximal solution. This distinction is a direct consequence of the comparison principle for reflected BDSDEs. In particular, using only the decreasing approximation would produce a decreasing sequence of auxiliary solutions converging to the maximal solution, while the increasing approximation generates an increasing sequence converging to the minimal solution.

It is also important to emphasize that the approximation parameter and the penalization parameter are treated independently throughout the analysis. This separation preserves the monotonicity of the auxiliary equations and allows the comparison theorem to be applied at each stage of the construction. Consequently, the two-parameter approximation strategy provides a unified framework for establishing both minimal and maximal solutions under the weaker assumption that the generator is merely continuous with linear growth.

7. Examples

These examples illustrate that the proposed assumptions are sufficiently general to cover stochastic models that naturally arise in constrained stochastic systems, while providing a theoretical basis for future applications to stochastic control, obstacle-type SPDEs, mathematical finance, and uncertainty-aware stochastic modeling.

We give some examples showing that the assumptions of Theorem 6.3 are non empty and that they cover genuinely non-Lipschitz drivers. The examples also illustrate both deterministic and random choices of the upper barrier.

Example 7.1 (Continuous non-Lipschitz driver with constant barriers)**.** *Let*

$$L_t = -1, \qquad U_t = 1, \qquad 0 \leq t \leq T.$$

Let $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ *satisfy*

$$-1 \leq \xi \leq 1, \qquad P\text{-a.s.}.$$

Define

$$f(t,y,z) = \gamma_t + \sqrt{|y|} + \frac{|z|}{1+|z|},$$

where γ *is a non-negative process in* $M^2(0,T;\mathbb{R})$.

The mapping $(y,z) \mapsto f(t,y,z)$ *is continuous. It is not Lipschitz in y, because the function*

$$y \longmapsto \sqrt{|y|}$$

is not Lipschitz near 0*. Moreover,*

$$|f(t,y,z)| \le \gamma_t + \sqrt{|y|} + 1 \le C(1+\gamma_t+|y|+|z|).$$

Thus f *satisfies the continuity and linear growth assumptions.*

Let

$$g(t,y,z) = a_t + \lambda y + \beta z,$$

where $a \in M^2(0,T;\mathbb{R})$*,* $\lambda \in \mathbb{R}$*, and* $|\beta|^2 < 1/2$*. Then*

$$|g(t,y,z) - g(t,y',z')|^2 \le 2\lambda^2 \left|y-y'\right|^2 + 2|\beta|^2 \left|z-z'\right|^2.$$

Hence Assumption 2.1 (H3) holds with

$$\alpha = 2|\beta|^2 < 1.$$

Finally, Assumption 2.2 is satisfied by taking

$$U_t^n = 1 + \frac{1}{n}, \qquad u_n(t) = 0, \qquad v_n(t) = 0.$$

Therefore all assumptions of Theorem 6.3 are satisfied.

Example 7.2 (Time-dependent deterministic barriers)**.** *Let*

$$L_t = -2 - \sin t, \qquad U_t = 2 + \cos^2 t, \qquad 0 \le t \le T.$$

Then L and U are deterministic continuous functions. Moreover, we have

$$L_t < U_t, \qquad 0 \le t \le T.$$

Let ξ *be any square-integrable* $\mathcal{F}_T$*-measurable random variable satisfying*

$$L_T \le \xi \le U_T, \qquad P\text{-a.s.}.$$

Define

$$f(t,y,z) = \gamma_t + \frac{y}{1+|y|} + \sqrt{|z|}.$$

The mapping $(y,z) \mapsto f(t,y,z)$ *is continuous. It is not globally Lipschitz in z, because*

$$z \mapsto \sqrt{|z|}$$

is not Lipschitz near zero. Moreover, we have

$$|f(t, y, z)| \leq \gamma_t + 1 + \sqrt{|z|} \leq C(1 + \gamma_t + |y| + |z|).$$

Thus f is continuous and satisfies the required linear growth condition.

Let

$$g(t, y, z) = \frac{1}{2}y + \frac{1}{3}z.$$

Then

$$|g(t, y, z) - g(t, y', z')|^2 \leq C\,|y - y'|^2 + \frac{2}{9}\,|z - z'|^2\,.$$

Hence Assumption 2.1 (H3) holds with some $\alpha < 1$.

The upper barrier satisfies Assumption 2.2. Indeed, define

$$U_t^n = U_t + \frac{1}{n} = 2 + \cos^2 t + \frac{1}{n}.$$

Then $U^n \downarrow U$ and

$$U_t^n = U_0^n + \int_0^t u_n(s)\,ds, \qquad u_n(s) = -\sin(2s), \qquad v_n(s) = 0.$$

Since (u_n) is uniformly bounded, Assumption 2.2 is satisfied. Therefore Theorem 6.3 applies.

