Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(8): 24691–24716.
DOI: 10.3934/math.2026994
Received: 29 May 2026
Revised: 26 July 2026
Accepted: 03 August 2026
Published: 12 August 2026

Research article

Long-time dynamics of a fractional viscoelastic equation with infinite memory

Junfeng Li, Yang Liu*, **Qiong Li, and Yan Xu**

College of Mathematics and Computer Science, Northwest Minzu University, Lanzhou 730030, China

* **Correspondence:** Email: liuyangnufn@163.com; 183091699@xbmu.edu.cn.

Abstract: This work was concerned with a fractional viscoelastic equation with infinite memory. We first proved the global well-posedness of solutions in a suitable function space. Moreover, we derived a stabilizability estimate by using the perturbed energy method. Then, by utilizing the quasi-stability of the corresponding gradient dynamical system, we found the existence of global attractors with finite fractal dimension.

Keywords: fractional viscoelastic equation; global well-posedness; global attractors; gradient dynamical system; quasi-stability
Mathematics Subject Classification: 35R11, 35A01, 35B30, 35B41, 37N15

1. Introduction

The interplay between spatial nonlocality and temporal memory is fundamental in the modeling of complex viscoelastic materials. While the fractional Laplacian captures long-range interactions and the memory kernel encodes the Boltzmann relaxation mechanism, their combined effect on long-time dynamics remains largely unexplored. In this paper, we study the following initial-boundary value problem for a fractional viscoelastic equation with infinite memory:

$$u_{tt} + (-\Delta)^s u - \int_{-\infty}^{t} g(t-\tau)(-\Delta)^s u(\tau)\,\mathrm{d}\tau + (-\Delta)^s u_t + f(u) = \varphi(x), \quad x \in \Omega,\ t > 0, \tag{1.1}$$

$$u(x,t) = 0, \quad x \in \mathbb{R}^N \backslash \Omega,\ t \in \mathbb{R}, \tag{1.2}$$

$$u(x,t) = u_0(x,t), \quad u_t(x,0) = u_1(x), \quad x \in \Omega,\ t \le 0, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ $(N \ge 1)$ is a bounded domain with the Lipschitz boundary. The fractional Laplacian $(-\Delta)^s$ is defined by

$$(-\Delta)^s u(x) := 2 \lim_{r\to 0^+} \int_{\mathbb{R}^N \backslash B_r(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,\mathrm{d}y$$

for any $u \in C_0^\infty\left(\mathbb{R}^N\right)$, where $0 < s < 1$, and $B_r(x)$ is a ball with $x \in \mathbb{R}^N$ as the center and $r > 0$ as the radius.

Problem (1.1)–(1.3) governs the small-amplitude vibrations of the viscoelastic string (see [1]). In this problem, spatial nonlocality and temporal memory appear simultaneously, which can provide an accurate characterization, in both space and time domains, for the nonlocal phenomena in the small-amplitude vibrations of the viscoelastic string. The unknown function $u := u(x, t)$ represents the vertical displacement, $-\int_{-\infty}^{t} g(t-\tau)(-\Delta)^s u(\tau)\,\mathrm{d}\tau$ is the memory term, $(-\Delta)^s u_t$ is the damping term, $u_0 : \Omega \times (-\infty, 0] \to \mathbb{R}$ is a prescribed past history, and $u_1(x) = \left.\frac{\partial u_0(x,t)}{\partial t}\right|_{t=0}$. The memory kernel g, the nonlinear function f, and the external force φ will be specified later.

Fractional partial differential equations (PDEs) have been widely applied in many fields, such as game theory [2], fractional quantum mechanics [3, 4], thin obstacle problems [5], and conformal geometry [6]. Next, we review some work on fractional PDEs. Yang and Lü [7] studied the following fractional equation:

$$(-\Delta)^s u = f(u). \tag{1.4}$$

Under some assumptions on the nonlinear function f, they proved the radial symmetry, monotonicity, and regularity of positive solutions by utilizing the method of moving planes in integral forms. Moreover, they obtained the nonexistence of positive solutions. In the case where $f(u) = u^p$, Eq (1.4) becomes

$$(-\Delta)^s u = u^p. \tag{1.5}$$

Chen et al. [8] studied Eq (1.5), where $p > 1$ is a growth exponent satisfying certain assumptions. They found the nonexistence of positive solutions by using the method of moving planes in integral forms. Subsequently, Zhuo and Li [9] obtained the existence and nonexistence of nontrivial solutions for Eq (1.5) with $p > 0$ satisfying some assumptions. In the case where $f(u) = -u^{-p}$, He et al. [10] investigated Eq (1.4), where $p > 1$. Utilizing blow-up analysis and Liouville-type theorems, they proved an optimal Hölder regularity estimate for the solutions. In addition, they established a convergence result of sequences of solutions with uniform Hölder continuity. Du et al. [11] considered a fractional Schrödinger equation with trapping potential,

$$(-\Delta)^s u + V(x)u = \omega u + af(u),$$

where $\omega \in \mathbb{R}$ and $a > 0$ are two constants, and V is an external potential function satisfying some assumptions. Applying the fractional Gagliardo-Nirenberg-Sobolev inequality and scaling techniques, they proved the existence and nonexistence of minimizers. Moreover, they obtained the concentration behavior of minimizers by utilizing the energy estimates. In the case where $f(u) = |u|^{2s}u$, Liu et al. [12] investigated the following fractional Schrödinger equation with trapping potential:

$$(-\Delta)^s u + V(x)u = \omega u + a|u|^{2s}u, \tag{1.6}$$

where $\frac{1}{2} < s < 1$ and V satisfies suitable assumptions. They derived the concentration behavior of each minimizer by utilizing some energy estimates and analyzing decay properties of solution sequences. In addition, by establishing a local Pohožaev identity and analyzing the blow-up estimates on the fractional Laplacian $(-\Delta)^s$, they found the local uniqueness of the minimizers. Liu et al. [13] also investigated Eq (1.6), where V is given by a specific form. Studying some estimates on the least energy of the mass critical fractional Schrödinger energy functional, they found the concentration behavior of each minimizer. In recent years, qualitative properties of solutions for fractional evolution equations have been extensively investigated. Younes et al. [14] studied a fractional evolution equation with a logarithmic source term,

$$u_{tt} + (-\Delta)^s u + (-\Delta)^s u_t = |u|^{p-2} u \ln |u|,$$

where $2 < p < \frac{2N}{N-2s}$ and $N > 2s$. By the potential well theory (see, e.g., [15, 16]) and the Galerkin approximations, they derived the global existence of solutions with positive initial energy. Under the effect of more general damping, Bui et al. [17] considered the following fractional evolution equation:

$$u_{tt} + (-\Delta)^s u + \nu(-\Delta)^\alpha u_t = 0,$$

where $s \geq 1$, $0 \leq \alpha \leq s$, and $\nu > 0$ is a constant. By discussing the relationship between s, α, and N, they not only obtained the asymptotic profile, optimal decay rate, and infinite time blow-up of the solutions, but also found the global existence of solutions with sufficiently small initial data. In the case where $\nu = 1$, Chen and Girardi [18] analyzed a fractional evolution equation with a nonlinear source term:

$$u_{tt} + (-\Delta)^s u + (-\Delta)^\alpha u_t = |u|^p \phi(|u|),$$

where $s \geq 1$, $0 \leq \alpha \leq s$, $p > 1$, and ϕ is a modulus of continuity. They provided a threshold condition on ϕ, namely, the Dini condition [19, p. 101], which separates the global existence of small data solutions from the finite time blow-up of local solutions. Additionally, they obtained the polynomial decay of solutions, and gave the estimates on the upper and lower bounds for the blow-up time. Boudjeriou [20] dealt with the following fractional q-Laplacian evolution equation with a logarithmic source term:

$$u_{tt} + (-\Delta)_q^s u = |u|^{p-2} u \ln |u|,$$

where $2 \leq q < \frac{Nq}{N-sq}$, $1 < p < \frac{Nq}{N-sq}$, $N > sq$, and the fractional q-Laplacian $(-\Delta)_q^s$ is defined by

$$(-\Delta)_q^s u(x) := 2 \lim_{r \to 0^+} \int_{\mathbb{R}^N \setminus B_r(x)} \frac{|u(x) - u(y)|^{q-2}(u(x) - u(y))}{|x - y|^{N+sq}} \, \mathrm{d}y$$

for any $u \in C_0^\infty\left(\mathbb{R}^N\right)$. When $q = 2$, the fractional q-Laplacian $(-\Delta)_q^s$ reduces to the fractional Laplacian $(-\Delta)^s$ considered in [14, 17, 18]. He found the local existence of solutions by utilizing the Galerkin approximations. Based on the potential well theory, he proved the global existence and decay of solutions with positive initial energy. Moreover, he derived the finite time blow-up of solutions with negative initial energy. Considering the damping effect, Wu and You [21] studied

$$u_{tt} + (-\Delta)_q^s u + (-\Delta)^\alpha u_t = |u|^{p-2} u \ln |u|,$$

where $2 \leq q \leq p \leq \frac{Nq}{N-sq}$, $N > sq$, and $0 < \alpha < 1$. They proved the local existence of solutions by using the Galerkin approximations. By the potential well theory, they obtained the global existence and finite time blow-up of solutions with positive initial energy. In addition, they derived the polynomial decay of energy, and provided the estimates on the upper bound for the blow-up time.

Nonlinear wave phenomena arise in a wide range of physical contexts, from coastal hydrodynamics [22] to the dynamics of viscoelastic materials. Viscoelastic materials possess the mechanical properties of both elastic solids and viscous fluids. Boltzmann theory shows that the stress depends not only on the present values of strain or the velocity gradient, but also on the entire temporal history of motion. This phenomenon is commonly called the memory effect. We observe that the past history u_0 accounts for all deformations the material has undergone before the initial time $t = 0$. Viscoelastic problems with past history are significant, as they offer a framework for characterizing materials with historical memory and revealing the essence of the memory effect. Accordingly, the study of vibrations of the viscoelastic string has important physical significance. In the past, under appropriate assumptions on the memory kernel g (and the nonlinear function f), the qualitative properties of solutions for many viscoelastic equations with infinite memory were investigated. Pata [23] considered an abstract viscoelastic equation:

$$u_{tt}(t) + Au(t) - \int_0^\infty g(\tau)Au(t-\tau)\,\mathrm{d}\tau = 0, \quad t > 0, \tag{1.7}$$

where A is a self-adjoint linear operator satisfying certain assumptions. He obtained the exponential decay of the solution semigroup. Guesmia and Messaoudi [24] derived the general decay of solutions for Eq (1.7). Pata [25] studied an abstract viscoelastic equation with a weak damping term:

$$u_{tt}(t) + \mu_1 Au(t) - \int_0^\infty g(\tau)Au(t-\tau)\,\mathrm{d}\tau + \mu_2 u_t(t) = 0,$$

where $\mu_1 > 0$ and $\mu_2 \geq 0$ are two constants. He found the exponential decay of the solution semigroup. Guesmia [26] dealt with an abstract viscoelastic equation with a time delay term:

$$u_{tt}(t) + Au(t) - \int_0^\infty g(\tau)Au(t-\tau)\,\mathrm{d}\tau + \mu_3 u_t(t-s) = 0,$$

where $\mu_3 \in \mathbb{R}$ is a constant. By semigroup theory, he derived the global well-posedness of solutions. Under a smallness condition on $|\mu_3|$, he established the exponential decay of the solutions. In the case where $A = -\Delta$, Guo et al. [27] studied the following viscoelastic equation with nonlinear weak damping and nonlinear source terms:

$$u_{tt} - \mu_1 \Delta u + \int_0^\infty g(\tau)\Delta u(t-\tau)\,\mathrm{d}\tau + h(u_t) = f(u), \tag{1.8}$$

where $\mu_1 > 1$, and h is a continuous and monotone increasing function satisfying certain assumptions. Based on the theory of monotone operators and nonlinear semigroups, combined with the energy method, they obtained the local well-posedness of the solutions. Moreover, they found the global existence of the solutions. In the case where $h(u_t) = |u_t|^{m-1} u_t$ and $f(u) = |u|^{p-1}u$, Guo et al. [28] derived the finite time blow-up of solutions for Eq (1.8), where $\mu_1 > 1$, $1 \leq p < 6$, and $m \geq 1$ is a growth exponent satisfying suitable assumptions. Subsequently, Guo et al. [29] studied Eq (1.8) again, where $h(u_t)$ and $f(u)$ are given by the same specific forms as in [28]. By the potential well theory, they obtained the global existence of solutions. In addition, they found the exponential and polynomial decay of the energy.

