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Research article

Convergence analysis of hybrid interpolative contractions with applications to RLC circuits and nonlinear hinged biharmonic problems

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Abstract: This work investigates a limitation associated with interpolative contraction mappings, where the contractive conditions are generally not imposed at fixed points, resulting in conditions that are not valid over the entire domain. To overcome this drawback, we introduce novel hybrid interpolative contractions, specifically the hybrid interpolative-Kannan and the hybrid interpolative Reich–Rus–Čirić types. By suitably modifying the interpolative structure through an additional control term, the proposed contractive conditions hold on the entire domain, including fixed points. Using this hybrid framework, we establish rigorous existence and uniqueness results for fixed points of the associated self-mappings in metric spaces. The strength and enrichment of the theory are demonstrated through explicit continuous and discontinuous examples that satisfy the hybrid conditions but fail the classical interpolative ones. Finally, we illustrate the applicability of our results by studying an Resistor-Inductor-Capacitor (RLC) circuit system and a nonlinear biharmonic problem with hinged boundary conditions, where the existence and uniqueness of solutions are guaranteed via the proposed hybrid interpolative contractions.

Keywords: fixed point; hybrid interpolative contraction; biharmonic problem; RLC circuit system
Mathematics Subject Classification: 34B27, 45B05, 47H10, 54H25

1. Introduction

Over the last decade, significant developments in fixed-point theory have been driven by contraction principles that incorporate interpolation techniques. Karapinar [1] proposed an interpolative method of contractions of the Kannan type, which extended the classical framework and did not exhibit some standard assumptions that made it more applicable. Subsequently, Karapinar et al. [2] introduced the interpolative contractions of Reich-Rus-Čirić to partial metric spaces, which allows one to analyze a situation in which the points can have positive self-distances. These contributions broadened the toolkit of the study of fixed points of generalized metric structures. In the same direction, Gaba and Karapinar [3] came up with a more adaptable framework of interpolative contractions, which introduced fresh informational characterizations that complement the interpretation of convergence tendencies in various contexts.

Aydi et al. [4] investigated the kind of contractions known as the Cirić-Reich-Rus contractions in the so-called ω interpolative form with added control functions in the analysis. Their approach used the Branciari distance to look at interpolative Cirić-Reich-Rus contractions and built on that work with a new study where they got better results in spaces, where the triangle inequality does not hold [5]. Then Karapinar and Agarwal [6] took it further by introducing the idea of simulation functions to the interpolation framework and were able to come up with a general method that includes many well-known fixed-point results as special cases. The development of interpolative versions of other kinds of contraction like the Hardy-Rogers, Boyd-Wong, and Matkowski contractions in [7, 8] has significantly enhanced in how frequently fixed-point methods can be used to solve nonlinear problems. Together, these studies give us a strong base for dealing with equations of function and dynamic networks in situations where more traditional fixed-point methods fails.

The interpolative framework has been further extended to accommodate specific situations, such as the (φ, ψ)-type Z-contractions discussed in [9], which truly highlights the remarkable flexibility and applicability of this theory. It lets you tailor the framework to suit different needs , and that means it can be adapted and improved if you want it to do something specific. Saeed et al. [10] took this further by coming up with a graph-based version of the interpolative contraction, which showed it could be a real game-changer for sorting out integral equations. Building on that work, Ahmad et al. [11] figured out a way to use the Ćirić–Reich–Rus fixed-point formula to prove whether chaotic systems with fractional-orders actually work or not. After that, Ahmad et al. [12] also developed an interpolative contraction approach specifically for the fractional King Cobra model, which turned out to be particularly useful for analyzing the complicated behavior in fractional-order dynamical systems. Dhanraj et al. [13] established fixed-point results for orthogonal extended interpolative Ćirić-Reich-Rus type ψF-contraction mappings on orthogonal complete b-metric spaces, with an application to analytical solutions of functional equations. Li et al. [14] provided an efficient framework for establishing the existence and rapid approximation of solutions to nonlinear integral impulsive differential equations through the monotone iterative and quasilinearization method. Furthermore, Hu and Liao [15] established convergence conditions for extreme solutions of impulsive differential systems, thereby strengthening the theoretical foundation for the qualitative analysis of such systems.

Motivated by the literature on the limitations of the interpolative contractions at fixed points, in which the classical definitions of Kannan-type and Reich–Rus–Čiric-type contractions do not satisfy the contractive conditions, this work attempts to create a more general framework resolving this inherent constraint. In classical interpolative contractions, the inequalities are required to hold only outside the set of fixed points, which limits both theoretical completeness and practical applicability in various mathematical and physical models. Due to these limitations, we propose hybrid interpolative contractions, involving more control terms, so that the contractive conditions are satisfied in the full domain, even at the fixed point. This extension does not only extend the theory of fixed-point mappings but also allows investigation of more complex and realistic systems, as can be illustrated by examples and application to RLC circuit systems and nonlinear biharmonic equations with hinged boundary conditions.

