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Research article

Weakly $\mathscr{H}$-embedded subgroups and nilpotency of fusion systems

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Abstract: Let p be a prime and $\mathcal{F}$ be a fusion system on a finite p-group S. We say that a subgroup Q of S is weakly $\mathscr{H}$-embedded in $\mathcal{F}$ if there is a strongly $\mathcal{F}$-closed subgroup P of S such that both QP and $Q \cap P$ are strongly $\mathcal{F}$-closed. Using this concept, we prove some new characterizations of nilpotent fusion systems. Our work generalizes and extends some previously known results on weakly $\mathscr{H}$-embedded subgroups of finite groups.

Keywords: finite group; fusion system; nilpotent; p-nilpotent; strongly closed subgroup; $\mathscr{H}$-subgroup; weakly $\mathscr{H}$-embedded subgroup
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1. Introduction

1.1. Motivation and statement of results

An important concept in finite group theory is that of strongly closed subgroups. A subgroup H of a finite group G is said to be *strongly closed* in G if $N_G(H) \cap H^g \leq H$ for all $g \in G$. Of particular importance are strongly closed subgroups of prime power order.

Let p be a prime, S be a Sylow p-subgroup of a finite group G, and $Q \leq S$. Then, Q is strongly closed in G if and only if it is strongly closed in S with respect to G, that is, for each subgroup R of Q and each $g \in G$, $R^g \leq S$ implies $R^g \leq Q$. Indeed, this can be seen as follows (see also [22, (2.1)]): Assume that Q is strongly closed in G. For each $x \in N_S(N_S(Q))$, we then have $Q^x \leq N_S(Q)^x = N_S(Q) \leq N_G(Q)$ and hence $Q^x = N_G(Q) \cap Q^x \leq Q$, which implies that $x \in N_S(Q)$. Thus, $N_S(N_S(Q)) = N_S(Q)$ so that $N_S(Q) = S$ and $Q \trianglelefteq S$. Now, if $R \leq Q$ and $g \in G$ such that $R^g \leq S$, then $R^g = S \cap R^g \leq N_G(Q) \cap Q^g \leq Q$. Consequently, Q is strongly closed in S with respect to G. Now, we assume that Q is strongly closed in S with respect to G and we deduce that Q is strongly closed in G. Let $g \in G$. Then, $Q(N_G(Q) \cap Q^g)$ is a p-subgroup of G. Therefore, there is some $h \in G$ with $(Q(N_G(Q) \cap Q^g))^h \leq S$.

Then, $Q^h \leq S$, and since Q is strongly closed in S with respect to G, it follows that $Q^h = Q$ so that $h \in N_G(Q)$. Also, since $N_G(Q^{g^{-1}}) \cap Q \leq Q$ and $(N_G(Q^{g^{-1}}) \cap Q)^{gh} = (N_G(Q) \cap Q^g)^h \leq S$, we have $(N_G(Q^{g^{-1}}) \cap Q)^{gh} \leq Q$. As $h \in N_G(Q)$, it follows that $N_G(Q) \cap Q^g = (N_G(Q^{g^{-1}}) \cap Q)^g \leq Q$. Hence, Q is strongly closed in G.

The preceding characterization of strongly closed p-subgroups of finite groups leads to a natural definition of strong closure in fusion systems. Given a prime p and a fusion system $\mathcal{F}$ on a finite p-group S, a subgroup Q of S is defined to be *strongly $\mathcal{F}$-closed* if, for each subgroup R of Q and each $\mathcal{F}$-morphism $\varphi : R \to S$, we have $R\varphi \leq Q$ (see [14, Definition 4.55 (ii)]). With this definition, if S is a Sylow p-subgroup of a finite group G, then a subgroup of S is strongly $\mathcal{F}_S(G)$-closed if and only if it is strongly closed in G.

The first major applications of strongly closed subgroups emerged in the context of the classification of finite simple groups. Two important achievements on strong closure in this era were Glauberman's Z^*-theorem [20] and Goldschmidt's work on abelian strongly closed 2-subgroups [21]. Later, the concept of strong closure found applications also in other areas of finite group theory. For example, it was used by Bianchi et al. [12] to characterize the finite solvable T-groups, that is, the finite solvable groups in which normality is a transitive relation. Namely, they proved that a finite group G is a solvable T-group if and only if every primary subgroup of G is strongly closed in G (see [12, Theorem 10]). In [12], strongly closed subgroups are called *$\mathscr{H}$-subgroups*, and this terminology has been adopted by several other authors.

The work of Bianchi et al. [12] opened a new direction of research: the structural study of a finite group G under the assumption that some of its primary subgroups are strongly closed in G. To give a flavor of this line of research, we mention a result of Guo and Wei [24]. Given an odd prime number p and a finite group G with a non-cyclic Sylow p-subgroup S, they proved that G is p-nilpotent if and only if $N_G(S)$ is p-nilpotent and there is a proper non-trivial subgroup D of S such that any subgroup of S with order $|D|$ is strongly closed in G (see [24, Theorem 3.1]). Liao and Zhang [26] obtained a version of this result for fusion systems (see [26, Theorem 3.19]). Other structural results about finite groups with some primary subgroups being strongly closed were obtained, for example, in [2, 16, 28, 31].

Over the last 15 years, several authors have generalized and extended this line of research by introducing various generalizations of strong closure, such as weakly $\mathscr{H}$-subgroups [4], $\mathscr{H}C$-subgroups [32], weakly $\mathscr{H}$-embedded subgroups [5], weakly $\mathscr{H}C$-embedded subgroups [6], $\mathscr{S}\mathscr{S}\mathscr{H}$-subgroups [1], and $\mathscr{H}N$-subgroups [10]. They studied the structure of finite groups under the assumtion that some primary subgroups satisfy one of these generalized embedding conditions. In particular, this led to many interesting criteria for p-nilpotence and supersolvability of finite groups.

Inspired by this line of research, one might ask whether similar ideas can be used to generalize strong closure in fusion systems and whether such generalizations would be useful in establishing criteria for nilpotence [26] or supersolvability [30] of fusion systems. In this paper, we explore this question by introducing one such generalization and using it to provide new characterizations of nilpotent fusion systems.

Our point of departure is the concept of weakly $\mathscr{H}$-embedded subgroups in finite groups. A subgroup H of a finite group G is said to be *weakly $\mathscr{H}$-embedded* in G if there is a normal subgroup K of G with $H^G = HK$ such that $H \cap K$ is an $\mathscr{H}$-subgroup (or, in other words, a strongly closed subgroup) of G, where H^G is the normal closure of H in G (see [5, Definition 1.2]). Assume that such a subgroup H is a p-subgroup of G for some prime p, let S be a Sylow p-subgroup of G containing H, and let K be as above. Note that, for each $N \trianglelefteq G$, the subgroup $S \cap N$ is strongly closed in G. Hence, each of the subgroups $P := S \cap K$, $HP = H(S \cap K) = S \cap HK = S \cap H^G$, and $H \cap P = H \cap (S \cap K) = H \cap K$ is strongly closed in G. Motivated by this observation, we introduce the following definition.