It is well-known that the attractors can effectively characterize the long-time dynamics of nonlinear evolution equations. To date, global attractors for viscoelastic equations with infinite memory have been investigated by several authors. Zhang et al. [30] handled a viscoelastic equation with dispersion, weak damping, nonlinear source, and external force terms:

$$u_{tt} - \Delta u_{tt} - \mu_1 \Delta u + \int_0^\infty g(\tau)\Delta u(t-\tau)\,\mathrm{d}\tau + \mu_2 u_t + f(u) = \varphi(x), \tag{1.9}$$

where $\mu_1 \geq 1$ and $0 \leq \mu_2 < 1$. They proved the existence of global attractors by verifying the existence of a bounded absorbing set and the asymptotic compactness of the semigroup. In addition, they obtained the upper semicontinuity of global attractors when $\mu_2 \to 0$. In fact, through a direct calculation, the memory term $\int_0^\infty g(\tau)\Delta u(t-\tau)\,\mathrm{d}\tau$ in Eq (1.9) can be rewritten as $\int_{-\infty}^t g(t-\tau)\Delta u(\tau)\,\mathrm{d}\tau$. Oliveira Araújo et al. [31] studied the following viscoelastic equation with dispersion, strong damping, nonlinear source, and external force terms:

$$|u_t|^\rho u_{tt} - \Delta u_{tt} - \mu_1 \Delta u + \int_{-\infty}^t g(t-\tau)\Delta u(\tau)\,\mathrm{d}\tau - \mu_4 \Delta u_t + f(u) = \varphi(x), \tag{1.10}$$

where $\mu_1 \geq 1$, $\rho > 0$ is an exponent satisfying certain assumptions, and $\mu_4 \geq 0$ is a constant. They obtained the global well-posedness of solutions and the exponential decay of the energy. Moreover, they derived the existence of global attractors by verifying the existence of a bounded absorbing set and the asymptotic smoothness. In the case where $\mu_1 = 1$ and $\mu_4 = 0$, Conti et al. [32] studied Eq (1.10) with $0 \leq \rho \leq 4$. They proved the exponential decay of the energy. In addition, they obtained the existence of global attractors by verifying the gradient property and the asymptotic compactness, and found the optimal regularity of global attractors by using a bootstrap result.

In the case where $\rho = 0$, $\mu_1 = 1$, and $\mu_4 = 1$, and neglecting the dispersion effect, Eq (1.10) reduces to

$$u_{tt} - \Delta u + \int_{-\infty}^t g(t-\tau)\Delta u(\tau)\,\mathrm{d}\tau - \Delta u_t + f(u) = \varphi(x). \tag{1.11}$$

Equation (1.1) can be regarded as a fractional version of Eq (1.11). The fractional Laplacian $(-\Delta)^s$ is nonlocal, which can provide a more accurate characterization for the nonlocal phenomena in the small-amplitude vibrations of the viscoelastic string. Thus, a natural question is whether the dynamical system corresponding to problem (1.1)–(1.3) possesses global attractors like [31, 32]. However, to the best of our knowledge, much less effort has been devoted to the study of the existence of global attractors of the dynamical system corresponding to problem (1.1)–(1.3). In the present paper, we would like to conduct this study. The nonlocality of the fractional Laplacian prevents us from investigating the long-time dynamics of problem (1.1)–(1.3) within the conventional integer-order Sobolev spaces adopted in [31, 32]. To address this issue, we introduce a dedicated function space where the nonlocal fractional derivative acts as a well-defined bounded operator, establishing a rigorous framework for the subsequent energy estimates. Nevertheless, this brings about a defect: the associated space embeddings lack sufficient compactness. As a consequence, the regularity of solutions is weaker than that of the solutions considered in [31, 32]. The regularity loss raises the difficulty in the proof of the existence of global attractors. To overcome this difficulty, we carry out the proof of the existence of global attractors by verifying the gradient property and quasi-stability of the dynamical system discussed by Chueshov and Lasiecka [33], which is the technical framework of this paper. This technical framework does not strongly rely on the regularity of solutions. Compared with [32], the regularity loss renders the operator decomposition technique adopted in [32] no longer applicable to the present work, which further makes the verification of the asymptotic compactness infeasible within our framework. In contrast to the approach in [31], a practical advantage of the technical framework used in this paper lies in its exemption from the requirement of demonstrating the existence of a bounded absorbing set. Therefore, our methodology exhibits excellent targeted adaptability to problem (1.1)–(1.3).

The rest of this paper is organized as follows. In Section 2, we display some preliminary knowledge on function space, notations, and assumptions on the memory kernel g and the nonlinear function f. Moreover, we transform problem (1.1)–(1.3) into an equivalent form in the history space framework [27, 34, 35]. In Section 3, we prove the global well-posedness of the solutions. In Section 4, we obtain the existence of global attractors with finite fractal dimension. Lastly, we summarize our work in Section 5.

2. Preliminaries

2.1. Function space

According to [36, 37], we define by Y the linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function u in Y belongs to $L^2(\Omega)$ and

$$\iint_G \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y < \infty, \tag{2.1}$$

where $G := \mathbb{R}^{2N}\backslash(C\Omega\times C\Omega)$ and $C\Omega := \mathbb{R}^N\backslash\Omega$. According to (2.1), the space Y is equipped with the norm

$$\|u\|_Y := \|u\|_{L^2(\Omega)} + \left(\iint_G \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}. \tag{2.2}$$

It can be easily shown that (2.2) is a norm on Y. Moreover, we introduce the space

$$Y_0 := \{u \in Y \mid u = 0 \text{ a.e. in } C\Omega\}, \tag{2.3}$$

the inner product

$$(u,\eta)_{Y_0} := \iint_G \frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y, \tag{2.4}$$

and the norm

$$\|u\|_{Y_0} := \left(\iint_G \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}. \tag{2.5}$$

By virtue of (2.4) and (2.5), we deduce that (2.3) is a Hilbert space. In fact, (2.5) is equivalent to (2.2).

Theorem 2.1 ([36, 37])**.** *The embedding $Y_0 \hookrightarrow L^q(\Omega)$ is continuous for any $1 \le q \le 2_s^*$ and compact for any $1 \le q < 2_s^*$, where*

$$2_s^* = \begin{cases} \dfrac{2N}{N-2s} & \text{if } N > 2s, \\ \infty & \text{if } N \le 2s. \end{cases}$$

2.2. Notations and assumptions

Throughout the paper, for simplicity, we denote

$$\|\cdot\|_q := \|\cdot\|_{L^q(\Omega)}, \ \ \|\cdot\| := \|\cdot\|_2, \ \ (u,\eta) := \int_\Omega u\eta \,\mathrm{d}x.$$

Moreover, $C > 0$ represents a generic constant whose value may differ at each occurrence even within the same formula, and $C(\cdot) > 0$ stands for a constant depending on the quantity appearing in the parentheses.

As in [32, 38, 39], we make the following assumptions on the memory kernel g.

(A_1) $g \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, $g(t) \geq 0$, and $g'(t) \leq 0$ for all $t \in [0,\infty)$.

$$m := 1 - \int_0^\infty g(t)\,\mathrm{d}t > 0. \tag{2.6}$$

We see from (A_1) that the memory kernel g is nonnegative and nonincreasing, which is consistent with the feature of the stress relaxation of viscoelastic strings. Moreover, (A_1) is the fundamental assumption under which the memory effect produces energy dissipation.

Inspired by [39–41], we adopt the following assumptions on the nonlinear function f.

(A_2) $f \in C^1(\mathbb{R})$, and there exists a constant $a > 0$ such that

$$|f'(u)| \leq a\left(|u|^{p-2} + 1\right), \ \ \forall u \in \mathbb{R}, \tag{2.7}$$

where

$$2 < p < \infty \text{ if } N \leq 2s, \text{ and } 2 < p < \frac{2N-2s}{N-2s} \text{ if } N > 2s.$$

In addition, there exist constants $0 \leq \kappa < \frac{m}{S^2}$ and $\lambda > 0$ such that

$$uf(u) \geq -\kappa u^2 - \lambda \tag{2.8}$$

and

$$F(u) \geq -\frac{\kappa}{2}u^2 - \lambda, \tag{2.9}$$

where S is the best Sobolev constant for the embedding $Y_0 \hookrightarrow L^2(\Omega)$ and $F(u) := \int_0^u f(s)\,\mathrm{d}s$.