The paper is organized as follows. Section 1 reviews the relevant literature on interpolative contractions and outlines the motivation for this study. Section 2 introduces the necessary definitions and preliminary concepts. In Section 3, we present our main results: Theorem 1 on hybrid interpolative Kannan contractions, illustrated with Examples 1 and 2, featuring continuous and discontinuous mappings that are valid on the entire domain where classical interpolative Kannan contractions fail; and Theorem 2 on hybrid interpolative Reich-Rus-Ćirić contractions, illustrated with Examples 3 and 4, demonstrating mappings that work globally where classical interpolative Reich-Rus-Ćirić contractions fail. Section 4 applies Theorem 1 to establish the existence and uniqueness for the RLC circuit problem (4.1). Section 5 applies Theorem 2 to the nonlinear biharmonic problem with hinged boundary conditions (5.1). Finally, Section 6 concludes the paper with a discussion of the results and future research directions.

2. Fundamental concepts

The following section provides the necessary background and definitions to support the development of our main results.

Definition 1. *[16] Let $\mathscr{U}$ be a nonempty set and $\ddot{\mathbb{Q}} : \mathscr{U} \times \mathscr{U} \to \mathbb{R}$ a function. The pair $(\mathscr{U}, \ddot{\mathbb{Q}})$ is called a metric space if for all $\dot{\varpi}, \dot{\vartheta}, \dot{\eta} \in \mathscr{U}$, we have*

(i) $\ddot{\mathbb{Q}}(\dot{\varpi}, \dot{\vartheta}) \geq 0$;

(ii) $\ddot{\mathbb{Q}}(\dot{\varpi}, \dot{\vartheta}) = 0$ *if and only if* $\dot{\varpi} = \dot{\vartheta}$;

(iii) $\ddot{\mathbb{Q}}(\dot{\varpi}, \dot{\vartheta}) = \ddot{\mathbb{Q}}(\dot{\vartheta}, \dot{\varpi})$;

(iv) $\ddot{\mathbb{Q}}(\dot{\varpi}, \dot{\eta}) \leq \ddot{\mathbb{Q}}(\dot{\varpi}, \dot{\vartheta}) + \ddot{\mathbb{Q}}(\dot{\vartheta}, \dot{\eta})$.

Definition 2. *[16] Let $(\mathscr{U}, \ddot{\mathbb{Q}})$ be a metric space and $\{\dot{\varpi}_n\}$ a sequence in $\mathscr{U}$. In this case, we have the following.*

(i) $\{\dot{\varpi}_n\}$ *is Cauchy if, for every* $\epsilon > 0$, $N \in \mathbb{N}$ *exists such that*

$$\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_m) < \epsilon \quad \text{for all } n, m > N.$$

Equivalently,

$$\lim_{n,m\to\infty} \ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_m) = 0.$$

(ii) $\{\dot{\varpi}_n\}$ *converges to* $\dot{\varpi} \in \mathscr{U}$ *if*

$$\lim_{n\to\infty} \ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}) = 0.$$

(iii) $(\mathscr{U}, \ddot{\mathbb{Q}})$ *is complete if every Cauchy sequence in* $\mathscr{U}$ *converges to some point in* $\mathscr{U}$*.*

Now, we recall some fundamental definitions and basic types of contractions in metric spaces that are frequently used in fixed-point theory. We summarize several classical contraction conditions, including Banach, Kannan, Chatterjea, Reich, and Hardy-Rogers contractions, which provide sufficient criteria for the existence and uniqueness of fixed points. These definitions will serve as a foundation for the subsequent discussion on more general and interpolative contraction mappings.