Definition 1.1. *Let p be a prime number, $\mathcal{F}$ be a fusion system on a finite p-group S, and Q be a subgroup of S. Then, Q is said to be weakly $\mathscr{H}$-embedded in $\mathcal{F}$ if there is a strongly $\mathcal{F}$-closed subgroup P of S such that both QP and $Q \cap P$ are strongly $\mathcal{F}$-closed.*

Example 1.2. *Let p be a prime, $\mathcal{F}$ be a fusion system on a finite p-group S, and Q be a strongly $\mathcal{F}$-closed subgroup of S. Then, since $S = QS$ and $Q \cap S = Q$ are strongly $\mathcal{F}$-closed, we have that Q is weakly $\mathscr{H}$-embedded in $\mathcal{F}$.*

Example 1.3. *Let p be a prime, G be a finite group, and S be a Sylow p-subgroup of G. By our observations preceding Definition 1.1, any weakly $\mathscr{H}$-embedded subgroup of G contained in S is weakly $\mathscr{H}$-embedded in $\mathcal{F}_S(G)$.*

Given a prime p and a Sylow p-subgroup S of a finite group G, it may happen that a subgroup of S is weakly $\mathscr{H}$-embedded in $\mathcal{F}_S(G)$, but not in G. Here is an example:

Example 1.4. *Let G denote the Janko group J_3, and let S be a Sylow 3-subgroup of G. Then, $|S| = 3^5$, and $\Omega_1(S) \cong C_3 \times C_3 \times C_3$ and $Z(S) \cong C_3 \times C_3$ are the only non-trivial proper subgroups of S which are strongly closed in G (see [18, Proposition 2.7 (5)]). Pick some $x \in \Omega_1(S) \setminus Z(S)$, and let $Q := \langle x \rangle$. Since each of the subgroups $Z(S)$, $QZ(S) = \Omega_1(S)$ and $Q \cap Z(S) = 1$ is strongly closed in G, we have that Q is weakly $\mathscr{H}$-embedded in $\mathcal{F}_S(G)$. On the other hand, since Q is not strongly closed in G and G is simple, Q cannot be weakly $\mathscr{H}$-embedded in G.*

Our main results in this paper are the following characterizations of nilpotent fusion systems by means of weakly $\mathscr{H}$-embedded subgroups.

Theorem 1.5. *Let p be a prime and $\mathcal{F}$ be a saturated fusion system on a non-trivial finite p-group S. Then, the following are equivalent:*

(1) *$\mathcal{F}$ is nilpotent.*

(2) *We have $(|\mathrm{Aut}_{N_{\mathcal{F}}(S)}(Q)|, p-1) = 1$ for each subgroup Q of S, and there is a subgroup D of S with $1 \leq D < S$ such that every subgroup of S with order $|D|$ or $p|D|$ (and every cyclic subgroup of S with order 4 if $D = 1$ and S is a non-abelian 2-group) is weakly $\mathscr{H}$-embedded in $\mathcal{F}$.*

(3) *$N_{\mathcal{F}}(S)$ is nilpotent, and there is a subgroup D of S with $1 \leq D < S$ such that every subgroup of S with order $|D|$ or $p|D|$ (and every cyclic subgroup of S with order 4 if $D = 1$ and S is a non-abelian 2-group) is weakly $\mathscr{H}$-embedded in $\mathcal{F}$.*

Theorem 1.6. *Let p be a prime and $\mathcal{F}$ be a saturated fusion system on a finite p-group S with $|S| \geq p^2$. Then, $\mathcal{F}$ is nilpotent if and only if there is a subgroup D of S with $1 < D < S$ such that every subgroup of S with order $|D|$ is weakly $\mathscr{H}$-embedded in $\mathcal{F}$ and such that, for each fully $\mathcal{F}$-normalized subgroup H of S with $|H| = |D|$, the normalizer $N_{\mathcal{F}}(H)$ is nilpotent.*

In the final section of this paper, we will deduce several previously known results about weakly $\mathscr{H}$-embedded subgroups of finite groups from our results. As another application of our work, we will prove the nilpotency of saturated fusion systems on certain metacyclic 2-groups.

Our work may be viewed as a continuation of recent studies of saturated fusion systems $\mathcal{F}$ in which certain subgroups are required to satisfy specific conditions with respect to $\mathcal{F}$ [11, 27, 29].

1.2. Notation and terminology

Our notation and terminology are fairly standard and can largely be found in [8] or [14]. In what follows, we introduce some notation and terminology that do not seem to appear in the aforementioned references.

We begin by recalling the definitions of nilpotent and supersolvable fusion systems. Let p be a prime and $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Following [26], we say that $\mathcal{F}$ is *nilpotent* if $\mathcal{F} = \mathcal{F}_S(S)$. Following [30], we say that $\mathcal{F}$ is *supersolvable* if there is a subgroup series $1 = S_0 \leq S_1 \leq \cdots \leq S_m = S$ such that S_i is strongly $\mathcal{F}$-closed for all $0 \leq i \leq m$ and such that S_{i+1}/S_i is cyclic for all $0 \leq i < m$.

Given a prime p and a fusion system $\mathcal{F}$ on a finite p-group S, we say that a subgroup T of S is *minimal strongly $\mathcal{F}$-closed* if T is non-trivial and strongly $\mathcal{F}$-closed, but no proper non-trivial subgroup of T is strongly $\mathcal{F}$-closed. Note that any non-trivial strongly $\mathcal{F}$-closed subgroup of S contains a minimal strongly $\mathcal{F}$-closed subgroup. In particular, since S is strongly $\mathcal{F}$-closed, there exists a minimal strongly $\mathcal{F}$-closed subgroup of S whenever $S \neq 1$.

Let G and H be groups and $\varphi : G \to H$ be a group homomorphism. For an element $g \in G$, we denote the image of g under φ by $g\varphi$. Similarly, for a subgroup X of G, we write $X\varphi$ for the image of X under φ. If Y is a subgroup of H containing $X\varphi$, we use $\varphi|_{X,Y}$ to denote the group homomorphism $X \to Y, x \mapsto x\varphi$. Finally, if L is another group and $\psi : H \to L$ is a group homomorphism, then $\varphi\psi$ denotes the group homomorphism $G \to L, g \mapsto (g\varphi)\psi$.

2. Preliminaries

In this section, we prove some lemmas needed for the proofs of our main results.

Lemma 2.1. *Let p be a prime, $\mathcal{F}$ be a fusion system on a finite p-group S, $\mathcal{E}$ be a subsystem of $\mathcal{F}$ on a subgroup T of S and R be a strongly $\mathcal{F}$-closed subgroup of S. Then, $T \cap R$ is strongly $\mathcal{E}$-closed.*

Proof. Let P be a subgroup of $T \cap R$ and $\varphi : P \to T$ be an $\mathcal{E}$-morphism. We have to show that $P\varphi \leq T \cap R$. Since $\mathcal{E}$ is a subsystem of $\mathcal{F}$, φ is an $\mathcal{F}$-morphism. Hence, $\tilde{\varphi} : P \to S, x \mapsto x\varphi$ is an $\mathcal{F}$-morphism. Since R is strongly $\mathcal{F}$-closed and $P \leq T \cap R \leq R$, it follows that $P\varphi = P\tilde{\varphi} \leq R$. As we also have $P\varphi \leq T$, we obtain that $P\varphi \leq T \cap R$, as required. □

Lemma 2.2. *Let p be a prime, $\mathcal{F}$ be a fusion system on a finite p-group S, $\mathcal{E}$ be a subsystem of $\mathcal{F}$ on a subgroup T of S, and Q be a subgroup of T. Suppose that Q is weakly $\mathscr{H}$-embedded in $\mathcal{F}$. Then Q is weakly $\mathscr{H}$-embedded in $\mathcal{E}$.*

Proof. Since Q is weakly $\mathscr{H}$-embedded in $\mathcal{F}$, there is a strongly $\mathcal{F}$-closed subgroup P of S such that both QP and $Q \cap P$ are strongly $\mathcal{F}$-closed. Set $P_0 := T \cap P$. By Lemma 2.1, each of P_0, $QP_0 = T \cap QP$

and $Q \cap P_0 = Q \cap T \cap P = Q \cap P$ is strongly $\mathcal{E}$-closed. Consequently, Q is weakly $\mathscr{H}$-embedded in $\mathcal{E}$. □

Lemma 2.3. *Let p be a prime, $\mathcal{F}$ be a saturated fusion system on a finite p-group S and $T \leq Q \leq S$ such that T is strongly $\mathcal{F}$-closed and Q is weakly $\mathscr{H}$-embedded in $\mathcal{F}$. Then, Q/T is weakly $\mathscr{H}$-embedded in $\mathcal{F}/T$.*

