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Research article

Hirota bilinear forms and multisoliton structures: A catalogue of classical integrable evolution equations

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Abstract: This paper develops and verifies Hirota bilinear forms for a broad catalogue of classical integrable evolution equations by deriving each transformation from explicit differential identities. The presentation separates scalar one-tau equations from coupled tau systems, hierarchy formulations, lattice equations, and potential forms, thereby preventing unsupported use of scalar interaction formulas. For the general fifth-order Korteweg–de Vries (KdV) family, coefficient matching under $u = c(\ln f)_{xx}$ distinguishes the Sawada–Kotera/Caudrey–Dodd–Gibbon, Kaup–Kupershmidt, and Lax branches. The first admits a scalar bilinear equation; the second requires an auxiliary tau function to represent the q_{xxx}^2 term; and the third requires an auxiliary hierarchy time. We also establish the normalization used for Kadomtsev–Petviashvili (KP)-type equations, derive the full spectral-vector two-soliton coefficient for multidimensional scalar bilinear equations, and clarify why one- and two-soliton solutions do not by themselves prove N-soliton integrability for seventh-order KdV-type models. Coupled bilinear structures for modified KdV (mKdV), Gardner, nonlinear Schrödinger (NLS), sine–Gordon, and Manakov systems are treated separately, with representative soliton and breather constructions. Finally, the local clock-fractional reparametrization $T = t^\alpha/\Gamma(1+\alpha)$ is introduced; it preserves the ordinary Hirota algebra and generates exact clock-fractional one-soliton solutions. The resulting catalogue provides a reproducible framework for symbolic verification and normalization control in soliton analysis.

Keywords: Hirota bilinear method; integrable evolution equations; tau functions; multisoliton solutions; fifth-order KdV equations; Kaup–Kupershmidt equation; KP equation; breather solutions; symbolic verification; clock-fractional models
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1. Introduction and contribution

Hirota's direct method is one of the central algebraic tools in soliton theory. It transforms nonlinear evolution equations into bilinear equations, usually through a tau function, and then constructs solutions by finite exponential expansions [1, 2]. Together with inverse scattering and the hierarchy viewpoint, this method forms part of the classical theory of integrable nonlinear waves [3–6]. For the Korteweg–de Vries (KdV) equation and its descendants, the historical line begins with the original long-wave model [7], the inverse-scattering solution of KdV [8], Lax's hierarchy formulation [9], and Miura's survey of KdV-type transformations [10].

The bilinear method is not merely a convenient notation; it imposes strong algebraic restrictions on the nonlinear equation. This is reflected in Hietarinta's classification program based on Hirota's three-soliton condition [11–15]. In particular, a scalar equation of the form

$$P(D)f \cdot f = 0$$

has a different algebraic structure from a coupled tau system. Consequently, a formula derived for scalar one-tau equations cannot be applied automatically to systems such as modified KdV (mKdV), nonlinear Schrödinger (NLS), Manakov, sine–Gordon, Gardner, or Kaup–Kupershmidt. This distinction is one of the main organizing principles of the present work.

Several bilinear forms discussed in this paper are classical. We therefore state explicitly that the purpose is not to claim originality for the known Hirota transformations of KdV, Kadomtsev–Petviashvili (KP), NLS, sine–Gordon, Toda, Sawada–Kotera (SK), Lax, or Kaup–Kupershmidt (KK) equations. The Sawada–Kotera equation and related higher-order KdV-type flows were introduced in the early construction of N-soliton solutions [16], and the Caudrey–Dodd–Gibbon hierarchy and related higher-order equations were developed in [17, 18]. The Kaup–Kupershmidt equation arises from a distinct cubic spectral problem [19]. Modern symbolic implementations of Hirota's method are surveyed in [20]. The NLS and Manakov systems have standard integrable origins in the inverse-scattering theory of nonlinear wave envelopes [21, 22]. The KP, Toda, and hierarchy formulations are also classical parts of the tau-function theory [23–27]. Broader introductions to soliton concepts and applications are given in [28, 29].

The contribution of this paper is instead methodological, computational, and organizational. The main objectives are as follows:

(i) to derive each bilinear form from explicit Hirota differential identities rather than listing it as a formal transformation;

(ii) to separate scalar one-tau bilinear equations from coupled tau systems, hierarchy bilinearizations, lattice forms, and potential equations;

(iii) to give a coefficient-matching derivation for the general fifth-order KdV family;

(iv) to clarify the different bilinear mechanisms behind the Sawada–Kotera/Caudrey–Dodd–Gibbon (CDG), Kaup–Kupershmidt, and Lax fifth-order branches;

(v) to record the correct full spectral-vector two-soliton coefficient for scalar bilinear equations with transverse variables;

(vi) to distinguish fully integrable seventh-order KdV hierarchy members from scalar seventh-order bilinear KdV-type equations that admit one- and two-soliton solutions but are not automatically N-soliton integrable;

(vii) to add a clock-fractional extension of the one-soliton formulae by means of a natural time reparametrization;

(viii) to provide a reproducible framework for symbolic verification.

The fifth-order KdV family illustrates why this careful separation is necessary. Starting from

$$u_t + \alpha u^2 u_x + \beta u_x u_{xx} + \gamma u u_{xxx} + u_{5x} = 0 \tag{1.1}$$

and using the logarithmic substitution

$$u = c(\ln f)_{xx} \tag{1.2}$$

one obtains three distinguished integrable branches. The Sawada–Kotera/Caudrey–Dodd–Gibbon branch is represented by a single scalar tau equation. The Kaup–Kupershmidt branch is not; it requires an auxiliary tau function g because the logarithmic form contains a q_{xxx}^2 term that cannot be generated from the scalar identity for $D_x^6(f \cdot f)/(2f^2)$. The Lax branch belongs to the KdV hierarchy and requires an auxiliary hierarchy time. Thus, all three branches are bilinearizable but not in the same sense.

A second source of confusion concerns normalization. For example, the KP equation

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0 \tag{1.3}$$

corresponds to

$$(D_x D_t + D_x^4 + 3\sigma^2 D_y^2) f \cdot f = 0 \tag{1.4}$$

under the convention used here. The alternative form containing $-4D_x D_t$ belongs to a different time normalization. Similarly, for multidimensional scalar bilinear equations, the two-soliton coefficient must be computed using the full spectral vector, not only a reduced pair (k, ω). This point is essential for KP-type and Fokas-type equations.

We also include a warning about seventh-order KdV equations. The phrase "seventh-order KdV" does not identify a unique equation. Some seventh-order equations are genuine hierarchy members with N-soliton solutions, whereas other scalar bilinear seventh-order KdV-type equations possess one- and two-soliton solutions without necessarily passing the three-soliton test. This distinction is consistent with the classification logic of Hirota's method and with the study of seventh-order KdV families [30–32].

Recent work continues to combine exact reductions, symbolic, semi-symbolic, and computational methods for nonlinear dispersive waves [33], including machine-learning-assisted approaches in engineering models [34]. In that context, a verified catalogue of Hirota identities is useful not as a new integrability theorem but as a reproducible reference that helps avoid incorrect scalar reductions, normalization errors, and unsupported soliton claims.

Finally, the paper closes with a clock-fractional variant. Instead of replacing the time derivative by a nonlocal Caputo or Riemann–Liouville derivative [35–37], we use a local clock derivative associated with the effective time

$$T = \Theta_\alpha(t) = \frac{t^\alpha}{\Gamma(1+\alpha)}. \tag{1.5}$$

This idea is related in spirit to local fractional or conformable-type differentiation [38], but here it is used only as a clock reparametrization. The construction preserves the Hirota algebra exactly for the corresponding clock-fractional models because differentiation with respect to the clock T satisfies the ordinary chain rule. Thus, every classical one-soliton tau function yields a clock-fractional one-soliton by the replacement

$$T \mapsto \Theta_\alpha(t).$$

For genuinely nonlocal fractional equations, however, the same formulae must be treated only as ansatz functions unless a direct residual verification is performed.

2. Hirota differential identities

Throughout the paper, we use a single notation: f denotes a tau function, g denotes an auxiliary tau function when needed, and

$$q = \ln f.$$

We avoid switching among f, F, and τ unless a lattice index is essential.

2.1. Definition of the D-operators

For functions $f = f(x, y, z, w, s, t)$ and $g = g(x, y, z, w, s, t)$, Hirota's differential operators are defined by

$$D_x^{m_1} D_y^{m_2} \cdots D_t^{m_r} f \cdot g = \prod_j (\partial_{x_j} - \partial_{x'_j})^{m_j} f(\mathbf{x}) g(\mathbf{x}') \Big|_{\mathbf{x}'=\mathbf{x}}. \tag{2.1}$$

For two variables, this reads

$$D_x^m D_t^n f \cdot g = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x,t) g(x', t')|_{x'=x, t'=t}. \tag{2.2}$$

Equivalently,

$$D_x^m D_t^n f \cdot g = \sum_{i=0}^{m} \sum_{j=0}^{n} (-1)^{i+j} \binom{m}{i} \binom{n}{j} \partial_x^{m-i} \partial_t^{n-j} f \, \partial_x^i \partial_t^j g. \tag{2.3}$$

2.2. Low-order identities for one tau function

The following identities are used repeatedly. Direct expansion gives

$$D_x D_t (f \cdot f) = 2(f f_{xt} - f_x f_t), \tag{2.4}$$
$$D_x^2 (f \cdot f) = 2(f f_{xx} - f_x^2), \tag{2.5}$$
$$D_x^4 (f \cdot f) = 2(f f_{4x} - 4 f_x f_{3x} + 3 f_{xx}^2), \tag{2.6}$$
$$D_x^6 (f \cdot f) = 2 f f_{6x} - 12 f_x f_{5x} + 30 f_{xx} f_{4x} - 20 f_{3x}^2. \tag{2.7}$$

Dividing by $2f^2$ and writing

$$q = \ln f,$$

one obtains

$$\frac{D_x D_t (f \cdot f)}{2f^2} = q_{xt}, \tag{2.8}$$

$$\frac{D_x^2(f\cdot f)}{2f^2}=q_{xx}, \tag{2.9}$$

$$\frac{D_x^4(f\cdot f)}{2f^2}=q_{4x}+6q_{xx}^2, \tag{2.10}$$

$$\frac{D_x^6(f\cdot f)}{2f^2}=q_{6x}+30q_{xx}q_{4x}+60q_{xx}^3. \tag{2.11}$$

It is important that (2.11) contains no term of the form

$$q_{xxx}^2.$$

This absence is precisely why the Kaup–Kupershmidt equation cannot be obtained from a single bilinear expression

$$(D_xD_t+aD_x^6)\,f\cdot f=0.$$

Mixed higher-order identities required for hierarchy flows are

$$\frac{D_s^2(f\cdot f)}{2f^2}=q_{ss}, \tag{2.12}$$

$$\frac{D_sD_x^3(f\cdot f)}{2f^2}=q_{xxxs}+6q_{xx}q_{xs}. \tag{2.13}$$

For KP-type equations,

$$\frac{D_y^2(f\cdot f)}{2f^2}=q_{yy}, \tag{2.14}$$

$$\frac{D_zD_w(f\cdot f)}{2f^2}=q_{zw}. \tag{2.15}$$

2.3. Auxiliary-tau identity

For a second tau function g, we shall use

$$D_x^2(f\cdot g)=f_{xx}g-2f_xg_x+fg_{xx}. \tag{2.16}$$

In the Kaup–Kupershmidt derivation, g is not arbitrary. It is defined by

$$D_x^4(f\cdot f)-4fg=0. \tag{2.17}$$

Using (2.10), this is equivalent to

$$g=\frac{D_x^4(f\cdot f)}{4f}=\frac{f}{2}\left(q_{4x}+6q_{xx}^2\right). \tag{2.18}$$

Substituting (2.18) into (2.16) gives the identity

$$\boxed{\frac{D_x^2(f\cdot g)}{f^2}=\frac{1}{2}q_{6x}+7q_{xx}q_{4x}+6q_{xxx}^2+6q_{xx}^3.} \tag{2.19}$$

This formula is the source of the q_{xxx}^2 term in the Kaup–Kupershmidt equation.

2.4. Exponential rule and sign convention

Let

$$\eta_i = k_i x + \ell_i y + m_i z + n_i w - \omega_i t + \delta_i. \tag{2.20}$$

Then,

$$D_x^a D_y^b D_z^c D_w^d D_t^e\, \mathrm{e}^{\eta_i} \cdot \mathrm{e}^{\eta_j} = (k_i - k_j)^a (\ell_i - \ell_j)^b (m_i - m_j)^c (n_i - n_j)^d (-\omega_i + \omega_j)^e \mathrm{e}^{\eta_i + \eta_j}. \tag{2.21}$$

Thus, if $P(D)$ is a scalar bilinear polynomial and $\mathbf{p}_i = (k_i, \ell_i, m_i, n_i, \omega_i)$, then

$$P(D)\mathrm{e}^{\eta_i} \cdot \mathrm{e}^{\eta_j} = P(\mathbf{p}_i - \mathbf{p}_j)\mathrm{e}^{\eta_i + \eta_j} \tag{2.22}$$

with the convention that the time component enters through $-\omega_i + \omega_j$. To avoid ambiguity, we define the dispersion polynomial P using the same phase convention as (2.21). If another convention $\eta = kx + \omega t + \delta$ is adopted, the sign of every odd D_t term changes accordingly.

3. Derivations from differential identities

3.1. KdV equation

Consider

$$u_t + 6uu_x + u_{xxx} = 0. \tag{3.1}$$

Integrating once in x gives

$$\partial_t \left(\int^x u \, \mathrm{d}x \right) + 3u^2 + u_{xx} = 0. \tag{3.2}$$

Set

$$u = 2q_{xx} = 2(\ln f)_{xx}. \tag{3.3}$$

Then,

$$2q_{xt} + 12q_{xx}^2 + 2q_{4x} = 0 \tag{3.4}$$

or

$$q_{xt} + q_{4x} + 6q_{xx}^2 = 0. \tag{3.5}$$

Using (2.8) and (2.10), this is

$$\frac{1}{2f^2}\left(D_x D_t + D_x^4\right) f \cdot f = 0. \tag{3.6}$$

Hence,

$$\boxed{(D_x D_t + D_x^4) f \cdot f = 0.} \tag{3.7}$$

4. Modified KdV equation

We consider the modified KdV equation

$$u_t + u_{xxx} + 6\sigma u^2 u_x = 0, \qquad \sigma = \pm 1. \tag{4.1}$$

The modified KdV equation is not naturally represented by a single scalar one-tau equation of the form

$$P(D)f \cdot f = 0. \tag{4.2}$$

Instead, we introduce two tau functions f and g through

$$u = \frac{g}{f}, \qquad f \neq 0. \tag{4.3}$$

Let

$$q = \ln f. \tag{4.4}$$

Then,

$$g = uf. \tag{4.5}$$

4.1. Differential identities

The Hirota differential operator is defined by

$$D_x^m D_t^n a \cdot b = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n a(x,t) b(x',t')|_{x'=x,\, t'=t}\,. \tag{4.6}$$

First, for the time derivative,

$$D_t g \cdot f = g_t f - g f_t. \tag{4.7}$$

Because $g = uf$, one has

$$g_t = u_t f + u f_t. \tag{4.8}$$

Therefore,

$$\begin{aligned} D_t g \cdot f &= (u_t f + u f_t) f - u f f_t \\ &= f^2 u_t. \end{aligned} \tag{4.9}$$

Thus,

$$\boxed{\frac{D_t g \cdot f}{f^2} = u_t.} \tag{4.10}$$

Next, by the definition of D_x^3,

$$D_x^3 g \cdot f = g_{xxx} f - 3 g_{xx} f_x + 3 g_x f_{xx} - g f_{xxx}. \tag{4.11}$$

Using $g = uf$, we compute

$$g_x = u_x f + u f_x, \tag{4.12}$$
$$g_{xx} = u_{xx} f + 2 u_x f_x + u f_{xx}, \tag{4.13}$$
$$g_{xxx} = u_{xxx} f + 3 u_{xx} f_x + 3 u_x f_{xx} + u f_{xxx}. \tag{4.14}$$

Substituting (4.12)–(4.14) into (4.11), we obtain

$$D_x^3 g \cdot f = (u_{xxx} f + 3 u_{xx} f_x + 3 u_x f_{xx} + u f_{xxx}) f$$

$$
\begin{aligned}
&- 3\left(u_{xx}f + 2u_xf_x + uf_{xx}\right)f_x \\
&+ 3\left(u_xf + uf_x\right)f_{xx} - uff_{xxx}.
\end{aligned} \tag{4.15}
$$

We now collect terms carefully. The u_{xxx}-term is

$$f^2u_{xxx}. \tag{4.16}$$

The u_{xx}-terms cancel because

$$3u_{xx}ff_x - 3u_{xx}ff_x = 0. \tag{4.17}$$

The u-terms cancel because

$$uff_{xxx} - 3uf_{xx}f_x + 3uf_xf_{xx} - uff_{xxx} = 0. \tag{4.18}$$

The u_x-terms are

$$3u_xff_{xx} - 6u_xf_x^2 + 3u_xff_{xx} = 6u_x(ff_{xx} - f_x^2). \tag{4.19}$$

Hence,

$$D_x^3g\cdot f = f^2u_{xxx} + 6u_x(ff_{xx} - f_x^2). \tag{4.20}$$

Because

$$ff_{xx} - f_x^2 = f^2(\ln f)_{xx} = f^2q_{xx} \tag{4.21}$$

we get

$$\boxed{\frac{D_x^3g\cdot f}{f^2} = u_{xxx} + 6q_{xx}u_x.} \tag{4.22}$$

Finally,

$$D_x^2f\cdot f = f_{xx}f - 2f_xf_x + ff_{xx}. \tag{4.23}$$

Therefore,

$$D_x^2f\cdot f = 2(ff_{xx} - f_x^2). \tag{4.24}$$

Dividing by $2f^2$, we obtain

$$\boxed{\frac{D_x^2f\cdot f}{2f^2} = (\ln f)_{xx} = q_{xx}.} \tag{4.25}$$

4.2. Derivation of the bilinear system

Using (4.10) and (4.22), we obtain

$$\frac{(D_t + D_x^3)g\cdot f}{f^2} = u_t + u_{xxx} + 6q_{xx}u_x. \tag{4.26}$$

The modified KdV equation is

$$u_t + u_{xxx} + 6\sigma u^2u_x = 0. \tag{4.27}$$

Comparing this equation with (4.26), we need

$$q_{xx} = \sigma u^2. \tag{4.28}$$

Because $u = g/f$, condition (4.28) becomes

$$q_{xx} = \sigma \frac{g^2}{f^2}. \tag{4.29}$$

Using (4.25), this is equivalent to

$$\frac{D_x^2 f \cdot f}{2f^2} = \sigma \frac{g^2}{f^2}. \tag{4.30}$$

Multiplying by $2f^2$, we obtain

$$D_x^2 f \cdot f = 2\sigma g^2. \tag{4.31}$$

Therefore, the Hirota bilinear system for the modified KdV equation is

$$\boxed{\begin{aligned} (D_t + D_x^3) g \cdot f &= 0, \\ D_x^2 f \cdot f &= 2\sigma g^2. \end{aligned}} \tag{4.32}$$

Indeed, the first bilinear equation gives

$$u_t + u_{xxx} + 6q_{xx}u_x = 0 \tag{4.33}$$

while the second gives

$$q_{xx} = \sigma u^2. \tag{4.34}$$

Combining both equations gives

$$u_t + u_{xxx} + 6\sigma u^2 u_x = 0. \tag{4.35}$$

4.3. Reverse verification

Conversely, assume f and g satisfy (4.32), and define

$$u = \frac{g}{f}. \tag{4.36}$$

Dividing the first bilinear equation by f^2 and using (4.26), gives

$$u_t + u_{xxx} + 6q_{xx}u_x = 0. \tag{4.37}$$

Dividing the second bilinear equation by $2f^2$ and using (4.25), gives

$$q_{xx} = \sigma \frac{g^2}{f^2} = \sigma u^2. \tag{4.38}$$

Substituting (4.38) into (4.37), we obtain

$$u_t + u_{xxx} + 6\sigma u^2 u_x = 0. \tag{4.39}$$

Thus, the bilinear system (4.32) is equivalent to the modified KdV equation (4.1), provided $f \neq 0$.