Definition 3. *For a metric space* $(\mathscr{U}, \ddot{\mathbb{Q}})$*, we have the following results. Then for all* $\dot{\varpi}, \dot{\vartheta} \in \mathscr{U}$*, the self-mapping* $\mathcal{P} : \mathscr{U} \to \mathscr{U}$ *is referred to as follows.*

1) ***Banach contraction*** *[17]: if* $\alpha_1 \in [0, 1)$ *exists such that*

$$\ddot{\mathbb{Q}}(\mathcal{P}\dot{\varpi}, \mathcal{P}\dot{\vartheta}) \le \alpha_1\, \ddot{\mathbb{Q}}(\dot{\varpi}, \dot{\vartheta});$$

2) ***Kannan contraction*** *[18]: if* $\alpha_1 \in [0, 1/2)$ *exists such that*

$$\ddot{\mathbb{Q}}(\mathcal{P}\dot{\varpi}, \mathcal{P}\dot{\vartheta}) \le \alpha_1\, (\ddot{\mathbb{Q}}(\dot{\varpi}, \mathcal{P}\dot{\varpi}) + \ddot{\mathbb{Q}}(\dot{\vartheta}, \mathcal{P}\dot{\vartheta}));$$

3) ***Chatterjea contraction*** *[19]: if* $\alpha_1 \in [0, 1/2)$ *exists such that*

$$\ddot{\mathbb{Q}}(\mathcal{P}\dot{\varpi}, \mathcal{P}\dot{\vartheta}) \le \alpha_1\, (\ddot{\mathbb{Q}}(\dot{\varpi}, \mathcal{P}\dot{\vartheta}) + \ddot{\mathbb{Q}}(\dot{\vartheta}, \mathcal{P}\dot{\varpi}));$$

4) ***Reich contraction*** *[20]: if* $\alpha_1, \alpha_2, \alpha_3 \in [0, 1)$ *with* $\alpha_1 + \alpha_2 + \alpha_3 < 1$ *exist such that*

$$\ddot{\mathbb{Q}}(\mathcal{P}\dot{\varpi}, \mathcal{P}\dot{\vartheta}) \le \alpha_1\, \ddot{\mathbb{Q}}(\dot{\varpi}, \dot{\vartheta}) + \alpha_2\, \ddot{\mathbb{Q}}(\dot{\varpi}, \mathcal{P}\dot{\varpi}) + \alpha_3\, \ddot{\mathbb{Q}}(\dot{\vartheta}, \mathcal{P}\dot{\vartheta});$$

5) ***Hardy-Rogers contraction*** *[21] : if* $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in [0, 1)$ *with* $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$ *exist such that*

$$\ddot{\mathbb{Q}}(\mathcal{P}\dot{\varpi}, \mathcal{P}\dot{\vartheta}) \le \alpha_1\, \ddot{\mathbb{Q}}(\dot{\varpi}, \dot{\vartheta}) + \alpha_2\, \ddot{\mathbb{Q}}(\dot{\varpi}, \mathcal{P}\dot{\varpi}) + \alpha_3\, \ddot{\mathbb{Q}}(\dot{\vartheta}, \mathcal{P}\dot{\vartheta}) + \alpha_4\, \ddot{\mathbb{Q}}(\dot{\vartheta}, \mathcal{P}\dot{\varpi}) + \alpha_5\, \ddot{\mathbb{Q}}(\dot{\varpi}, \mathcal{P}\dot{\vartheta}).$$

The foundation for interpolative contractions in metric spaces was laid and expanded upon by Karapinar [1, 2, 7]. These contractions generalize classical fixed-point results by combining various types of distance comparisons in a single inequality. The following definitions summarize the main types of interpolative contractions used in this work.

Definition 4. *Let* $\mathcal{P}$ *be a self-map on* $(\mathscr{U}, \ddot{\mathbb{Q}})$*, i.e.,* $\mathcal{P} : \mathscr{U} \to \mathscr{U}$*. For any* $u, v \in \mathscr{U}$ *such that* $u \neq \mathcal{P}u$*, the ensuing results are referred to as follows.*

1) ***Interpolative Kannan type contraction*** *[1]: if the constants* $\lambda \in [0, 1)$ *and* $\alpha \in (0, 1)$ *exist such that*

$$\ddot{\mathbb{Q}}(\mathcal{P}\dot{\varpi}, \mathcal{P}\dot{\vartheta}) \le \lambda\, [\, \ddot{\mathbb{Q}}(\mathcal{P}\dot{\varpi}, \dot{\varpi})\,]^{\alpha}\, [\, \ddot{\mathbb{Q}}(\mathcal{P}\dot{\vartheta}, \dot{\vartheta})\,]^{1-\alpha};$$

2) ***Interpolative Reich-Rus-Čirić type contraction*** *[2]: if $\lambda \in [0,1)$ and $\alpha,\beta \in (0,1)$ satisfying $\alpha+\beta<1$ exist such that*

$$\ddot{\mathbb{Q}}(\mathcal{P}\dot{\varpi},\mathcal{P}\dot{\vartheta}) \leq \lambda\,[\,\ddot{\mathbb{Q}}(\dot{\varpi},\mathcal{P}\dot{\varpi})\,]^{\alpha}\,[\,\ddot{\mathbb{Q}}(\dot{\varpi},\dot{\vartheta})\,]^{\beta}\,[\,\ddot{\mathbb{Q}}(\dot{\vartheta},\mathcal{P}\dot{\vartheta})\,]^{1-\alpha-\beta};$$