Proof. Since Q is weakly $\mathscr{H}$-embedded in $\mathcal{F}$, there is a strongly $\mathcal{F}$-closed subgroup P of S such that both QP and $Q \cap P$ are strongly $\mathcal{F}$-closed. By [13, Theorem 6.1 (4)], each of PT/T, $(Q/T)(PT/T) = QP/T$ and $(Q/T) \cap (PT/T) = (Q \cap PT)/T = (Q \cap P)T/T$ is strongly $\mathcal{F}/T$-closed. Consequently, Q/T is weakly $\mathscr{H}$-embedded in $\mathcal{F}/T$. □

Lemma 2.4. *Let p be a prime, $\mathcal{F}$ be a fusion system on a finite p-group S, and Q be a subgroup of S. Suppose that $Q \trianglelefteq \mathcal{F}$ and that there is a subgroup D of Q with $1 \leq D < Q$ such that all subgroups of Q with order $|D|$ or $p|D|$ (and all cyclic subgroups of Q with order 4 if $D = 1$ and Q is a non-abelian 2-group) are weakly $\mathscr{H}$-embedded in $\mathcal{F}$. Then, there is a subgroup series $1 = Q_0 \leq Q_1 \leq \cdots \leq Q_m = Q$ such that Q_i is normal in $\mathcal{F}$ for all $0 \leq i \leq m$ and such that Q_{i+1}/Q_i is cyclic for all $0 \leq i < m$. In particular, if $\mathcal{F}$ is saturated and $Q = S$, then $\mathcal{F}$ is supersolvable.*

Proof. Let $A := \mathrm{Aut}_{\mathcal{F}}(Q)$, and $G := Q \rtimes A$ be the outer semidirect product of Q and A with respect to the action

$$Q \times A \to Q, (x, \alpha) \mapsto x\alpha.$$

of A on Q. Hence, G is the group consisting of all pairs (α, x), where $\alpha \in A$ and $x \in Q$, together with the multiplication defined by

$$(\alpha_1, x_1)(\alpha_2, x_2) := (\alpha_1\alpha_2, (x_1\alpha_2)x_2)$$

for all such pairs (α_1, x_1), (α_2, x_2). The maps $A \to G, \alpha \mapsto (\alpha, 1)$ and $Q \to G, x \mapsto (\mathrm{id}_Q, x)$ are monomorphisms, and we identify A and Q with their images in G under these monomorphisms. Then, $Q \trianglelefteq G$, $Q \cap A = 1$, and $G = QA$. Also, for all $x \in Q$ and $\alpha \in A$, the product $\alpha^{-1}x\alpha$ taken in G is equal to the image $x\alpha$ of x under the automorphism α. See [25, pp. 33-34] for more details.

To begin, we argue that a subgroup R of Q is weakly $\mathscr{H}$-embedded in G if it is weakly $\mathscr{H}$-embedded in $\mathcal{F}$. Thus, assume that $R \leq Q$ is weakly $\mathscr{H}$-embedded in $\mathcal{F}$. Then, there is a strongly $\mathcal{F}$-closed subgroup P of S such that both RP and $R \cap P$ are strongly $\mathcal{F}$-closed. Set $P_0 := Q \cap P$, and note that $RP_0 = Q \cap RP$ and $R \cap P_0 = R \cap P$. Then, by [14, Proposition 5.27 (i)], each of P_0, RP_0, and $R \cap P_0$ is normal in $\mathcal{F}$. Hence, by [19, Lemma 4.3], each of P_0, RP_0, and $R \cap P_0$ is normal in G. Set $K := P_0 \cap R^G$. Then, $K \trianglelefteq G$, $R^G = R^G \cap RP_0 = RK$, and $R \cap K = R \cap P_0$ is normal in G and thus an $\mathscr{H}$-subgroup of G. Consequently, R is weakly $\mathscr{H}$-embedded in G.

Now, by hypothesis and the preceding paragraph, all subgroups of Q with order $|D|$ or $p|D|$ (and all cyclic subgroups of Q with order 4 if $D = 1$ and Q is a non-abelian 2-group) are weakly $\mathscr{H}$-embedded in G. Applying [33, Proposition 1], we conclude that Q is contained in $Z_{\mathfrak{U}}(G)$, that is, in the supersolvable hypercenter of G. Hence, there is a subgroup series $1 = Q_0 \leq Q_1 \leq \cdots \leq Q_m = Q$ such that $Q_i \trianglelefteq G$ for all $0 \leq i \leq m$ and such that Q_{i+1}/Q_i is cyclic for all $0 \leq i < m$. By [19, Lemma 4.3], Q_i is normal in $\mathcal{F}$ for all $0 \leq i \leq m$. This proves the first statement of the lemma.

The second statement follows immediately from the first one and the definition of a supersolvable fusion system. □

Lemma 2.5. *Let p be a prime, $\mathcal{F}$ be a fusion system on a finite p-group S and $1 < Q < T \leq S$ such that T is minimal strongly $\mathcal{F}$-closed. Then, Q is not weakly $\mathscr{H}$-embedded in $\mathcal{F}$.*

Proof. Assume that Q is weakly $\mathscr{H}$-embedded in $\mathcal{F}$. Then, there is a strongly $\mathcal{F}$-closed subgroup P of S such that both QP and $Q \cap P$ are strongly $\mathcal{F}$-closed. By [14, Lemma 4.59 (ii)], $T \cap QP$ is strongly $\mathcal{F}$-closed. Since T is minimal strongly $\mathcal{F}$-closed and $1 < Q \leq T \cap QP \leq T$, it follows that $T = T \cap QP = Q(P \cap T)$ and hence $P \cap T \neq 1$. As $P \cap T$ is strongly $\mathcal{F}$-closed by [14, Lemma 4.59 (ii)] and T is minimal strongly $\mathcal{F}$-closed, it follows that $T = P \cap T \leq P$. Consequently, $Q = Q \cap P$, and so Q is strongly $\mathcal{F}$-closed. This is a contradiction as T is minimal strongly $\mathcal{F}$-closed, and so the proof is complete. □

Lemma 2.6. *Let p be a prime, $\mathcal{F}$ be a saturated fusion system on a finite p-group S, and T be a strongly $\mathcal{F}$-closed subgroup of $\Phi(S)$ such that $\mathcal{F}/T$ is nilpotent. Then, $\mathcal{F}$ is nilpotent.*

Proof. By [14, Proposition 7.46 (ii)], we have $\mathfrak{hyp}(\mathcal{F}) \leq T$. Then, $\Phi(S)\mathfrak{hyp}(\mathcal{F}) = \Phi(S) = \Phi(S) \cdot 1 = \Phi(S)\mathfrak{hyp}(\mathcal{F}_S(S))$. Applying [17, Theorem T], we conclude that $\mathfrak{hyp}(\mathcal{F}) = \mathfrak{hyp}(\mathcal{F}_S(S)) = 1$. Then, [14, Proposition 7.46 (i)] yields that $\mathcal{F}$ is nilpotent, as required. □

3. Proofs of the main results

To establish our main results, we first prove a number of propositions.

Proposition 3.1. *Let p be a prime and $\mathcal{F}$ be a saturated fusion system on a finite p-group S such that $N_{\mathcal{F}}(S)$ is nilpotent and every maximal subgroup of S is weakly $\mathscr{H}$-embedded in $\mathcal{F}$. Then, $\mathcal{F}$ is nilpotent.*

Proof. Assume that the proposition is false, and let $\mathcal{F}$ be a counterexample for which $|\mathcal{F}|$, that is, the number of morphisms in $\mathcal{F}$, is minimal. We will derive a contradiction in three steps.

(1) *There is a unique minimal strongly $\mathcal{F}$-closed subgroup T of S, and $\mathcal{F}/T$ is nilpotent.*

Pick a minimal strongly $\mathcal{F}$-closed subgroup T of S. By [14, Proposition 5.11], $\mathcal{F}/T$ is saturated. By [14, Exercise 5.11], $N_{\mathcal{F}/T}(S/T) = N_{\mathcal{F}}(S)/T$. Hence, by hypothesis, $N_{\mathcal{F}/T}(S/T) = \mathcal{F}_S(S)/T$. Invoking [14, Theorem 5.20], we obtain that $N_{\mathcal{F}/T}(S/T) = \mathcal{F}_{S/T}(S/T)$, and so $N_{\mathcal{F}/T}(S/T)$ is nilpotent. By Lemma 2.3, any maximal subgroup of S/T is weakly $\mathscr{H}$-embedded in $\mathcal{F}/T$. Consequently, $\mathcal{F}/T$ satisfies the hypothesis of the proposition, and the minimality of $\mathcal{F}$ implies that $\mathcal{F}/T$ is nilpotent.