4.4. Sign convention

For the positive-sign modified KdV equation

$$u_t + u_{xxx} + 6u^2u_x = 0 \tag{4.40}$$

we set $\sigma = 1$, and the bilinear system is

$$\boxed{\begin{aligned}(D_t + D_x^3)g \cdot f &= 0,\\ D_x^2 f \cdot f &= 2g^2.\end{aligned}} \tag{4.41}$$

For the negative-sign modified KdV equation

$$u_t + u_{xxx} - 6u^2u_x = 0 \tag{4.42}$$

we set $\sigma = -1$, and the bilinear system is

$$\boxed{\begin{aligned}(D_t + D_x^3)g \cdot f &= 0,\\ D_x^2 f \cdot f &= -2g^2.\end{aligned}} \tag{4.43}$$

4.5. Why the scalar one-tau formula is not applicable

The scalar two-soliton coefficient formula

$$A_{12} = -\frac{P(\mathbf{p}_1 - \mathbf{p}_2)}{P(\mathbf{p}_1 + \mathbf{p}_2)} \tag{4.44}$$

applies to scalar one-tau bilinear equations of the form

$$P(D)f \cdot f = 0. \tag{4.45}$$

The modified KdV equation derived above is not represented in that form. It is represented by the coupled bilinear system

$$(D_t + D_x^3)g \cdot f = 0, \qquad D_x^2 f \cdot f = 2\sigma g^2. \tag{4.46}$$

Therefore, the interaction coefficients for the mKdV equation must be derived from the coupled Hirota expansion, not from a scalar $P(D)f \cdot f$ shortcut.

4.5.1. Breather solutions of the modified KdV equation

We now derive breather solutions of the modified KdV equation from the two-soliton tau functions. We work with the positive-sign modified KdV equation

$$u_t + u_{xxx} + 6u^2u_x = 0 \tag{4.47}$$

whose Hirota bilinear system is

$$\boxed{\begin{aligned}(D_t + D_x^3)g \cdot f &= 0,\\ D_x^2 f \cdot f &= 2g^2.\end{aligned}} \tag{4.48}$$

The physical field is

$$u = \frac{g}{f}. \tag{4.49}$$

4.6. Starting point: Two-soliton tau functions

From the two-soliton construction, the tau functions are

$$g = E_1 + E_2 + \frac{(k_1 - k_2)^2}{4k_1^2(k_1 + k_2)^2} E_1^2 E_2 + \frac{(k_1 - k_2)^2}{4k_2^2(k_1 + k_2)^2} E_1 E_2^2, \tag{4.50}$$

$$f = 1 + \frac{E_1^2}{4k_1^2} + \frac{2E_1 E_2}{(k_1 + k_2)^2} + \frac{E_2^2}{4k_2^2} + \frac{(k_1 - k_2)^4}{16k_1^2 k_2^2 (k_1 + k_2)^4} E_1^2 E_2^2, \tag{4.51}$$

where

$$E_j = \exp(\eta_j), \qquad \eta_j = k_j x - k_j^3 t + \delta_j, \qquad j = 1, 2. \tag{4.52}$$

These tau functions satisfy the bilinear system (4.48). Hence, $u = g/f$ satisfies (4.47) whenever $f \neq 0$.

4.7. Complex-conjugate reduction

A breather is obtained by taking a complex-conjugate pair of wave numbers. Let

$$k_1 = K = a + ib, \qquad k_2 = \overline{K} = a - ib \tag{4.53}$$

where

$$a > 0, \qquad b \neq 0. \tag{4.54}$$

Also, take

$$\delta_1 = \delta_R + i\delta_I, \qquad \delta_2 = \overline{\delta_1} = \delta_R - i\delta_I. \tag{4.55}$$

Then,

$$E_2 = \overline{E_1}. \tag{4.56}$$

For simplicity, write

$$E_1 = E, \qquad E_2 = \overline{E}. \tag{4.57}$$

The basic algebraic quantities are

$$K + \overline{K} = 2a, \qquad K - \overline{K} = 2ib, \qquad K\overline{K} = a^2 + b^2. \tag{4.58}$$

Define

$$S = a^2 + b^2. \tag{4.59}$$

Then,

$$\frac{(K - \overline{K})^2}{(K + \overline{K})^2} = -\frac{b^2}{a^2}. \tag{4.60}$$

4.8. Real phase variables

We now write E in real amplitude-phase form. Because

$$K^3 = (a + ib)^3 = a^3 - 3ab^2 + i(3a^2 b - b^3) \tag{4.61}$$

the phase

$$\eta = Kx - K^3 t + \delta_1 \tag{4.62}$$

can be written as

$$\eta = \Xi + i\Theta \tag{4.63}$$

where

$$\boxed{\Xi = ax - (a^3 - 3ab^2)t + \delta_R = a\left[x - (a^2 - 3b^2)t\right] + \delta_R,} \tag{4.64}$$

and

$$\boxed{\Theta = bx - (3a^2b - b^3)t + \delta_I = b\left[x - (3a^2 - b^2)t\right] + \delta_I.} \tag{4.65}$$

Therefore,

$$E = e^{\eta} = e^{\Xi}e^{i\Theta}. \tag{4.66}$$

Set

$$R = e^{\Xi}. \tag{4.67}$$

Then,

$$E = Re^{i\Theta}, \qquad \overline{E} = Re^{-i\Theta}. \tag{4.68}$$

4.9. Derivation of the real numerator

Substituting $k_1 = K$, $k_2 = \overline{K}$, $E_1 = E$, and $E_2 = \overline{E}$ into (4.50), we get

$$g = E + \overline{E} - \frac{b^2}{4a^2}\left(\frac{E^2\overline{E}}{K^2} + \frac{E\overline{E}^2}{\overline{K}^2}\right). \tag{4.69}$$

Because

$$E + \overline{E} = 2R\cos\Theta \tag{4.70}$$

we only need to simplify the cubic terms. First,

$$E^2\overline{E} = R^3e^{i\Theta}, \qquad E\overline{E}^2 = R^3e^{-i\Theta}. \tag{4.71}$$

Also,

$$\frac{1}{K^2} = \frac{\overline{K}^2}{(K\overline{K})^2} = \frac{(a-ib)^2}{S^2} = \frac{a^2 - b^2 - 2iab}{S^2}. \tag{4.72}$$

Therefore,

$$\begin{aligned} \frac{E^2\overline{E}}{K^2} + \frac{E\overline{E}^2}{\overline{K}^2} &= 2R^3\,\mathrm{Re}\left(\frac{e^{i\Theta}}{K^2}\right) \\ &= \frac{2R^3}{S^2}\left[(a^2 - b^2)\cos\Theta + 2ab\sin\Theta\right]. \end{aligned} \tag{4.73}$$

Substituting this into (4.69), we obtain

$$\boxed{g_B = 2R\cos\Theta - \frac{b^2R^3}{2a^2S^2}\left[(a^2 - b^2)\cos\Theta + 2ab\sin\Theta\right].} \tag{4.74}$$

4.10. Derivation of the real denominator

Now, substitute the same complex-conjugate reduction into (4.51). We obtain

$$f = 1 + \frac{E^2}{4K^2} + \frac{2E\overline{E}}{(K+\overline{K})^2} + \frac{\overline{E}^2}{4\overline{K}^2} + \frac{(K-\overline{K})^4}{16K^2\overline{K}^2(K+\overline{K})^4}E^2\overline{E}^2. \tag{4.75}$$

The middle term is

$$\frac{2E\overline{E}}{(K+\overline{K})^2} = \frac{2R^2}{(2a)^2} = \frac{R^2}{2a^2}. \tag{4.76}$$

The pair of quadratic terms gives

$$\begin{aligned}\frac{E^2}{4K^2} + \frac{\overline{E}^2}{4\overline{K}^2} &= \frac{R^2}{2}\,\mathrm{Re}\left(\frac{e^{2i\Theta}}{K^2}\right)\\ &= \frac{R^2}{2S^2}\left[(a^2-b^2)\cos(2\Theta) + 2ab\sin(2\Theta)\right].\end{aligned} \tag{4.77}$$

For the quartic term, we use

$$(K-\overline{K})^4 = (2ib)^4 = 16b^4 \tag{4.78}$$

$$(K+\overline{K})^4 = (2a)^4 = 16a^4 \tag{4.79}$$

and

$$K^2\overline{K}^2 = S^2. \tag{4.80}$$

Thus,

$$\frac{(K-\overline{K})^4}{16K^2\overline{K}^2(K+\overline{K})^4}E^2\overline{E}^2 = \frac{b^4R^4}{16a^4S^2}. \tag{4.81}$$

Therefore, the real denominator is

$$\boxed{f_B = 1 + \frac{R^2}{2a^2} + \frac{R^2}{2S^2}\left[(a^2-b^2)\cos(2\Theta) + 2ab\sin(2\Theta)\right] + \frac{b^4R^4}{16a^4S^2}.} \tag{4.82}$$

4.11. Breather field

The mKdV breather generated by the complex-conjugate reduction is

$$\boxed{u_B(x,t) = \frac{g_B(x,t)}{f_B(x,t)},} \tag{4.83}$$

where g_B and f_B are given by (4.74) and (4.82). Explicitly,

$$\boxed{u_B(x,t) = \frac{2R\cos\Theta - \dfrac{b^2R^3}{2a^2S^2}\left[(a^2-b^2)\cos\Theta + 2ab\sin\Theta\right]}{1 + \dfrac{R^2}{2a^2} + \dfrac{R^2}{2S^2}\left[(a^2-b^2)\cos(2\Theta) + 2ab\sin(2\Theta)\right] + \dfrac{b^4R^4}{16a^4S^2}}.} \tag{4.84}$$

Here,

$$S = a^2 + b^2, \qquad R = e^{\Xi} \tag{4.85}$$

with

$$\Xi = a\left[x - (a^2 - 3b^2)t\right] + \delta_R \tag{4.86}$$

and

$$\Theta = b\left[x - (3a^2 - b^2)t\right] + \delta_I. \tag{4.87}$$

4.12. Interpretation

The variable Ξ controls the localization envelope. Because

$$R = e^{\Xi} \tag{4.88}$$

the solution decays as $x \to -\infty$ because $R \to 0$, and it also decays as $x \to +\infty$ because the denominator grows as R^4, whereas the numerator grows only as R^3. Thus, the solution is spatially localized.

The variable Θ controls the internal oscillation. Hence, the solution is a localized oscillatory bound state, that is, a breather. The envelope velocity is

$$V_{\text{env}} = a^2 - 3b^2 \tag{4.89}$$

and the carrier phase velocity is

$$V_{\text{car}} = 3a^2 - b^2. \tag{4.90}$$

4.13. Why the construction works

The original two-soliton tau functions (4.50) and (4.51) satisfy the bilinear system

$$(D_t + D_x^3)g \cdot f = 0, \qquad D_x^2 f \cdot f = 2g^2. \tag{4.91}$$

The replacement

$$k_2 = \overline{k_1}, \qquad \delta_2 = \overline{\delta_1} \tag{4.92}$$

is an algebraic reduction of the same tau functions. Therefore, the reduced real functions g_B and f_B also satisfy the bilinear system. Consequently,

$$u_B = \frac{g_B}{f_B} \tag{4.93}$$

satisfies

$$u_t + u_{xxx} + 6u^2 u_x = 0. \tag{4.94}$$

The only required nondegeneracy condition is

$$f_B(x, t) \neq 0 \tag{4.95}$$

on the spacetime region under consideration.

5. Gardner equation

We consider the Gardner-type equation

$$v_t + avv_x + bv^2 v_x + v_{xxx} = 0, \qquad a \neq 0, \quad b \neq 0. \tag{5.1}$$

The purpose of this section is twofold. First, we derive the normalized Gardner equation from a Hirota-type mKdV bilinear structure. Second, we prove that the breather–brother solutions of the normalized Gardner equation generate solutions of the general equation (5.1) by scaling.

5.1. Normalized Gardner equation

We start from the normalized Gardner equation

$$\boxed{U_t + \partial_x \left(U_{xx} + 6\mu U^2 + 2U^3 \right) = 0,} \tag{5.2}$$

where $\mu \neq 0$. Equivalently,

$$\boxed{U_t + U_{xxx} + 12\mu U U_x + 6U^2 U_x = 0.} \tag{5.3}$$

This is the form naturally connected with the focusing mKdV equation by a constant shift.

5.2. Connection with mKdV by a constant shift

Define

$$W = U + \mu. \tag{5.4}$$

Then, $U = W - \mu$. Because $U_x = W_x$, $U_t = W_t$, and $U_{xxx} = W_{xxx}$, we compute

$$\begin{aligned} 12\mu U U_x + 6U^2 U_x &= 12\mu(W-\mu)W_x + 6(W-\mu)^2 W_x \\ &= 12\mu W W_x - 12\mu^2 W_x + 6(W^2 - 2\mu W + \mu^2)W_x \\ &= 6W^2 W_x - 6\mu^2 W_x. \end{aligned} \tag{5.5}$$

Therefore, (5.3) becomes

$$\boxed{W_t + W_{xxx} + 6W^2 W_x - 6\mu^2 W_x = 0.} \tag{5.6}$$

This is a focusing mKdV equation with a linear transport term.

Equivalently,

$$W_t + \partial_x \left(W_{xx} + 2W^3 - 6\mu^2 W \right) = 0. \tag{5.7}$$

5.3. Removing the linear transport term

Set

$$Y = x + 6\mu^2 t, \qquad W(x,t) = M(Y,t). \tag{5.8}$$

Then,

$$W_t = M_t + 6\mu^2 M_Y, \qquad W_x = M_Y, \qquad W_{xxx} = M_{YYY}. \tag{5.9}$$

Substituting into (5.6) gives

$$0 = M_t + 6\mu^2 M_Y + M_{YYY} + 6M^2 M_Y - 6\mu^2 M_Y$$

$$= M_t + M_{YYY} + 6M^2 M_Y. \tag{5.10}$$

Thus, M satisfies the focusing mKdV equation

$$\boxed{M_t + M_{xxx} + 6M^2 M_x = 0.} \tag{5.11}$$

Hence, the normalized Gardner equation is obtained from focusing mKdV by

$$\boxed{U(x,t) = M(x + 6\mu^2 t, t) - \mu.} \tag{5.12}$$

5.4. Bilinear form through the mKdV arctangent representation

The focusing mKdV equation (5.11) admits the arctangent tau representation

$$\boxed{M = 2\partial_x \arctan\left(\frac{G}{F}\right),} \tag{5.13}$$

where $F = F(x,t)$ and $G = G(x,t)$ are real tau functions. Equivalently, if

$$\tau = F + iG, \qquad \overline{\tau} = F - iG \tag{5.14}$$

then

$$\arctan\left(\frac{G}{F}\right) = \frac{1}{2i} \ln\left(\frac{\tau}{\overline{\tau}}\right) \tag{5.15}$$

and hence,

$$M = \frac{1}{i}\partial_x \ln\left(\frac{\tau}{\overline{\tau}}\right). \tag{5.16}$$

A standard Hirota form for τ and $\overline{\tau}$ is

$$\boxed{(D_t + D_x^3)\tau \cdot \overline{\tau} = 0.} \tag{5.17}$$

Writing $\tau = F + iG$, the real and imaginary parts of (5.17) give a coupled real bilinear system for F and G. The arctangent field

$$M = 2\partial_x \arctan(G/F)$$

then satisfies

$$M_t + M_{xxx} + 6M^2 M_x = 0.$$

Consequently, the Gardner field

$$\boxed{U(x,t) = 2\partial_x \arctan\left(\frac{G(x + 6\mu^2 t, t)}{F(x + 6\mu^2 t, t)}\right) - \mu} \tag{5.18}$$

satisfies (5.3). This gives the bilinear origin of the normalized Gardner equation.

5.5. *Breather–brother solution of the normalized Gardner equation*

We now record the explicit real breather–brother solution of (5.3). Let

$$\Delta = \alpha^2 + \beta^2 - 4\mu^2, \qquad \alpha > 0, \quad \beta > 0, \quad \Delta > 0. \tag{5.19}$$

Define

$$y_1 = x + x_1 + t(\alpha^2 - 3\beta^2), \qquad y_2 = x + x_2 + t(3\alpha^2 - \beta^2). \tag{5.20}$$

Let

$$\mathcal{N} = \frac{\beta\sqrt{\alpha^2+\beta^2}\,\sin(\alpha y_1)}{\alpha\sqrt{\Delta}} - \frac{2\beta\mu\,(\cosh(\beta y_2) + \sinh(\beta y_2))}{\Delta} \tag{5.21}$$

and

$$\mathcal{D} = \cosh(\beta y_2) - \frac{2\beta\mu\,[\alpha\cos(\alpha y_1) - \beta\sin(\alpha y_1)]}{\alpha\sqrt{\alpha^2+\beta^2}\,\sqrt{\Delta}}. \tag{5.22}$$

Define

$$\Phi = \arctan\left(\frac{\mathcal{N}}{\mathcal{D}}\right). \tag{5.23}$$

Then, the breather–brother field is

$$\boxed{U(x,t) = 2\partial_x\Phi(x,t).} \tag{5.24}$$

Equivalently,

$$\boxed{U(x,t) = 2\partial_x \arctan\left[\frac{\dfrac{\beta\sqrt{\alpha^2+\beta^2}\sin(\alpha y_1)}{\alpha\sqrt{\Delta}} - \dfrac{2\beta\mu\,(\cosh(\beta y_2)+\sinh(\beta y_2))}{\Delta}}{\cosh(\beta y_2) - \dfrac{2\beta\mu\,[\alpha\cos(\alpha y_1) - \beta\sin(\alpha y_1)]}{\alpha\sqrt{\alpha^2+\beta^2}\,\sqrt{\Delta}}}\right].} \tag{5.25}$$

5.6. *Scaling from the normalized Gardner equation to the general Gardner equation*

We now prove that any solution of the normalized Gardner equation generates a solution of the general Gardner equation

$$v_t + avv_x + bv^2v_x + v_{xxx} = 0.$$

Let $U = U(X,T)$ be any solution of

$$U_T + U_{XXX} + 12\mu UU_X + 6U^2U_X = 0. \tag{5.26}$$

Define

$$\boxed{v(x,t) = p\,U(qx, rt).} \tag{5.27}$$

Then,

$$v_t = pr\,U_T, \qquad v_x = pq\,U_X, \qquad v_{xxx} = pq^3\,U_{XXX}. \tag{5.28}$$

Also,

$$avv_x = ap^2q\,UU_X, \qquad bv^2v_x = bp^3q\,U^2U_X. \tag{5.29}$$

Therefore, the residual of the general Gardner equation is

$$v_t + avv_x + bv^2 v_x + v_{xxx} = pr\,U_T + pq^3 U_{XXX} + ap^2 q\,UU_X + bp^3 q\,U^2 U_X. \tag{5.30}$$

Using (5.26),

$$U_T = -U_{XXX} - 12\mu UU_X - 6U^2 U_X. \tag{5.31}$$

Substitution into (5.30) gives

$$\begin{aligned} v_t + avv_x + bv^2 v_x + v_{xxx} = {} & p(q^3 - r)U_{XXX} + \left(ap^2 q - 12\mu pr\right) UU_X \\ & + \left(bp^3 q - 6pr\right) U^2 U_X. \end{aligned} \tag{5.32}$$

Hence, the residual vanishes for every solution U if

$$q^3 - r = 0 \tag{5.33}$$

$$ap^2 q - 12\mu pr = 0 \tag{5.34}$$

and

$$bp^3 q - 6pr = 0. \tag{5.35}$$

Assuming $pqr \neq 0$, the first condition gives

$$r = q^3. \tag{5.36}$$

The second condition gives

$$ap = 12\mu q^2. \tag{5.37}$$

The third condition gives

$$bp^2 = 6q^2. \tag{5.38}$$

Solving (5.37) and (5.38), we obtain

$$\boxed{q^2 = \frac{a^2}{24b\mu^2}, \qquad p = \frac{a}{2b\mu}.} \tag{5.39}$$

Choosing the branch

$$\boxed{q = -\frac{a}{2\sqrt{6}\sqrt{b}\,\mu},} \tag{5.40}$$

we have

$$\boxed{r = q^3 = -\frac{a^3}{48\sqrt{6}\,b^{3/2}\mu^3}.} \tag{5.41}$$

Thus,

$$\boxed{p = \frac{a}{2b\mu}, \qquad q = -\frac{a}{2\sqrt{6}\sqrt{b}\,\mu}, \qquad r = -\frac{a^3}{48\sqrt{6}\,b^{3/2}\mu^3}.} \tag{5.42}$$

Therefore, if $U(X, T)$ solves the normalized Gardner equation (5.26), then

$$\boxed{v(x,t) = \frac{a}{2b\mu}\,U\left(-\frac{ax}{2\sqrt{6}\sqrt{b}\,\mu}, -\frac{a^3 t}{48\sqrt{6}\,b^{3/2}\mu^3}\right)} \tag{5.43}$$

solves

$$\boxed{v_t + avv_x + bv^2 v_x + v_{xxx} = 0.} \tag{5.44}$$

5.7. Brother solutions of the general Gardner equation

Combining the breather–brother U in (5.25) with the scaling (5.43), we obtain the corresponding brother solution of the general Gardner equation:

$$\boxed{v(x,t) = \frac{a}{2b\mu}\, U\left(-\frac{ax}{2\sqrt{6}\sqrt{b}\,\mu}, -\frac{a^3 t}{48\sqrt{6}\,b^{3/2}\mu^3}\right)}, \tag{5.45}$$

where U is given by

$$U(X,T) = 2\partial_X \arctan\left[\frac{\dfrac{\beta\sqrt{\alpha^2+\beta^2}\sin(\alpha Y_1)}{\alpha\sqrt{\Delta}} - \dfrac{2\beta\mu\,(\cosh(\beta Y_2)+\sinh(\beta Y_2))}{\Delta}}{\cosh(\beta Y_2) - \dfrac{2\beta\mu\,[\alpha\cos(\alpha Y_1) - \beta\sin(\alpha Y_1)]}{\alpha\sqrt{\alpha^2+\beta^2}\sqrt{\Delta}}}\right] \tag{5.46}$$

with

$$Y_1 = X + x_1 + T(\alpha^2 - 3\beta^2), \qquad Y_2 = X + x_2 + T(3\alpha^2 - \beta^2) \tag{5.47}$$

and

$$\Delta = \alpha^2 + \beta^2 - 4\mu^2. \tag{5.48}$$

The parameter restrictions are

$$\alpha > 0, \qquad \beta > 0, \qquad \mu \neq 0, \qquad \Delta > 0, \qquad b \neq 0. \tag{5.49}$$

If one wants all scalings to remain real with the chosen square root $\sqrt{b}$, one assumes $b > 0$. Otherwise, a complex branch of the square root must be specified.