2.3. Reformulation of the problem

Following [27, 34, 35], we introduce the auxiliary variable

$$v := v^t(x,\tau) := u(x,t) - u(x,t-\tau), \ \ x \in \Omega, \ \tau > 0, \ t \geq 0. \tag{2.10}$$

It follows from (2.6) in (A_1) and (2.10) that the memory term in Eq (1.1) can be rewritten as

$$\begin{aligned} -\int_{-\infty}^t g(t-\tau)(-\Delta)^s u(\tau)\,\mathrm{d}\tau &= -\int_0^\infty g(\tau)(-\Delta)^s u(t-\tau)\,\mathrm{d}\tau \\ &= (m-1)(-\Delta)^s u + \int_0^\infty g(\tau)(-\Delta)^s v(\tau)\,\mathrm{d}\tau. \end{aligned}$$

Hence, problem (1.1)–(1.3) is transformed into the following equivalent problem:

$$\begin{cases} u_{tt} + m(-\Delta)^s u + \int_0^\infty g(\tau)(-\Delta)^s v(\tau)\, \mathrm{d}\tau + (-\Delta)^s u_t + f(u) = \varphi(x), \quad x \in \Omega,\ t > 0, \\ v_t = u_t - v_\tau, \quad x \in \Omega,\ \tau > 0,\ t > 0, \end{cases} \tag{2.11}$$

$$\begin{cases} u(x,t) = 0, \quad x \in \mathbb{R}^N \backslash \Omega,\ t > 0, \\ v^t(x,\tau) = 0, \quad x \in \mathbb{R}^N \backslash \Omega,\ \tau > 0,\ t > 0, \end{cases} \tag{2.12}$$

$$\begin{cases} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ v^0(x,\tau) = v_0(x,\tau), \quad x \in \Omega,\ \tau > 0, \end{cases} \tag{2.13}$$

where

$$u_0(x) := u_0(x,0), \quad x \in \Omega,$$

$$v_0(x,\tau) := u_0(x,0) - u_0(x,-\tau), \quad x \in \Omega,\ \tau > 0.$$

We introduce the weighted L^2-space

$$L_g := L_g^2(\mathbb{R}^+; Y_0) := \left\{ v : \mathbb{R}^+ \to Y_0 \ \middle|\ \int_0^\infty g(\tau)\|v(\tau)\|_{Y_0}^2 \,\mathrm{d}\tau < \infty \right\},$$

which is a Hilbert space equipped with the inner product

$$(v,\eta)_g := \int_0^\infty g(\tau)(v(\tau),\eta(\tau))_{Y_0}\, \mathrm{d}\tau$$

and the norm

$$\|v\|_g^2 := \int_0^\infty g(\tau)\|v(\tau)\|_{Y_0}^2\, \mathrm{d}\tau.$$

Definition 2.2 (Weak solutions)**.** *For given $T > 0$, a triple of functions (u, u_t, v) is called a weak solution of problem* (2.11)–(2.13)*, if $u \in C([0,T]; Y_0)$, $u_t \in C\left([0,T]; L^2(\Omega)\right) \cap L^2(0,T;Y_0)$, $v \in C\left([0,T]; L_g\right)$, $u(0) = u_0$ in Y_0, $u_t(0) = u_1$ in $L^2(\Omega)$, $v^0 = v_0$ in L_g, and*

$$\begin{cases} (u_t(t),\eta_1) + m\int_0^t (u(s),\eta_1)_{Y_0}\ \mathrm{d}s + \int_0^t (v^s,\eta_1)_g\ \mathrm{d}s \\ \qquad + (u(t),\eta_1)_{Y_0} + \int_0^t (f(u(s)),\eta_1)\ \mathrm{d}s \\ \qquad = \int_0^t (\varphi,\eta_1)\ \mathrm{d}s + (u_1,\eta_1) + (u_0,\eta_1)_{Y_0}, \\ (v^t,\eta_2)_g = (u(t),\eta_2)_g - (u_0,\eta_2)_g - \int_0^t (v_\tau^s,\eta_2)_g\ \mathrm{d}s + (v_0,\eta_2)_g \end{cases} \tag{2.14}$$

for any $\eta_1 \in Y_0$, $\eta_2 \in L_g$, and $t \in (0,T]$.

Remark 2.3. *System* (2.14) *implies that*

$$\begin{cases} \langle u_{tt}(t),\eta_1\rangle + m\,(u(t),\eta_1)_{Y_0} + (v^t,\eta_1)_g + (u_t(t),\eta_1)_{Y_0} + (f(u(t)),\eta_1) = (\varphi,\eta_1), \\ (v_t^t,\eta_2)_g = (u_t(t),\eta_2)_g - (v_\tau^t,\eta_2)_g \end{cases} \tag{2.15}$$

for a.e. $t \in (0,T]$, where $\langle\cdot,\cdot\rangle$ represents the duality pairing between Y_0 and its dual space.

For the convenience of the reader, we summarize in Table 1 the main symbols and notations used in the paper.

Table 1. List of main symbols.

Symbol	Description
g	Memory kernel
f	Nonlinear function
φ	External force term
C	Positive generic constant whose value may differ at each occurrence even within the same formula
$C(\cdot)$	Positive generic constant depending on the quantity appearing in the parentheses
m	Positive constant $m := 1 - \int_0^\infty g(t)\,\mathrm{d}t$
p	Growth exponent
S	Best Sobolev constant for the embedding $Y_0 \hookrightarrow L^2(\Omega)$
F	Primitive of f, i.e., $F(u) := \int_0^u f(s)\,\mathrm{d}s$
v	Auxiliary variable $v := v^t(x,\tau) := u(x,t) - u(x,t-\tau)$
L_g	$L_g := \left\{v : \mathbb{R}^+ \to Y_0 \;\middle|\; \int_0^\infty g(\tau)\|v(\tau)\|_{Y_0}^2\,\mathrm{d}\tau < \infty\right\}$

3. Global well-posedness

The aim of this section is to establish the global well-posedness of solutions for problem (2.11)–(2.13), namely, the global existence, uniqueness, and continuous dependence on initial data.

To handle the nonlinearity in problem (2.11)–(2.13), we have the following lemma.

Lemma 3.1. *Let f satisfy* (A_2). *Then, for any $u(x,t)$ and $v(x,t)$ with $(x,t) \in \Omega \times [0,T)$, it holds that*

$$|f(u) - f(v)| \le a\left((|u| + |v|)^{p-2} + 1\right)|u - v|.$$

Proof. We write $\tilde{u} := u - v$. Then, by the property of the Gâteaux differential, we have

$$f(u) - f(v) = f(v + \tilde{u}) - f(v) = \int_0^1 Df(v + s\tilde{u}; \tilde{u})\,\mathrm{d}s,$$

where $Df(v + s\tilde{u}; \tilde{u})$ is the Gâteaux differential of f at $v + s\tilde{u}$ in the direction $\tilde{u}$. By virtue of

$$Df(v + s\tilde{u}; \tilde{u}) = \lim_{\tau\to 0} \frac{f(v + s\tilde{u} + \tau\tilde{u}) - f(v + s\tilde{u})}{\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau} f(v + s\tilde{u} + \tau\tilde{u})\Big|_{\tau=0},$$

we further infer from (2.7) that

$$\begin{aligned}
|f(u) - f(v)| &= \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau} f(v + s\tilde{u} + \tau\tilde{u})\Big|_{\tau=0}\,\mathrm{d}s\right| \\
&\le a\int_0^1 \left(|v + s\tilde{u}|^{p-2} + 1\right)|u - v|\,\mathrm{d}s \\
&= a\int_0^1 \left(|su + (1-s)v|^{p-2} + 1\right)|u - v|\,\mathrm{d}s \\
&\le a\left((|u| + |v|)^{p-2} + 1\right)|u - v|.
\end{aligned}$$

$\square$

Next, we give the global well-posedness of solutions for problem (2.11)–(2.13).

Theorem 3.2 (Global well-posedness). *Let* (A_1) *and* (A_2) *be fulfilled. Assume that* $\varphi \in L^2(\Omega)$ *and* $(u_0, u_1, v_0) \in \mathcal{H} := Y_0 \times L^2(\Omega) \times L_g$*. Then, for any* $T > 0$*, problem* (2.11)–(2.13) *admits a unique solution* (u, u_t, v) *depending continuously on the initial data.*

Proof. We divide the proof of this theorem into four steps.

Step 1. Galerkin approximations.

Let $\left\{e_j\right\}_{j=1}^{\infty}$ be an orthogonal basis of Y_0 and an orthonormal basis of $L^2(\Omega)$ given by eigenfunctions of $(-\Delta)^s$ with (2.12) (see [36, Proposition 9] for details). As in [42,43], we select $\left\{w_j\right\}_{j=1}^{\infty}$ of the form $\left\{l_k\tilde{e}_j\right\}_{k,j=1}^{\infty}$, where $\{l_k\}_{k=1}^{\infty}$ is an orthonormal basis of $L_g^2(\mathbb{R}^+) \cap C_0^{\infty}(\mathbb{R}^+)$ and $\tilde{e}_j = \frac{e_j}{\|e_j\|_{Y_0}}$. Then, $\left\{w_j\right\}_{j=1}^{\infty}$ is an orthonormal basis of L_g.

We construct the approximate solutions of problem (2.11)–(2.13):

$$u_n(t) := \sum_{j=1}^{n} \xi_{jn}(t)e_j, \quad v_n^t(\tau) := \sum_{j=1}^{n} \zeta_{jn}(t)w_j(\tau), \quad n = 1, 2, \cdots,$$

which satisfy

$$\begin{cases} \left\langle u_{ntt}(t), e_j\right\rangle + m\left(u_n(t), e_j\right)_{Y_0} + \left(v_n^t, e_j\right)_g + \left(u_{nt}(t), e_j\right)_{Y_0} + \left(f(u_n(t)), e_j\right) = \left(\varphi, e_j\right), \\ \left(v_{nt}^t, w_j\right)_g = \left(u_{nt}(t), w_j\right)_g - \left(v_{n\tau}^t, w_j\right)_g, \quad j = 1, 2, \cdots, n, \end{cases} \tag{3.1}$$

with

$$\begin{cases} u_n(0) = \displaystyle\sum_{j=1}^{n} \xi_{jn}(0)e_j \to u_0 \text{ in } Y_0, \\ u_{nt}(0) = \displaystyle\sum_{j=1}^{n} \xi'_{jn}(0)e_j \to u_1 \text{ in } L^2(\Omega), \\ v_n^0 = \displaystyle\sum_{j=1}^{n} \zeta_{jn}(0)w_j(\tau) \to v_0 \text{ in } L_g \end{cases} \tag{3.2}$$

as $n \to \infty$. The approximate problem (3.1)–(3.2) can be transformed into an ordinary differential system with respect to the variables $\xi_{jn}(t)$ and $\zeta_{jn}(t)$. According to the standard theory for ordinary differential equations (ODEs), there exists a solution $(u_n(t), u_{nt}(t), v_n^t)$ on some interval $[0, T_n)$ with $T_n \le T$.

Step 2. A priori estimates.

Multiplying $\xi'_{jn}(t)$ in $(3.1)_1$ and $\zeta_{jn}(t)$ in $(3.1)_2$, summing j, and adding the two results, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_n(t) + \|u_{nt}(t)\|_{Y_0}^2 = -\left(v_{n\tau}^t, v_n^t\right)_g, \tag{3.3}$$

where

$$E_n(t) := \frac{1}{2}\|u_{nt}(t)\|^2 + \frac{m}{2}\|u_n(t)\|_{Y_0}^2 + \frac{1}{2}\left\|v_n^t\right\|_g^2 + \int_{\Omega} F\left(u_n(t)\right) \mathrm{d}x - \left(\varphi, u_n(t)\right). \tag{3.4}$$

In view of (2.10), we obtain

$$\lim_{\tau\to 0} v_n^t(x,\tau) = 0. \tag{3.5}$$

From the assumptions $g'(t) \le 0$ and $g(t) \ge 0$ in (A_1), we have

$$\lim_{t\to\infty} g(t) = 0. \tag{3.6}$$

Hence, from (3.5), (3.6), and the assumption $g'(t) \le 0$ in (A_1), we arrive at

$$-\left(v_{n\tau}^t, v_n^t\right)_g = -\frac{1}{2}\int_0^\infty \frac{\partial}{\partial\tau}\left(g(\tau)\left\|v_n^t(\tau)\right\|_{Y_0}^2\right) \mathrm{d}\tau + \frac{1}{2}\int_0^\infty g'(\tau)\left\|v_n^t(\tau)\right\|_{Y_0}^2 \mathrm{d}\tau \le 0.$$

Consequently, by integrating (3.3) with respect to t, we obtain

$$E_n(t) + \int_0^t \|u_{ns}(s)\|_{Y_0}^2 \,\mathrm{d}s \le E_n(0) \tag{3.7}$$

for all $t \in [0, T_n)$.