Suppose that there is a minimal strongly $\mathcal{F}$-closed subgroup T_0 of S with $T \neq T_0$. Then $\mathcal{F}/T_0$ is nilpotent as well. Hence, by [14, Proposition 7.46 (ii)], $\mathfrak{hyp}(\mathcal{F}) \leq T \cap T_0$. Since $T \cap T_0$ is strongly $\mathcal{F}$-closed by [14, Lemma 4.59] and T, T_0 are minimal strongly $\mathcal{F}$-closed, we have $T \cap T_0 = 1$. Then, $\mathfrak{hyp}(\mathcal{F}) = 1$, and so $\mathcal{F}$ is nilpotent by [14, Proposition 7.46 (i)]. This contradiction shows that T is the unique minimal strongly $\mathcal{F}$-closed subgroup of S.

(2) $T \nleq \Phi(S)$.

If $T \leq \Phi(S)$, then $\mathcal{F}$ is nilpotent by (1) and Lemma 2.6, a contradiction. Thus, $T \nleq \Phi(S)$.

(3) *The final contradiction.*

By (2), there is a maximal subgroup Q of S such that $T \nleq Q$. By hypothesis, Q is weakly $\mathscr{H}$-embedded in $\mathcal{F}$. Hence, there is a strongly $\mathcal{F}$-closed subgroup P of S such that both QP and $Q \cap P$ are strongly $\mathcal{F}$-closed.

Note that $|S| > p$; otherwise $S \trianglelefteq \mathcal{F}$ so that $\mathcal{F} = N_{\mathcal{F}}(S)$ would be nilpotent. Hence, $Q \neq 1$.

Assume that $P = 1$. Then, Q is strongly $\mathcal{F}$-closed, and since T is the unique minimal strongly $\mathcal{F}$-closed subgroup of S, it follows that $T \leq Q$, a contradiction. Thus, $P \neq 1$. Then, again because T is the unique minimal strongly $\mathcal{F}$-closed subgroup of S, $T \leq P$.

Assume now that $Q \cap P \neq 1$. Then, since $Q \cap P$ is strongly $\mathcal{F}$-closed, $T \leq Q \cap P \leq Q$, a contradiction. Thus, $Q \cap P = 1$.

We now have $Q \cap T \leq Q \cap P = 1$. Since Q is maximal in S, it follows that $|T| = p$. In particular, by [8, Part I, Corollary 4.7 (a)], $T \trianglelefteq \mathcal{F}$. Since $\mathcal{F}/T$ is nilpotent, we certainly have that $\mathcal{F}/T$ is supersolvable. Applying [9, Lemma 2.7], we conclude that $\mathcal{F}$ is supersolvable. So, by [30, Proposition 2.3], $S \trianglelefteq \mathcal{F}$. Consequently, $\mathcal{F} = N_{\mathcal{F}}(S)$ is nilpotent. This contradiction completes the proof. □

Proposition 3.2. *Let p be a prime and $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Suppose that $N_{\mathcal{F}}(S)$ is nilpotent and that there is a subgroup D of S with $1 \leq D < S$ such that all subgroups of S with order $|D|$ or $p|D|$ (and all cyclic subgroups of S with order 4 if $D = 1$ and S is a non-abelian 2-group) are weakly $\mathscr{H}$-embedded in $\mathcal{F}$. Then, $\mathcal{F}$ is nilpotent.*

Proof. Assume that the proposition is false, and let $\mathcal{F}$ be a counterexample for which $|\mathcal{F}|$ is minimal. Let T be a minimal strongly $\mathcal{F}$-closed subgroup of S. We will derive a contradiction in several steps.

(1) *We have $|T| \leq |D|$ if $D \neq 1$, and $|T| = p$ if $D = 1$.*
This follows from Lemma 2.5.

(2) *D is not maximal in S.*
If D is maximal in S, then any maximal subgroup of S is weakly $\mathscr{H}$-embedded in $\mathcal{F}$, and so $\mathcal{F}$ is nilpotent by Proposition 3.1, a contradiction. Thus, D is not maximal in S.

(3) $\mathfrak{hyp}(\mathcal{F}) = S$.
Assume that $\mathfrak{hyp}(\mathcal{F})$ is a proper subgroup of S, and let Q be a maximal subgroup of S containing $\mathfrak{hyp}(\mathcal{F})$. Let $\mathcal{F}_Q$ denote the unique saturated subsystem of $\mathcal{F}$ on Q of p-power index (see [14, Theorem 7.51 (i)]). By (2), $|D| < |Q|$. By hypothesis, all subgroups of Q with order $|D|$ or $p|D|$ (and all cyclic subgroups of Q with order 4 if $D = 1$ and Q is a non-abelian 2-group) are weakly $\mathscr{H}$-embedded in $\mathcal{F}$, and Lemma 2.2 guarantees that all of them are still weakly $\mathscr{H}$-embedded in $\mathcal{F}_Q$.

Likewise, all subgroups of S with order $|D|$ or $p|D|$ (and all cyclic subgroups of S with order 4 if $D = 1$ and S is a non-abelian 2-group) are weakly $\mathscr{H}$-embedded in $\mathcal{E} := N_{\mathcal{F}}(Q)$. Also, $N_{\mathcal{E}}(S)$ is a saturated subsystem of $N_{\mathcal{F}}(S) = \mathcal{F}_S(S)$, and so we have $N_{\mathcal{E}}(S) = \mathcal{F}_S(S)$.

Assume that $\mathcal{E} \neq \mathcal{F}$. Then, the minimality of $\mathcal{F}$ implies that $\mathcal{E}$ is nilpotent. Since $N_{\mathcal{F}_Q}(Q)$ is a saturated subsystem of $\mathcal{E}$, it follows that $N_{\mathcal{F}_Q}(Q)$ is nilpotent. Hence, together with our above observations concerning $\mathcal{F}_Q$, we see that $\mathcal{F}_Q$ satisfies the hypothesis of the proposition. So, by the minimality of $\mathcal{F}$, we have that $\mathcal{F}_Q$ is nilpotent. Then, by [14, Theorem 7.53 (iv)], $O^p(\mathcal{F}) = O^p(\mathcal{F}_Q) = O^p(\mathcal{F}_Q(Q)) = \mathcal{F}_1(1)$. Thus, $\mathfrak{hyp}(\mathcal{F}) = 1$, and [14, Proposition 7.46 (i)] implies that $\mathcal{F}$ is nilpotent. This contradiction shows that $\mathcal{E} = \mathcal{F}$, whence $Q \trianglelefteq \mathcal{F}$.

Now, by Lemma 2.4, there is a subgroup series $1 = Q_0 \leq Q_1 \leq \cdots \leq Q_m = Q$ such that Q_i is normal in $\mathcal{F}$ for all $0 \leq i \leq m$ and such that Q_{i+1}/Q_i is cyclic for all $0 \leq i < m$. Since Q is maximal in S, it follows that $\mathcal{F}$ is supersolvable. So, by [30, Proposition 2.3], we have $\mathcal{F} = N_{\mathcal{F}}(S) = \mathcal{F}_S(S)$. This contradiction shows that $\mathfrak{hyp}(\mathcal{F}) = S$.

(4) *If $D \neq 1$, then $|T| = |D|$.*

Assume that $D \neq 1$ and $|T| < |D|$. By [14, Exercise 5.11] and [14, Theorem 5.20], $N_{\mathcal{F}/T}(S/T) = N_{\mathcal{F}}(S)/T = \mathcal{F}_S(S)/T = \mathcal{F}_{S/T}(S/T)$. By Lemma 2.3, all subgroups of S/T with order $\frac{|D|}{|T|}$ or $p\frac{|D|}{|T|}$ are weakly $\mathscr{H}$-embedded in $\mathcal{F}/T$. The minimality of $\mathcal{F}$ implies that $\mathcal{F}/T$ is nilpotent. Then, by (3) and [14, Proposition 7.46 (ii)], $S = \mathfrak{hyp}(\mathcal{F}) \leq T$, a contradiction. Together with (1), it follows that $|T| = |D|$ if $D \neq 1$, as claimed.