3) ***interpolative Hardy-Rogers type contraction*** *[7]: if $\lambda \in [0,1)$ and $\alpha,\beta,\gamma \in (0,1)$ with $\alpha+\beta+\gamma<1$ exist such that*

$$\ddot{\mathbb{Q}}(\mathcal{P}\dot{\varpi},\mathcal{P}\dot{\vartheta}) \leq \lambda\,[\,\ddot{\mathbb{Q}}(\dot{\varpi},\dot{\vartheta})\,]^{\beta}\,[\,\ddot{\mathbb{Q}}(\dot{\varpi},\mathcal{P}\dot{\varpi})\,]^{\alpha}\,[\,\ddot{\mathbb{Q}}(\dot{\vartheta},\mathcal{P}\dot{\vartheta})\,]^{\gamma}\left[\frac{1}{2}(\ddot{\mathbb{Q}}(\dot{\varpi},\mathcal{P}\dot{\vartheta})+\ddot{\mathbb{Q}}(\dot{\vartheta},\mathcal{P}\dot{\varpi}))\right]^{1-\alpha-\beta-\gamma}.$$

3. Results on hybrid interpolative contractions

In this section, we introduce hybrid interpolative contractions that act on the entire domain and we present examples showing that these contractions are more general than classical interpolative ones.

To address the limitation that interpolative Kannan contractions are required to hold only for points $\dot{\varpi},\dot{\vartheta} \in \mathscr{U} \setminus \mathrm{Fix}(\mathcal{P})$ and therefore do not act on the entire domain, in Definition 5, we introduce a new class of mappings termed hybrid interpolative Kannan contractions. This notion unifies the essential features of the classical Kannan contraction (No. 3) and the interpolative Kannan contraction (No. 1), while removing the restriction to $\mathscr{U} \setminus \mathrm{Fix}(\mathcal{P})$ by ensuring that the contractive condition is valid for all $\dot{\varpi},\dot{\vartheta} \in \mathscr{U}$. Within this generalized framework, Theorem 1 establishes a fixed-point result guaranteeing both the existence and uniqueness of a fixed point for mappings satisfying the proposed hybrid interpolative Kannan condition.

Definition 5. *Let $(\mathscr{U},\ddot{\mathbb{Q}})$ be a metric space and let $\mathcal{P}:\mathscr{U}\to\mathscr{U}$ be a self-map. We say that $\mathcal{P}$ is a hybrid interpolative Kannan contraction if the constants $\lambda \in [0,1)$, $\gamma \in [0,\frac{1}{2})$ with $\lambda+2\gamma<1$ and $\alpha \in (0,1)$ exist such that for all $\dot{\varpi},\dot{\vartheta} \in \mathscr{U}$,*

$$\ddot{\mathbb{Q}}(\mathcal{P}\dot{\varpi},\mathcal{P}\dot{\vartheta}) \leq \lambda\,[\ddot{\mathbb{Q}}(\dot{\varpi},\mathcal{P}\dot{\varpi})]^{\alpha}[\ddot{\mathbb{Q}}(\dot{\vartheta},\mathcal{P}\dot{\vartheta})]^{1-\alpha}+\gamma(\ddot{\mathbb{Q}}(\dot{\varpi},\mathcal{P}\dot{\varpi})+\ddot{\mathbb{Q}}(\dot{\vartheta},\mathcal{P}\dot{\vartheta})). \tag{3.1}$$

Remark 1. *It is worth noting that the proposed hybrid interpolative Kannan contraction unifies several known contraction notions. In particular, by setting $\lambda=0$, Condition* (3.1) *reduces to the classical Kannan contraction [18]. On the other hand, if $\gamma=0$, the hybrid condition coincides with the interpolative Kannan contraction introduced in [1].*

Theorem 1. *Suppose $\mathcal{P}:\mathscr{U}\to\mathscr{U}$ is defined on a complete metric space $(\mathscr{U},\ddot{\mathbb{Q}})$ and satisfies the hybrid interpolative Kannan contraction condition. Then $\mathcal{P}$ admits a unique fixed point in $\mathscr{U}$, and the iterative sequence $\dot{\varpi}_{n+1}=\mathcal{P}\dot{\varpi}_n$, $n \geq 0$, converges to this fixed point for any initial point $\dot{\varpi}_0 \in \mathscr{U}$.*