For the fourth term on the right-hand side of (3.4), we deduce from (2.9) in (A_2) and Theorem 2.1 that

$$\begin{aligned}\int_\Omega F\left(u_n(t)\right) \mathrm{d}x &\ge -\frac{\kappa}{2}\int_\Omega |u_n(t)|^2 \,\mathrm{d}x - \lambda|\Omega| \\ &\ge -\frac{\kappa S^2}{2}\|u_n(t)\|_{Y_0}^2 - \lambda|\Omega|. \end{aligned} \tag{3.8}$$

Concerning the last term on the right-hand side of (3.4), it follows from Schwarz's inequality, Theorem 2.1, and Cauchy's inequality with $\epsilon > 0$ that

$$\begin{aligned}-(\varphi, u_n(t)) &\ge -\|\varphi\|\,\|u_n(t)\| \\ &\ge -\frac{1}{4\epsilon}\|\varphi\|^2 - \epsilon S^2\|u_n(t)\|_{Y_0}^2. \end{aligned} \tag{3.9}$$

Inserting (3.8) and (3.9) into (3.4), we obtain

$$E_n(t) \ge \frac{1}{2}\|u_{nt}(t)\|^2 + \left(\frac{m}{2} - \frac{\kappa S^2}{2} - \epsilon S^2\right)\|u_n(t)\|_{Y_0}^2 + \frac{1}{2}\left\|v_n^t\right\|_g^2 - \lambda|\Omega| - \frac{1}{4\epsilon}\|\varphi\|^2$$

for all $t \in [0, T_n)$. Choosing ϵ small enough such that

$$M_1 := \frac{m}{2} - \frac{\kappa S^2}{2} - \epsilon S^2 > 0$$

and

$$M_2 := \lambda|\Omega| + \frac{1}{4\epsilon}\|\varphi\|^2 > 0,$$

we obtain

$$E_n(t) \ge \frac{1}{2}\|u_{nt}(t)\|^2 + M_1\|u_n(t)\|_{Y_0}^2 + \frac{1}{2}\left\|v_n^t\right\|_g^2 - M_2 \tag{3.10}$$

for all $t \in [0, T_n)$. Consequently, from (3.7), (3.10), and (3.2), we have

$$\|u_{nt}(t)\|^2 + \|u_n(t)\|_{Y_0}^2 + \left\|v_n^t\right\|_g^2 + \int_0^t \|u_{ns}(s)\|_{Y_0}^2 \, \mathrm{d}s \le C \tag{3.11}$$

for all $t \in [0, T_n)$, where C is independent of n and t. It follows from (3.11) that the approximate solution $\left(u_n(t), u_{nt}(t), v_n^t\right)$ can be extended to the interval $[0, T]$ for any $T > 0$.

Step 3. Passage to the limit.

From estimate (3.11), there exist subsequences of $\{u_n\}$, $\{v_n\}$ (always relabeled as the same notation) and functions u, v such that, for any $T > 0$,

$$u_n \rightharpoonup u \text{ weakly star in } L^\infty(0, T; Y_0),$$

$$u_{nt} \rightharpoonup u_t \text{ weakly star in } L^\infty\left(0, T; L^2(\Omega)\right) \cap L^2(0, T; Y_0),$$

and

$$v_n \rightharpoonup v \text{ weakly star in } L^\infty\left(0, T; L_g\right)$$

as $n \to \infty$. Applying the Aubin-Lions lemma [44, Chapter 1, Theorem 5.1], we obtain

$$u_n \to u \text{ in } L^2(0, T; Y_0). \tag{3.12}$$

We now claim that, for all $t \in [0, T]$ and fixed j,

$$\int_0^t \left(f(u_n(s)), e_j\right) \mathrm{d}s \to \int_0^t \left(f(u(s)), e_j\right) \mathrm{d}s \tag{3.13}$$

as $n \to \infty$. Indeed, from Lemma 3.1, Hölder's inequality, Theorem 2.1, and (3.11), we obtain

$$\begin{aligned}
&\left|\left(f(u_n(t)) - f(u(t)), e_j\right)\right| \\
\le& C \int_\Omega \left(|u_n(t)|^{p-2} + |u(t)|^{p-2} + 1\right) |u_n(t) - u(t)| \left|e_j\right| \mathrm{d}x \\
\le& C \left(\|u_n(t)\|_{2p-2}^{p-2} + \|u(t)\|_{2p-2}^{p-2} + |\Omega|^{\frac{p-2}{2p-2}}\right) \|u_n(t) - u(t)\|_{2p-2} \left\|e_j\right\| && (3.14) \\
\le& C \left(\|u_n(t)\|_{Y_0}^{p-2} + \|u(t)\|_{Y_0}^{p-2} + |\Omega|^{\frac{p-2}{2p-2}}\right) \|u_n(t) - u(t)\|_{Y_0} \left\|e_j\right\| && (3.15) \\
\le& C \|u_n(t) - u(t)\|_{Y_0}.
\end{aligned}$$

Thus,

$$\left|\int_0^t \left(f(u_n(s)) - f(u(s)), e_j\right) \mathrm{d}s\right| \le C \int_0^t \|u_n(s) - u(s)\|_{Y_0} \, \mathrm{d}s,$$

which together with (3.12) yields assertion (3.13).

For fixed j, we infer from (3.5) and (3.6) that

$$\left(v_{n\tau}^t, w_j\right)_g = -\int_0^\infty g'(\tau) \left(v_n^t(\tau), w_j(\tau)\right)_{Y_0} \mathrm{d}\tau - \int_0^\infty g(\tau) \left(v_n^t(\tau), w_{j\tau}(\tau)\right)_{Y_0} \mathrm{d}\tau.$$

Further,

$$\lim_{n\to\infty} \left(v_{n\tau}^t, w_j\right)_g = \left(v_\tau^t, w_j\right)_g.$$

Consequently, by integrating (3.1) with respect to t and taking $n \to \infty$, we have

$$
\begin{cases}
\left(u_t(t), e_j\right) + m \int_0^t \left(u(s), e_j\right)_{Y_0} \mathrm{d}s + \int_0^t \left(v^s, e_j\right)_g \mathrm{d}s \\
\qquad + \left(u(t), e_j\right)_{Y_0} + \int_0^t \left(f(u(s)), e_j\right) \mathrm{d}s \\
\qquad = \int_0^t \left(\varphi, e_j\right) \mathrm{d}s + \left(u_1, e_j\right) + \left(u_0, e_j\right)_{Y_0}, \\
\left(v^t, w_j\right)_g = \left(u(t), w_j\right)_g - \left(u_0, w_j\right)_g - \int_0^t \left(v^s_\tau, w_j\right)_g \mathrm{d}s + \left(v_0, w_j\right)_g
\end{cases}
$$

for all $t \in [0, T]$. By virtue of $u \in L^\infty(0, T; Y_0)$ and $u_t \in L^\infty\left(0, T; L^2(\Omega)\right) \cap L^2(0, T; Y_0)$, we derive $u \in C([0, T]; Y_0)$ and $u_t \in C\left([0, T]; L^2(\Omega)\right) \cap L^2(0, T; Y_0)$. In view of [45, Theorem 3.2], we have $v \in C\left([0, T]; L_g\right)$. In addition, we infer from (3.2) that $u(0) = u_0$ in Y_0, $u_t(0) = u_1$ in $L^2(\Omega)$, and $v^0 = v_0$ in L_g. Thus, (u, u_t, v) is a global solution of problem (2.11)–(2.13).

Step 4. Continuous dependence and uniqueness.

Suppose that (u, u_t, v) and $(\bar{u}, \bar{u}_t, \bar{v})$ are two solutions of problem (2.11)–(2.13) with initial data (u_0, u_1, v_0) and $(\bar{u}_0, \bar{u}_1, \bar{v}_0)$, respectively. Denote $\tilde{u}(t) := \bar{u}(t) - u(t)$ and $\tilde{v}^t := \bar{v}^t - v^t$. Then, $(\tilde{u}, \tilde{u}_t, \tilde{v})$ satisfies

$$
\begin{cases}
\langle \tilde{u}_{tt}(t), \eta_1 \rangle + m\left(\tilde{u}(t), \eta_1\right)_{Y_0} + \left(\tilde{v}^t, \eta_1\right)_g + \left(\tilde{u}_t(t), \eta_1\right)_{Y_0} + \left(f(\bar{u}(t)) - f(u(t)), \eta_1\right) = 0, \\
\left(\tilde{v}^t_t, \eta_2\right)_g = \left(\tilde{u}_t(t), \eta_2\right)_g - \left(\tilde{v}^t_\tau, \eta_2\right)_g,
\end{cases}
\tag{3.16}
$$

$$
\begin{cases}
\tilde{u}(x, 0) = \tilde{u}_0(x), \quad \tilde{u}_t(x, 0) = \tilde{u}_1(x), \quad x \in \Omega, \\
\tilde{v}^0(x, \tau) = \tilde{v}_0(x, \tau), \quad x \in \Omega, \ \tau > 0.
\end{cases}
\tag{3.17}
$$

Taking $\eta_1 = \tilde{u}_t(t)$ in $(3.16)_1$ and $\eta_2 = \tilde{v}^t$ in $(3.16)_2$, and adding the two results, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{u}_t(t)\|^2 + m\|\tilde{u}(t)\|^2_{Y_0} + \left\|\tilde{v}^t\right\|^2_g\right) + \|\tilde{u}_t(t)\|^2_{Y_0} \\
=& \frac{1}{2}\int_0^\infty g'(\tau)\left\|\tilde{v}^t(\tau)\right\|^2_{Y_0} \mathrm{d}\tau - \left(f(\bar{u}(t)) - f(u(t)), \tilde{u}_t(t)\right).
\end{aligned}
\tag{3.18}
$$

Concerning the first term on the right-hand side of (3.18), it follows from the assumption $g'(t) \le 0$ in (A_1) that

$$
\int_0^\infty g'(\tau)\left\|\tilde{v}^t(\tau)\right\|^2_{Y_0} \mathrm{d}\tau \le 0.
\tag{3.19}
$$

For the last term on the right-hand side of (3.18), we conclude from the arguments similar to the proof of (3.15) that

$$
-\left(f(\bar{u}(t)) - f(u(t)), \tilde{u}_t(t)\right) \le C\left(\|\bar{u}(t)\|^{p-2}_{Y_0} + \|u(t)\|^{p-2}_{Y_0} + |\Omega|^{\frac{p-2}{2p-2}}\right)\|\tilde{u}(t)\|_{Y_0}\|\tilde{u}_t(t)\|.
$$

From Cauchy's inequality, we further derive

$$
-\left(f(\bar{u}(t)) - f(u(t)), \tilde{u}_t(t)\right) \le C\|\tilde{u}(t)\|^2_{Y_0} + \frac{1}{2}\|\tilde{u}_t(t)\|^2.
\tag{3.20}
$$

Substituting (3.19) and (3.20) into (3.18), we infer from (2.6) in (A_1) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{u}_t(t)\|^2 + m\,\|\tilde{u}(t)\|_{Y_0}^2 + \left\|\tilde{v}^t\right\|_g^2\right) \le C\left(\|\tilde{u}_t(t)\|^2 + m\,\|\tilde{u}(t)\|_{Y_0}^2 + \left\|\tilde{v}^t\right\|_g^2\right).$$

It follows from Gronwall's inequality and (3.17) that

$$\|\tilde{u}_t(t)\|^2 + m\,\|\tilde{u}(t)\|_{Y_0}^2 + \left\|\tilde{v}^t\right\|_g^2 \le \mathrm{e}^{CT}\left(\|\tilde{u}_1\|^2 + m\,\|\tilde{u}_0\|_{Y_0}^2 + \|\tilde{v}_0\|_g^2\right)$$

for all $t \in [0, T]$.