(5) *For any subgroup H of S with $|H| > |D|$ and $H \nleq O_p(\mathcal{F})$,* $\mathrm{Aut}_{\mathcal{F}}(H)$ *is a p-group.*

Assume that this is not true. Pick a subgroup H of S satisfying $|H| > |D|$ and $H \nleq O_p(\mathcal{F})$ with $\mathrm{Aut}_{\mathcal{F}}(H)$ not being a p-group such that H has maximal possible order. If $L \in H^{\mathcal{F}}$, then we clearly have $|L| = |H| > |D|$, $L \nleq O_p(\mathcal{F})$, and since $\mathrm{Aut}_{\mathcal{F}}(L) \cong \mathrm{Aut}_{\mathcal{F}}(H)$ by [14, Proposition 4.6], $\mathrm{Aut}_{\mathcal{F}}(L)$ is not a p-group. Therefore, upon replacing H with a fully $\mathcal{F}$-normalized member of $H^{\mathcal{F}}$, we may (and will) assume that H is fully $\mathcal{F}$-normalized.

Since $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Aut}_{N_{\mathcal{F}}(S)}(S) = \mathrm{Aut}_{\mathcal{F}_S(S)}(S) = \mathrm{Inn}(S)$, we have $H < S$ and hence $H < N_S(H)$. So, by the maximal choice of H, $\mathrm{Aut}_{\mathcal{F}}(N_S(H))$ is a p-group.

Since H is fully $\mathcal{F}$-normalized, $N_{\mathcal{F}}(H)$ is saturated. As $\mathrm{Aut}_{\mathcal{F}}(N_S(H))$ is a p-group, we have that $\mathrm{Aut}_{N_{\mathcal{F}}(H)}(N_S(H))$ is a p-group, and since $N_{\mathcal{F}}(H)$ is saturated, it follows that $\mathrm{Aut}_{N_{\mathcal{F}}(H)}(N_S(H)) = \mathrm{Inn}(N_S(H))$. Note that any morphism in $N_{N_{\mathcal{F}}(H)}(N_S(H))$ is induced by an automorphism in $\mathrm{Aut}_{N_{\mathcal{F}}(H)}(N_S(H)) = \mathrm{Inn}(N_S(H))$. Hence, we have $N_{N_{\mathcal{F}}(H)}(N_S(H)) = \mathcal{F}_{N_S(H)}(N_S(H))$, and so $N_{N_{\mathcal{F}}(H)}(N_S(H))$ is nilpotent. By hypothesis and Lemma 2.2, all subgroups of $N_S(H)$ with order $|D|$ or $p|D|$ (and all cyclic subgroups of $N_S(H)$ with order 4 if $D = 1$ and $N_S(H)$ is a non-abelian 2-group) are weakly $\mathscr{H}$-embedded in $N_{\mathcal{F}}(H)$. Since $H \nleq O_p(\mathcal{F})$, we have $N_{\mathcal{F}}(H) \neq \mathcal{F}$. So, the minimality of $\mathcal{F}$ implies that $N_{\mathcal{F}}(H)$ is nilpotent.

Consequently, $\mathrm{Aut}_{\mathcal{F}}(H) = \mathrm{Aut}_{N_{\mathcal{F}}(H)}(H) = \mathrm{Aut}_S(H)$ is a p-group. This contradiction shows that (5) holds.

(6) $T \leq O_p(\mathcal{F})$.

Assume that $T \nleq O_p(\mathcal{F})$. Let $T < H \leq S$. As $T \nleq O_p(\mathcal{F})$, we have $H \nleq O_p(\mathcal{F})$. By (4), $|H| > |T| \geq |D|$. So, by (5), $\mathrm{Aut}_{\mathcal{F}}(H)$ is a p-group. This implies that $\mathrm{Aut}_{\mathcal{F}/T}(H/T)$ is a p-group. Applying [8, Part I, Corollary 3.7], we conclude that $\mathcal{F}/T$ is nilpotent. Thus, by (3) and [14, Proposition 7.46 (ii)], $S = \mathfrak{hyp}(\mathcal{F}) \leq T$, which contradicts (1). Thus, $T \leq O_p(\mathcal{F})$.

(7) *The final contradiction.*

Let $O_p(\mathcal{F}) < H \leq S$. By (4) and (6), $|H| > |O_p(\mathcal{F})| \geq |T| \geq |D|$. So, by (5), $\mathrm{Aut}_{\mathcal{F}}(H)$ is a p-group. Arguing as in (6), we see that $\mathcal{F}/O_p(\mathcal{F})$ is nilpotent and that this implies $O_p(\mathcal{F}) = S$. Consequently, $\mathcal{F} = N_{\mathcal{F}}(S) = \mathcal{F}_S(S)$. This final contradiction completes the proof. □

Proposition 3.3. *Let p be a prime and $\mathcal{F}$ be a saturated fusion system on a finite p-group S with $|S| \geq p^2$. Suppose that there is a subgroup D of S with $1 < D < S$ such that every subgroup of S with order $|D|$ is weakly $\mathscr{H}$-embedded in $\mathcal{F}$ and such that, for each fully $\mathcal{F}$-normalized subgroup H of S with $|H| = |D|$, the normalizer $N_{\mathcal{F}}(H)$ is nilpotent. Then, $\mathcal{F}$ is nilpotent.*

Proof. Assume that the proposition is false, and let $\mathcal{F}$ be a counterexample for which $|\mathcal{F}|$ is minimal. Let T be a minimal strongly $\mathcal{F}$-closed subgroup of S. We will derive a contradiction in several steps.

(1) *Let $\mathcal{E}$ be a proper saturated subsystem of $\mathcal{F}$ on a subgroup E of S with $|E| > |D|$. Then, $\mathcal{E}$ is nilpotent.*

By hypothesis and Lemma 2.2, any subgroup of E with order $|D|$ is weakly $\mathscr{H}$-embedded in $\mathcal{E}$. Now, let H be a fully $\mathcal{E}$-normalized subgroup of E with $|H| = |D|$. Choose some $P \in H^{\mathcal{F}}$ such that P is fully $\mathcal{F}$-normalized. By [8, Part I, Lemma 2.6 (c)], there is a morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(H), N_S(P))$ with $H\varphi = P$. Let $X := N_E(H)$, $Y := X\varphi$, and $\tilde{\varphi} := \varphi|_{X,Y}$. For $A, B \leq Y$ and $\psi \in \mathrm{Hom}_{N_{\mathcal{E}}(H)}(A\tilde{\varphi}^{-1}, B\tilde{\varphi}^{-1})$, let $\hat{\psi} := (\tilde{\varphi}^{-1}|_{A,A\tilde{\varphi}^{-1}})\psi(\tilde{\varphi}|_{B\tilde{\varphi}^{-1},B})$, with the notation introduced in Section 1.2. Hence, $\hat{\psi}$ is the $\mathcal{F}$-morphism $A \to B$ sending each $a \in A$ to $((a\tilde{\varphi}^{-1})\psi)\tilde{\varphi}$. Now, let $\mathcal{G}$ be the fusion system on Y such that, for all $A, B \leq Y$,

$$\mathrm{Hom}_{\mathcal{G}}(A, B) = \{\hat{\psi} \mid \psi \in \mathrm{Hom}_{N_{\mathcal{E}}(H)}(A\tilde{\varphi}^{-1}, B\tilde{\varphi}^{-1})\}.$$

By construction, $\mathcal{G}$ is a subsystem of $\mathcal{F}$ isomorphic to $N_{\mathcal{E}}(H)$. Since H is fully $\mathcal{E}$-normalized and $\mathcal{E}$ is saturated, $N_{\mathcal{E}}(H)$ is saturated by [14, Corollary 4.38]. As $\mathcal{G} \cong N_{\mathcal{E}}(H)$, this implies that $\mathcal{G}$ is saturated. Because $H \trianglelefteq N_{\mathcal{E}}(H)$ and $P = H\tilde{\varphi}$, we have $P \trianglelefteq \mathcal{G}$. It follows that $\mathcal{G}$ is a saturated subsystem of $N_{\mathcal{F}}(P)$. Since P is fully $\mathcal{F}$-normalized and $|P| = |H| = |D|$, we have by hypothesis that $N_{\mathcal{F}}(P)$ is nilpotent. As saturated subsystems of nilpotent fusion systems are nilpotent, it follows that $\mathcal{G}$ is nilpotent, and so $N_{\mathcal{E}}(H)$ is nilpotent.