5.8. Summary of the proof

The proof has two independent parts.

First, the field U in (5.25) satisfies the normalized Gardner equation

$$U_T + U_{XXX} + 12\mu U U_X + 6U^2 U_X = 0. \tag{5.50}$$

This follows by direct substitution after rewriting the trigonometric and hyperbolic functions through

$$w = e^{i\alpha Y_1}, \qquad z = e^{\beta Y_2},$$

which reduces the residual to a rational function whose numerator is identically zero.

Second, the scaling

$$v(x,t) = pU(qx, rt)$$

maps the normalized equation to the general Gardner equation exactly when

$$r = q^3, \qquad ap = 12\mu q^2, \qquad bp^2 = 6q^2.$$

The choice

$$p = \frac{a}{2b\mu}, \qquad q = -\frac{a}{2\sqrt{6}\sqrt{b}\,\mu}, \qquad r = -\frac{a^3}{48\sqrt{6}\,b^{3/2}\mu^3}$$

satisfies these identities. Therefore, the scaled field

$$v(x,t) = \frac{a}{2b\mu}\, U\left(-\frac{ax}{2\sqrt{6}\sqrt{b}\,\mu}, -\frac{a^3 t}{48\sqrt{6}\, b^{3/2}\mu^3}\right)$$

solves

$$v_t + avv_x + bv^2 v_x + v_{xxx} = 0.$$

5.9. KP equation

Consider the KP equation in the normalization

$$\boxed{(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0,} \tag{5.51}$$

where $\sigma^2 = \pm 1$. Introduce the logarithmic tau transformation

$$\boxed{u = 2(\ln f)_{xx} = 2q_{xx}, \qquad q = \ln f.} \tag{5.52}$$

We derive the bilinear form directly from the differential identities, without guessing the Hirota operator.

The required identities are

$$\frac{D_x D_t (f \cdot f)}{2f^2} = q_{xt} \tag{5.53}$$

$$\frac{D_x^4 (f \cdot f)}{2f^2} = q_{4x} + 6q_{xx}^2 \tag{5.54}$$

and

$$\frac{D_y^2 (f \cdot f)}{2f^2} = q_{yy}. \tag{5.55}$$

Therefore, the bilinear expression

$$\left(D_x D_t + D_x^4 + 3\sigma^2 D_y^2\right) f \cdot f \tag{5.56}$$

satisfies

$$\frac{\left(D_x D_t + D_x^4 + 3\sigma^2 D_y^2\right) f \cdot f}{2f^2} = q_{xt} + q_{4x} + 6q_{xx}^2 + 3\sigma^2 q_{yy}. \tag{5.57}$$

Thus, the bilinear equation

$$\boxed{\left(D_x D_t + D_x^4 + 3\sigma^2 D_y^2\right) f \cdot f = 0} \tag{5.58}$$

is equivalent to

$$q_{xt} + q_{4x} + 6q_{xx}^2 + 3\sigma^2 q_{yy} = 0. \tag{5.59}$$

Differentiating (5.59) with respect to x gives

$$q_{xxt} + q_{5x} + 12q_{xx}q_{xxx} + 3\sigma^2 q_{xyy} = 0. \tag{5.60}$$

Using

$$u = 2q_{xx}, \qquad u_t = 2q_{xxt}, \qquad u_{xxx} = 2q_{5x}, \qquad u_{yy} = 2q_{xxyy},$$

and observing that

$$6uu_x = 6(2q_{xx})(2q_{xxx}) = 24q_{xx}q_{xxx},$$

equation (5.60), after multiplication by 2 and one x-differentiation in the y-term, gives exactly

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0.$$

Hence, (5.58) is the correct bilinear form for the KP normalization (5.51).

Remark on the common $-4D_xD_t$ convention. The frequently used bilinear form

$$\left(D_x^4 + 3\sigma^2 D_y^2 - 4D_xD_t\right) f \cdot f = 0$$

corresponds instead to the normalization

$$(-4u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0,$$

or to an equivalent rescaling/reversal of the time variable. It should not be written together with (5.51) unless the time normalization is explicitly changed.

One-soliton condition

Let

$$f = 1 + e^{\eta_1}, \qquad \eta_1 = k_1 x + \ell_1 y - \omega_1 t + \delta_1. \tag{5.61}$$

For the bilinear operator

$$P(D_x, D_y, D_t) = D_xD_t + D_x^4 + 3\sigma^2 D_y^2,$$

we have

$$P(D_x, D_y, D_t)e^{\eta_i} \cdot e^{\eta_j} = P(\mathbf{p}_i - \mathbf{p}_j)e^{\eta_i+\eta_j} \tag{5.62}$$

where the full spectral vector is

$$\mathbf{p}_i = (k_i, \ell_i, \omega_i) \tag{5.63}$$

and, because the phase contains $-\omega_i t$, the scalar polynomial is

$$\boxed{P(k, \ell, \omega) = k^4 - k\omega + 3\sigma^2 \ell^2.} \tag{5.64}$$

Substitution of $f = 1 + e^{\eta_1}$ into (5.58) gives

$$2P(k_1, \ell_1, \omega_1)e^{\eta_1} = 0. \tag{5.65}$$

Therefore, the one-soliton dispersion relation is

$$\boxed{P(k_1, \ell_1, \omega_1) = k_1^4 - k_1\omega_1 + 3\sigma^2 \ell_1^2 = 0.} \tag{5.66}$$

Equivalently,

$$\boxed{\omega_1 = k_1^3 + \frac{3\sigma^2 \ell_1^2}{k_1}, \qquad k_1 \neq 0.} \tag{5.67}$$

The corresponding one-soliton solution is

$$u_1 = 2\partial_x^2 \ln(1 + e^{\eta_1}) = \frac{k_1^2}{2} \operatorname{sech}^2\left(\frac{\eta_1}{2}\right). \tag{5.68}$$

Two-soliton condition and interaction coefficient

For the two-soliton tau function, we take

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2} \tag{5.69}$$

where

$$\eta_i = k_i x + \ell_i y - \omega_i t + \delta_i, \qquad i = 1, 2. \tag{5.70}$$

For the bilinear KP equation

$$\left(D_x D_t + D_x^4 + 3\sigma^2 D_y^2\right) f \cdot f = 0 \tag{5.71}$$

the Hirota polynomial associated with the phase convention $\eta = kx + \ell y - \omega t + \delta$ is

$$P(k, \ell, \omega) = k^4 - k\omega + 3\sigma^2\ell^2. \tag{5.72}$$

Therefore, the coefficients of e^{η_i} give the dispersion relations

$$P(k_i, \ell_i, \omega_i) = 0, \qquad i = 1, 2 \tag{5.73}$$

or, equivalently,

$$\omega_i = k_i^3 + \frac{3\sigma^2\ell_i^2}{k_i}, \qquad i = 1, 2. \tag{5.74}$$

We now collect the coefficient of $e^{\eta_1+\eta_2}$. There are two types of contributions:

$$e^{\eta_1} \cdot e^{\eta_2}, \qquad 1 \cdot A_{12}e^{\eta_1+\eta_2}.$$

Introduce the spectral vectors

$$\mathbf{p}_i = (k_i, \ell_i, \omega_i), \qquad i = 1, 2. \tag{5.75}$$

Because the Hirota polynomial is even under simultaneous reversal of the full spectral vector,

$$P(-k, -\ell, -\omega) = P(k, \ell, \omega),$$

the coefficient of $e^{\eta_1+\eta_2}$ is proportional to

$$P(\mathbf{p}_1 - \mathbf{p}_2) + A_{12}P(\mathbf{p}_1 + \mathbf{p}_2). \tag{5.76}$$

Thus, the two-soliton condition is

$$P(\mathbf{p}_1 - \mathbf{p}_2) + A_{12}P(\mathbf{p}_1 + \mathbf{p}_2) = 0. \tag{5.77}$$

Solving for A_{12} gives

$$\boxed{A_{12} = -\frac{P(\mathbf{p}_1 - \mathbf{p}_2)}{P(\mathbf{p}_1 + \mathbf{p}_2)}.} \tag{5.78}$$

In explicit KP variables, this becomes

$$\boxed{A_{12} = -\frac{(k_1 - k_2)^4 - (k_1 - k_2)(\omega_1 - \omega_2) + 3\sigma^2(\ell_1 - \ell_2)^2}{(k_1 + k_2)^4 - (k_1 + k_2)(\omega_1 + \omega_2) + 3\sigma^2(\ell_1 + \ell_2)^2}.} \tag{5.79}$$

Using the dispersion relations (5.74), the same coefficient can be written in the factorized form

$$\boxed{A_{12} = \frac{k_1^2 k_2^2 (k_1 - k_2)^2 - \sigma^2 (k_1\ell_2 - k_2\ell_1)^2}{k_1^2 k_2^2 (k_1 + k_2)^2 - \sigma^2 (k_1\ell_2 - k_2\ell_1)^2}.} \tag{5.80}$$

This is the form used in the direct symbolic verification.

The remaining exponential coefficients vanish as follows. The terms $e^{2\eta_i}$ vanish because $P(\mathbf{0}) = 0$. The terms $e^{2\eta_1+\eta_2}$ and $e^{\eta_1+2\eta_2}$ are proportional to $P(\mathbf{p}_2)$ and $P(\mathbf{p}_1)$, respectively, and therefore, vanish by the dispersion relations. Hence, (5.69), together with (5.79) or equivalently (5.80), satisfies the bilinear KP equation.

Finally, the KP field

$$\boxed{u^{(2)} = 2(\ln f)_{xx}} \tag{5.81}$$

with f given by (5.69) is the two-soliton solution of (5.51). In particular, direct substitution gives

$$\Big[(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy}\Big]_{u=u^{(2)}} = 0. \tag{5.82}$$

6. The general fifth-order KdV family

Consider

$$\boxed{u_t + \alpha u^2 u_x + \beta u_x u_{xx} + \gamma u u_{xxx} + u_{5x} = 0.} \tag{6.1}$$

The normalized integrable branches in this family include Lax, Sawada–Kotera/Caudrey–Dodd–Gibbon, and Kaup–Kupershmidt. We now derive the coefficient conditions from the logarithmic substitution.

6.1. Integrated form and logarithmic substitution

Integrating (6.1) once with respect to x gives

$$\partial_t\left(\int^x u \, \mathrm{d}x\right) + \frac{\alpha}{3}u^3 + \frac{\beta-\gamma}{2}u_x^2 + \gamma u u_{xx} + u_{4x} = 0. \tag{6.2}$$

Set

$$u = cq_{xx} = c(\ln f)_{xx}. \tag{6.3}$$

Then, (6.2) becomes

$$cq_{xt} + \frac{\alpha c^3}{3}q_{xx}^3 + \frac{(\beta-\gamma)c^2}{2}q_{xxx}^2 + \gamma c^2 q_{xx}q_{4x} + cq_{6x} = 0. \tag{6.4}$$

Dividing by $c \neq 0$,

$$\boxed{q_{xt} + q_{6x} + \gamma c\, q_{xx}q_{4x} + \frac{\beta-\gamma}{2}c\, q_{xxx}^2 + \frac{\alpha c^2}{3}q_{xx}^3 = 0.} \tag{6.5}$$

This is the master identity for the classification below.

6.2. *Single scalar bilinear case: Sawada–Kotera/CDG*

Assume the transformed equation is generated by a single scalar equation

$$(D_x D_t + D_x^6) f \cdot f = 0. \tag{6.6}$$

Dividing by $2f^2$ and using (2.8) and (2.11) gives

$$q_{xt} + q_{6x} + 30 q_{xx} q_{4x} + 60 q_{xx}^3 = 0. \tag{6.7}$$

Comparing with (6.5) gives

$$\gamma c = 30, \tag{6.8}$$

$$\frac{\beta - \gamma}{2} c = 0, \tag{6.9}$$

$$\frac{\alpha c^2}{3} = 60. \tag{6.10}$$

Therefore,

$$\boxed{\alpha = \frac{1}{5}\gamma^2, \qquad \beta = \gamma, \qquad c = \frac{30}{\gamma}.} \tag{6.11}$$

Thus, the Sawada–Kotera/CDG branch is

$$u_t + \frac{1}{5}\gamma^2 u^2 u_x + \gamma u_x u_{xx} + \gamma u u_{xxx} + u_{5x} = 0 \tag{6.12}$$

with

$$u = \frac{30}{\gamma} (\ln f)_{xx} \tag{6.13}$$

and bilinear form

$$\boxed{(D_x D_t + D_x^6) f \cdot f = 0.} \tag{6.14}$$

For $\gamma = 15$ one obtains the common normalization

$$u_t + 45 u^2 u_x + 15 u_x u_{xx} + 15 u u_{xxx} + u_{5x} = 0. \tag{6.15}$$

The two-soliton coefficient follows from the scalar Hirota formula

$$A_{12} = -\frac{P(k_1 - k_2, \omega_1 - \omega_2)}{P(k_1 + k_2, \omega_1 + \omega_2)}$$

with

$$P(k, \omega) = k^6 - k\omega \tag{6.16}$$

and the convention $\eta = kx - \omega t + \delta$. The dispersion relation is $\omega = k^5$, and

$$\boxed{A_{12} = \frac{(k_1 - k_2)^2 (k_1^2 - k_1 k_2 + k_2^2)}{(k_1 + k_2)^2 (k_1^2 + k_1 k_2 + k_2^2)}.} \tag{6.17}$$

6.3. Kaup–Kupershmidt branch: Why an auxiliary function is necessary

For the Kaup–Kupershmidt branch, the coefficient conditions are

$$\boxed{\alpha = \frac{1}{5}\gamma^2, \qquad \beta = \frac{5}{2}\gamma, \qquad c = \frac{15}{\gamma}.} \tag{6.18}$$

Thus, the potential equation (6.5) becomes

$$\boxed{q_{xt} + q_{6x} + 15q_{xx}q_{4x} + \frac{45}{4}q_{xxx}^2 + 15q_{xx}^3 = 0.} \tag{6.19}$$

This is the decisive difference with the scalar Sawada–Kotera/CDG case: the identity (2.11) contains no q_{xxx}^2 term. Hence, (6.19) cannot be produced from a single scalar bilinear equation of the form

$$(D_xD_t + aD_x^6)f \cdot f = 0.$$

An auxiliary tau function is therefore necessary.

Introduce g by

$$\boxed{D_x^4(f \cdot f) - 4fg = 0.} \tag{6.20}$$

Equivalently,

$$g = \frac{D_x^4(f \cdot f)}{4f}. \tag{6.21}$$

Using $q = \ln f$, this gives the identity

$$\frac{D_x^2(f \cdot g)}{f^2} = \frac{1}{2}q_{6x} + 7q_{xx}q_{4x} + 6q_{xxx}^2 + 6q_{xx}^3. \tag{6.22}$$

Now, assume the coupled bilinear equation

$$\left(D_xD_t + aD_x^6\right) f \cdot f + bD_x^2(f \cdot g) = 0. \tag{6.23}$$

Dividing (6.23) by f^2, and using the one-tau identities together with (6.22), gives

$$\begin{aligned} 0 = 2q_{xt} &+ a\left(2q_{6x} + 60q_{xx}q_{4x} + 120q_{xx}^3\right) \\ &+ b\left(\frac{1}{2}q_{6x} + 7q_{xx}q_{4x} + 6q_{xxx}^2 + 6q_{xx}^3\right). \end{aligned} \tag{6.24}$$

To match twice (6.19), namely

$$2q_{xt} + 2q_{6x} + 30q_{xx}q_{4x} + \frac{45}{2}q_{xxx}^2 + 30q_{xx}^3 = 0 \tag{6.25}$$

we require

$$6b = \frac{45}{2}, \qquad 2a + \frac{b}{2} = 2. \tag{6.26}$$

Therefore,

$$b = \frac{15}{4}, \qquad a = \frac{1}{16}. \tag{6.27}$$

The verified Kaup–Kupershmidt bilinear system is

$$\boxed{\left(D_xD_t + \frac{1}{16}D_x^6\right) f \cdot f + \frac{15}{4}D_x^2(f \cdot g) = 0,} \tag{6.28}$$

together with

$$\boxed{D_x^4(f \cdot f) - 4fg = 0.} \tag{6.29}$$

Eliminating g from (6.28) and (6.29) gives the quartic homogeneous equation

$$\begin{aligned} 0 = {} & 4f^3(f_{xt} + f_{6x}) - f^2\left(4f_tf_x - 5f_{3x}^2 + 24f_xf_{5x}\right) \\ & - 30ff_x\,(f_{xx}f_{3x} - 2f_xf_{4x}) + 15f_x^2\left(3f_{xx}^2 - 4f_xf_{3x}\right). \end{aligned} \tag{6.30}$$

For the dependent-variable transformation

$$\boxed{u = \frac{15}{\gamma}(\ln f)_{xx},} \tag{6.31}$$

the corresponding partial differential equation (PDE) is

$$\boxed{u_t + \frac{\gamma^2}{5}u^2u_x + \frac{5\gamma}{2}u_xu_{xx} + \gamma uu_{xxx} + u_{5x} = 0.} \tag{6.32}$$

For $\gamma = 10$, this becomes

$$u_t + 20u^2u_x + 25u_xu_{xx} + 10uu_{xxx} + u_{5x} = 0, \qquad u = \frac{3}{2}(\ln f)_{xx}. \tag{6.33}$$

The alternative scalar-looking expression

$$(D_x^6 + 5D_x^3D_t \pm 5D_t^2)f \cdot f = 0$$

is not used here as the bilinearization of (6.30). If such a form appears in another convention, it must be tied to a different dependent-variable transformation and a different sign convention. For the logarithmic transformation (6.31), the coupled system (6.28) and (6.29) is the verified form.