Proof. Let $\dot{\varpi}_0 \in \mathscr{U}$ be arbitrary and define $\dot{\varpi}_{n+1}=\mathcal{P}\dot{\varpi}_n$ for $n \geq 0$. For each $n \geq 1$, apply (3.1) with $\dot{\varpi}=\dot{\varpi}_{n-1}$ and $\dot{\vartheta}=\dot{\varpi}_n$, we have

$$\begin{aligned}\ddot{\mathbb{Q}}(\dot{\varpi}_n,\dot{\varpi}_{n+1}) &= \ddot{\mathbb{Q}}(\mathcal{P}\dot{\varpi}_{n-1},\mathcal{P}\dot{\varpi}_n)\\ &\leq \lambda\,[\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1},\dot{\varpi}_n)]^{\alpha}[\ddot{\mathbb{Q}}(\dot{\varpi}_n,\dot{\varpi}_{n+1})]^{1-\alpha}+\gamma(\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1},\dot{\varpi}_n)+\ddot{\mathbb{Q}}(\dot{\varpi}_n,\dot{\varpi}_{n+1})).\end{aligned} \tag{3.2}$$

If $\ddot{\mathbb{Q}}(\dot{\varpi}_n,\dot{\varpi}_{n+1})=0$ for some n, then $\dot{\varpi}_n=\dot{\varpi}_{n+1}$ and the sequence becomes constant; the constant value is a fixed point of $\mathcal{P}$. Thus we may assume $\ddot{\mathbb{Q}}(\dot{\varpi}_k,\dot{\varpi}_{k+1})>0$ for all k, we consider the following.

Case I: If we choose $\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n) \leq \ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1})$ in the equation (3.2) and we know $0 < \alpha < 1$ then above inequality becomes

$$[\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n)]^{\alpha}[\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1})]^{1-\alpha} \leq \ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}).$$

Putting this into Eq (3.2) gives

$$\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}) \leq \lambda\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}) + \gamma(\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n) + \ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1})).$$

Using $\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n) \leq \ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1})$ on the right-hand side, we have

$$\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}) \leq (\lambda + 2\gamma)\, \ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}).$$

Since $\lambda + 2\gamma < 1$, the factor $1 - (\lambda + 2\gamma) > 0$, which contradicts our nontrivial assumption.
Case II: If we pick $\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n) \geq \ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1})$ in the equation (3.2), we arrive at

$$[\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n)]^{\alpha}[\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1})]^{1-\alpha} \leq \ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n),$$

and substituting into (3.2) gives

$$\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}) \leq \lambda\, \ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n) + \gamma(\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n) + \ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1})).$$

Hence

$$(1 - \gamma)\, \ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}) \leq (\lambda + \gamma)\, \ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n),$$

and since $1 - \gamma > 0$, we have

$$\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}) \leq q\, \ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n), \qquad q := \frac{\lambda + \gamma}{1 - \gamma}. \tag{3.3}$$

The condition $\lambda + 2\gamma < 1$ implies $\lambda + \gamma < 1 - \gamma$, and hence $q \in (0, 1)$. Iterating the inequality (3.3) yields the geometric estimate

$$\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}) \leq q^n\, \ddot{\mathbb{Q}}(\dot{\varpi}_0, \dot{\varpi}_1), \qquad n \in \mathbb{N}.$$

Now let $m > n$, so by the triangle inequality, we have

$$\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_m) \leq \sum_{k=n}^{m-1} \ddot{\mathbb{Q}}(\dot{\varpi}_k, \dot{\varpi}_{k+1}).$$

Thus, it can be expressed as follows:

$$\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_m) \leq \frac{q^n}{1 - q}\, \ddot{\mathbb{Q}}(\dot{\varpi}_0, \dot{\varpi}_1).$$

Since $0 < q < 1$, we have $q^n \to 0$ as $n \to \infty$, and therefore

$$\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_m) \to 0 \qquad \text{as } m > n \to \infty.$$

Hence, the sequence $\{\dot{\varpi}_n\}$ is Cauchy in $(\mathscr{U}, \ddot{\mathbb{Q}})$. The completeness of $\mathscr{U}$ yields a limit point $\varpi \in \mathscr{U}$. Now we use the triangular inequality for verifying that ϖ is a fixed point

$$\ddot{\mathbb{Q}}(\varpi, \mathcal{P}\varpi) \leq \ddot{\mathbb{Q}}(\varpi, \dot{\varpi}_{n+1}) + \ddot{\mathbb{Q}}(\dot{\varpi}_{n+1}, \mathcal{P}\varpi).$$