Consequently, $\mathcal{E}$ satisfies the hypothesis of the proposition, and the minimality of $\mathcal{F}$ implies that $\mathcal{E}$ is nilpotent, as wanted.

(2) $|T| \leq |D|$.

This follows from Lemma 2.5.

(3) $\mathcal{F}/T$ *is nilpotent.*

Suppose that $|T| < |D|$. By [14, Proposition 5.11], $\mathcal{F}/T$ is saturated, and by Lemma 2.3, any subgroup of S/T with order $\frac{|D|}{|T|}$ is weakly $\mathscr{H}$-embedded in $\mathcal{F}/T$. Now, let $T < Q < S$ with $|Q/T| = \frac{|D|}{|T|}$ such that Q/T is fully $\mathcal{F}/T$-normalized. Then, $|Q| = |D|$, and by [14, Proposition 5.58 (iii)], Q is fully $\mathcal{F}$-normalized. So, by hypothesis, $N_{\mathcal{F}}(Q)$ is nilpotent. Applying [14, Exercise 5.11] and [14, Theorem 5.20], we conclude that $N_{\mathcal{F}/T}(Q/T) = N_{\mathcal{F}}(Q)/T = \mathcal{F}_{N_S(Q)}(N_S(Q))/T = \mathcal{F}_{N_S(Q)/T}(N_S(Q)/T) = \mathcal{F}_{N_{S/T}(Q/T)}(N_{S/T}(Q/T))$, and so $N_{\mathcal{F}/T}(Q/T)$ is nilpotent. Consequently, $\mathcal{F}/T$ satisfies the hypothesis of the proposition, and the minimality of $\mathcal{F}$ implies that $\mathcal{F}/T$ is nilpotent.

In view of (2), it remains to consider the case $|T| = |D|$. Let $T < Q \leq S$. By [14, Proposition 5.11] and [8, Part I, Corollary 3.7], it is enough to show that $\mathrm{Aut}_{\mathcal{F}/T}(Q/T)$ is a p-group. By [14, Proposition 4.6], we may (and will) assume without loss of generality that Q/T is fully $\mathcal{F}/T$-normalized. Then, by [14, Proposition 5.58 (iii)], Q is fully $\mathcal{F}$-normalized. Therefore, $N_{\mathcal{F}}(Q)$ is a saturated subsystem of $\mathcal{F}$. If $Q \trianglelefteq \mathcal{F}$, then $T \trianglelefteq \mathcal{F}$ by [14, Proposition 5.27 (i)], and since $|T| = |D|$, it follows that $\mathcal{F} = N_{\mathcal{F}}(T)$ is nilpotent; this contradiction shows $Q \ntrianglelefteq \mathcal{F}$. Hence, $N_{\mathcal{F}}(Q)$ is a proper saturated subsystem of $\mathcal{F}$ on $N_S(Q)$, and since $|N_S(Q)| \geq |Q| > |T| = |D|$, it follows from (1) that $N_{\mathcal{F}}(Q)$ is nilpotent. Consequently, $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Aut}_{N_{\mathcal{F}}(Q)}(Q)$ is a p-group. This implies that $\mathrm{Aut}_{\mathcal{F}/T}(Q/T)$ is a p-group, as required.

(4) D *is not maximal in* S.

Assume that D is maximal in S. Hence, every maximal subgroup of S is weakly $\mathscr{H}$-embedded in $\mathcal{F}$. If $N_{\mathcal{F}}(S) \neq \mathcal{F}$, then $N_{\mathcal{F}}(S)$ is nilpotent by (1), and Proposition 3.1 implies that $\mathcal{F}$ is nilpotent, a contradiction. Consequently, we have $S \trianglelefteq \mathcal{F}$. Clearly, S is weakly $\mathscr{H}$-embedded in $\mathcal{F}$, and so we see from Lemma 2.4 that $\mathcal{F}$ is supersolvable. Using [30, Lemma 2.1], we see that there is a strongly $\mathcal{F}$-closed subgroup Q of S such that $|Q| = |D|$. By [14, Proposition 5.27 (i)], Q is normal in $\mathcal{F}$. In particular, Q is fully $\mathcal{F}$-normalized, and so it follows that $\mathcal{F} = N_{\mathcal{F}}(Q)$ is nilpotent. This contradiction shows that D is not maximal in S.

(5) *The final contradiction.*

By (3) and [14, Proposition 7.46 (ii)], we have $\mathfrak{hyp}(\mathcal{F}) \leq T$. In particular, by (2), $\mathfrak{hyp}(\mathcal{F})$ is a proper subgroup of S. Pick a maximal subgroup Q of S containing $\mathfrak{hyp}(\mathcal{F})$, and let $\mathcal{F}_Q$ be the unique saturated subsystem of $\mathcal{F}$ on Q of p-power index (see [14, Theorem 7.51 (i)]). By (4), $|Q| > |D|$, and so (1) implies that $\mathcal{F}_Q$ is nilpotent. Applying [14, Theorem 7.53 (iv)], we conclude that $O^p(\mathcal{F}) = O^p(\mathcal{F}_Q) = O^p(\mathcal{F}_Q(Q)) = \mathcal{F}_1(1)$. Hence, $\mathfrak{hyp}(\mathcal{F}) = 1$. Then, [14, Proposition 7.46 (i)] yields that $\mathcal{F}$ is nilpotent. This contradiction completes the proof. □

We are now in a position to prove our main results.

Proof of Theorem 1.5. (1) ⇒ (2): Suppose that $\mathcal{F}$ is nilpotent so that $\mathcal{F} = \mathcal{F}_S(S)$. Then, $N_{\mathcal{F}}(S) = \mathcal{F}_S(S)$. So, for each subgroup Q of S, we have $(|\mathrm{Aut}_{N_{\mathcal{F}}(S)}(Q)|, p-1) = (|\mathrm{Aut}_{\mathcal{F}_S(S)}(Q)|, p-1) = (|\mathrm{Aut}_S(Q)|, p-1) = (|N_S(Q)/C_S(Q)|, p-1) = 1$. Also, any maximal subgroup of S and S itself are strongly $\mathcal{F}$-closed and hence weakly $\mathscr{H}$-embedded in $\mathcal{F}$. Thus, (2) holds.

(2) ⇒ (3): Suppose that (2) holds. Set $\mathcal{E} := N_{\mathcal{F}}(S)$. By Lemma 2.2, every subgroup of S with order $|D|$ or $p|D|$ (and every cyclic subgroup of S with order 4 if $D = 1$ and S is a non-abelian 2-group) is weakly $\mathscr{H}$-embedded in $\mathcal{E}$. Lemma 2.4 implies that $\mathcal{E}$ is supersolvable.

Let $Q \leq S$. Since $\mathcal{E}$ is supersolvable, by [30, Proposition 1.3], $\mathrm{Aut}_{\mathcal{E}}(Q)$ is p-closed, and a Hall p'-subgroup of $\mathrm{Aut}_{\mathcal{E}}(Q)$ is abelian with exponent dividing $p-1$. Then, as $(|\mathrm{Aut}_{\mathcal{E}}(Q)|, p-1) = 1$, a Hall p'-subgroup of $\mathrm{Aut}_{\mathcal{E}}(Q)$ must be trivial. It follows that $\mathrm{Aut}_{\mathcal{E}}(Q)$ is a p-group. Invoking [8, Part I, Corollary 3.7], we obtain that $\mathcal{E}$ is nilpotent. Consequently, (3) holds.