In particular, by taking $(u_0, u_1, v_0) = (\bar{u}_0, \bar{u}_1, \bar{v}_0)$, we know that (u, u_t, v) is the unique solution of problem (2.11)–(2.13). □

Define the operator $S(t) : \mathcal{H} \to \mathcal{H}$ by

$$S(t)y_0 := \left(u(t), u_t(t), v^t\right), \quad y_0 := (u_0, u_1, v_0).$$

Then, we deduce from Theorem 3.2 that $\{S(t)\}_{t\ge 0}$ is a C^0-semigroup generated by problem (2.11)–(2.13).

4. Global attractors

4.1. Preliminaries on global attractors

We will prove the existence of global attractors for problem (1.1)–(1.3) by verifying the gradient property and quasi-stability of the corresponding dynamical system $(\mathcal{H}, S(t))$. For the convenience of the reader, we give several definitions and properties of a gradient dynamical system and the quasi-stability from [33].

According to [33, Definition 7.5.3], we have the following definition of a gradient dynamical system.

Definition 4.1 ([33]). *A dynamical system $(\mathcal{H}, S(t))$ is said to be a gradient if there exists a strict Lyapunov function L for $(\mathcal{H}, S(t))$ on the whole phase space $\mathcal{H}$, that is,*

(i) the function $t \mapsto L(S(t)y_0)$ is nonincreasing for any $y_0 \in \mathcal{H}$;
(ii) the equation $L(S(t)y_0) = L(y_0)$ for all $t \in (0, \infty)$ and some $y_0 \in \mathcal{H}$ means that $S(t)y_0 = y_0$ for all $t \in (0, \infty)$.

Under suitable conditions, we give the existence and structure of global attractors for a gradient and asymptotically smooth dynamical system from [33, Corollary 7.5.7].

Theorem 4.2 ([33]). *Assume that $(\mathcal{H}, S(t))$ is a gradient and asymptotically smooth dynamical system, and its Lyapunov function $L(\chi)$ is bounded from above on any bounded subset of $\mathcal{H}$. Moreover, assume that the set $L_R := \{\chi \in \mathcal{H} \mid L(\chi) \le R\}$ is bounded for every R. If the set $\mathcal{N}$ of stationary points of $(\mathcal{H}, S(t))$, that is,*

$$\mathcal{N} := \{v \in \mathcal{H} \mid S(t)v = v \text{ for all } t \in [0, \infty)\},$$

is bounded, then $(\mathcal{H}, S(t))$ has a compact global attractor $\mathcal{A} = \mathcal{M}^z(\mathcal{N})$, where $\mathcal{M}^z(\mathcal{N})$ is an unstable manifold emanating from the set $\mathcal{N}$ as a set of all $y_0 \in \mathcal{H}$ such that there exists a full trajectory $\{z(t) \mid t \in \mathbb{R}\}$ with the properties $z(0) = y_0$ and $\lim\limits_{t\to-\infty} \mathrm{dist}_{\mathcal{H}}(z(t), \mathcal{N}) = 0$.

For the purpose of introducing the quasi-stability of a dynamical system, we provide the following assumptions.

(H) Let U, V, and W be three reflexive Banach spaces with U compactly embedded in V. We endow the space $\mathcal{H} := U \times V \times W$ with the norm

$$\|(u(t), u_t(t), v(t))\|^2_{\mathcal{H}} := \|u(t)\|^2_U + \|u_t(t)\|^2_V + \|v(t)\|^2_W, \quad (u(t), u_t(t), v(t)) \in \mathcal{H}.$$

We suppose that $(\mathcal{H}, S(t))$ is a dynamical system on $\mathcal{H}$ with an evolution operator of the form

$$S(t)y_0 := (u(t), u_t(t), v(t)), \quad y_0 := (u_0, u_1, v_0) \in \mathcal{H},$$

where

$$u \in C(\mathbb{R}^+; U) \cap C^1(\mathbb{R}^+; V), \quad v \in C(\mathbb{R}^+; W).$$

According to [33, Definition 7.9.2], we state the definition of the quasi-stability for a dynamical system.

Definition 4.3 ([33]). *A dynamical system* $(\mathcal{H}, S(t))$ *fulfilling* (H) *is said to be quasi-stable on a set* $B \subset \mathcal{H}$ *if there exist a compact seminorm* $n_U(\cdot)$ *on the space* U *and nonnegative functions* $\psi_i(t)$ $(i = 1, 2, 3)$ *such that*

(i) $\psi_1(t)$ *and* $\psi_3(t)$ *are locally bounded on* $[0, \infty)$*;*
(ii) $\psi_2 \in L^1(\mathbb{R}^+)$ *and* $\lim_{t\to\infty} \psi_2(t) = 0$*;*
(iii) the relations

$$\|S(t)\bar{y}_0 - S(t)y_0\|^2_{\mathcal{H}} \leq \psi_1(t)\, \|\bar{y}_0 - y_0\|^2_{\mathcal{H}} \tag{4.1}$$

and

$$\|S(t)\bar{y}_0 - S(t)y_0\|^2_{\mathcal{H}} \leq \psi_2(t)\, \|\bar{y}_0 - y_0\|^2_{\mathcal{H}} + \psi_3(t) \sup_{0<s<t} n^2_U(\bar{u}(s) - u(s)) \tag{4.2}$$

hold for every $y, \bar{y}_0 \in B$ *and* $t \in (0, \infty)$, *where* $S(t)\bar{y}_0 := (\bar{u}(t), \bar{u}_t(t), \bar{v}(t))$ *and* $\bar{y}_0 := (\bar{u}_0, \bar{u}_1, \bar{v}_0)$.

For a quasi-stable dynamical system, we give two properties from [33, Proposition 7.9.4 and Theorem 7.9.6], namely, Proposition 4.4 and Theorem 4.5.

Proposition 4.4 ([33]). *Let* (H) *be fulfilled. Assume that a dynamical system* $(\mathcal{H}, S(t))$ *is quasi-stable on every bounded positively invariant set* B *of* $\mathcal{H}$*. Then,* $(\mathcal{H}, S(t))$ *is asymptotically smooth.*

Theorem 4.5 ([33]). *Let* (H) *be fulfilled. Assume that a dynamical system* $(\mathcal{H}, S(t))$ *has a compact global attractor* $\mathcal{A}$ *and is quasi-stable on* $\mathcal{A}$*. Then,* $\mathcal{A}$ *has finite fractal dimension.*

By Theorem 4.2, it can be seen that Proposition 4.4 plays a crucial role in in the proof of the existence of global attractors. We emphasize that the stringency of assumption (H) restricts the scope of application of Proposition 4.4, so this proposition cannot be applied to a large class of other evolution equations. Besides the viscoelastic equations considered in this paper, this proposition is also available for hyperbolic-hyperbolic coupled systems with infinite memory, as well as hyperbolic-parabolic coupled systems (either with or without infinite memory).

4.2. Existence of global attractors

We define the total energy functional

$$E(t) := E\left(u(t), u_t(t), v^t\right)$$
$$:= \frac{1}{2}\left\|u_t(t)\right\|^2 + \frac{m}{2}\|u(t)\|_{Y_0}^2 + \frac{1}{2}\left\|v^t\right\|_g^2 + \int_\Omega F(u(t))\,\mathrm{d}x - (\varphi, u(t)), \quad t \geq 0. \tag{4.3}$$

The following lemma shows that $E(t)$ is nonincreasing.

Lemma 4.6. *Under the assumptions of Theorem 3.2, $E(t)$ is nonincreasing, and*

$$E'(t) = -\left\|u_t(t)\right\|_{Y_0}^2 + \frac{1}{2}\int_0^\infty g'(\tau)\left\|v^t(\tau)\right\|_{Y_0}^2\,\mathrm{d}\tau. \tag{4.4}$$

Proof. Taking $\eta_1 = u_t(t)$ in $(2.15)_1$ and $\eta_2 = v^t$ in $(2.15)_2$, and adding the two results, we derive (4.4). In view of the assumption $g'(t) \leq 0$ in (A_1), we have

$$\int_0^\infty g'(\tau)\left\|v^t(\tau)\right\|_{Y_0}^2\,\mathrm{d}\tau \leq 0. \tag{4.5}$$

Consequently, $E'(t) \leq 0$, that is, $E(t)$ is nonincreasing. □

Next, under the additional assumption that the memory kernel g decays exponentially, we demonstrate that the dynamical system $(\mathcal{H}, S(t))$ corresponding to problem (2.11)–(2.13) is gradient.

Lemma 4.7 (Gradient property)**.** *In addition to the assumptions of Theorem 3.2, assume that there exists a constant $\alpha > 0$ such that $g'(t) + \alpha g(t) \leq 0$ for all $t \in [0, \infty)$. Then, the dynamical system $(\mathcal{H}, S(t))$ corresponding to problem (2.11)–(2.13) is gradient.*

Proof. For any $y_0 \in \mathcal{H}$, we take $L(S(t)y_0)$ as $E(t)$. Then, we infer from Lemma 4.6 that $L(S(t)y_0)$ is nonincreasing.

Let $L(S(t)y_0) = L(y_0)$ for all $t \in (0, \infty)$ and some $y_0 \in \mathcal{H}$. Then, by integrating (4.4) with respect to t, we derive

$$E(t) + \int_0^t \|u_s(s)\|_{Y_0}^2\,\mathrm{d}s - \frac{1}{2}\int_0^t\int_0^\infty g'(\tau)\,\|v^s(\tau)\|_{Y_0}^2\,\mathrm{d}\tau\mathrm{d}s = E(0) \tag{4.6}$$

for all $t \in [0, \infty)$. Hence,

$$\int_0^t \|u_s(s)\|_{Y_0}^2\,\mathrm{d}s - \frac{1}{2}\int_0^t\int_0^\infty g'(\tau)\,\|v^s(\tau)\|_{Y_0}^2\,\mathrm{d}\tau\mathrm{d}s = 0,$$

which together with the assumption $g'(t) \leq 0$ in (A_1) yields

$$\int_0^t \|u_s(s)\|_{Y_0}^2\,\mathrm{d}s = 0$$

and

$$-\frac{1}{2}\int_0^t\int_0^\infty g'(\tau)\,\|v^s(\tau)\|_{Y_0}^2\,\mathrm{d}\tau\mathrm{d}s = 0. \tag{4.7}$$

From the assumption $g'(t) + \alpha g(t) \leq 0$, we obtain

$$-\frac{1}{2}\int_0^t \int_0^\infty g'(\tau)\,\|v^s(\tau)\|_{Y_0}^2\ \mathrm{d}\tau\mathrm{d}s \geq \frac{\alpha}{2}\int_0^t \|v^s\|_g^2\ \mathrm{d}s,$$

which combined with (4.7) yields $v^t = 0$ for all $t \in [0,\infty)$. From (2.10), we further derive $u(t) = u_0$ for all $t \in [0,\infty)$. Consequently, $(u(t), u_t(t), v^t) = (u_0, 0, 0)$, i.e., $S(t)y_0 = y_0$ for all $t \in [0,\infty)$. By virtue of Definition 4.1, we know that $(\mathcal{H}, S(t))$ is gradient. □

For problem (2.11)–(2.13), we denote

$$\|S(t)y_0\|_{\mathcal{H}}^2 := \|u_t(t)\|^2 + m\|u(t)\|_{Y_0}^2 + \left\|v^t\right\|_g^2.$$

Next, we establish a stabilizability estimate by using the perturbed energy method [39,41,46].