The first term tends to 0. For the second term, we apply the hybrid interpolative Kannan contraction (3.1) with $\dot{\varpi} = \dot{\varpi}_n$ and $\dot{\vartheta} = \varpi$, so that

$$\ddot{\mathbb{Q}}(\dot{\varpi}_{n+1}, \mathcal{P}\varpi) \leq \lambda[\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1})]^{\alpha}[\ddot{\mathbb{Q}}(\varpi, \mathcal{P}\varpi)]^{1-\alpha} + \gamma(\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}) + \ddot{\mathbb{Q}}(\varpi, \mathcal{P}\varpi)).$$

Letting $n \to \infty$ and using $\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}) \to 0$ gives

$$\ddot{\mathbb{Q}}(\varpi, \mathcal{P}\varpi) \leq \gamma\ddot{\mathbb{Q}}(\varpi, \mathcal{P}\varpi).$$

Since $\gamma < 1$, this implies $\ddot{\mathbb{Q}}(\varpi, \mathcal{P}\varpi) = 0$, i.e., $\mathcal{P}\varpi = \varpi$.

Uniqueness: Suppose that $\bar{\sigma}$ and $\dot{\theta}$ are two fixed points of $\mathcal{P}$; that is, $\mathcal{P}\bar{\sigma} = \bar{\sigma}$ and $\mathcal{P}\dot{\theta} = \dot{\theta}$. Applying the hybrid interpolative Kannan contraction (3.1) to $\bar{\sigma}$ and $\dot{\theta}$ gives

$$\ddot{\mathbb{Q}}(\mathcal{P}\bar{\sigma}, \mathcal{P}\dot{\theta}) \leq \lambda\,[\ddot{\mathbb{Q}}(\bar{\sigma}, \mathcal{P}\bar{\sigma})]^{\alpha}[\ddot{\mathbb{Q}}(\dot{\theta}, \mathcal{P}\dot{\theta})]^{1-\alpha} + \gamma(\ddot{\mathbb{Q}}(\bar{\sigma}, \mathcal{P}\bar{\sigma}) + \ddot{\mathbb{Q}}(\dot{\theta}, \mathcal{P}\dot{\theta})).$$

Substituting $\mathcal{P}\bar{\sigma} = \bar{\sigma}$ and $\mathcal{P}\dot{\theta} = \dot{\theta}$, we get

$$\ddot{\mathbb{Q}}(\bar{\sigma}, \dot{\theta}) \leq \lambda\,[\ddot{\mathbb{Q}}(\bar{\sigma}, \bar{\sigma})]^{\alpha}[\ddot{\mathbb{Q}}(\dot{\theta}, \dot{\theta})]^{1-\alpha} + \gamma(\ddot{\mathbb{Q}}(\bar{\sigma}, \bar{\sigma}) + \ddot{\mathbb{Q}}(\dot{\theta}, \dot{\theta})).$$

Since $\ddot{\mathbb{Q}}(\bar{\sigma}, \bar{\sigma}) = \ddot{\mathbb{Q}}(\dot{\theta}, \dot{\theta}) = 0$, this simplifies to

$$\ddot{\mathbb{Q}}(\bar{\sigma}, \dot{\theta}) \leq 0 \implies \ddot{\mathbb{Q}}(\bar{\sigma}, \dot{\theta}) = 0 \implies \bar{\sigma} = \dot{\theta}.$$

Hence, the fixed point is unique. □

Example 1. *Let* $\mathscr{U} = [0, 1]$ *be equipped with the usual metric* $\ddot{\mathbb{Q}}(\dot{\varpi}, \dot{\vartheta}) = |\dot{\varpi} - \dot{\vartheta}|$*, for all* $\dot{\varpi}, \dot{\vartheta} \in [0, 1]$*. Define a mapping* $\mathcal{P} : \mathscr{U} \to \mathscr{U}$ *such that*

$$\mathcal{P}(\dot{\varpi}) = \frac{\dot{\varpi}}{4}, \quad \dot{\varpi} \in [0, 1].$$

Choose the following constants for the hybrid interpolative Kannan contraction:

$$\lambda = 0.19, \quad \gamma = 0.4, \quad \alpha = 0.5,$$

which satisfy $\lambda + \gamma < 1$*,* $\lambda, \gamma \in [0, 1)$*, and* $\alpha \in (0, 1)$*. For any* $\dot{\varpi}, \dot{\vartheta} \in [0, 1]$*, we have:*

$$\ddot{\mathbb{Q}}(\mathcal{P}\dot{\varpi}, \mathcal{P}\dot{\vartheta}) = \frac{1}{4}|\dot{\varpi} - \dot{\vartheta}|,\ \ddot{\mathbb{Q}}(\dot{\varpi}, \mathcal{P}\dot{\varpi}) = \frac{3\dot{\varpi}}{4},\ \ddot{\mathbb{Q}}(\dot{\vartheta}, \mathcal{P}\dot{\vartheta}) = \frac{3\dot{\vartheta}}{4}.$$