One-soliton solution

Let

$$\theta = kx - k^5t + \delta, \qquad E = e^{\theta}. \tag{6.34}$$

Unlike the scalar KdV/SK tau function, the KK one-soliton tau function is not $1 + E$. Instead,

$$\boxed{f = 1 + E + \frac{1}{16}E^2.} \tag{6.35}$$

Using (6.31), this yields

$$\boxed{u(x,t) = \frac{240}{\gamma}k^2\frac{e^{\theta}\left(16 + 4e^{\theta} + e^{2\theta}\right)}{(16 + 16e^{\theta} + e^{2\theta})^2}.} \tag{6.36}$$

Equivalently, the same profile may be written in hyperbolic form as

$$u(x,t) = \frac{240}{\gamma}k^2\frac{4 + 17\cosh\theta - 15\sinh\theta}{(16 + 17\cosh\theta - 15\sinh\theta)^2}. \tag{6.37}$$

Direct substitution of (6.36) into (6.32) gives zero.

Two-soliton solution

For two solitons, put

$$\theta_i = k_i x - k_i^5 t + \delta_i, \qquad E_i = e^{\theta_i}, \qquad i = 1, 2. \tag{6.38}$$

Define

$$\boxed{a_{12} = \frac{2k_1^4 - k_1^2 k_2^2 + 2k_2^4}{2(k_1 + k_2)^2(k_1^2 + k_1 k_2 + k_2^2)},} \tag{6.39}$$

and

$$\boxed{b_{12} = \frac{(k_1 - k_2)^2(k_1^2 - k_1 k_2 + k_2^2)}{16(k_1 + k_2)^2(k_1^2 + k_1 k_2 + k_2^2)}.} \tag{6.40}$$

The two-soliton tau function is

$$\boxed{\begin{aligned} f = 1 + E_1 + E_2 + \frac{1}{16}E_1^2 + \frac{1}{16}E_2^2 + a_{12}E_1E_2 \\ + b_{12}\left(E_1^2E_2 + E_1E_2^2\right) + b_{12}^2E_1^2E_2^2. \end{aligned}} \tag{6.41}$$

Then,

$$\boxed{u^{(2)}(x,t) = \frac{15}{\gamma}(\ln f)_{xx}} \tag{6.42}$$

satisfies (6.32).

A direct symbolic check is obtained by replacing

$$\delta_1 = \log\xi - k_1 x + k_1^5 t, \qquad \delta_2 = \log\eta - k_2 x + k_2^5 t.$$

After this substitution, the numerator of the residual of (6.32) vanishes coefficient by coefficient as a polynomial in ξ and η.

Three-soliton solution

For three solitons, let

$$\theta_i = k_i x - k_i^5 t + \delta_i, \qquad E_i = e^{\theta_i}, \qquad i = 1, 2, 3. \tag{6.43}$$

For $1 \le i < j \le 3$, define

$$\boxed{a_{ij} = \frac{2k_i^4 - k_i^2 k_j^2 + 2k_j^4}{2(k_i + k_j)^2(k_i^2 + k_i k_j + k_j^2)},} \tag{6.44}$$

and

$$\boxed{b_{ij} = \frac{(k_i - k_j)^2(k_i^2 - k_i k_j + k_j^2)}{16(k_i + k_j)^2(k_i^2 + k_i k_j + k_j^2)}.} \tag{6.45}$$

Furthermore, set

$$\boxed{\mathcal{D} = 4\prod_{1\le i<j\le 3}(k_i + k_j)^2(k_i^2 + k_i k_j + k_j^2).} \tag{6.46}$$

The coefficient of $E_1E_2E_3$ is

$$\boxed{\begin{aligned} c_{123} = \frac{1}{\mathcal{D}}\Big[&(2k_1^4 - k_1^2k_2^2 + 2k_2^4)(k_3^8 + k_1^4k_2^4) \\ &+ (2k_1^4 - k_1^2k_3^2 + 2k_3^4)(k_2^8 + k_1^4k_3^4) \\ &+ (2k_2^4 - k_2^2k_3^2 + 2k_3^4)(k_1^8 + k_2^4k_3^4)\Big] \\ - \frac{1}{2\mathcal{D}}\Big[&(k_1^2 + k_2^2)(k_1^4 + k_2^4)(k_3^6 + k_1^2k_2^2k_3^2) \\ &+ (k_1^2 + k_3^2)(k_1^4 + k_3^4)(k_2^6 + k_1^2k_2^2k_3^2) \\ &+ (k_2^2 + k_3^2)(k_2^4 + k_3^4)(k_1^6 + k_1^2k_2^2k_3^2) \\ &+ 12k_1^4k_2^4k_3^4\Big]. \end{aligned}} \tag{6.47}$$

The full three-soliton tau function is

$$\boxed{f = 1 + f^{(1)} + f^{(2)} + f^{(3)} + f^{(4)} + f^{(5)} + f^{(6)}.} \tag{6.48}$$

The successive parts are

$$f^{(1)} = E_1 + E_2 + E_3 \tag{6.49}$$

$$f^{(2)} = \frac{1}{16}\left(E_1^2 + E_2^2 + E_3^2\right) + a_{12}E_1E_2 + a_{13}E_1E_3 + a_{23}E_2E_3 \tag{6.50}$$

$$\begin{aligned} f^{(3)} = b_{12}\left(E_1^2E_2 + E_1E_2^2\right) + b_{13}\left(E_1^2E_3 + E_1E_3^2\right) \\ + b_{23}\left(E_2^2E_3 + E_2E_3^2\right) + c_{123}E_1E_2E_3, \end{aligned} \tag{6.51}$$

$$\begin{aligned} f^{(4)} = b_{12}^2E_1^2E_2^2 + b_{13}^2E_1^2E_3^2 + b_{23}^2E_2^2E_3^2 \\ + 16\Big(a_{23}b_{12}b_{13}E_1^2E_2E_3 + a_{13}b_{12}b_{23}E_1E_2^2E_3 \\ + a_{12}b_{13}b_{23}E_1E_2E_3^2\Big), \end{aligned} \tag{6.52}$$

$$f^{(5)} = 256b_{12}b_{13}b_{23}\Big(b_{12}E_1^2E_2^2E_3 + b_{13}E_1^2E_2E_3^2 + b_{23}E_1E_2^2E_3^2\Big), \tag{6.53}$$

and

$$f^{(6)} = 16(16b_{12}b_{13}b_{23})^2E_1^2E_2^2E_3^2. \tag{6.54}$$

Finally,

$$\boxed{u^{(3)}(x,t) = \frac{15}{\gamma}(\ln f)_{xx}} \tag{6.55}$$

with f given by (6.48) gives the three-soliton solution of the Kaup–Kupershmidt branch.

Verification and normalization remarks

The one- and two-soliton formulas above have been checked by direct substitution into (6.32). In the two-soliton case, the residual becomes identically zero after the exponential substitutions

$$\delta_1 = \log\xi - k_1x + k_1^5t, \qquad \delta_2 = \log\eta - k_2x + k_2^5t.$$

Equivalently, the numerator of the residual vanishes coefficient by coefficient in ξ and η. The three-soliton expression is the corresponding terminating perturbative tau polynomial; no nonzero terms occur beyond $f^{(6)}$.

The restrictions are

$$k_i + k_j \neq 0, \qquad k_i^2 + k_i k_j + k_j^2 \neq 0, \qquad \gamma \neq 0,$$

for the relevant pairs $i < j$.

6.4. Lax branch and the auxiliary hierarchy time

The Lax branch is obtained from the coefficient conditions

$$\boxed{\alpha = \frac{3}{10}\gamma^2, \qquad \beta = 2\gamma, \qquad c = \frac{20}{\gamma}.} \tag{6.56}$$

Then, the potential equation (6.5) becomes

$$\boxed{q_{xt} + q_{6x} + 20 q_{xx} q_{4x} + 10 q_{xxx}^2 + 40 q_{xx}^3 = 0.} \tag{6.57}$$

Equivalently, with

$$\boxed{u = \frac{20}{\gamma}(\ln f)_{xx},} \tag{6.58}$$

the physical fifth-order equation is

$$\boxed{u_t + \frac{3\gamma^2}{10} u^2 u_x + 2\gamma u_x u_{xx} + \gamma u u_{xxx} + u_{5x} = 0.} \tag{6.59}$$

In terms of f, Eq (6.57) is the homogeneous trilinear equation

$$\begin{aligned} 0 &= f^2(f_{xt} + f_{6x}) - f(f_x f_t - 5 f_{xx} f_{4x} + 6 f_x f_{5x}) \\ &\quad + 10(f_{xx}^3 - 2 f_x f_{xx} f_{3x} + f_x^2 f_{4x}). \end{aligned} \tag{6.60}$$

It is not the same as the single scalar equation

$$(D_x D_t + D_x^6) f \cdot f = 0.$$

The accepted bilinear representation introduces the auxiliary KdV hierarchy variable s:

$$\boxed{(D_x D_s + D_x^4) f \cdot f = 0,} \tag{6.61}$$

and

$$\boxed{(D_x D_t + D_x^6) f \cdot f - \frac{5}{3}(D_s^2 + D_s D_x^3) f \cdot f = 0.} \tag{6.62}$$

Here, s is not an additional physical variable of the original (x, t)-equation. It is instead the time of the lower KdV flow. Eliminating s from (6.61) and (6.62) recovers the trilinear form (6.60), and hence, the Lax potential equation (6.57).

One- and two-soliton tau functions

Let

$$E_i = e^{\eta_i}, \qquad \eta_i = k_i x - k_i^3 s - k_i^5 t + \delta_i. \tag{6.63}$$

The one-soliton tau function is

$$\boxed{f = 1 + E_1.} \tag{6.64}$$

After setting $s = 0$, the corresponding physical one-soliton field is

$$\boxed{u^{(1)} = \frac{20}{\gamma}(\ln f)_{xx}, \qquad E_1 = e^{k_1 x - k_1^5 t + \delta_1}.} \tag{6.65}$$

For two solitons, the hierarchy tau function is

$$\boxed{f = 1 + E_1 + E_2 + A_{12}E_1E_2,} \tag{6.66}$$

where

$$\boxed{A_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2.} \tag{6.67}$$

This is the KdV hierarchy interaction coefficient. It is not the Sawada–Kotera/CDG fifth-order scalar coefficient. Setting $s = 0$, we obtain

$$\boxed{u^{(2)} = \frac{20}{\gamma}(\ln f)_{xx}, \qquad E_i = e^{k_i x - k_i^5 t + \delta_i}, \quad i = 1, 2.} \tag{6.68}$$

Direct substitution gives

$$\left[u_t + \frac{3\gamma^2}{10}u^2 u_x + 2\gamma u_x u_{xx} + \gamma u u_{xxx} + u_{5x}\right]_{u=u^{(2)}} = 0. \tag{6.69}$$

Verification note

The tau function (6.66) satisfies the two hierarchy bilinear equations (6.61) and (6.62). In the symbolic check one replaces

$$\delta_i = \log \xi_i - k_i x + k_i^3 s + k_i^5 t, \qquad i = 1, 2,$$

and both bilinear residuals vanish identically. Setting $s = 0$ gives the physical fifth-order Lax soliton field.

7. Nonlinear Schrödinger equation

We now derive the Hirota bilinear form of the cubic nonlinear Schrödinger equation directly from differential identities. This section is written separately because the NLS equation is not a single-tau scalar equation of the KdV type. It requires a real tau function f and a complex numerator tau function g.

Consider the cubic NLS equation

$$\boxed{iu_t + u_{xx} + 2\sigma|u|^2u = 0, \qquad \sigma = \pm 1.} \tag{7.1}$$

The case $\sigma = 1$ is the focusing NLS equation, while $\sigma = -1$ is the defocusing NLS equation. We introduce

$$\boxed{u = \frac{g}{f}, \qquad \bar{u} = \frac{\bar{g}}{f},} \tag{7.2}$$

where $f = f(x,t)$ is real-valued and $g = g(x,t)$ is complex-valued. The complex conjugate of g is denoted by $\bar{g}$.

7.1. *Differential identities for $u = g/f$*

Because

$$u = \frac{g}{f},$$

we have

$$u_t = \frac{g_t f - g f_t}{f^2} = \frac{D_t g \cdot f}{f^2}. \tag{7.3}$$

Next,

$$D_x^2 g \cdot f = g_{xx} f - 2g_x f_x + g f_{xx}. \tag{7.4}$$

Dividing by f^2 and writing $g = uf$, gives

$$\begin{aligned} \frac{D_x^2 g \cdot f}{f^2} &= u_{xx} + 2u\left(\frac{f f_{xx} - f_x^2}{f^2}\right) \\ &= u_{xx} + 2u(\ln f)_{xx}. \end{aligned} \tag{7.5}$$

Also,

$$D_x^2 f \cdot f = 2(f f_{xx} - f_x^2) \tag{7.6}$$

and hence,

$$\boxed{\frac{D_x^2 f \cdot f}{2f^2} = (\ln f)_{xx}.} \tag{7.7}$$

7.2. *Derivation of the bilinear system*

Using (7.3) and (7.5), we obtain

$$\frac{(iD_t + D_x^2) g \cdot f}{f^2} = iu_t + u_{xx} + 2u(\ln f)_{xx}. \tag{7.8}$$

Thus, the equation

$$\boxed{(iD_t + D_x^2) g \cdot f = 0} \tag{7.9}$$

is equivalent to

$$iu_t + u_{xx} + 2u(\ln f)_{xx} = 0. \tag{7.10}$$

To recover the NLS equation, we impose

$$(\ln f)_{xx} = \sigma |u|^2. \tag{7.11}$$

Because

$$|u|^2 = u\bar{u} = \frac{g\bar{g}}{f^2},$$

condition (7.11) becomes

$$\frac{D_x^2 f \cdot f}{2f^2} = \sigma \frac{g\bar{g}}{f^2}.$$

Multiplying by $2f^2$, we obtain

$$\boxed{D_x^2 f \cdot f = 2\sigma g\bar{g}.} \tag{7.12}$$

The complex conjugate of (7.9) is

$$\boxed{(-iD_t + D_x^2)\bar{g} \cdot f = 0.} \tag{7.13}$$

Therefore, a closed bilinear system for NLS is

$$\boxed{\begin{aligned} (iD_t + D_x^2)g \cdot f &= 0, \\ (-iD_t + D_x^2)\bar{g} \cdot f &= 0, \\ D_x^2 f \cdot f &= 2\sigma g\bar{g}. \end{aligned}} \tag{7.14}$$

Conversely, if $f \neq 0$ and $g, \bar{g}, f$ satisfy (7.14), then division by f^2 gives

$$iu_t + u_{xx} + 2u(\ln f)_{xx} = 0,$$

while division of the third equation by $2f^2$ gives

$$(\ln f)_{xx} = \sigma \frac{g\bar{g}}{f^2} = \sigma |u|^2.$$

Hence,

$$iu_t + u_{xx} + 2\sigma |u|^2 u = 0.$$

Thus, (7.14) is equivalent to the NLS equation (7.1) under the transformation (7.2).

7.3. Focusing bright soliton expansion

The zero-background bright solitons correspond to the focusing case

$$\sigma = 1.$$

In this case, the bilinear system becomes

$$\boxed{\begin{aligned} (iD_t + D_x^2)g \cdot f &= 0, \\ (-iD_t + D_x^2)\bar{g} \cdot f &= 0, \\ D_x^2 f \cdot f &= 2g\bar{g}. \end{aligned}} \tag{7.15}$$

The standard Hirota parity expansion is

$$g = \varepsilon g_1 + \varepsilon^3 g_3 + \varepsilon^5 g_5 + \cdots, \qquad \bar{g} = \varepsilon \bar{g}_1 + \varepsilon^3 \bar{g}_3 + \cdots \tag{7.16}$$

and

$$f = 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4 + \cdots. \tag{7.17}$$

The odd powers occur in $g, \bar{g}$, while the even powers occur in f. This already shows why a scalar one-tau coefficient of the form

$$A_{12} = -\frac{P(\mathbf{p}_1 - \mathbf{p}_2)}{P(\mathbf{p}_1 + \mathbf{p}_2)}$$

does not apply directly to NLS.

7.4. One-soliton solution

For one soliton, the expansion truncates as

$$g = g_1, \qquad \bar{g} = \bar{g}_1, \qquad f = 1 + f_2. \tag{7.18}$$

Take

$$g_1 = E, \qquad \bar{g}_1 = \bar{E} \tag{7.19}$$

where

$$E = e^{\eta}, \qquad \bar{E} = e^{\bar{\eta}} \tag{7.20}$$

with

$$\eta = px + ip^2 t + \delta, \qquad \bar{\eta} = qx - iq^2 t + \bar{\delta}. \tag{7.21}$$

Here, $p, q, \delta, \bar{\delta}$ are treated as independent parameters in the algebraic verification. The physical complex-conjugate reduction is obtained later by imposing

$$q = \bar{p}, \qquad \bar{\delta} = \overline{\delta}.$$

The first bilinear equation gives

$$(iD_t + D_x^2)E \cdot 1 = \left(i(ip^2) + p^2\right)E = 0.$$

Similarly,

$$(-iD_t + D_x^2)\bar{E} \cdot 1 = \left(-i(-iq^2) + q^2\right)\bar{E} = 0.$$

The third bilinear equation at order $E\bar{E}$ gives

$$D_x^2 f \cdot f = 2E\bar{E}.$$

Because

$$D_x^2(1 \cdot f_2 + f_2 \cdot 1) = 2f_{2,xx},$$

we need

$$f_{2,xx} = E\bar{E}.$$

However,

$$\partial_x^2(E\bar{E}) = (p+q)^2 E\bar{E}.$$

Therefore,

$$\boxed{f_2 = \frac{E\bar{E}}{(p+q)^2}.} \tag{7.22}$$

Setting the bookkeeping parameter equal to one gives the one-soliton tau functions

$$\boxed{g = E, \qquad \bar{g} = \bar{E}, \qquad f = 1 + \frac{E\bar{E}}{(p+q)^2}.} \tag{7.23}$$

Therefore,

$$\boxed{u^{(1)}(x,t) = \frac{E}{1 + \dfrac{E\bar{E}}{(p+q)^2}}.} \tag{7.24}$$

Equivalently,

$$\boxed{\bar{u}^{(1)}(x,t) = \frac{\bar{E}}{1 + \dfrac{E\bar{E}}{(p+q)^2}}.} \tag{7.25}$$

For the physical bright soliton, impose

$$q = \bar{p}, \qquad \bar{\delta} = \overline{\delta},$$

and write

$$p = a + ib, \qquad a > 0, \qquad b \in \mathbb{R}.$$

Then,

$$p + \bar{p} = 2a,$$

and the exponential solution (7.24) reduces to the standard sech profile

$$\boxed{u^{(1)}(x,t) = a\ \mathrm{sech}(a(x - 2bt - x_0)) \exp\Big(i\left[bx + (a^2 - b^2)t + \phi_0\right]\Big),} \tag{7.26}$$

where x_0 and ϕ_0 are real constants.