Example 7.3 (Bounded random upper barrier)**.** *Let*

$$L_t = -1, \qquad U_t = 2 + \frac{1}{2}\sin(W_t), \qquad 0 \leq t \leq T.$$

Then U is continuous and satisfies

$$\frac{3}{2} \leq U_t \leq \frac{5}{2}, \qquad 0 \leq t \leq T.$$

In particular, we have

$$L_t \leq U_t, \qquad 0 \leq t \leq T, \quad P\text{-a.s.}.$$

Let $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ satisfy

$$-1 \leq \xi \leq 2 + \frac{1}{2}\sin(W_T), \qquad P\text{-a.s.}.$$

By Itô's formula, we have

$$dU_t = -\frac{1}{2}\sin(W_t)\,dt + \frac{1}{2}\cos(W_t)\,dW_t.$$

Therefore

$$U_t = U_0 + \int_0^t u(s)\,ds + \int_0^t v(s)\,dW_s,$$

where

$$u(s) = -\frac{1}{2}\sin(W_s), \qquad v(s) = \frac{1}{2}\cos(W_s).$$

Both u and v are bounded. Hence Assumption 2.2 is satisfied by taking

$$U_t^n = U_t + \frac{1}{n}.$$

Indeed, $U^n \downarrow U$ *and*

$$U_t^n = U_0^n + \int_0^t u(s)\,ds + \int_0^t v(s)\,dW_s.$$

Now define

$$f(t,y,z) = \gamma_t + \log(1+|y|) + \frac{|z|}{1+|z|}.$$

The mapping $(y,z) \mapsto f(t,y,z)$ *is continuous. Moreover, we have*

$$|f(t,y,z)| \le C(1+\gamma_t+|y|+|z|),$$

so f satisfies the linear growth condition. It is not required to be Lipschitz.

Let

$$g(t,y,z) = a_t + \lambda y + \beta z,$$

where $a \in M^2(0,T;\mathbb{R})$*,* $\lambda \in \mathbb{R}$*, and* $|\beta|^2 < 1/2$*. Then*

$$|g(t,y,z) - g(t,y',z')|^2 \le 2\lambda^2 |y-y'|^2 + 2|\beta|^2 |z-z'|^2.$$

Thus Assumption 2.1 (H3) holds with

$$\alpha = 2|\beta|^2 < 1.$$

Therefore all assumptions of Theorem 6.3 are satisfied.

Remark 7.4. *The third example uses a bounded random upper barrier. This choice avoids the possible violation of the condition* $L \le U$ *that may occur when the upper barrier contains an unbounded Gaussian martingale term. Hence the ordering of the two barriers holds pathwise.*

Remark 7.5. *These examples show that the theorem applies to continuous drivers with linear growth which are not necessarily Lipschitz. They also show that the assumptions allow deterministic barriers as well as bounded random upper barriers.*

8. Conclusions

In this paper, we established the existence of minimal and maximal adapted solutions for a class of doubly reflected BDSDEs with continuous reflecting barriers. Unlike the classical framework, where the generator is assumed to be globally Lipschitz, the drift coefficient considered here is only required to be continuous with respect to (y,z) and to satisfy a linear growth condition. Under an additional semimartingale approximation assumption on the upper barrier, we proved the existence of extremal solutions despite the possible lack of uniqueness.

The proposed analysis is based on a constructive approximation procedure combining monotone Lipschitz regularization of the continuous generator with a penalization technique for the upper obstacle. One of the main methodological contributions of this work is the introduction of two independent approximation parameters, one associated with the regularization of the generator and the other with the penalization of the upper barrier. Treating these parameters separately preserves the monotonicity required by the comparison theorem and allows the limiting procedure to be carried out in a rigorous manner. This strategy leads naturally to the construction of the limiting reflecting processes and the verification of the corresponding Skorokhod conditions.

A fundamental ingredient of the proof is the uniform estimate for the upper penalization term. This estimate shows that the excess of the penalized solution above the upper obstacle decreases at the rate $O(m^{-1})$, making it possible to recover the upper reflecting process in the limit. Combined with standard a priori estimates and stability arguments, it ensures the convergence of the solution sequence, the martingale integrands, and the reflecting processes, thereby establishing the existence of both minimal and maximal adapted solutions.

The examples presented in this paper illustrate that the proposed assumptions are sufficiently general to include genuinely non-Lipschitz generators together with deterministic and stochastic continuous barriers. Consequently, the present work extends the existing existence theory for doubly reflected BDSDEs beyond the classical Lipschitz setting while preserving the construction of extremal solutions under considerably weaker assumptions.

Although the present analysis is restricted to scalar doubly reflected BDSDEs with continuous generators of linear growth, the proposed methodology opens several interesting directions for future research. From a theoretical viewpoint, it would be worthwhile to investigate multidimensional doubly reflected BDSDEs, generators satisfying monotone or superlinear growth conditions, irregular barriers, and more general stochastic frameworks involving jumps, Lévy processes, or nonlinear expectations. From an application perspective, the existence theory developed here provides a rigorous mathematical foundation for constrained SPDEs, obstacle problems, stochastic control, Dynkin games, and mathematical finance. It may also contribute to future developments in uncertainty quantification and Bayesian inverse problems involving stochastic systems with state constraints or obstacle-type conditions. Extending the present framework to these settings remains an interesting and promising direction for future investigation.

Author contributions

Badreddine Mansouri: Supervision, methodology, validation, formal analysis, writing—original draft; Halim Zeghdoudi: Methodology, formal analysis, writing—review and editing; Muhammad Ameeq: Investigation, validation, writing—review and editing; Abdullah M. Almarashi: Conceptualization, validation, writing—review and editing. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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