*Now by substituting all the values in hybrid interpolative Kannan condition (*3.1*), we conclude*

$$\frac{1}{4}|\dot{\varpi} - \dot{\vartheta}| \leq \lambda\left[\frac{3\dot{\varpi}}{4}\right]^{\alpha}\left[\frac{3\dot{\vartheta}}{4}\right]^{1-\alpha} + \gamma\left(\frac{3\dot{\varpi}}{4} + \frac{3\dot{\vartheta}}{4}\right).$$

The inequality above holds for all $\dot{\varpi}, \dot{\vartheta} \in [0, 1]$*, as shown in Figure* 1.

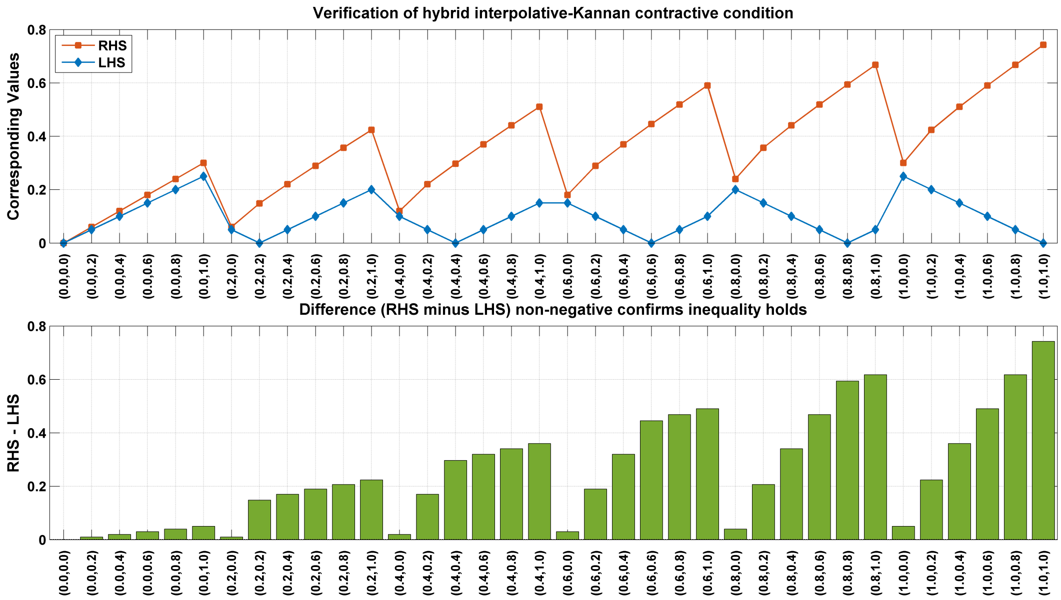

Figure 1. Comparison of the left-hand side and right-hand side of 1.

For the interpolative Kannan condition mentioned in Definition 1*, if we take* $\dot{\varpi} = 1$ *and* $\dot{\vartheta} = 0$*, we arrive at*

$$\frac{1}{4} \leq \lambda \left[\frac{3}{4}\right]^{\alpha} [0]^{1-\alpha}.$$

From the inequality, we can observe that no λ *and* α *exists such that interpolative Kannan contractions holds. Therefore, all the conditions of Theorem* 1 *is satisfied and 0 is the unique fixed point of the mapping* $\mathcal{P}$.

This example demonstrates that the hybrid interpolative Kannan contraction genuinely extends the original definition by removing the restrictive condition $\dot{\varpi} \neq \mathcal{P}\dot{\varpi}$*. The fixed point* $\dot{\varpi} = 0$ *causes the original inequality to fail, while the additional linear term in the hybrid definition allows the inequality to hold globally. Furthermore, if we take any initial guess from* $[0, 1]$*, it will converge towards the fixed point shown in Figure* 2.

In addition to continuous mappings, it is of interest to study whether the hybrid interpolative Kannan contraction can accommodate discontinuous functions while still guaranteeing the existence of a unique fixed point. To illustrate this, we provide an example of a discontinuous self-mapping on a complete metric space that satisfies the hybrid interpolative Kannan contraction globally but fails to satisfy the classical interpolative Kannan contraction. This demonstrates that the hybrid formulation genuinely extends the applicability of the contraction framework beyond continuous maps and the restrictive conditions required by the original interpolative Kannan definition.