Lemma 4.8 (Stabilizability estimate)**.** *Under the assumptions of Lemma 4.7, for a given bounded set $B \subset \mathcal{H}$, there exist constants $\beta, \gamma > 0$ and $\theta > 0$ depending on B such that*

$$\|S(t)\bar{y}_0 - S(t)y_0\|_{\mathcal{H}}^2 \leq \beta \mathrm{e}^{-\gamma t}\|\bar{y}_0 - y_0\|_{\mathcal{H}}^2 + \theta\int_0^t \mathrm{e}^{-\gamma(t-s)}\|\bar{u}(s) - u(s)\|_{2p-2}^2\ \mathrm{d}s \tag{4.8}$$

for every $y_0, \bar{y}_0 \in B$ and $t \in (0,\infty)$, where $S(t)\bar{y}_0 := (\bar{u}(t), \bar{u}_t(t), \bar{v}^t)$ and $\bar{y}_0 := (\bar{u}_0, \bar{u}_1, \bar{v}_0)$.

Proof. Set $\tilde{u} := \bar{u} - u$, $\tilde{v} := \bar{v} - v$, and

$$\Phi(t) := \widetilde{E}(t) + \varepsilon\phi(t), \tag{4.9}$$

where

$$\begin{aligned}\widetilde{E}(t) &:= \|\tilde{u}_t(t)\|^2 + m\,\|\tilde{u}(t)\|_{Y_0}^2 + \left\|\tilde{v}^t\right\|_g^2,\\ \phi(t) &:= (\tilde{u}(t), \tilde{u}_t(t)),\end{aligned} \tag{4.10}$$

and $\varepsilon > 0$ is a constant to be determined later.

We first claim that there exist two constants $\sigma_1, \sigma_2 > 0$, depending on ε, such that

$$\sigma_1\widetilde{E}(t) \leq \Phi(t) \leq \sigma_2\widetilde{E}(t). \tag{4.11}$$

To demonstrate this, we infer from Schwarz's inequality, Theorem 2.1, and Cauchy's inequality that

$$\begin{aligned}|\phi(t)| &\leq \|\tilde{u}(t)\|\,\|\tilde{u}_t(t)\|\\ &\leq \frac{S^2}{2}\|\tilde{u}(t)\|_{Y_0}^2 + \frac{1}{2}\|\tilde{u}_t(t)\|^2.\end{aligned}$$

From this inequality and (4.10), we deduce that there exists a constant $Q > 0$ such that $|\phi(t)| \leq Q\widetilde{E}(t)$, which combined with (4.9) gives

$$(1 - \varepsilon Q)\widetilde{E}(t) \leq \Phi(t) \leq (1 + \varepsilon Q)\widetilde{E}(t).$$

Consequently, assertion (4.11) is proved and $\sigma_1 > 0$ will be guaranteed by the selection of ε later.

Next, we claim that there exist constants $\sigma_3 > 0$ and $\sigma_4 > 0$ depending on B such that

$$\Phi'(t) \le -\sigma_3 \widetilde{E}(t) + \sigma_4 \|\tilde{u}(t)\|_{2p-2}^2 . \tag{4.12}$$

Indeed, by taking $\eta_1 = \tilde{u}_t(t)$ in $(3.16)_1$ and $\eta_2 = \tilde{v}^t$ in $(3.16)_2$, and adding the two results, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{u}_t(t)\|^2 + m\|\tilde{u}(t)\|_{Y_0}^2 + \left\|\tilde{v}^t\right\|_g^2\right) + \|\tilde{u}_t(t)\|_{Y_0}^2 = \frac{1}{2}\int_0^\infty g'(\tau)\left\|\tilde{v}^t(\tau)\right\|_{Y_0}^2 \,\mathrm{d}\tau - (f(\bar{u}(t)) - f(u(t)), \tilde{u}_t(t)),$$

i.e.,

$$\widetilde{E}'(t) = -2\|\tilde{u}_t(t)\|_{Y_0}^2 + \int_0^\infty g'(\tau)\left\|\tilde{v}^t(\tau)\right\|_{Y_0}^2 \,\mathrm{d}\tau - 2(f(\bar{u}(t)) - f(u(t)), \tilde{u}_t(t)). \tag{4.13}$$

For the second term on the right-hand side of (4.13), we conclude from the assumption $g'(t)+\alpha g(t) \le 0$ that

$$\int_0^\infty g'(\tau)\left\|\tilde{v}^t(\tau)\right\|_{Y_0}^2 \,\mathrm{d}\tau \le -\alpha\left\|\tilde{v}^t\right\|_g^2 . \tag{4.14}$$

Concerning the last term on the right-hand side of (4.13), it follows from the arguments similar to the proof of (3.14) that

$$-2(f(\bar{u}(t)) - f(u(t)), \tilde{u}_t(t)) \le C\left(\|\bar{u}(t)\|_{2p-2}^{p-2} + \|u(t)\|_{2p-2}^{p-2} + |\Omega|^{\frac{p-2}{2p-2}}\right)\|\tilde{u}(t)\|_{2p-2}\|\tilde{u}_t(t)\| .$$

By virtue of Theorem 2.1 and Cauchy's inequality, we further have

$$-2(f(\bar{u}(t)) - f(u(t)), \tilde{u}_t(t)) \le C(B)\|\tilde{u}(t)\|_{2p-2}^2 + \frac{1}{2}\|\tilde{u}_t(t)\|_{Y_0}^2 . \tag{4.15}$$

Substituting (4.14) and (4.15) into (4.13), we have

$$\widetilde{E}'(t) \le -\frac{3}{2}\|\tilde{u}_t(t)\|_{Y_0}^2 - \alpha\left\|\tilde{v}^t\right\|_g^2 + C(B)\|\tilde{u}(t)\|_{2p-2}^2 . \tag{4.16}$$

Since

$$\phi'(t) = \|\tilde{u}_t(t)\|^2 + \langle \tilde{u}_{tt}(t), \tilde{u}(t)\rangle,$$

we infer from $(2.15)_1$ that

$$\phi'(t) = \|\tilde{u}_t(t)\|^2 - m\|\tilde{u}(t)\|_{Y_0}^2 - \left(\tilde{v}^t, \tilde{u}(t)\right)_g - (\tilde{u}_t(t), \tilde{u}(t))_{Y_0} - (f(\bar{u}(t)) - f(u(t)), \tilde{u}(t)).$$

Hence, there exists a constant $0 < \delta < 1$ such that

$$\begin{aligned}\phi'(t) = &- \delta\widetilde{E}(t) + (1+\delta)\|\tilde{u}_t(t)\|^2 - m(1-\delta)\|\tilde{u}(t)\|_{Y_0}^2 + \delta\left\|\tilde{v}^t\right\|_g^2 \\ &- \left(\tilde{v}^t, \tilde{u}(t)\right)_g - (\tilde{u}_t(t), \tilde{u}(t))_{Y_0} - (f(\bar{u}(t)) - f(u(t)), \tilde{u}(t)).\end{aligned} \tag{4.17}$$

For the fifth and sixth terms on the right-hand side of (4.17), we deduce from Schwarz's inequality and Cauchy's inequalities with $\epsilon_1, \epsilon_2 > 0$ that

$$-\left(\tilde{v}^t, \tilde{u}(t)\right)_g \le \int_0^\infty g(\tau)\left\|\tilde{v}^t(\tau)\right\|_{Y_0} \,\mathrm{d}\tau\, \|\tilde{u}(t)\|_{Y_0}$$

$$\leq \frac{1}{4\epsilon_1} \left\|\tilde{v}^t\right\|_g^2 + \epsilon_1 \|\tilde{u}(t)\|_{Y_0}^2 \tag{4.18}$$

and

$$\begin{aligned} -(\tilde{u}_t(t), \tilde{u}(t))_{Y_0} &\leq \|\tilde{u}_t(t)\|_{Y_0} \|\tilde{u}(t)\|_{Y_0} \\ &\leq \frac{1}{4\epsilon_2} \|\tilde{u}_t(t)\|_{Y_0}^2 + \epsilon_2 \|\tilde{u}(t)\|_{Y_0}^2 . \end{aligned} \tag{4.19}$$

Concerning the last term on the right-hand side of (4.17), it follows from the arguments similar to the proof of (3.14) that

$$-(f(\bar{u}(t)) - f(u(t)), \tilde{u}(t)) \leq C\left(\|\bar{u}(t)\|_{2p-2}^{p-2} + \|u(t)\|_{2p-2}^{p-2} + |\Omega|^{\frac{p-2}{2p-2}}\right)\|\tilde{u}(t)\|_{2p-2} \|\tilde{u}(t)\| .$$

By Theorem 2.1 and Cauchy's inequality with ϵ_3, we further obtain

$$-(f(\bar{u}(t)) - f(u(t)), \tilde{u}(t)) \leq C(B, \epsilon_3) \|\tilde{u}(t)\|_{2p-2}^2 + \epsilon_3 \|\tilde{u}(t)\|_{Y_0}^2 . \tag{4.20}$$

Consequently, by plugging (4.18)–(4.20) into (4.17), we deduce from Theorem 2.1 that

$$\begin{aligned} \phi'(t) \leq &- \delta\widetilde{E}(t) - (m(1-\delta) - \epsilon_1 - \epsilon_2 - \epsilon_3) \|\tilde{u}(t)\|_{Y_0}^2 + \left((1+\delta)S^2 + \frac{1}{4\epsilon_2}\right)\|\tilde{u}_t(t)\|_{Y_0}^2 \\ &+ \left(\delta + \frac{1}{4\epsilon_1}\right)\left\|\tilde{v}^t\right\|_g^2 + C(B, \epsilon_3) \|\tilde{u}(t)\|_{2p-2}^2 . \end{aligned}$$

Choosing ϵ_i $(i = 1, 2, 3)$ small enough such that

$$m(1-\delta) - \epsilon_1 - \epsilon_2 - \epsilon_3 > 0,$$

we arrive at

$$\phi'(t) \leq -\delta\widetilde{E}(t) + \left((1+\delta)S^2 + \frac{1}{4\epsilon_2}\right)\|\tilde{u}_t(t)\|_{Y_0}^2 + \left(\delta + \frac{1}{4\epsilon_1}\right)\left\|\tilde{v}^t\right\|_g^2 + C(B) \|\tilde{u}(t)\|_{2p-2}^2 . \tag{4.21}$$

Therefore, we conclude from (4.9), (4.16), and (4.21) that

$$\begin{aligned} \Phi'(t) \leq &- \varepsilon\delta\widetilde{E}(t) - \left(\frac{3}{2} - \varepsilon\left((1+\delta)S^2 + \frac{1}{4\epsilon_2}\right)\right)\|\tilde{u}_t(t)\|_{Y_0}^2 \\ &- \left(\alpha - \varepsilon\left(\delta + \frac{1}{4\epsilon_1}\right)\right)\left\|\tilde{v}^t\right\|_g^2 + (C(B) + \varepsilon C(B)) \|\tilde{u}(t)\|_{2p-2}^2 . \end{aligned} \tag{4.22}$$

For fixed ϵ_1 and ϵ_2, we choose

$$\varepsilon < \min\left\{\frac{1}{Q}, \frac{6\epsilon_2}{4\epsilon_2(1+\delta)S^2 + 1}, \frac{4\epsilon_1\alpha}{4\epsilon_1\delta + 1}\right\},$$

which combined with (4.22) gives assertion (4.12), where $\sigma_3 := \varepsilon\delta$ and $\sigma_4 := C(B) + \varepsilon C(B)$.