7.5. Two-soliton solution

For two bright solitons, define

$$E_i = e^{\eta_i}, \qquad \bar{E}_i = e^{\bar{\eta}_i}, \qquad i = 1, 2 \tag{7.27}$$

where

$$\eta_i = p_i x + i p_i^2 t + \delta_i, \qquad \bar{\eta}_i = q_i x - i q_i^2 t + \bar{\delta}_i. \tag{7.28}$$

Again, p_i and q_i are kept independent for the algebraic calculation. The physical complex-conjugate reduction is obtained by taking

$$q_i = \bar{p}_i, \qquad \bar{\delta}_i = \overline{\delta_i}, \qquad i = 1, 2.$$

Define

$$\boxed{a_{ij} = \frac{1}{(p_i + q_j)^2}, \qquad i, j = 1, 2.} \tag{7.29}$$

Next, define

$$\boxed{A_{1122} = \frac{(p_1 - p_2)^2 (q_1 - q_2)^2}{(p_1 + q_1)^2 (p_1 + q_2)^2 (p_2 + q_1)^2 (p_2 + q_2)^2}.} \tag{7.30}$$

The cubic coefficients in g are

$$\boxed{c_1 = \frac{(p_1 - p_2)^2}{(p_1 + q_1)^2 (p_2 + q_1)^2}, \qquad c_2 = \frac{(p_1 - p_2)^2}{(p_1 + q_2)^2 (p_2 + q_2)^2}.} \tag{7.31}$$

The cubic coefficients in $\bar{g}$ are

$$\boxed{d_1 = \frac{(q_1 - q_2)^2}{(p_1 + q_1)^2 (p_1 + q_2)^2}, \qquad d_2 = \frac{(q_1 - q_2)^2}{(p_2 + q_1)^2 (p_2 + q_2)^2}.} \tag{7.32}$$

The two-soliton tau functions are

$$\boxed{\begin{aligned} f = 1 &+ a_{11} E_1 \bar{E}_1 + a_{12} E_1 \bar{E}_2 + a_{21} E_2 \bar{E}_1 + a_{22} E_2 \bar{E}_2 \\ &+ A_{1122} E_1 E_2 \bar{E}_1 \bar{E}_2, \end{aligned}} \tag{7.33}$$

$$\boxed{g = E_1 + E_2 + c_1 E_1 E_2 \bar{E}_1 + c_2 E_1 E_2 \bar{E}_2,} \tag{7.34}$$

and

$$\boxed{\bar{g} = \bar{E}_1 + \bar{E}_2 + d_1 E_1 \bar{E}_1 \bar{E}_2 + d_2 E_2 \bar{E}_1 \bar{E}_2.} \tag{7.35}$$

Hence,

$$\boxed{u^{(2)}(x, t) = \frac{g}{f}, \qquad \bar{u}^{(2)}(x, t) = \frac{\bar{g}}{f}.} \tag{7.36}$$

With $f, g, \bar{g}$ given by (7.33)–(7.35), direct substitution gives

$$\boxed{iu_t + u_{xx} + 2u^2 \bar{u} = 0,} \tag{7.37}$$

and

$$\boxed{-i\bar{u}_t + \bar{u}_{xx} + 2\bar{u}^2 u = 0.} \tag{7.38}$$

Equivalently, the tau functions satisfy

$$\boxed{\begin{aligned} (iD_t + D_x^2) g \cdot f &= 0, \\ (-iD_t + D_x^2) \bar{g} \cdot f &= 0, \\ D_x^2 f \cdot f &= 2g\bar{g}. \end{aligned}} \tag{7.39}$$

7.6. *Why the scalar A_{12} formula is not used for NLS*

For scalar one-tau equations of the form

$$P(D)f \cdot f = 0,$$

the two-soliton coefficient is often obtained from

$$A_{12} = -\frac{P(\mathbf{p}_1 - \mathbf{p}_2)}{P(\mathbf{p}_1 + \mathbf{p}_2)}.$$

This shortcut does not apply directly to NLS because NLS is not represented by a single scalar equation $P(D)f \cdot f = 0$. Instead, it has the coupled system

$$(iD_t + D_x^2)g \cdot f = 0, \qquad (-iD_t + D_x^2)\bar{g} \cdot f = 0, \qquad D_x^2 f \cdot f = 2\sigma g\bar{g}.$$

Therefore, the two-soliton solution must be computed from the coupled Hirota expansion, not from a single scalar bilinear polynomial. This distinction is essential: the NLS interaction coefficients couple g, $\bar{g}$, and f, and the algebra is determinant-like rather than a single scalar A_{12} ratio.

7.7. *Verification note*

For the one-soliton solution, substitution of

$$\delta = \log\xi - px - ip^2 t, \qquad \bar{\delta} = \log\rho - qx + iq^2 t$$

into the residuals gives

$$iu_t + u_{xx} + 2u^2\bar{u} = 0, \qquad -i\bar{u}_t + \bar{u}_{xx} + 2\bar{u}^2 u = 0.$$

For the two-soliton solution, the substitutions

$$\delta_i = \log\xi_i - p_i x - ip_i^2 t, \qquad \bar{\delta}_i = \log\rho_i - q_i x + iq_i^2 t, \qquad i = 1, 2,$$

turn the residuals into rational expressions whose numerators vanish identically. Thus, the one- and two-soliton formulae above have been checked directly in the physical NLS equations.

7.8. *Remark on the defocusing case*

The algebraic bilinear system

$$(iD_t + D_x^2)g \cdot f = 0, \qquad (-iD_t + D_x^2)\bar{g} \cdot f = 0, \qquad D_x^2 f \cdot f = 2\sigma g\bar{g}$$

is valid for both $\sigma = 1$ and $\sigma = -1$. However, the zero-background bright-soliton tau functions above apply to the focusing case $\sigma = 1$. The defocusing case $\sigma = -1$ naturally supports dark solitons on a nonzero background, and its Hirota construction uses a background phase together with a modified tau ansatz. Therefore, the bright-soliton formulae in this section should be read as the focusing, zero-background case.

8. Fokas-type bilinear equation: Derivation and normalization

We now treat the multidimensional Fokas-type bilinear equation carefully. The purpose of this subsection is not to quote a bilinear form but to derive the nonlinear equation associated with it through differential identities.

Consider the scalar Hirota equation

$$\boxed{\left(4D_xD_t - D_x^3D_y + D_xD_y^3 - 6D_zD_w\right) f \cdot f = 0.} \tag{8.1}$$

Let

$$q = \ln f. \tag{8.2}$$

We use the identities

$$\frac{D_xD_t(f \cdot f)}{2f^2} = q_{xt} \tag{8.3}$$

$$\frac{D_zD_w(f \cdot f)}{2f^2} = q_{zw} \tag{8.4}$$

$$\frac{D_x^3D_y(f \cdot f)}{2f^2} = q_{xxxy} + 6q_{xx}q_{xy} \tag{8.5}$$

and

$$\frac{D_xD_y^3(f \cdot f)}{2f^2} = q_{xyyy} + 6q_{yy}q_{xy}. \tag{8.6}$$

Dividing (8.1) by $2f^2$, we obtain

$$0 = 4q_{xt} - \left(q_{xxxy} + 6q_{xx}q_{xy}\right) + \left(q_{xyyy} + 6q_{yy}q_{xy}\right) - 6q_{zw}. \tag{8.7}$$

Therefore, the bilinear equation (8.1) is equivalent to the potential equation

$$\boxed{4q_{xt} - q_{xxxy} + q_{xyyy} - 6q_{zw} + 6(q_{yy} - q_{xx})q_{xy} = 0.} \tag{8.8}$$

This is the equation obtained directly from the Hirota differential identities. No dependent-variable transformation has been assumed beyond $q = \ln f$.

8.1. Important warning about the field transformation

The equation

$$4u_{xt} - u_{xxxy} + u_{xyyy} + 6(u^2)_{xy} - 6u_{zw} = 0 \tag{8.9}$$

does not follow from (8.1) by the simple substitution

$$u = C(\ln f)_{xx} = Cq_{xx}.$$

Indeed, differentiating (8.8) twice with respect to x and setting $u = Cq_{xx}$ produces nonlinear terms of the form

$$\partial_x^2\left[(q_{yy} - q_{xx})q_{xy}\right],$$

not the term

$$(u^2)_{xy}.$$

Hence, the statement

$$u = C(\ln f)_{xx} \quad \Longrightarrow \quad 4u_{xt} - u_{xxxy} + u_{xyyy} + 6(u^2)_{xy} - 6u_{zw} = 0$$

is not justified by (8.1). It must be removed unless an additional reduction, change of variables, or different dependent-variable transformation is explicitly supplied and verified.

Thus, in this paper, the verified statement is the following:

> The bilinear equation (8.1) is equivalent to the potential equation (8.8). Any further claim that it is equivalent to (8.9) requires a separate derivation.

8.2. One-soliton dispersion relation

Let

$$f = 1 + e^{\eta_1}, \qquad \eta_1 = k_1 x + \ell_1 y + m_1 z + n_1 w - \omega_1 t + \delta_1. \tag{8.10}$$

For the phase convention containing $-\omega t$, the bilinear polynomial associated with (8.1) is

$$\boxed{P(k, \ell, m, n, \omega) = -4k\omega - k^3\ell + k\ell^3 - 6mn.} \tag{8.11}$$

Indeed,

$$D_t e^{\eta_i} \cdot e^{\eta_j} = (-\omega_i + \omega_j)e^{\eta_i+\eta_j},$$

so the contribution of $4D_xD_t$ is $-4k\omega$, not $+4k\omega$.

Substitution of $f = 1 + e^{\eta_1}$ into (8.1) gives

$$2P(k_1, \ell_1, m_1, n_1, \omega_1)e^{\eta_1} = 0. \tag{8.12}$$

Therefore, the one-soliton dispersion relation is

$$\boxed{-4k_1\omega_1 - k_1^3\ell_1 + k_1\ell_1^3 - 6m_1n_1 = 0.} \tag{8.13}$$

Equivalently, for $k_1 \neq 0$,

$$\boxed{\omega_1 = \frac{-k_1^3\ell_1 + k_1\ell_1^3 - 6m_1n_1}{4k_1}.} \tag{8.14}$$

8.3. Two-soliton coefficient

For two solitons, take

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2} \tag{8.15}$$

where

$$\eta_i = k_i x + \ell_i y + m_i z + n_i w - \omega_i t + \delta_i, \qquad i = 1, 2. \tag{8.16}$$

The full spectral vector is

$$\mathbf{p}_i = (k_i, \ell_i, m_i, n_i, \omega_i). \tag{8.17}$$

The coefficient of e^{η_i} gives

$$P(\mathbf{p}_i) = 0, \qquad i = 1, 2. \tag{8.18}$$

The coefficient of $e^{\eta_1+\eta_2}$ comes from

$$e^{\eta_1} \cdot e^{\eta_2}, \qquad 1 \cdot A_{12} e^{\eta_1+\eta_2}.$$

Hence,

$$P(\mathbf{p}_1 - \mathbf{p}_2) + A_{12} P(\mathbf{p}_1 + \mathbf{p}_2) = 0. \tag{8.19}$$

Solving for A_{12}, we obtain

$$\boxed{A_{12} = -\frac{P(\mathbf{p}_1 - \mathbf{p}_2)}{P(\mathbf{p}_1 + \mathbf{p}_2)}.} \tag{8.20}$$

Explicitly,

$$\boxed{A_{12} = -\frac{P(k_1 - k_2, \ell_1 - \ell_2, m_1 - m_2, n_1 - n_2, \omega_1 - \omega_2)}{P(k_1 + k_2, \ell_1 + \ell_2, m_1 + m_2, n_1 + n_2, \omega_1 + \omega_2)},} \tag{8.21}$$

where P is the polynomial in (8.11).

9. Boussinesq family

We now treat the Boussinesq family in a way that avoids normalization ambiguities. The same scalar bilinear operator may correspond to different nonlinear coefficients depending on the logarithmic transformation.

9.1. Scalar bilinear equation

Consider the scalar Hirota equation

$$\boxed{(D_t^2 + aD_x^2 + bD_x^4) f \cdot f = 0,} \tag{9.1}$$

where a and b are constants. Let

$$q = \ln f. \tag{9.2}$$

The required identities are

$$\frac{D_t^2(f \cdot f)}{2f^2} = q_{tt}, \qquad \frac{D_x^2(f \cdot f)}{2f^2} = q_{xx} \tag{9.3}$$

and

$$\frac{D_x^4(f \cdot f)}{2f^2} = q_{4x} + 6q_{xx}^2. \tag{9.4}$$

Dividing (9.1) by $2f^2$, we obtain

$$\boxed{q_{tt} + aq_{xx} + bq_{4x} + 6bq_{xx}^2 = 0.} \tag{9.5}$$

This is the potential Boussinesq equation associated with (9.1).

9.2. Recovering the nonlinear Boussinesq equation

Introduce

$$\boxed{u = \lambda q_{xx} = \lambda(\ln f)_{xx},} \tag{9.6}$$

where $\lambda \neq 0$. Applying $\lambda \partial_x^2$ to (9.5) gives

$$\boxed{u_{tt} + au_{xx} + bu_{4x} + \frac{6b}{\lambda}(u^2)_{xx} = 0.} \tag{9.7}$$

Thus, if the PDE is written as

$$\boxed{u_{tt} + Au_{xx} + Bu_{4x} + C(u^2)_{xx} = 0,} \tag{9.8}$$

then it is generated by

$$\boxed{(D_t^2 + AD_x^2 + BD_x^4)f \cdot f = 0,} \tag{9.9}$$

with

$$\boxed{u = \frac{6B}{C}(\ln f)_{xx}, \qquad BC \neq 0.} \tag{9.10}$$

In the common normalization $C = 3$, this becomes

$$\boxed{u = 2B(\ln f)_{xx},} \tag{9.11}$$

and the PDE is

$$\boxed{u_{tt} + Au_{xx} + Bu_{4x} + 3(u^2)_{xx} = 0.} \tag{9.12}$$

9.3. Normalization table

The following table records common Boussinesq conventions. The transform must be matched to the coefficient of $(u^2)_{xx}$.

Nonlinear PDE	Bilinear equation	Transform
$u_{tt} - u_{xx} - u_{4x} - (u^2)_{xx} = 0$	$(D_t^2 - D_x^2 - D_x^4)f \cdot f = 0$	$u = 6(\ln f)_{xx}$
$u_{tt} - u_{xx} - u_{4x} - 3(u^2)_{xx} = 0$	$(D_t^2 - D_x^2 - D_x^4)f \cdot f = 0$	$u = 2(\ln f)_{xx}$
$u_{tt} + u_{4x} + 3(u^2)_{xx} = 0$	$(D_t^2 + D_x^4)f \cdot f = 0$	$u = 2(\ln f)_{xx}$
$u_{tt} + Au_{xx} + Bu_{4x} + C(u^2)_{xx} = 0$	$(D_t^2 + AD_x^2 + BD_x^4)f \cdot f = 0$	$u = \frac{6B}{C}(\ln f)_{xx}$

9.4. Equivalent homogeneous form

Expanding the bilinear equation gives

$$\begin{aligned} 0 &= \frac{1}{2}(D_t^2 + aD_x^2 + bD_x^4)f \cdot f \\ &= f(f_{tt} + af_{xx} + bf_{4x}) - f_t^2 - af_x^2 + b(3f_{xx}^2 - 4f_xf_{3x}). \end{aligned} \tag{9.13}$$

Equivalently,

$$\boxed{f(f_{tt} + af_{xx} + bf_{4x}) - f_t^2 - af_x^2 + b(3f_{xx}^2 - 4f_xf_{3x}) = 0.} \tag{9.14}$$

9.5. Linear dispersion relation

Let

$$f = 1 + \varepsilon e^{\theta} + \cdots, \qquad \theta = kx - \omega t + \delta. \tag{9.15}$$

The corresponding Hirota polynomial is

$$\boxed{P(k, \omega) = \omega^2 + ak^2 + bk^4.} \tag{9.16}$$

Therefore, the dispersion relation is

$$\boxed{\omega^2 + ak^2 + bk^4 = 0.} \tag{9.17}$$

9.6. One-soliton solution

Let

$$\theta_1 = k_1 x - \omega_1 t + \delta_1, \qquad E_1 = e^{\theta_1} \tag{9.18}$$

where

$$\boxed{\omega_1^2 + ak_1^2 + bk_1^4 = 0.} \tag{9.19}$$

Then,

$$\boxed{f = 1 + E_1} \tag{9.20}$$

satisfies (9.1). In the normalization $u = 2b(\ln f)_{xx}$, the corresponding field is

$$\boxed{u^{(1)}(x,t) = 2b(\ln f)_{xx} = \frac{bk_1^2}{2}\operatorname{sech}^2\left(\frac{k_1 x - \omega_1 t + \delta_1}{2}\right).} \tag{9.21}$$

9.7. Two-soliton solution

For two solitons, set,

$$\theta_i = k_i x - \omega_i t + \delta_i, \qquad E_i = e^{\theta_i}, \qquad i = 1, 2 \tag{9.22}$$

with

$$\boxed{\omega_i^2 + ak_i^2 + bk_i^4 = 0, \qquad i = 1, 2.} \tag{9.23}$$

The two-soliton tau function has the scalar form

$$\boxed{f = 1 + E_1 + E_2 + A_{12}E_1E_2.} \tag{9.24}$$

Because (9.1) is a scalar one-tau equation, the interaction coefficient is

$$\boxed{A_{12} = -\frac{P(k_1 - k_2, \omega_1 - \omega_2)}{P(k_1 + k_2, \omega_1 + \omega_2)}.} \tag{9.25}$$

Thus,

$$\boxed{A_{12} = -\frac{(\omega_1 - \omega_2)^2 + a(k_1 - k_2)^2 + b(k_1 - k_2)^4}{(\omega_1 + \omega_2)^2 + a(k_1 + k_2)^2 + b(k_1 + k_2)^4}.} \tag{9.26}$$

In the normalization $u = 2b(\ln f)_{xx}$, the two-soliton solution is

$$\boxed{u^{(2)}(x,t) = 2b(\ln f)_{xx},} \tag{9.27}$$

with f given by (9.24). A direct symbolic check gives zero after imposing the two dispersion relations (9.17).

10. Sine–Gordon

We now derive the Hirota form of the sine–Gordon equation directly from differential identities. This equation is fundamentally different from KdV, KP, and Boussinesq scalar one-tau equations. It is naturally written with two real tau functions f and g.

Consider the normalized sine–Gordon family

$$\boxed{\phi_{xt} = \kappa \sin\phi,} \tag{10.1}$$

where $\kappa \neq 0$ is a real constant. The standard sine–Gordon equation corresponds to $\kappa = 1$. The sign of κ may be changed by reversing one light-cone variable.

We introduce the tau transformation

$$\boxed{\phi = 4\arctan\left(\frac{g}{f}\right),} \tag{10.2}$$

where $f = f(x,t)$ and $g = g(x,t)$ are real tau functions. Define

$$z = \frac{g}{f}. \tag{10.3}$$

Then,

$$\phi = 4\arctan z. \tag{10.4}$$

10.1. Elementary identities

We first compute ϕ_{xt} and $\sin\phi$ in terms of z.

Because

$$\phi_x = \frac{4z_x}{1+z^2},$$

we obtain

$$\phi_{xt} = 4\frac{(1+z^2)z_{xt} - 2zz_xz_t}{(1+z^2)^2}. \tag{10.5}$$

Also,

$$\sin(4\arctan z) = \frac{4z(1-z^2)}{(1+z^2)^2}. \tag{10.6}$$

Therefore, $\phi_{xt} = \kappa\sin\phi$ is equivalent to

$$\boxed{(1+z^2)z_{xt} - 2zz_xz_t = \kappa z(1-z^2).} \tag{10.7}$$

Thus, our task is to rewrite (10.7) in bilinear form.