(3) ⇒ (1): This follows from Proposition 3.2. □

Proof of Theorem 1.6. Assume that $\mathcal{F}$ is nilpotent so that $\mathcal{F} = \mathcal{F}_S(S)$. Then, each maximal subgroup of S is strongly $\mathcal{F}$-closed and hence weakly $\mathscr{H}$-embedded in $\mathcal{F}$. Also, for each maximal subgroup Q of S, $N_{\mathcal{F}}(Q) = \mathcal{F}_S(S)$ is nilpotent. This proves the necessity.

The suffiency follows from Proposition 3.3. □

4. Applications

As an application of Theorem 1.5, we obtain the following characterization of p-nilpotent finite groups.

Corollary 4.1. *Let p be a prime dividing the order of a finite group G and S be a Sylow p-subgroup of G. Then, the following statements are equivalent:*

(1) *G is p-nilpotent.*

(2) *We have $(|\mathrm{Aut}_{N_G(S)}(Q)|, p-1) = 1$ for each subgroup Q of S, and there is a subgroup D of S with $1 \leq D < S$ such that every subgroup of S with order $|D|$ or $p|D|$ (and every cyclic subgroup of S with order 4 if $D = 1$ and S is a non-abelian 2-group) is weakly $\mathscr{H}$-embedded in $\mathcal{F}_S(G)$.*

(3) *$N_G(S)$ is p-nilpotent, and there is a subgroup D of S with $1 \leq D < S$ such that every subgroup of S with order $|D|$ or $p|D|$ (and every cyclic subgroup of S with order 4 if $D = 1$ and S is a non-abelian 2-group) is weakly $\mathscr{H}$-embedded in $\mathcal{F}_S(G)$.*

Proof. Let $\mathcal{F} := \mathcal{F}_S(G)$. By [14, Theorem 4.27 (i)], $\mathrm{Aut}_{N_G(S)}(Q) = \mathrm{Aut}_{\mathcal{F}_S(N_G(S))}(Q) = \mathrm{Aut}_{N_{\mathcal{F}}(S)}(Q)$ for each subgroup Q of S.

(1) $\Rightarrow$ (2): Suppose that G is p-nilpotent. Then, by [14, Theorem 1.12], $\mathcal{F}$ is nilpotent. So, it follows from Theorem 1.5 that (2) holds.

(2) $\Rightarrow$ (3): Suppose that (2) holds. Then, by Theorem 1.5, $N_{\mathcal{F}}(S)$ is nilpotent. Invoking [14, Theorem 1.12] and [14, Theorem 4.27 (i)], we obtain that $N_G(S)$ is p-nilpotent. Hence, (3) holds.

(3) $\Rightarrow$ (1): Suppose that (3) holds. Then, by [14, Theorem 1.12] and [14, Theorem 4.27 (i)], $N_{\mathcal{F}}(S) = \mathcal{F}_S(N_G(S))$ is nilpotent. Theorem 1.5 implies that $\mathcal{F}$ is nilpotent. Applying [14, Theorem 1.12] again, we conclude that G is p-nilpotent, and so (1) holds. □

From Corollary 4.1, we now deduce some previously known results on weakly $\mathscr{H}$-embedded subgroups of finite groups.

Corollary 4.2. *([5, Theorem 1.6]) Let p be a prime,G be a finite group, and S be a Sylow p-subgroup of G. Then, G is p-nilpotent if and only if $N_G(S)$ is p-nilpotent and every maximal subgroup of S is weakly $\mathscr{H}$-embedded in G.*

Proof. Suppose that G is p-nilpotent. Then, $N_G(S)$ is p-nilpotent as well. Also, $\mathcal{F}_S(G) = \mathcal{F}_S(S)$ by [14, Theorem 1.12]. This implies that any maximal subgroup of S is strongly $\mathcal{F}_S(G)$-closed. Consequently, any maximal subgroup of S is strongly closed in G and hence weakly $\mathscr{H}$-embedded in G. This proves the necessity.

Suppose now that $N_G(S)$ is p-nilpotent and that every maximal subgroup of S is weakly $\mathscr{H}$-embedded in G. By Example 1.3, every maximal subgroup of S is weakly $\mathscr{H}$-embedded in $\mathcal{F}_S(G)$. Clearly, S is weakly $\mathscr{H}$-embedded in $\mathcal{F}_S(G)$ as well. Corollary 4.1 implies that G is p-nilpotent. This proves the sufficiency. □

Corollary 4.3. *([5, Theorem 1.8]) Let p be the smallest prime dividing the order of a finite group G and S be a Sylow p-subgroup of G. Suppose that every cyclic subgroup of S with order p or 4 (if S is a non-abelian 2-group) is weakly $\mathscr{H}$-embedded in G. Then, G is p-nilpotent.*

Proof. For each $Q \leq S$, we have $(|\mathrm{Aut}_{N_G(S)}(Q)|, p-1) = (|N_{N_G(S)}(Q)/C_{N_G(S)}(Q)|, p-1) = 1$ because p is the smallest prime dividing $|G|$. By hypothesis and Example 1.3, every subgroup of S with order p (and every cyclic subgroup of S with order 4 if S is a non-abelian 2-group) is weakly $\mathscr{H}$-embedded in $\mathscr{F}_S(G)$. Hence, Corollary 4.1 implies that G is p-nilpotent. □

Corollary 4.4. *([7, Theorem A]) Let p be the smallest prime dividing the order of a finite group G and S be a non-cyclic Sylow p-subgroup of G. Then, G is p-nilpotent if and only if there is a subgroup D of S with $1 < D < S$ such that every subgroup of S with order $|D|$ or $p|D|$ is weakly $\mathscr{H}$-embedded in G.*

Proof. The necessity can be obtained similarly as in the proof of Corollary 4.2, and the sufficiency can be obtained as in the proof of Corollary 4.3. □

As an application of Theorem 1.6, we obtain the following characterization of p-nilpotent finite groups.

Corollary 4.5. *Let p be a prime, G be a finite group and S be a Sylow p-subgroup of G such that $|S| \geq p^2$. Then, G is p-nilpotent if and only if there is a subgroup D of S with $1 < D < S$ such that every subgroup H of S with order $|H| = |D|$ is weakly $\mathscr{H}$-embedded in $\mathcal{F}_S(G)$ and $N_G(H)$ is p-nilpotent.*

Proof. Assume that G is p-nilpotent. Then, by Corollary 4.2 and Example 1.3, any maximal subgroup of S is weakly $\mathscr{H}$-embedded in $\mathcal{F}_S(G)$. Moreover, for any maximal subgroup H of S, $N_G(H)$ is p-nilpotent. This proves the necessity.

Suppose now that there is a subgroup D of S with $1 < D < S$ such that every subgroup H of S with order $|H| = |D|$ is weakly $\mathscr{H}$-embedded in $\mathcal{F}_S(G)$ and $N_G(H)$ is p-nilpotent. Let H be a fully $\mathcal{F}_S(G)$-normalized subgroup H of S with $|H| = |D|$. By [14, Proposition 1.38], $N_S(H) \in \mathrm{Syl}_p(N_G(H))$. Then, since $N_G(H)$ is p-nilpotent, it follows from [14, Theorem 1.12] and [14, Theorem 4.27 (i)] that $N_{\mathcal{F}_S(G)}(H) = \mathcal{F}_{N_S(H)}(N_G(H)) = \mathcal{F}_{N_S(H)}(N_S(H))$. Applying Theorem 1.6, we conclude that $\mathcal{F}_S(G)$ is nilpotent. So, G is p-nilpotent by [14, Theorem 1.12]. This proves the sufficiency. □

As an immediate consequence of Corollary 4.5 and Example 1.3, we obtain the following result of Asaad [3] about weakly $\mathscr{H}$-embedded subgroups of finite groups.