It follows from (4.12) and (4.11) that

$$\Phi'(t) \leq -\frac{\sigma_3}{\sigma_2}\Phi(t) + \sigma_4 \|\tilde{u}(t)\|_{2p-2}^2 .$$

Hence,

$$\Phi(t) \leq \mathrm{e}^{-\gamma t}\Phi(0) + \sigma_4 \int_0^t \mathrm{e}^{-\gamma(t-s)} \|\tilde{u}(s)\|_{2p-2}^2 \, \mathrm{d}s, \tag{4.23}$$

where $\gamma := \frac{\sigma_3}{\sigma_2}$. Again by (4.11), we derive $\Phi(0) \leq \sigma_2 \widetilde{E}(0)$, which together with (4.23) and (4.11) yields

$$\widetilde{E}(t) \leq \beta \mathrm{e}^{-\gamma t}\widetilde{E}(0) + \theta \int_0^t \mathrm{e}^{-\gamma(t-s)} \|\tilde{u}(s)\|_{2p-2}^2 \, \mathrm{d}s,$$

where $\beta := \frac{\sigma_2}{\sigma_1}$ and $\theta := \frac{\sigma_4}{\sigma_1}$. Therefore, (4.8) follows from (4.10) immediately. □

The following lemma shows that the dynamical system $(\mathcal{H}, S(t))$ corresponding to problem (2.11)–(2.13) is quasi-stable.

Lemma 4.9 (Quasi-stability)**.** *Under the assumptions of Lemma 4.7, the dynamical system* $(\mathcal{H}, S(t))$ *corresponding to problem* (2.11)–(2.13) *is quasi-stable on any bounded positively invariant set* $B \subset \mathcal{H}$*.*

Proof. Let $U := Y_0$, $V := L^2(\Omega)$, and $W := L_g$. Then, Theorem 3.2 shows that the dynamical system $(\mathcal{H}, S(t))$ fulfills (H). Since the solution depends continuously on initial data, we have (4.1). In addition, by taking

$$n_U^2(\tilde{u}) := \|\tilde{u}\|_{2p-2}^2, \quad \psi_2(t) := \beta \mathrm{e}^{-\gamma t}, \quad \psi_3(t) := \theta \int_0^t \mathrm{e}^{-\gamma(t-s)} \, \mathrm{d}s,$$

we infer from Lemma 4.8 that (4.2) holds. Consequently, in view of Definition 4.3, the proof of Lemma 4.9 is complete. □

Now, we complete the proof of the existence of global attractors.

Theorem 4.10 (Existence of global attractors)**.** *Under the assumptions of Lemma 4.7, the dynamical system* $(\mathcal{H}, S(t))$ *corresponding to problem* (2.11)–(2.13) *has a compact global attractor* $\mathscr{A} = \mathcal{M}^z(\mathcal{N})$ *with finite fractal dimension, where*

$$\mathcal{N} = \{(u, 0, 0) \mid m(-\Delta)^s u + f(u) = \varphi\} .$$

Proof. Denote $y(t) := (u(t), u_t(t), v^t)$. Since L is defined as E given by (4.3), we know that $L(y)$ is bounded from above on any bounded subset of $\mathcal{H}$. Concerning the solution (u, u_t, v) of problem (2.11)–(2.13) such that $L(y_0) \leq R$, we deduce from (4.6) and (4.5) that

$$E(t) + \int_0^t \|u_s(s)\|_{Y_0}^2 \, \mathrm{d}s \leq E(0). \tag{4.24}$$

By the arguments similar to the proof of (3.10), we obtain

$$E(t) \geq \frac{1}{2} \|u_t(t)\|^2 + M_1 \|u(t)\|_{Y_0}^2 + \frac{1}{2} \left\|v^t\right\|_g^2 - M_2.$$

Substituting this inequality into (4.24), we have

$$\|u_t(t)\|^2 + \|u(t)\|_{Y_0}^2 + \left\|v^t\right\|_g^2 \le C,$$

i.e., $\left\|(u(t), u_t(t), v^t)\right\|_{\mathcal{H}}^2 \le C$. Hence, L_R is bounded.

Now, we show that the stationary set $\mathcal{N}$ is non-empty. Indeed, it can be demonstrated that the functional

$$I(u) := \frac{m}{2}\|u\|_{Y_0}^2 + \int_\Omega F(u)\,\mathrm{d}x - (\varphi, u)$$

is coercive and weakly lower semicontinuous. Hence, $I(u)$ admits a minimizer on Y_0, which implies that the equation $m(-\Delta)^s u + f(u) = \varphi$ has a solution. Next, we prove that $\mathcal{N}$ is bounded. In fact, regarding the stationary solution $(u, 0, 0)$ of problem (2.11)–(2.13), we have

$$m\|u\|_{Y_0}^2 = -(f(u), u) + (\varphi, u). \tag{4.25}$$

For the first term on the right-hand side of (4.25), we deduce from (2.8) in (A_2) and Theorem 2.1 that

$$\begin{aligned} -(f(u), u) &\le \kappa\|u\|^2 + \lambda|\Omega| \\ &\le \kappa S^2\|u\|_{Y_0}^2 + \lambda|\Omega|. \end{aligned} \tag{4.26}$$

Concerning the last term on the right-hand side of (4.25), it follows from Schwarz's inequality, Theorem 2.1, and Cauchy's inequality with $\epsilon > 0$ that

$$\begin{aligned} (\varphi, u) &\le \|\varphi\|\|u\| \\ &\le \frac{1}{4\epsilon}\|\varphi\|^2 + \epsilon S^2\|u\|_{Y_0}^2. \end{aligned} \tag{4.27}$$

Inserting (4.26) and (4.27) into (4.25), we derive

$$\left(m - \kappa S^2 - \epsilon S^2\right)\|u\|_{Y_0}^2 \le \lambda|\Omega| + \frac{1}{4\epsilon}\|\varphi\|^2.$$

Choosing ϵ small enough such that

$$m - \kappa S^2 - \epsilon S^2 > 0,$$

we further have $\|u\|_{Y_0}^2 \le C$. Consequently, $\mathcal{N}$ is bounded. We conclude from Lemma 4.9 and Proposition 4.4 that $(\mathcal{H}, S(t))$ is asymptotically smooth. By virtue of Lemma 4.7 and Theorem 4.2, we know that $(\mathcal{H}, S(t))$ possesses a compact global attractor $\mathcal{A} = \mathcal{M}^z(\mathcal{N})$. Finally, we infer from Lemma 4.9 and Theorem 4.5 that $\mathcal{A}$ has finite fractal dimension. □

From a physical perspective, Theorem 4.10 illustrates that if the memory kernel g (reflecting the viscoelasticity of the string), nonlinear function f, external force φ, initial displacement u_0, initial velocity u_1, and relative displacement v_0 satisfy appropriate assumptions, then the long-time motion of the viscoelastic string exhibits the dynamical features characterized by the global attractor $\mathcal{A} = \mathcal{M}^z(\mathcal{N})$.

5. Conclusions

In the present paper, we investigated the initial-boundary value problem for a fractional viscoelastic equation with infinite memory, namely, problem (1.1)–(1.3). For the purpose of dealing with the long-time dynamics of problem (1.1)–(1.3), we transformed problem (1.1)–(1.3) into the equivalent problem (2.11)–(2.13) in the history space framework. We first used the Galerkin approximations to derive the global well-posedness of problem (2.11)–(2.13), namely, Theorem 3.2. Moreover, by utilizing the properties of the total energy functional, we found the gradient property of the dynamical system $(\mathcal{H}, S(t))$ corresponding to problem (2.11)–(2.13). Subsequently, by virtue of the perturbed energy method, we derived a stabilizability estimate, which enabled us to verify the quasi-stability of the dynamical system $(\mathcal{H}, S(t))$ corresponding to problem (2.11)–(2.13). Finally, according to the gradient property and quasi-stability of the dynamical system $(\mathcal{H}, S(t))$, we found the existence of global attractors with finite fractal dimension, namely, Theorem 4.10.

In contrast to the viscoelastic equations studied in [31, 32] (which involve the standard Laplacian $-\Delta$), although the regularity of solutions for our fractional model is lower due to spatial nonlocality, the corresponding dynamical system still admits a global attractor. This shows that spatial nonlocality does not prevent problem (1.1)–(1.3) from exhibiting well-behaved long-time dynamics.

The main theoretical contribution of this paper is that we lay a theoretical framework for the existence of global attractors for problem (1.1)–(1.3), and provide a reference for the qualitative study on the long-time dynamics of fractional viscoelastic equations with infinite memory. In addition, we point out several limitations of the present work. First, the main results of this paper are only applicable to Eq (1.1) with the special fractional damping term $(-\Delta)^s u_t$ rather than the more general fractional damping term $(-\Delta)^\alpha u_t$ $(0 \leq \alpha \leq s)$. Second, the analysis is entirely theoretical without numerical simulations. Third, although we prove the existence of global attractors, we do not investigate further properties such as the upper semicontinuity and regularity.

Motivated by the above limitations, possible directions for future research are outlined as follows.

- One may attempt to consider the existence of global attractors for the modified equation obtained by replacing the fractional damping term $(-\Delta)^s u_t$ in Eq (1.1) with the more general term $(-\Delta)^\alpha u_t$. Some energy estimates from [17, 18] may inspire the theoretical exploration.
- It is meaningful to perform numerical simulations on the attractor structure.
- We will devote ourselves to the upper semicontinuity and regularity of global attractors, as well as the existence of exponential attractors.

Author contributions

Junfeng Li: Investigation, Writing–original draft, Funding acquisition; Yang Liu: Investigation, Methodology, Writing-review and editing, Funding acquisition; Qiong Li: Investigation; Yan Xu: Investigation. All authors have read and agreed to the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors express gratitude to the anonymous reviewers, whose constructive feedback contributed to the improvement of the manuscript. This work was supported by the Fundamental Research Funds for the Central Universities (Grant No. 31920260001) and the Discipline Construction Fund Project of Northwest Minzu University (Grant No. 1110130131).