10.2. Differential identities for $z = g/f$

Let

$$z = \frac{g}{f}.$$

The first required identity is

$$\boxed{\frac{D_x D_t(g \cdot f)}{f^2} = z_{xt} + 2z(\ln f)_{xt}.} \tag{10.8}$$

Indeed, since $g = zf$, we have

$$g_x = z_x f + z f_x, \qquad g_t = z_t f + z f_t,$$

and

$$g_{xt} = z_{xt} f + z_x f_t + z_t f_x + z f_{xt}.$$

Hence,

$$\begin{aligned} D_x D_t(g \cdot f) &= g_{xt} f - g_x f_t - g_t f_x + g f_{xt} \\ &= f^2 z_{xt} + 2z f f_{xt} - 2z f_x f_t \\ &= f^2 \left[z_{xt} + 2z(\ln f)_{xt} \right]. \end{aligned} \tag{10.9}$$

Next,

$$D_x D_t(f \cdot f) = 2(f f_{xt} - f_x f_t) \tag{10.10}$$

and therefore,

$$\boxed{\frac{D_x D_t(f \cdot f)}{2f^2} = (\ln f)_{xt}.} \tag{10.11}$$

We also need $D_x D_t(g \cdot g)$. Because $g = zf$,

$$g g_{xt} - g_x g_t = f^2 \left[z z_{xt} - z_x z_t + z^2 (\ln f)_{xt} \right]. \tag{10.12}$$

Thus,

$$\boxed{\frac{D_x D_t(g \cdot g)}{2f^2} = z z_{xt} - z_x z_t + z^2 (\ln f)_{xt}.} \tag{10.13}$$

Combining (10.11) and (10.13), we obtain

$$\boxed{\frac{D_x D_t(f \cdot f - g \cdot g)}{2f^2} = (1 - z^2)(\ln f)_{xt} - z z_{xt} + z_x z_t.} \tag{10.14}$$

10.3. Derivation of the bilinear system

We now impose the first bilinear equation

$$\boxed{(D_x D_t - \kappa) g \cdot f = 0.} \tag{10.15}$$

Using (10.8), Eq (10.15) becomes

$$z_{xt} + 2z(\ln f)_{xt} = \kappa z. \tag{10.16}$$

Therefore,

$$(\ln f)_{xt} = \frac{\kappa z - z_{xt}}{2z}. \tag{10.17}$$

Now, impose the second bilinear equation

$$\boxed{D_x D_t(f \cdot f - g \cdot g) = 0.} \tag{10.18}$$

Using (10.14), this gives

$$(1 - z^2)(\ln f)_{xt} - z z_{xt} + z_x z_t = 0. \tag{10.19}$$

Substituting (10.17) into (10.19), we get

$$(1 - z^2)\frac{\kappa z - z_{xt}}{2z} - z z_{xt} + z_x z_t = 0. \tag{10.20}$$

Multiplying by $2z$,

$$(1 - z^2)(\kappa z - z_{xt}) - 2z^2 z_{xt} + 2z z_x z_t = 0. \tag{10.21}$$

Expanding,

$$\kappa z(1 - z^2) - (1 - z^2)z_{xt} - 2z^2 z_{xt} + 2z z_x z_t = 0. \tag{10.22}$$

Because

$$-(1 - z^2)z_{xt} - 2z^2 z_{xt} = -(1 + z^2)z_{xt},$$

we obtain

$$\kappa z(1 - z^2) - (1 + z^2)z_{xt} + 2z z_x z_t = 0. \tag{10.23}$$

Equivalently,

$$\boxed{(1 + z^2)z_{xt} - 2z z_x z_t = \kappa z(1 - z^2).} \tag{10.24}$$

This is exactly (10.7). Hence, using $\phi = 4\arctan z$, we recover

$$\phi_{xt} = \kappa \sin\phi.$$

Therefore, the sine–Gordon family (10.1) is equivalent to the bilinear system

$$\boxed{\begin{aligned} (D_x D_t - \kappa)g \cdot f &= 0,\\ D_x D_t(f \cdot f - g \cdot g) &= 0, \end{aligned}} \tag{10.25}$$

with

$$\phi = 4\arctan\left(\frac{g}{f}\right).$$

10.4. Reverse verification

Conversely, suppose f and g satisfy (10.25), with $f \neq 0$, and define

$$z = \frac{g}{f}, \qquad \phi = 4 \arctan z.$$

The first bilinear equation gives

$$z_{xt} + 2z(\ln f)_{xt} = \kappa z.$$

The second gives

$$(1 - z^2)(\ln f)_{xt} - z z_{xt} + z_x z_t = 0.$$

Eliminating $(\ln f)_{xt}$ yields

$$(1 + z^2) z_{xt} - 2z z_x z_t = \kappa z(1 - z^2).$$

Using

$$\phi_{xt} = 4 \frac{(1 + z^2) z_{xt} - 2z z_x z_t}{(1 + z^2)^2}$$

and

$$\sin \phi = \frac{4z(1 - z^2)}{(1 + z^2)^2},$$

we obtain

$$\phi_{xt} = \kappa \sin \phi.$$

Thus, the bilinear system is exactly equivalent to the sine–Gordon family.

10.5. One-kink solution

Let

$$f = 1, \qquad g = e^{\eta}, \qquad \eta = kx + \Omega t + \delta. \tag{10.26}$$

The first bilinear equation gives

$$(D_x D_t - \kappa) e^{\eta} \cdot 1 = (k\Omega - \kappa) e^{\eta} = 0. \tag{10.27}$$

Thus, the dispersion relation is

$$\boxed{k\Omega = \kappa.} \tag{10.28}$$

The second bilinear equation is automatically satisfied because

$$D_x D_t (1 \cdot 1) = 0, \qquad D_x D_t (e^{\eta} \cdot e^{\eta}) = 0.$$

Therefore,

$$\boxed{\phi(x, t) = 4 \arctan \left(e^{kx + \frac{\kappa}{k} t + \delta} \right)} \tag{10.29}$$

is a one-kink solution of

$$\phi_{xt} = \kappa \sin \phi.$$

Changing g to $-g$ produces the antikink.

10.6. Two-soliton construction

For two sine–Gordon solitons, the tau functions have even–odd structure:

$$\boxed{g = s_1 E_1 + s_2 E_2, \qquad f = 1 - s_1 s_2 A_{12} E_1 E_2,} \tag{10.30}$$

where

$$E_i = e^{\eta_i}, \qquad \eta_i = k_i x + \Omega_i t + \delta_i, \qquad k_i \Omega_i = \kappa,$$

and

$$s_i = \pm 1.$$

The signs s_i encode kink or antikink orientation.

We now derive A_{12}. The first bilinear equation is

$$(D_x D_t - \kappa) g \cdot f = 0.$$

The terms $E_i \cdot 1$ vanish because $k_i \Omega_i = \kappa$. The terms $E_i \cdot E_1 E_2$ also vanish. For example,

$$D_x D_t E_1 \cdot (E_1 E_2) = (k_1 - k_1 - k_2)(\Omega_1 - \Omega_1 - \Omega_2) E_1^2 E_2 = k_2 \Omega_2 E_1^2 E_2 = \kappa E_1^2 E_2.$$

Hence,

$$(D_x D_t - \kappa) E_1 \cdot (E_1 E_2) = 0.$$

The same argument applies to $E_2 \cdot (E_1 E_2)$. Thus, the first bilinear equation imposes only one-soliton dispersions.

Now, use the second bilinear equation

$$D_x D_t (f \cdot f - g \cdot g) = 0.$$

The coefficient of $E_1 E_2$ in $D_x D_t (f \cdot f)$ is

$$-2 s_1 s_2 A_{12} (k_1 + k_2)(\Omega_1 + \Omega_2) E_1 E_2.$$

The coefficient of $E_1 E_2$ in $D_x D_t (g \cdot g)$ is

$$2 s_1 s_2 (k_1 - k_2)(\Omega_1 - \Omega_2) E_1 E_2.$$

Therefore, the coefficient of $E_1 E_2$ in $D_x D_t (f \cdot f - g \cdot g)$ is

$$-2 s_1 s_2 \left[A_{12} (k_1 + k_2)(\Omega_1 + \Omega_2) + (k_1 - k_2)(\Omega_1 - \Omega_2) \right] E_1 E_2.$$

Thus,

$$A_{12} = -\frac{(k_1 - k_2)(\Omega_1 - \Omega_2)}{(k_1 + k_2)(\Omega_1 + \Omega_2)}. \tag{10.31}$$

Using

$$\Omega_i = \frac{\kappa}{k_i},$$

we get

$$\Omega_1 - \Omega_2 = \kappa\left(\frac{1}{k_1} - \frac{1}{k_2}\right) = -\kappa\frac{k_1 - k_2}{k_1 k_2},$$

and

$$\Omega_1 + \Omega_2 = \kappa\left(\frac{1}{k_1} + \frac{1}{k_2}\right) = \kappa\frac{k_1 + k_2}{k_1 k_2}.$$

Therefore,

$$\boxed{A_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2.} \tag{10.32}$$

Thus, the two-soliton tau functions are

$$\boxed{g = s_1 e^{\eta_1} + s_2 e^{\eta_2}, \qquad f = 1 - s_1 s_2 \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 e^{\eta_1+\eta_2}.} \tag{10.33}$$

The associated field is

$$\boxed{\phi = 4\arctan\left(\frac{g}{f}\right).} \tag{10.34}$$

10.7. Comments on soliton polarity

If $s_1 s_2 = 1$, the two objects have the same polarity, and

$$f = 1 - A_{12}E_1E_2.$$

If $s_1 s_2 = -1$, the pair is a kink–antikink configuration, and

$$f = 1 + A_{12}E_1E_2.$$

Complex conjugate choices of k_i and δ_i yield breather-type solutions. These follow from the same tau structure after imposing reality of ϕ.

10.8. Why the scalar A_{12} formula is not used

The sine–Gordon equation is not represented by a single scalar equation

$$P(D)f \cdot f = 0.$$

It is instead represented by the coupled system

$$(D_xD_t - \kappa)g \cdot f = 0, \qquad D_xD_t(f \cdot f - g \cdot g) = 0.$$

Therefore, the usual scalar shortcut

$$A_{12} = -\frac{P(\mathbf{p}_1 - \mathbf{p}_2)}{P(\mathbf{p}_1 + \mathbf{p}_2)}$$

does not apply directly. The interaction coefficient

$$A_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2$$

must be derived from the coupled equations, as done above.

11. Manakov system

We consider the two-component Manakov system

$$\boxed{iu_{1,t} + u_{1,xx} + 2\sigma\left(|u_1|^2 + |u_2|^2\right)u_1 = 0,} \tag{11.1}$$

$$\boxed{iu_{2,t} + u_{2,xx} + 2\sigma\left(|u_1|^2 + |u_2|^2\right)u_2 = 0,} \tag{11.2}$$

where

$$\sigma = \pm 1.$$

The case $\sigma = 1$ is the focusing Manakov system, while $\sigma = -1$ is the defocusing Manakov system.

The Manakov system is not represented by one scalar tau equation

$$P(D)f \cdot f = 0.$$

Instead, it requires one real tau function f and two complex numerator tau functions g_1, g_2. We introduce

$$\boxed{u_j = \frac{g_j}{f}, \qquad \bar{u}_j = \frac{\bar{g}_j}{f}, \qquad j = 1, 2,} \tag{11.3}$$

where

$$f = \bar{f}, \qquad g_j \in \mathbb{C}.$$

11.1. Differential identities

Let

$$u_j = \frac{g_j}{f}, \qquad q = \ln f.$$

Because $g_j = u_j f$, we compute the Hirota identities component by component. First,

$$D_t g_j \cdot f = g_{j,t} f - g_j f_t. \tag{11.4}$$

Using $g_{j,t} = u_{j,t} f + u_j f_t$, we obtain

$$\begin{aligned} D_t g_j \cdot f &= (u_{j,t} f + u_j f_t) f - u_j f f_t \\ &= f^2 u_{j,t}. \end{aligned} \tag{11.5}$$

Thus,

$$\boxed{\frac{D_t g_j \cdot f}{f^2} = u_{j,t}.} \tag{11.6}$$

Next,

$$D_x^2 g_j \cdot f = g_{j,xx} f - 2 g_{j,x} f_x + g_j f_{xx}. \tag{11.7}$$

Because

$$g_{j,x} = u_{j,x} f + u_j f_x, \qquad g_{j,xx} = u_{j,xx} f + 2u_{j,x} f_x + u_j f_{xx},$$

substitution gives

$$\begin{aligned} D_x^2 g_j \cdot f &= (u_{j,xx} f + 2u_{j,x} f_x + u_j f_{xx}) f \\ &\quad - 2(u_{j,x} f + u_j f_x) f_x + u_j f f_{xx} \\ &= f^2 u_{j,xx} + 2u_j (f f_{xx} - f_x^2). \end{aligned} \tag{11.8}$$

Because $f f_{xx} - f_x^2 = f^2 (\ln f)_{xx} = f^2 q_{xx}$,

$$\boxed{\frac{D_x^2 g_j \cdot f}{f^2} = u_{j,xx} + 2q_{xx} u_j.} \tag{11.9}$$

Finally,

$$D_x^2 f \cdot f = 2(f f_{xx} - f_x^2) \tag{11.10}$$

and therefore,

$$\boxed{\frac{D_x^2 f \cdot f}{2f^2} = q_{xx}.} \tag{11.11}$$

11.2. Derivation of the bilinear system

Using (11.6) and (11.9), we get

$$\frac{(iD_t + D_x^2) g_j \cdot f}{f^2} = iu_{j,t} + u_{j,xx} + 2q_{xx} u_j. \tag{11.12}$$

The Manakov system requires

$$iu_{j,t} + u_{j,xx} + 2\sigma \left(|u_1|^2 + |u_2|^2 \right) u_j = 0, \qquad j = 1, 2. \tag{11.13}$$

Thus, we need

$$q_{xx} = \sigma \left(|u_1|^2 + |u_2|^2 \right). \tag{11.14}$$

Because

$$|u_1|^2 + |u_2|^2 = \frac{g_1 \bar{g}_1 + g_2 \bar{g}_2}{f^2},$$

condition (11.14) becomes

$$q_{xx} = \sigma \frac{g_1 \bar{g}_1 + g_2 \bar{g}_2}{f^2}. \tag{11.15}$$

Using (11.11), this is equivalent to

$$\boxed{D_x^2 f \cdot f = 2\sigma \left(g_1 \bar{g}_1 + g_2 \bar{g}_2 \right).} \tag{11.16}$$

Therefore, the closed Hirota bilinear system is

$$\boxed{\begin{aligned} (iD_t + D_x^2) g_1 \cdot f &= 0, \\ (iD_t + D_x^2) g_2 \cdot f &= 0, \\ (-iD_t + D_x^2) \bar{g}_1 \cdot f &= 0, \\ (-iD_t + D_x^2) \bar{g}_2 \cdot f &= 0, \\ D_x^2 f \cdot f &= 2\sigma \left(g_1 \bar{g}_1 + g_2 \bar{g}_2 \right). \end{aligned}} \tag{11.17}$$

11.3. Reverse verification

Assume that $f, g_1, g_2, \bar{g}_1, \bar{g}_2$ satisfy (11.17), with $f \neq 0$, and define

$$u_j = \frac{g_j}{f}, \qquad \bar{u}_j = \frac{\bar{g}_j}{f}, \qquad j = 1, 2.$$

Dividing the first two bilinear equations by f^2 and using (11.12), gives

$$iu_{j,t} + u_{j,xx} + 2q_{xx}u_j = 0, \qquad j = 1, 2. \tag{11.18}$$

Dividing the fifth bilinear equation by $2f^2$ and using (11.11), gives

$$q_{xx} = \sigma\frac{g_1\bar{g}_1 + g_2\bar{g}_2}{f^2} = \sigma\left(|u_1|^2 + |u_2|^2\right). \tag{11.19}$$

Substituting (11.19) into (11.18) yields the Manakov system. Thus, the bilinear system is equivalent to the PDE under the transformation (11.3).

11.4. Focusing bright one-soliton solution

We now take $\sigma = 1$. Let

$$E = e^{\eta}, \qquad \bar{E} = e^{\bar{\eta}} \tag{11.20}$$

where

$$\eta = px + ip^2t + \delta, \qquad \bar{\eta} = qx - iq^2t + \bar{\delta}. \tag{11.21}$$

For the algebraic calculation, $p, q, \delta, \bar{\delta}$ are kept independent. The physical reduction is imposed later by taking

$$q = \bar{p}, \qquad \bar{\delta} = \overline{\delta}.$$

Let $\alpha^{(1)}, \alpha^{(2)}$ be the polarization constants in g_1, g_2, and let $\beta^{(1)}, \beta^{(2)}$ be the corresponding constants in $\bar{g}_1, \bar{g}_2$. Define

$$\boxed{S = \alpha^{(1)}\beta^{(1)} + \alpha^{(2)}\beta^{(2)}.} \tag{11.22}$$

Then, the one-soliton tau functions are

$$\boxed{g_1 = \alpha^{(1)}E, \qquad g_2 = \alpha^{(2)}E,} \tag{11.23}$$

$$\boxed{\bar{g}_1 = \beta^{(1)}\bar{E}, \qquad \bar{g}_2 = \beta^{(2)}\bar{E},} \tag{11.24}$$

and

$$\boxed{f = 1 + \frac{S}{(p+q)^2}E\bar{E}.} \tag{11.25}$$

Therefore,

$$\boxed{u_j^{(1)}(x,t) = \frac{\alpha^{(j)}E}{1 + \dfrac{S}{(p+q)^2}E\bar{E}}, \qquad j = 1, 2.} \tag{11.26}$$

For the physical bright soliton, impose

$$q = \bar{p}, \qquad \bar{\delta} = \overline{\delta}, \qquad \beta^{(j)} = \overline{\alpha^{(j)}}, \qquad j = 1, 2. \tag{11.27}$$

Then, $S = |\alpha^{(1)}|^2 + |\alpha^{(2)}|^2$.

11.5. Real amplitude form of the one-soliton

Write

$$p = a + ib, \qquad a > 0, \qquad b \in \mathbb{R}. \tag{11.28}$$

Let $\eta = \eta_R + i\eta_I$. From $\eta = px + ip^2 t + \delta$, we obtain

$$\eta_R = ax - 2ab\, t + \delta_R \tag{11.29}$$

and

$$\eta_I = bx + (a^2 - b^2)t + \delta_I. \tag{11.30}$$

Under the physical reduction,

$$p + \bar{p} = 2a, \qquad S = |\alpha^{(1)}|^2 + |\alpha^{(2)}|^2.$$

Define

$$e^R = \frac{S}{4a^2}, \qquad \Theta = \eta_R + \frac{R}{2}. \tag{11.31}$$

Then,

$$\boxed{u_j^{(1)}(x,t) = ac_j \operatorname{sech}\Theta\, e^{i\eta_I}, \qquad j = 1, 2,} \tag{11.32}$$

where

$$\boxed{c_j = \frac{\alpha^{(j)}}{\sqrt{S}}, \qquad |c_1|^2 + |c_2|^2 = 1.} \tag{11.33}$$

Thus, the focusing Manakov bright one-soliton is a scalar bright envelope shared by two components through a constant polarization vector (c_1, c_2).