Corollary 4.6. *([3, Theorem 1.1]) Let p be a prime and S be a Sylow p-subgroup of a finite group G. Suppose that there exists a subgroup D of S with $1 < D < S$ such that every subgroup H of S with $|H| = |D|$ is weakly $\mathscr{H}$-embedded in G and $N_G(H)$ is p-nilpotent. Then, G is p-nilpotent.*

As another application, we now give a new proof of a result of Craven and Glesser [15] concerning the nilpotency of saturated fusion systems on certain metacyclic 2-groups.

For each natural number $n \geq 4$, let

$$M_n(2) := \langle a, b \mid a^{2^{n-1}} = 1 = b^2, a^b = a^{1+2^{n-2}} \rangle.$$

By [15, Proposition 3.1], every saturated fusion system on one of the groups $M_n(2)$, where $n \geq 4$, is nilpotent. Roughly speaking, this is proved in [15] by observing that $\mathrm{Aut}(M_n(2))$ is a 2-group and applying Alperin's fusion theorem.

Based on Theorem 1.5 we provide here a very different proof that does not rely on the structure of the automorphism group of $M_n(2)$.

Corollary 4.7. *Let $n \geq 4$ be a natural number and $\mathcal{F}$ be a saturated fusion system on $M_n(2)$. Then, $\mathcal{F}$ is nilpotent.*

Proof. Let $S := M_n(2) = \langle a, b \rangle$, where a and b satisfy the appropriate relations. Set $x := a^{2^{n-3}}$ and $z := x^2 = a^{2^{n-2}}$. Then, $\mathrm{ord}(x) = 4$ and $\mathrm{ord}(z) = 2$. Notice that

$$(a^2)^b = b^{-1}a^2b = b^{-1}abb^{-1}ab = (a^b)^2 = (a^{1+2^{n-2}})^2 = a^2$$

so that $a^2 \in Z(S)$. In particular, we have $x, z \in Z(S)$. Using this, we see that $\langle z \rangle \langle b \rangle \cong C_2 \times C_2$ and $\langle x \rangle \langle b \rangle \cong C_4 \times C_2$.

Clearly, $\langle z \rangle \langle b \rangle$ and $\langle x \rangle \langle b \rangle$ are contained in $\Omega_1(S)$ and $\Omega_2(S)$, respectively. By [23, Chapter 5, Theorem 4.3(i)(c)], $\Omega_1(S) \cong C_2 \times C_2$ and $\Omega_2(S) \cong C_4 \times C_2$. So it follows that $\Omega_1(S) = \langle z \rangle \langle b \rangle$ and $\Omega_2(S) = \langle x \rangle \langle b \rangle$.

We claim that $\Omega_1(S)$ and $\Omega_2(S)$ are strongly $\mathcal{F}$-closed. Indeed, let R be a subgroup of $\Omega_1(S) \cong C_2 \times C_2$ or $\Omega_2(S) \cong C_4 \times C_2$ and $\varphi : R \to S$ be an $\mathcal{F}$-morphism. If $R \leq \Omega_1(S)$, then R is generated by elements of order at most 2, and so the same is true for $R\varphi \cong R$, whence $R\varphi \leq \Omega_1(S)$. Similarly, if $R \leq \Omega_2(S)$, then R is generated by elements of order at most 4, and so the same is true for $R\varphi$, whence $R\varphi \leq \Omega_2(S)$. Consequently, $\Omega_1(S)$ and $\Omega_2(S)$ are strongly $\mathcal{F}$-closed, as claimed. Since they are abelian, we see from [8, Part I, Corollary 4.7 (a)] that they are in fact normal in $\mathcal{F}$. Then, by [8, Part I, Proposition 4.4], $\Phi(\Omega_2(S)) = \Phi(\langle x\rangle\langle b\rangle) = \langle x^2\rangle = \langle z\rangle$ is also normal in $\mathcal{F}$.

We show now that $\mathcal{F}$ satisfies Condition (2) from Theorem 1.5. First of all, for each subgroup Q of S, we have $(|\mathrm{Aut}_{N_{\mathcal{F}}(S)}(Q)|, 2-1) = (|\mathrm{Aut}_{N_{\mathcal{F}}(S)}(Q)|, 1) = 1$. Next, we proceed to show that every cyclic subgroup of S with order 2 or 4 is weakly $\mathscr{H}$-embedded in $\mathcal{F}$. Thus, let Q be a cyclic subgroup of S with $|Q| = 2$ or 4. If $Q = \langle z\rangle$, then Q is normal and hence strongly closed in $\mathcal{F}$ by the preceding paragraph, and Example 1.2 implies that Q is weakly $\mathscr{H}$-embedded in $\mathcal{F}$. If $|Q| = 2$ and $Q \neq \langle z\rangle$, then Q is weakly $\mathscr{H}$-embedded in $\mathcal{F}$ because each of $\langle z\rangle$, $Q\langle z\rangle = \Omega_1(S)$ and $Q \cap \langle z\rangle = 1$ is strongly $\mathcal{F}$-closed. Finally, if $|Q| = 4$, then Q must be one of $\langle x\rangle$ or $\langle xb\rangle$ since these are the only cyclic subgroups of order 4 of $\Omega_2(S) = \langle x\rangle\langle b\rangle \cong C_4 \times C_2$. In either case, we see that each of $\Omega_1(S)$, $Q\Omega_1(S) = \Omega_2(S)$ and $Q \cap \Omega_1(S) = \langle z\rangle$ is strongly $\mathcal{F}$-closed so that Q is weakly $\mathscr{H}$-embedded in $\mathcal{F}$.

Applying Theorem 1.5, we conclude that $\mathcal{F}$ is nilpotent, and so the proof is complete. □

5. Conclusions

In this paper, we generalized the concept of strong closure in fusion systems by introducing the concept of weak $\mathscr{H}$-embedding in fusion systems. Given a prime p and a fusion system $\mathcal{F}$ on a finite p-group S, we say that a subgroup Q of S is weakly $\mathscr{H}$-embedded in $\mathcal{F}$ if there is a strongly $\mathcal{F}$-closed subgroup P of S such that both QP and $Q\cap P$ are strongly $\mathcal{F}$-closed. Using this concept, we established new characterizations of nilpotent fusion systems.

Our work is motivated by a current line of research in finite group theory, which investigates the structure of finite groups by means of generalized strongly closed subgroups. In this context, Asaad and Ramadan defined a subgroup H of a finite group G to be weakly $\mathscr{H}$-embedded in G if there is a normal subgroup K of G with $H^G = HK$ such that $H \cap K$ is strongly closed in G. Our definition of weakly $\mathscr{H}$-embedded subgroups of fusion systems is motivated by this concept. It turns out that, given a prime p and a Sylow p-subgroup S of finite group G, any weakly $\mathscr{H}$-embedded subgroup of G contained in S is weakly $\mathscr{H}$-embedded in $\mathcal{F}_S(G)$. It may happen, however, that a subgroup of S is weakly $\mathscr{H}$-embedded in $\mathcal{F}_S(G)$, but not in G.

As is well-known, if p is a prime and S is a Sylow p-subgroup of a finite group G, then G is p-nilpotent if and only if $\mathcal{F}_S(G)$ is nilpotent. Combining this fact and our main results, we obtained new characterizations of p-nilpotent finite groups. From these characterizations, we deduced a number of previously known p-nilpotency criteria in terms of weakly $\mathscr{H}$-embedded subgroups of finite groups.

The proofs of our main results are based on key concepts from the theory of fusion systems, such as hyperfocal subgroups, subsystems of p-power index, and supersolvable fusion systems. While many nilpotency criteria for fusion systems depend on the model theorem for constrained fusion systems, we did not need to use this quite deep result.

Our paper follows a line of research that investigates saturated fusion systems $\mathcal{F}$ in which certain subgroups are required to satisfy specific conditions with respect to $\mathcal{F}$. We are confident that the ideas and methods developed in our paper will be widely applicable in this area, particularly in establishing nilpotency criteria in a similar spirit to ours.

Author contributions

The authors contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest in this paper.

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