Conflict of interest

The authors declare no conflicts of interest.

References

1. A. Fiscella, E. Valdinoci, A critical Kirchhoff type problem involving a nonlocal operator, *Nonlinear Anal.*, **94** (2014), 156–170. https://doi.org/10.1016/j.na.2013.08.011
2. L. Caffarelli, Non-local diffusions, drifts and games, In: *Nonlinear Partial Differential Equations*, Berlin: Springer, 2012, 37–52. https://doi.org/10.1007/978-3-642-25361-4
3. N. Laskin, Fractional quantum mechanics and Lévy path integrals, *Phys. Lett. A*, **268** (2000), 298–305. https://doi.org/10.1016/s0375-9601(00)00201-2
4. N. Laskin, Fractional Schrödinger equation, *Phys. Rev. E*, **66** (2002), 056108. https://doi.org/10.1103/physreve.66.056108
5. L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Commun. Pure Appl. Math.*, **60** (2007), 67–112. https://doi.org/10.1002/cpa.20153
6. S. Y. A. Chang, M. M. González, Fractional Laplacian in conformal geometry, *Adv. Math.*, **226** (2011), 1410–1432. https://doi.org/10.1016/j.aim.2010.07.016
7. R. R. Yang, Z. X. Lü, The properties of positive solutions to semilinear equations involving the fractional Laplacian, *Commun. Pure Appl. Anal.*, **18** (2019), 1073–1089. https://doi.org/10.3934/cpaa.2019052
8. W. X. Chen, Y. Q. Fang, R. Yang, Liouville theorems involving the fractional Laplacian on a half space, *Adv. Math.*, **274** (2015), 167–198. https://doi.org/10.1016/j.aim.2014.12.013
9. R. Zhuo, C. M. Li, Classification of anti-symmetric solutions to nonlinear fractional Laplace equations, *Calc. Var. PDE*, **61** (2022), 17. https://doi.org/10.1007/s00526-021-02128-w
10. E. Y. He, Z. H. Liu, S. Zhang, Qualitative analysis of rupture set for a semilinear elliptic equation involving the fractional Laplacian, *Calc. Var. PDE*, **65** (2026), 79. https://doi.org/10.1007/s00526-026-03257-w
11. M. Du, L. X. Tian, J. Wang, F. B. Zhang, Existence of normalized solutions for nonlinear fractional Schrödinger equations with trapping potentials, *Proc. Roy. Soc. Edinburgh Sect. A*, **149** (2019), 617–653. https://doi.org/10.1017/prm.2018.41
12. L. T. Liu, S. Yao, K. M. Teng, H. B. Chen, Concentration and uniqueness of minimizers for fractional Schrödinger energy functionals, *Acta Math. Sci. Ser. B Engl. Ed.*, **45** (2025), 1981–2009. https://doi.org/10.1007/s10473-025-0511-1

13. L. T. Liu, K. M. Teng, J. Yang, H. B. Chen, Concentration behaviour of normalized ground states of the mass critical fractional Schrödinger equations with ring-shaped potentials, *Proc. Roy. Soc. Edinburgh Sect. A*, **153** (2023), 1993–2024. https://doi.org/10.1017/prm.2022.81

14. B. Younes, A. Beniani, K. Zennir, Z. Hajjej, H. W. Zhang, Global solution for wave equation involving the fractional Laplacian with logarithmic nonlinearity, *Elec. Res. Arch.*, **32** (2024), 5268–5286. https://doi.org/10.3934/era.2024243

15. C. Duojie, Y. Liu, X. Long, Qualitative properties of solutions to a class of fourth-order nonlinear parabolic equations modeling the epitaxial growth of thin films, *Adv. Nonlinear Anal.*, **15** (2026), 20250136. https://doi.org/10.1515/anona-2025-0136

16. L. E. Payne, D. H. Sattinger, Saddle points and instability of nonlinear hyperbolic equations, *Israel J. Math.*, **22** (1975), 273–303. https://doi.org/10.1007/bf02761595

17. T. N. Bui, T. A. Dao, X. Y. Li, Some remarks on the asymptotic profile of solutions to structurally damped σ-evolution equations, *Math. Methods Appl. Sci.*, in press, 2026. https://doi.org/10.1002/mma.70818

18. W. H. Chen, G. Girardi, Sharp lifespan estimates for semilinear fractional evolution equations with critical nonlinearity, *J. Differ. Equ.*, **443** (2025), 113568. https://doi.org/10.1016/j.jde.2025.113568

19. U. Dini, *Serie di Fourier e Altre Rappresentazioni Analitiche Delle Funzioni di una Variabile Reale*, Pisa: Nistri, 1880.

20. T. Boudjeriou, Global well-posedness and finite time blow-up for a class of wave equation involving fractional p-Laplacian with logarithmic nonlinearity, *Math. Nachr.*, **296** (2023), 938–956. https://doi.org/10.1002/mana.202000266

21. X. L. Wu, J. H. You, Decay estimate and blow-up for fractional p-Laplacian wave equations involving damping and logarithmic source, *J. Evol. Equ.*, **26** (2026), 46. https://doi.org/10.1007/s00028-026-01192-z

22. S. C. Mohapatra, R. B. Fonseca, C. Guedes Soares, Comparison of analytical and numerical simulations of long nonlinear internal solitary waves in shallow water, *J. Coast. Res.*, **34** (2018), 928–938. https://doi.org/10.2112/JCOASTRES-D-16-00193.1

23. V. Pata, Stability and exponential stability in linear viscoelasticity, *Milan J. Math.*, **77** (2009), 333–360. https://doi.org/10.1007/s00032-009-0098-3

24. A. Guesmia, S. A. Messaoudi, A new approach to the stability of an abstract system in the presence of infinite history, *J. Math. Anal. Appl.*, **416** (2014), 212–228. https://doi.org/10.1016/j.jmaa.2014.02.030

25. V. Pata, Exponential stability in linear viscoelasticity, *Quart. Appl. Math.*, **64** (2006), 499–513. https://doi.org/10.1090/s0033-569x-06-01010-4

26. A. Guesmia, Well-posedness and exponential stability of an abstract evolution equation with infinite memory and time delay, *IMA J. Math. Control Inform.*, **30** (2013), 507–526. https://doi.org/10.1093/imamci/dns039

27. Y. Q. Guo, M. A. Rammaha, S. Sakuntasathien, E. S. Titi, D. Toundykov, Hadamard well-posedness for a hyperbolic equation of viscoelasticity with supercritical sources and damping, *J. Differ. Equ.*, **257** (2014), 3778–3812. https://doi.org/10.1016/j.jde.2014.07.009

28. Y. Q. Guo, M. A. Rammaha, S. Sakuntasathien, Blow-up of a hyperbolic equation of viscoelasticity with supercritical nonlinearities, *J. Differ. Equ.*, **262** (2017), 1956–1979. https://doi.org/10.1016/j.jde.2016.10.037

29. Y. Q. Guo, M. A. Rammaha, S. Sakuntasathien, Energy decay of a viscoelastic wave equation with supercritical nonlinearities, *Z. Angew. Math. Phys.*, **69** (2018), 65. https://doi.org/10.1007/s00033-018-0961-6

30. J. W. Zhang, Z. M. Liu, J. H. Huang, Upper semicontinuity of optimal attractors for viscoelastic equations lacking strong damping, *Appl. Anal.*, **102** (2023), 3609–3628. https://doi.org/10.1080/00036811.2022.2088532

31. R. Oliveira Araújo, T. F. Ma, Y. M. Qin, Long-time behavior of a quasilinear viscoelastic equation with past history, *J. Differ. Equ.*, **254** (2013), 4066–4087. https://doi.org/10.1016/j.jde.2013.02.010

32. M. Conti, T. F. Ma, E. M. Marchini, P. N. Seminario Huertas, Asymptotics of viscoelastic materials with nonlinear density and memory effects, *J. Differ. Equ.*, **264** (2018), 4235–4259. https://doi.org/10.1016/j.jde.2017.12.010

33. I. Chueshov, I. Lasiecka, *Von Karman Evolution Equations: Well-posedness and Long-Time Dynamics*, New York: Springer-Verlag, 2010. https://doi.org/10.1007/978-0-387-87712-9

34. C. M. Dafermos, Asymptotic stability in viscoelasticity, *Arch. Ration. Mech. Anal.*, **37** (1970), 297–308. https://doi.org/10.1007/bf00251609

35. M. Grasselli, V. Pata, Uniform attractors of nonautonomous dynamical systems with memory, In: *Evolution Equations, Semigroups and Functional Analysis*, Birkhäuser, 2002, 155–178. https://doi.org/10.1007/978-3-0348-8221-7_9

36. R. Servadei, E. Valdinoci, Variational methods for non-local operators of elliptic type, *Discr. Contin. Dyn. Syst.*, **33** (2013), 2105–2137. https://doi.org/10.3934/dcds.2013.33.2105

37. R. Servadei, E. Valdinoci, The Brezis-Nirenberg result for the fractional Laplacian, *Trans. Amer. Math. Soc.*, **367** (2015), 67–102. https://doi.org/10.1090/s0002-9947-2014-05884-4

38. J. F. Li, Y. Liu, J. Mu, S. G. Zhang, Local existence and finite time blow-up of solutions for a viscoelastic Kirchhoff-like plate equation with logarithmic source term, *Adv. Nonlinear Anal.*, **15** (2026), 20250155. https://doi.org/10.1515/anona-2025-0155

39. Q. Zhou, Y. Liu, D. Yang, Global attractors for a class of viscoelastic plate equations with past history, *AIMS Math.*, **9** (2024), 24887–24907. https://doi.org/10.3934/math.20241212

40. L. H. Fatori, M. A. Jorge Silva, T. F. Ma, Z. J. Yang, Long-time behavior of a class of thermoelastic plates with nonlinear strain, *J. Differ. Equ.*, **259** (2015), 4831–4862. https://doi.org/10.1016/j.jde.2015.06.026

41. Y. Liu, X. Long, L. Zhang, Long-time dynamics for a coupled system modeling the oscillations of suspension bridges, *Commun. Anal. Mech.*, **17** (2025), 15–40. https://doi.org/10.3934/cam.2025002

42. C. Giorgi, V. Pata, A. Marzocchi, Asymptotic behavior of a semilinear problem in heat conduction with memory, *NoDEA Nonlinear Differ. Equ. Appl.*, **5** (1998), 333–354. https://doi.org/10.1007/s000300050049

43. M. A. Jorge Silva, T. F. Ma, On a viscoelastic plate equation with history setting and perturbation of p-Laplacian type, *IMA J. Appl. Math.*, **78** (2013), 1130–1146. https://doi.org/10.1093/imamat/hxs011

44. J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Paris: Dunod, 1969.

45. V. Pata, A. Zucchi, Attractors for a damped hyperbolic equation with linear memory, *Adv. Math. Sci. Appl.*, **11** (2001), 505–529.

46. A. Haraux, E. Zuazua, Decay estimates for some semilinear damped hyperbolic problems, *Arch. Ration. Mech. Anal.*, **100** (1988), 191–206. https://doi.org/10.1007/BF00282203

AIMS Press