11.6. Focusing bright two-soliton solution

For two bright solitons, define

$$E_i = e^{\eta_i}, \qquad \bar{E}_i = e^{\bar{\eta}_i}, \qquad i = 1, 2 \tag{11.34}$$

where

$$\eta_i = p_i x + ip_i^2 t + \delta_i, \qquad \bar{\eta}_i = q_i x - iq_i^2 t + \bar{\delta}_i. \tag{11.35}$$

Again, $p_i, q_i, \delta_i, \bar{\delta}_i$ are treated as independent parameters during the algebraic verification. Let the polarization vectors be

$$\boldsymbol{\alpha}_i = (\alpha_i^{(1)}, \alpha_i^{(2)}), \qquad \boldsymbol{\beta}_i = (\beta_i^{(1)}, \beta_i^{(2)}), \qquad i = 1, 2.$$

Define the contractions

$$\boxed{S_{ij} = \alpha_i^{(1)}\beta_j^{(1)} + \alpha_i^{(2)}\beta_j^{(2)}, \qquad i, j = 1, 2} \tag{11.36}$$

and

$$\boxed{K_{ij} = p_i + q_j, \qquad F_{ij} = \frac{S_{ij}}{K_{ij}^2}, \qquad i, j = 1, 2.} \tag{11.37}$$

Let

$$\Delta_p = p_1 - p_2, \qquad \Delta_q = q_1 - q_2.$$

The denominator tau function is

$$\boxed{\begin{aligned} f = 1 &+ F_{11}E_1\bar{E}_1 + F_{12}E_1\bar{E}_2 + F_{21}E_2\bar{E}_1 + F_{22}E_2\bar{E}_2 \\ &+ F_{1122}E_1E_2\bar{E}_1\bar{E}_2. \end{aligned}} \tag{11.38}$$

The numerator tau functions are

$$\boxed{g_r = \alpha_1^{(r)}E_1 + \alpha_2^{(r)}E_2 + C_{r1}E_1E_2\bar{E}_1 + C_{r2}E_1E_2\bar{E}_2, \qquad r = 1, 2,} \tag{11.39}$$

and

$$\boxed{\bar{g}_r = \beta_1^{(r)}\bar{E}_1 + \beta_2^{(r)}\bar{E}_2 + D_{r1}E_1\bar{E}_1\bar{E}_2 + D_{r2}E_2\bar{E}_1\bar{E}_2, \qquad r = 1, 2.} \tag{11.40}$$

The cubic coefficients in g_r are

$$\boxed{C_{r1} = \Delta_p\left[\frac{\alpha_1^{(r)}S_{21}}{K_{11}K_{21}^2} - \frac{\alpha_2^{(r)}S_{11}}{K_{11}^2K_{21}}\right], \qquad r = 1, 2,} \tag{11.41}$$

and

$$\boxed{C_{r2} = \Delta_p\left[\frac{\alpha_1^{(r)}S_{22}}{K_{12}K_{22}^2} - \frac{\alpha_2^{(r)}S_{12}}{K_{12}^2K_{22}}\right], \qquad r = 1, 2.} \tag{11.42}$$

The cubic coefficients in $\bar{g}_r$ are

$$\boxed{D_{r1} = \Delta_q\left[\frac{\beta_1^{(r)}S_{12}}{K_{11}K_{12}^2} - \frac{\beta_2^{(r)}S_{11}}{K_{11}^2K_{12}}\right], \qquad r = 1, 2,} \tag{11.43}$$

and

$$\boxed{D_{r2} = \Delta_q\left[\frac{\beta_1^{(r)}S_{22}}{K_{21}K_{22}^2} - \frac{\beta_2^{(r)}S_{21}}{K_{21}^2K_{22}}\right], \qquad r = 1, 2.} \tag{11.44}$$

It remains to define F_{1122}. Set

$$\boxed{K = p_1 + p_2 + q_1 + q_2.} \tag{11.45}$$

Define

$$\boxed{H = (K_{11} - K_{22})^2F_{11}F_{22} + (K_{12} - K_{21})^2F_{12}F_{21},} \tag{11.46}$$

and

$$\boxed{R = \sum_{r=1}^{2}\left(\alpha_1^{(r)}D_{r2} + \alpha_2^{(r)}D_{r1} + C_{r1}\beta_2^{(r)} + C_{r2}\beta_1^{(r)}\right).} \tag{11.47}$$

Then,

$$\boxed{F_{1122} = \frac{R - H}{K^2}.} \tag{11.48}$$

Finally, the focusing Manakov two-soliton solution is

$$\boxed{u_r^{(2)}(x,t) = \frac{g_r}{f}, \qquad r = 1,2,} \tag{11.49}$$

where $f, g_r, \bar{g}_r$ are given by (11.38)–(11.48). For the physical complex-conjugate reduction, take

$$q_i = \bar{p}_i, \qquad \bar{\delta}_i = \overline{\delta_i}, \qquad \beta_i^{(r)} = \overline{\alpha_i^{(r)}}, \qquad i, r = 1,2. \tag{11.50}$$

11.7. Verification note

The one-soliton tau functions satisfy

$$(iD_t + D_x^2)g_r \cdot f = 0, \qquad (-iD_t + D_x^2)\bar{g}_r \cdot f = 0, \qquad r = 1,2,$$

and

$$D_x^2 f \cdot f = 2(g_1\bar{g}_1 + g_2\bar{g}_2).$$

The two-soliton tau functions were verified symbolically in the same bilinear system. The phase shifts were replaced by

$$\delta_i = \log \xi_i - p_i x - ip_i^2 t, \qquad \bar{\delta}_i = \log \rho_i - q_i x + iq_i^2 t, \qquad i = 1,2. \tag{11.51}$$

After this substitution, the five bilinear residuals become rational functions in $\xi_1, \xi_2, \rho_1, \rho_2$, and every numerator coefficient vanishes. Therefore, the fields $u_r = g_r/f$, $r = 1,2$, solve the focusing Manakov system

$$iu_{r,t} + u_{r,xx} + 2\left(|u_1|^2 + |u_2|^2\right)u_r = 0, \qquad r = 1,2.$$

11.8. Remark on the defocusing case

The bilinear system

$$(iD_t + D_x^2)g_r \cdot f = 0, \qquad (-iD_t + D_x^2)\bar{g}_r \cdot f = 0, \qquad r = 1,2,$$

and

$$D_x^2 f \cdot f = 2\sigma(g_1\bar{g}_1 + g_2\bar{g}_2)$$

is valid algebraically for $\sigma = \pm 1$. However, the zero-background bright tau functions above correspond to the focusing case $\sigma = 1$. The defocusing case $\sigma = -1$ requires a nonzero-background dark-soliton construction and is not obtained by merely changing the sign in these bright tau functions.

11.9. Why the scalar interaction formula is not applicable

The scalar formula

$$A_{12} = -\frac{P(\mathbf{p}_1 - \mathbf{p}_2)}{P(\mathbf{p}_1 + \mathbf{p}_2)} \tag{11.52}$$

applies to scalar one-tau bilinear equations $P(D)f \cdot f = 0$. The Manakov system is not of this form. Its Hirota representation is a coupled vector system, and the two-soliton coefficients depend on the polarization contractions

$$S_{ij} = \alpha_i^{(1)}\beta_j^{(1)} + \alpha_i^{(2)}\beta_j^{(2)}.$$

Therefore, the scalar A_{12}-shortcut is not valid for the Manakov system.

11.10. Remark on seventh-order KdV equations

The phrase "seventh-order KdV equation" is not unique. In general, one may consider the family

$$u_t + au^3u_x + bu_x^3 + cuu_xu_{xx} + du^2u_{xxx} + \alpha u_{xx}u_{xxx} + \beta u_xu_{xxxx} + \gamma uu_{xxxxx} + u_{xxxxxxx} = 0. \tag{11.53}$$

Only special choices of the parameters lead to soliton equations.

Classical seventh-order integrable cases include the seventh-order Sawada–Kotera–Itô equation, the seventh-order Lax equation, and the seventh-order Kaup–Kupershmidt equation. These equations pass the multisoliton test and admit N-soliton solutions.

However, there also exists a scalar Hirota-bilinear seventh-order KdV-type equation generated by

$$\boxed{(D_xD_t + D_x^8)f \cdot f = 0,} \tag{11.54}$$

with

$$\boxed{u = 2(\ln f)_{xx}.} \tag{11.55}$$

This equation has one- and two-soliton solutions, but it does not necessarily belong to the fully integrable KdV hierarchy because it fails the three-soliton test.

For the scalar bilinear equation (11.54), the one-soliton phase is

$$\eta_i = k_ix - k_i^7t + \delta_i, \qquad i = 1, 2 \tag{11.56}$$

and the two-soliton tau function is

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2} \tag{11.57}$$

where

$$\boxed{A_{12} = \frac{(k_1 - k_2)^2(k_1^2 - k_1k_2 + k_2^2)^2}{(k_1 + k_2)^2(k_1^2 + k_1k_2 + k_2^2)^2}.} \tag{11.58}$$

Therefore,

$$\boxed{u(x,t) = 2\partial_x^2 \ln\left[1 + e^{\eta_1} + e^{\eta_2} + \frac{(k_1 - k_2)^2(k_1^2 - k_1k_2 + k_2^2)^2}{(k_1 + k_2)^2(k_1^2 + k_1k_2 + k_2^2)^2}e^{\eta_1+\eta_2}\right].} \tag{11.59}$$

More generally, the scalar bilinear equation

$$\boxed{(D_xD_t + pD_x^4 + qD_x^6 + D_x^8)f \cdot f = 0} \tag{11.60}$$

has the dispersion relation

$$\boxed{\omega_i = pk_i^3 + qk_i^5 + k_i^7, \qquad \eta_i = k_ix - \omega_it + \delta_i.} \tag{11.61}$$

The corresponding scalar two-soliton coefficient is

$$\boxed{A_{12} = \frac{(k_1 - k_2)^2\left[7k_1^4 - 14k_1^3k_2 + 21k_1^2k_2^2 - 14k_1k_2^3 + 7k_2^4 + 5q(k_1^2 - k_1k_2 + k_2^2) + 3p\right]}{(k_1 + k_2)^2\left[7k_1^4 + 14k_1^3k_2 + 21k_1^2k_2^2 + 14k_1k_2^3 + 7k_2^4 + 5q(k_1^2 + k_1k_2 + k_2^2) + 3p\right]}.} \tag{11.62}$$

Thus, the correct conclusion is the following. Some seventh-order KdV equations are fully integrable and admit N-soliton solutions. Other seventh-order KdV-type equations are bilinearizable in a scalar Hirota form and admit one- and two-soliton solutions, but they may fail the three-soliton test. Therefore, for KdV7, one must distinguish between the following:

1) the integrable seventh-order KdV hierarchy members;
2) scalar bilinear seventh-order KdV-type equations with two solitons;
3) generalized seventh-order KdV equations that admit only special traveling-wave or restricted soliton solutions.

12. Fractional clock variants and one-soliton solutions

We close the paper with a fractional-clock extension of the one-soliton solutions considered above. The construction is not the standard Caputo replacement. It is instead a natural time-clock reparametrization. Namely, every classical solution written in an evolution variable T is converted into a clock-fractional solution by replacing

$$T \longmapsto \Theta_\alpha(t), \qquad \Theta_\alpha(t) = \frac{t^\alpha}{\Gamma(1+\alpha)}, \qquad 0 < \alpha \leq 1. \tag{12.1}$$

12.1. Clock derivative and chain rule

Define the clock derivative by

$$\boxed{\mathcal{D}_t^\alpha F(t) = \frac{1}{\Theta'_\alpha(t)} \frac{dF}{dt}.} \tag{12.2}$$

Because

$$\Theta'_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \tag{12.3}$$

we may also write

$$\boxed{\mathcal{D}_t^\alpha F(t) = \Gamma(\alpha) t^{1-\alpha} \frac{dF}{dt}.} \tag{12.4}$$

If

$$F(t) = \widehat{F}(\Theta_\alpha(t)) \tag{12.5}$$

then

$$\frac{dF}{dt} = \widehat{F}_T(\Theta_\alpha(t))\Theta'_\alpha(t). \tag{12.6}$$

Therefore,

$$\boxed{\mathcal{D}_t^\alpha F(t) = \widehat{F}_T(\Theta_\alpha(t)).} \tag{12.7}$$

This identity is the basis of all formulae in this section.

Important distinction. The operator $\mathcal{D}_t^\alpha$ is a local derivative with respect to the clock $T = \Theta_\alpha(t)$. It is not the standard Caputo derivative. Hence, the formulae below are exact for the clock-fractional equations written with $\mathcal{D}_t^\alpha$. For a nonlocal Caputo equation, the same expressions must be checked by direct residual substitution.

12.2. Master reparametrization principle

Let $U(\mathbf{x}, T)$ solve a classical evolution equation

$$U_T = \mathcal{K}[U] \tag{12.8}$$

where $\mathcal{K}$ contains only spatial derivatives, shifts, lattice operators, or algebraic nonlinearities. Define

$$u(\mathbf{x}, t) = U(\mathbf{x}, \Theta_\alpha(t)). \tag{12.9}$$

Using (12.7), one gets

$$\mathcal{D}_t^\alpha u(\mathbf{x}, t) = U_T(\mathbf{x}, \Theta_\alpha(t)) = \mathcal{K}[U](\mathbf{x}, \Theta_\alpha(t)) = \mathcal{K}[u](\mathbf{x}, t). \tag{12.10}$$

Therefore, u solves

$$\boxed{\mathcal{D}_t^\alpha u = \mathcal{K}[u].} \tag{12.11}$$

Equivalently, every classical one-soliton phase

$$\eta = kx - \omega T + \delta \tag{12.12}$$

becomes

$$\boxed{\eta_\alpha = kx - \omega\Theta_\alpha(t) + \delta.} \tag{12.13}$$

12.3. KdV equation

The clock-fractional KdV equation is

$$\boxed{\mathcal{D}_t^\alpha u + 6uu_x + u_{xxx} = 0.} \tag{12.14}$$

The one-soliton phase is

$$\eta_\alpha = kx - k^3\Theta_\alpha(t) + \delta. \tag{12.15}$$

With

$$f = 1 + \exp(\eta_\alpha) \tag{12.16}$$

and

$$u = 2(\ln f)_{xx} \tag{12.17}$$

one obtains

$$\boxed{u(x, t) = \frac{k^2}{2}\operatorname{sech}^2\left(\frac{kx - k^3\Theta_\alpha(t) + \delta}{2}\right).} \tag{12.18}$$

12.4. Modified KdV equation

For the positive-sign modified KdV equation, the clock-fractional model is

$$\boxed{\mathcal{D}_t^\alpha u + u_{xxx} + 6u^2u_x = 0.} \tag{12.19}$$

The one-soliton tau functions are

$$g = \exp(\eta_\alpha), \qquad f = 1 + \frac{\exp(2\eta_\alpha)}{4k^2} \tag{12.20}$$

where

$$\eta_\alpha = kx - k^3\Theta_\alpha(t) + \delta. \tag{12.21}$$

Therefore,

$$\boxed{u(x,t) = \frac{g}{f} = k\,\mathrm{sech}\left(kx - k^3\Theta_\alpha(t) + \delta - \ln(2k)\right).} \tag{12.22}$$

12.5. Normalized Gardner equation

The normalized Gardner equation is

$$U_T + \partial_X\left(U_{XX} + 6\mu U^2 + 2U^3\right) = 0. \tag{12.23}$$

The clock-fractional version is

$$\boxed{\mathcal{D}_t^\alpha U + \partial_x\left(U_{xx} + 6\mu U^2 + 2U^3\right) = 0.} \tag{12.24}$$

Using the shift relation

$$W = U + \mu \tag{12.25}$$

and the transport variable

$$Y = x + 6\mu^2 T \tag{12.26}$$

a one-soliton solution is obtained from the mKdV one-soliton. After replacing T by $\Theta_\alpha(t)$, we obtain

$$\boxed{U(x,t) = k\,\mathrm{sech}\left(kx + \left(6k\mu^2 - k^3\right)\Theta_\alpha(t) + \delta - \ln(2k)\right) - \mu.} \tag{12.27}$$

12.6. General Gardner equation

Consider the clock-fractional Gardner equation

$$\boxed{\mathcal{D}_t^\alpha v + avv_x + bv^2 v_x + v_{xxx} = 0.} \tag{12.28}$$

Let $U(X,T)$ solve the normalized Gardner equation. Define

$$v(x,t) = pU(qx, r\Theta_\alpha(t)). \tag{12.29}$$

Then, v solves (12.28) provided

$$r = q^3, \qquad ap = 12\mu q^2, \qquad bp^2 = 6q^2. \tag{12.30}$$

One convenient branch is

$$p = \frac{a}{2b\mu}, \qquad q = -\frac{a}{2\sqrt{6}\sqrt{b}\,\mu}, \qquad r = -\frac{a^3}{48\sqrt{6}\,b^{3/2}\mu^3}. \tag{12.31}$$

Therefore,

$$\boxed{v(x,t) = \frac{a}{2b\mu}U\left(-\frac{ax}{2\sqrt{6}\sqrt{b}\,\mu}, -\frac{a^3\Theta_\alpha(t)}{48\sqrt{6}\,b^{3/2}\mu^3}\right)} \tag{12.32}$$

solves (12.28). Inserting (12.27) gives the corresponding one-soliton member of the general Gardner family.

12.7. Sawada–Kotera/CDG branch

For the Sawada–Kotera/CDG branch,

$$\alpha_5 = \frac{\gamma^2}{5}, \qquad \beta_5 = \gamma, \qquad u = \frac{30}{\gamma}(\ln f)_{xx} \tag{12.33}$$

the scalar bilinear equation is

$$(D_x D_T + D_x^6) f \cdot f = 0. \tag{12.34}$$

The one-soliton phase is

$$\eta_\alpha = kx - k^5 \Theta_\alpha(t) + \delta. \tag{12.35}$$

With

$$f = 1 + \exp(\eta_\alpha) \tag{12.36}$$

we obtain

$$\boxed{u(x,t) = \frac{30}{\gamma}\partial_x^2 \ln f = \frac{15k^2}{2\gamma}\operatorname{sech}^2\left(\frac{\eta_\alpha}{2}\right).} \tag{12.37}$$

12.8. Kaup–Kupershmidt branch

For the Kaup–Kupershmidt branch,

$$\alpha_5 = \frac{\gamma^2}{5}, \qquad \beta_5 = \frac{5\gamma}{2}, \qquad u = \frac{15}{\gamma}(\ln f)_{xx}, \tag{12.38}$$

the bilinearization is coupled:

$$D_x^4(f \cdot f) - 4fg = 0 \tag{12.39}$$

$$\left(D_x D_T + \frac{1}{16} D_x^6\right) f \cdot f + \frac{15}{4} D_x^2 (f \cdot g) = 0. \tag{12.40}$$

For one soliton,

$$\eta_\alpha = kx - k^5 \Theta_\alpha(t) + \delta, \qquad f = 1 + \exp(\eta_\alpha). \tag{12.41}$$

Hence,

$$\boxed{u(x,t) = \frac{15}{\gamma}\partial_x^2 \ln f = \frac{15k^2}{4\gamma}\operatorname{sech}^2\left(\frac{\eta_\alpha}{2}\right).} \tag{12.42}$$

The auxiliary tau function is determined by

$$g = \frac{D_x^4(f \cdot f)}{4f}. \tag{12.43}$$

12.9. Lax fifth-order branch

For the Lax fifth-order branch,

$$\alpha_5 = \frac{3\gamma^2}{10}, \qquad \beta_5 = 2\gamma, \qquad u = \frac{20}{\gamma}(\ln f)_{xx}, \tag{12.44}$$

the hierarchy bilinearization is

$$(D_x D_s + D_x^4) f \cdot f = 0 \tag{12.45}$$

$$(D_x D_T + D_x^6) f \cdot f - \frac{5}{3}\left(D_s^2 + D_s D_x^3\right) f \cdot f = 0. \tag{12.46}$$

A one-soliton hierarchy phase is

$$\eta_\alpha = kx - k^3 s - k^5 \Theta_\alpha(t) + \delta. \tag{12.47}$$

For a fixed hierarchy time $s = 0$, this reduces to

$$\eta_\alpha = kx - k^5 \Theta_\alpha(t) + \delta. \tag{12.48}$$

With

$$f = 1 + \exp(\eta_\alpha) \tag{12.49}$$

one obtains

$$\boxed{u(x,t) = \frac{20}{\gamma}\partial_x^2 \ln f = \frac{5k^2}{\gamma}\,\mathrm{sech}^2\left(\frac{\eta_\alpha}{2}\right).} \tag{12.50}$$

12.10. Scalar seventh-order KdV-type equation

Consider

$$(D_x D_T + pD_x^4 + qD_x^6 + D_x^8) f \cdot f = 0, \qquad u = 2(\ln f)_{xx}. \tag{12.51}$$

The one-soliton phase is

$$\eta_\alpha = kx - \omega(k)\Theta_\alpha(t) + \delta \tag{12.52}$$

where

$$\boxed{\omega(k) = pk^3 + qk^5 + k^7.} \tag{12.53}$$

Thus,

$$f = 1 + \exp(\eta_\alpha) \tag{12.54}$$

and

$$\boxed{u(x,t) = \frac{k^2}{2}\,\mathrm{sech}^2\left(\frac{\eta_\alpha}{2}\right).} \tag{12.55}$$

This is a clock-fractional one-soliton of the scalar seventh-order KdV-type equation. It does not by itself prove complete integrability.

12.11. KP equation

The clock-fractional KP equation in the normalization used in this paper is

$$\boxed{(\mathcal{D}_t^{\alpha} u + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0.} \tag{12.56}$$

Let

$$u = 2(\ln f)_{xx}. \tag{12.57}$$

The one-soliton phase is

$$\eta_\alpha = kx + \ell y - \omega \Theta_\alpha(t) + \delta \tag{12.58}$$

where

$$\boxed{\omega = k^3 + \frac{3\sigma^2 \ell^2}{k}, \qquad k \neq 0.} \tag{12.59}$$

Thus,

$$f = 1 + \exp(\eta_\alpha) \tag{12.60}$$

and

$$\boxed{u(x, y, t) = \frac{k^2}{2} \operatorname{sech}^2\left(\frac{\eta_\alpha}{2}\right).} \tag{12.61}$$

12.12. Boussinesq family

Consider the Boussinesq family

$$u_{TT} + Au_{xx} + Bu_{4x} + C(u^2)_{xx} = 0. \tag{12.62}$$

The clock-fractional version is

$$\boxed{(\mathcal{D}_t^{\alpha})^2 u + Au_{xx} + Bu_{4x} + C(u^2)_{xx} = 0.} \tag{12.63}$$

The tau transformation is

$$u = \frac{6B}{C}(\ln f)_{xx}. \tag{12.64}$$

The one-soliton phase is

$$\eta_\alpha = kx - \omega \Theta_\alpha(t) + \delta \tag{12.65}$$

with

$$\boxed{\omega^2 + Ak^2 + Bk^4 = 0.} \tag{12.66}$$

Thus,

$$f = 1 + \exp(\eta_\alpha) \tag{12.67}$$

and

$$\boxed{u(x, t) = \frac{3Bk^2}{2C} \operatorname{sech}^2\left(\frac{\eta_\alpha}{2}\right).} \tag{12.68}$$

12.13. Fokas (4 + 1)-dimensional equation

For the Fokas-type bilinear equation

$$(4D_xD_T - D_x^3D_y + D_xD_y^3 - 6D_zD_w)f \cdot f = 0 \tag{12.69}$$

let

$$q = \ln f. \tag{12.70}$$

The one-soliton phase is

$$\eta_\alpha = kx + \ell y + mz + nw - \omega\Theta_\alpha(t) + \delta. \tag{12.71}$$

The dispersion polynomial is

$$P(k, \ell, m, n, \omega) = -4k\omega - k^3\ell + k\ell^3 - 6mn. \tag{12.72}$$

Hence,

$$\boxed{\omega = \frac{-k^3\ell + k\ell^3 - 6mn}{4k}, \qquad k \neq 0.} \tag{12.73}$$

The potential one-soliton is

$$\boxed{q(x, y, z, w, t) = \ln\left(1 + \exp(\eta_\alpha)\right).} \tag{12.74}$$

It solves the clock-fractional potential equation obtained from

$$4q_{xT} - q_{xxxy} + q_{xyyy} - 6q_{zw} + 6(q_{yy} - q_{xx})q_{xy} = 0 \tag{12.75}$$

by replacing

$$q_{xT} \longmapsto \partial_x\mathcal{D}_t^\alpha q. \tag{12.76}$$

12.14. Nonlinear Schrödinger equation

The focusing clock-fractional NLS equation is

$$\boxed{i\mathcal{D}_t^\alpha u + u_{xx} + 2|u|^2u = 0.} \tag{12.77}$$

Let

$$u = \frac{g}{f}. \tag{12.78}$$

Set

$$\eta_\alpha = px + ip^2\Theta_\alpha(t) + \eta_0, \qquad p = a + ib, \qquad a > 0. \tag{12.79}$$

The one-soliton tau functions are

$$g = \exp(\eta_\alpha), \qquad f = 1 + \frac{\exp(\eta_\alpha + \overline{\eta}_\alpha)}{(p + \overline{p})^2}. \tag{12.80}$$

Therefore,

$$\boxed{u(x, t) = a\,\mathrm{sech}\left(a[x - 2b\Theta_\alpha(t) - x_0]\right)\exp\left(i[bx + (a^2 - b^2)\Theta_\alpha(t) + \phi_0]\right).} \tag{12.81}$$

12.15. Manakov system

The focusing clock-fractional Manakov system is

$$\boxed{i\mathcal{D}_t^{\alpha} u_j + u_{j,xx} + 2\left(|u_1|^2 + |u_2|^2\right) u_j = 0, \qquad j = 1, 2.} \tag{12.82}$$

Let

$$u_j = \frac{g_j}{f}, \qquad j = 1, 2. \tag{12.83}$$

Set

$$\eta_\alpha = px + ip^2\Theta_\alpha(t) + \eta_0, \qquad p = a + ib, \qquad a > 0. \tag{12.84}$$

Choose polarization constants $\alpha_1, \alpha_2 \in \mathbb{C}$, and define

$$S_0 = |\alpha_1|^2 + |\alpha_2|^2. \tag{12.85}$$

The tau functions are

$$g_j = \alpha_j \exp(\eta_\alpha), \qquad j = 1, 2 \tag{12.86}$$

and

$$f = 1 + \frac{S_0}{(p + \overline{p})^2} \exp(\eta_\alpha + \overline{\eta}_\alpha). \tag{12.87}$$

Writing

$$c_j = \frac{\alpha_j}{\sqrt{S_0}}, \qquad |c_1|^2 + |c_2|^2 = 1 \tag{12.88}$$

we get

$$\boxed{u_j(x, t) = ac_j \operatorname{sech}\left(a[x - 2b\Theta_\alpha(t) - x_0]\right) \exp\left(i[bx + (a^2 - b^2)\Theta_\alpha(t) + \phi_0]\right), \quad j = 1, 2.} \tag{12.89}$$

12.16. Sine–Gordon equation

The clock-fractional sine–Gordon equation is

$$\boxed{\partial_x \left(\mathcal{D}_t^{\alpha} \phi\right) = \kappa \sin \phi.} \tag{12.90}$$

The two-tau transformation is

$$\phi = 4 \arctan\left(\frac{g}{f}\right). \tag{12.91}$$

For a kink, take

$$f = 1, \qquad g = \exp(\eta_\alpha) \tag{12.92}$$

where

$$\eta_\alpha = kx + \Omega\Theta_\alpha(t) + \delta, \qquad k\Omega = \kappa. \tag{12.93}$$

Hence,

$$\boxed{\phi(x, t) = 4 \arctan\left[\exp\left(kx + \frac{\kappa}{k}\Theta_\alpha(t) + \delta\right)\right].} \tag{12.94}$$

12.17. Toda lattice

For the Toda lattice bilinear form

$$(D_x D_T - 1)\tau_n \cdot \tau_n = -\tau_{n+1}\tau_{n-1} \tag{12.95}$$

the clock-fractional version is obtained by replacing T with $\Theta_\alpha(t)$. Let

$$\tau_n = 1 + \exp(\eta_{n,\alpha}) \tag{12.96}$$

where

$$\eta_{n,\alpha} = kx - \omega\Theta_\alpha(t) + n\rho + \delta. \tag{12.97}$$

Substitution gives the dispersion relation

$$\boxed{k\omega = \cosh(\rho) - 1.} \tag{12.98}$$

Therefore,

$$\boxed{\tau_n(x,t) = 1 + \exp\left(kx - \frac{\cosh(\rho) - 1}{k}\Theta_\alpha(t) + n\rho + \delta\right).} \tag{12.99}$$

With

$$u_n = \ln\left(\frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}\right) \tag{12.100}$$

this gives the corresponding clock-fractional Toda one-soliton.

13. Conclusions

This work presented a differential-identity-based verification of Hirota bilinear forms for several classical nonlinear evolution equations. The central point is that Hirota bilinearization must be handled equation by equation. A single scalar tau equation of the form

$$P(D)f \cdot f = 0 \tag{13.1}$$

is powerful, but it is not universal. Several important integrable equations require coupled tau functions, auxiliary fields, lattice tau functions, or additional hierarchy times.

For scalar one-tau equations such as KdV, KP in the correctly normalized form, the Sawada–Kotera/Caudrey–Dodd–Gibbon branch, the Boussinesq family, and certain scalar seventh-order KdV-type equations, the bilinear form follows directly from logarithmic substitutions such as

$$u = 2(\ln f)_{xx} \tag{13.2}$$

or its coefficient-adjusted variants. In these cases, the derivation is controlled by identities such as

$$\frac{D_x D_t(f \cdot f)}{2f^2} = q_{xt}, \qquad \frac{D_x^4(f \cdot f)}{2f^2} = q_{4x} + 6q_{xx}^2, \qquad q = \ln f. \tag{13.3}$$

Thus, the bilinear equation is not guessed; it is obtained by matching the differential identity with the transformed nonlinear PDE.

The fifth-order KdV family provides the clearest test case. Starting from

$$u_t + \alpha u^2 u_x + \beta u_x u_{xx} + \gamma u u_{xxx} + u_{5x} = 0 \tag{13.4}$$

the logarithmic substitution

$$u = c(\ln f)_{xx} \tag{13.5}$$

separates three distinct bilinear mechanisms. The Sawada–Kotera/CDG branch is represented by the single scalar equation

$$(D_x D_t + D_x^6) f \cdot f = 0. \tag{13.6}$$

The Kaup–Kupershmidt branch requires an auxiliary tau function g because the transformed equation contains a q_{xxx}^2 term that cannot be produced by $D_x^6(f \cdot f)/(2f^2)$. The Lax branch belongs to the KdV hierarchy and requires an auxiliary hierarchy time s. Therefore, these three branches are all bilinearizable but not in the same sense.

This distinction is important for soliton formulae. The scalar interaction coefficient

$$A_{12} = -\frac{P(\mathbf{p}_1 - \mathbf{p}_2)}{P(\mathbf{p}_1 + \mathbf{p}_2)} \tag{13.7}$$

is valid only for scalar one-tau equations

$$P(D) f \cdot f = 0. \tag{13.8}$$

It must not be applied directly to coupled systems such as mKdV, NLS, Manakov, sine–Gordon, Gardner, or Kaup–Kupershmidt. In those cases, the interaction coefficients must be derived from the full coupled Hirota expansion or from the corresponding tau representation.

The paper also clarified normalization issues. For example, the KP equation

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0 \tag{13.9}$$

is associated with the bilinear equation

$$(D_x D_t + D_x^4 + 3\sigma^2 D_y^2) f \cdot f = 0. \tag{13.10}$$

The alternative form

$$(D_x^4 + 3\sigma^2 D_y^2 - 4 D_x D_t) f \cdot f = 0 \tag{13.11}$$

corresponds to a different time normalization. Similarly, the Fokas (4+1)-dimensional bilinear equation gives a verified potential equation for $q = \ln f$; it should not be converted into a physical equation for a new variable unless the transformation is separately justified.

For seventh-order KdV-type equations, the analysis shows that the phrase "KdV7" is not unique. Some seventh-order equations are completely integrable hierarchy members and admit N-soliton solutions. Other scalar bilinear seventh-order equations admit one- and two-soliton solutions through the Hirota calculation, but the existence of two solitons alone does not prove complete integrability. A three-soliton test or an explicit hierarchy structure is still required.

Finally, the fractional-clock extension gives an exact way to generate fractional-time one-soliton solutions without destroying the algebraic Hirota structure. The essential clock is

$$T = \Theta_\alpha(t) = \frac{t^\alpha}{\Gamma(1+\alpha)}, \qquad 0 < \alpha \leq 1 \tag{13.12}$$

and the associated clock derivative is

$$\mathcal{D}_t^\alpha F(t) = \frac{1}{\Theta'_\alpha(t)} \frac{dF}{dt}. \tag{13.13}$$

If $F(t) = \widehat{F}(\Theta_\alpha(t))$, then

$$\mathcal{D}_t^\alpha F(t) = \widehat{F}_T(\Theta_\alpha(t)). \tag{13.14}$$

Consequently, every classical one-soliton tau function depending on the evolution time T produces an exact clock-fractional one-soliton by the replacement

$$T \mapsto \Theta_\alpha(t). \tag{13.15}$$

This clock construction is exact for the local clock-fractional models written with $\mathcal{D}_t^\alpha$. It should not be confused with the standard Caputo or Riemann–Liouville replacement. For genuinely nonlocal fractional models, the same reparametrized soliton formulae are useful ansatz functions, but they are not automatically exact. In that case, a direct residual calculation is necessary.

In summary, the reliable use of Hirota's method requires three safeguards: First, derive the bilinear form from explicit differential identities; second, separate scalar, coupled, hierarchy, lattice, and potential bilinearizations; and third, distinguish exact clock-fractional reparametrizations from nonlocal fractional equations. With these safeguards, the catalogue becomes a verified computational reference rather than a formal list of untested bilinear formulae.

Author contributions

A.H.S.: Conceptualization, methodology, formal analysis, investigation, validation, writing–original draft, and writing–review and editing. C.A.G.S.: Methodology, formal analysis, validation, and writing–review and editing. S.C.T.: Conceptualization, methodology, validation, supervision, and writing–review and editing. All authors have read and approved the final version of the manuscript.

Use of generative AI tools declaration

The authors used generative artificial intelligence tools for language editing, grammatical refinement, and LaTeX formatting assistance during manuscript preparation. The mathematical derivations, equations, proofs, interpretations, reference selection, and final scientific content were independently checked and approved by the authors, who take full responsibility for the manuscript.

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Conflict of interest

The authors declare that they have no conflicts of interest in this paper.

References

1. R. Hirota, *The direct method in soliton theory*, Cambridge: Cambridge University Press, 2004.
2. Y. Matsuno, *Bilinear transformation method*, Orlando: Academic Press, 1984.
3. M. J. Ablowitz, H. Segur, *Solitons and the inverse scattering transform*, Philadelphia: SIAM, 1981.
4. M. J. Ablowitz, P. A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*, Cambridge: Cambridge University Press, 1991.
5. P. G. Drazin, R. S. Johnson, *Solitons: An introduction*, Cambridge: Cambridge University Press, 1989.
6. L. D. Faddeev, L. A. Takhtajan, *Hamiltonian methods in the theory of solitons*, Springer, 1987. https://doi.org/10.1007/978-3-540-69969-9
7. D. J. Korteweg, G. de Vries, On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, *Phil. Mag.*, **39** (1895), 422–443.
8. C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura, Method for solving the Korteweg–de Vries equation, *Phys. Rev. Lett.*, **19** (1967), 1095. https://doi.org/10.1103/PhysRevLett.19.1095
9. P. D. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Commun. Pure Appl. Math.*, **21** (1968), 467–490. https://doi.org/10.1002/cpa.3160210503
10. R. M. Miura, The Korteweg–de Vries equation: A survey of results, *SIAM Rev.*, **18** (1976), 412–459. https://doi.org/10.1137/1018076
11. J. Hietarinta, A search for bilinear equations passing Hirota's three-soliton condition. I. KdV-type bilinear equations, *J. Math. Phys.*, **28** (1987), 1732–1742.
12. J. Hietarinta, A search for bilinear equations passing Hirota's three-soliton condition. II. mKdV-type bilinear equations, *J. Math. Phys.*, **28** (1987), 2094–2101.
13. J. Hietarinta, A search for bilinear equations passing Hirota's three-soliton condition. III. Sine-Gordon-type bilinear equations, *J. Math. Phys.*, **28** (1987), 2586–2592.
14. J. Hietarinta, A search for bilinear equations passing Hirota's three-soliton condition. IV. Complex bilinear equations, *J. Math. Phys.*, **29** (1988), 628–635.
15. J. Hietarinta, Introduction to the Hirota bilinear method, 1997, arXiv: solv-int/9708006. https://doi.org/10.48550/arXiv.solv-int/9708006
16. K. Sawada, T. Kotera, A method for finding N-soliton solutions of the K.d.V. equation and K.d.V.-like equation, *Prog. Theor. Phys.*, **51** (1974), 1355–1367. https://doi.org/10.1143/PTP.51.1355
17. P. J. Caudrey, R. K. Dodd, J. D. Gibbon, A new hierarchy of Korteweg–de Vries equations, *Proc. R. Soc. A*, **351** (1976), 407–422. https://doi.org/10.1098/rspa.1976.0149
18. R. K. Dodd, J. D. Gibbon, The prolongation structure of a higher order Korteweg–de Vries equation, *Proc. R. Soc. A*, **358** (1978), 287–296. https://doi.org/10.1098/rspa.1978.0011

19. D. J. Kaup, On the inverse scattering problem for cubic eigenvalue problems of the class $\psi_{xxx} + 6Q\psi_x + 6R\psi = \lambda\psi$, *Stud. Appl. Math.*, **62** (1980), 189–216. https://doi.org/10.1002/sapm1980623189

20. W. Hereman, U. Göktaş, Symbolic computation of solitary wave solutions and solitons through homogenization of degree, 2023, arXiv: 2307.08888. https://doi.org/10.48550/arXiv.2307.08888

21. V. E. Zakharov, A. B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, *Sov. Phys. JETP*, **34** (1972), 62–69.

22. S. V. Manakov, On the theory of two-dimensional stationary self-focusing of electromagnetic waves, *Sov. Phys. JETP*, **38** (1974), 248–253.

23. Y. Kodama, *Solitons in two-dimensional shallow water*, Philadelphia: SIAM, 2018.

24. M. Jimbo, T. Miwa, Solitons and infinite dimensional Lie algebras, *Publ. Res. Inst. Math. Sci.*, **19** (1983), 943–1001. https://doi.org/10.2977/prims/1195182017

25. E. Date, M. Jimbo, M. Kashiwara, T. Miwa, Transformation groups for soliton equations, In *Nonlinear integrable systems—Classical theory and quantum theory*, World Scientific, Singapore, 1983.

26. T. Miwa, On Hirota's difference equations, *Proc. Japan Acad. Ser. A Math. Sci.*, **58** (1982), 9–12. https://doi.org/10.3792/pjaa.58.9

27. M. Toda, Vibration of a chain with nonlinear interaction, *J. Phys. Soc. Japan*, **22** (1967), 431–436. https://doi.org/10.1143/JPSJ.22.431

28. A. C. Scott, F. Y. F. Chu, D. W. McLaughlin, The soliton: A new concept in applied science, *Proc. IEEE*, **61** (1973), 1443–1483. https://doi.org/10.1109/PROC.1973.9296

29. G. L. Lamb Jr, *Elements of soliton theory*, New York: Wiley, 1980.

30. A. H. Salas, Perspective chapter: Families of seventh-order KdV equations having traveling wave and soliton solutions, In: *Nonlinear systems and matrix analysis–Recent advances in theory and applications*, 2024. https://doi.org/10.5772/intechopen.1004789

31. R. X. Yao, Z. B. Li, Conservation laws and new exact solutions for the generalized seventh order KdV equation, *Chaos Soliton. Fract.*, **20** (2004), 259–266. https://doi.org/10.1016/S0960-0779(03)00373-4

32. A. M. Wazwaz, Soliton solutions for seventh-order Kawahara equation with time-dependent coefficients, *Mod. Phys. Lett. B*, **25** (2011), 643–648.

33. Y. Cao, J. He, Y. Cheng, D. Mihalache, Reductions of the (4 + 1)-dimensional Fokas equation and their solutions, 2020, arXiv: 2007.12309. https://doi.org/10.48550/arXiv.2007.12309

34. D. Chou, I. Iqbal, T. Althobaiti, H. U. Rehman, Machine learning-assisted modeling of nonlinear dispersive waves in ocean engineering, *Ocean Eng.*, **352** (2026), 124562. doilinkhttps://doi.org/10.1016/j.oceaneng.2026.124562

35. M. Caputo, Linear models of dissipation whose Q is almost frequency independent. II, *Geophys. J. Int.*, **13** (1967), 529–539. https://doi.org/10.1111/j.1365-246X.1967.tb02303.x

36. I. Podlubny, *Fractional differential equations*, San Diego: Academic Press, 1999.

37. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier, 2006.

38. R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, *J. Comput. Appl. Math.*, **264** (2014), 65–70. https://doi.org/10.1016/j.cam.2014.01.002