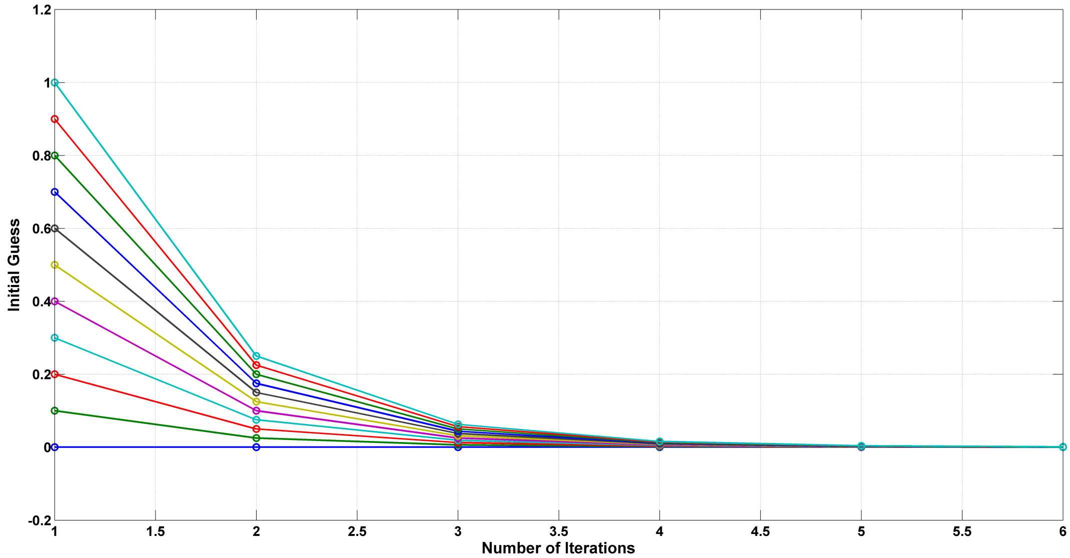

Figure 2. Convergence of different initial guesses to the fixed point.

Example 2. *Let* $\mathscr{U} = [0,1]$ *be equipped with the usual metric* $\ddot{\mathbb{Q}}(\dot{\varpi},\dot{\vartheta}) = |\dot{\varpi}-\dot{\vartheta}|$ *for all* $\dot{\varpi},\dot{\vartheta} \in [0,1]$. *Define a mapping* $\mathcal{P}:\mathscr{U}\to\mathscr{U}$ *such that*

$$\mathcal{P}(\dot{\varpi}) = \begin{cases} \dfrac{\dot{\varpi}+1}{5}, & \dot{\varpi}\in[0,1/2],\\ \dfrac{\dot{\varpi}}{5}, & \dot{\varpi}\in(1/2,1]. \end{cases}$$

Choose the following constants for the hybrid interpolative Kannan contraction:

$$\lambda = 0.19, \quad \gamma = 0.4, \quad \alpha = 0.5,$$

which satisfy $\lambda + 2\gamma < 1$, $\lambda,\gamma \in [0,1)$, *and* $\alpha\in(0,1)$. *The expression above is not Banach contraction but it is discontinuous at* $\dot{\varpi} = \frac{1}{2}$.
Case (i): *For* $\dot{\varpi},\dot{\vartheta}\in[0,1/2]$, *then hybrid interpolative Kannan* (3.1) *inequality becomes*

$$\frac{|\dot{\varpi}-\dot{\vartheta}|}{5} \le \lambda\left|\frac{3\dot{\varpi}}{5}\right|^{\alpha}\left|\frac{3\dot{\vartheta}}{5}\right|^{1-\alpha} + \gamma\left(\frac{3\dot{\varpi}}{5}+\frac{3\dot{\vartheta}}{5}\right)$$

which holds for all $\dot{\varpi},\dot{\vartheta}\in[0,1/2]$, *as shown in Figure* 3.
Case (ii): *For* $\dot{\varpi},\dot{\vartheta}\in(1/2,1]$, *the hybrid interpolative Kannan* (3.1) *inequality becomes*

$$\frac{|\dot{\varpi}-\dot{\vartheta}|}{5} \le \lambda\left(\frac{4\dot{\varpi}}{5}\right)^{\alpha}\left(\frac{4\dot{\vartheta}}{5}\right)^{1-\alpha} + \gamma\left(\frac{4\dot{\varpi}}{5}+\frac{4\dot{\vartheta}}{5}\right),$$

which is satisfied for all $\dot{\varpi},\dot{\vartheta}\in(1/2,1]$, *as shown in Figure* 4.

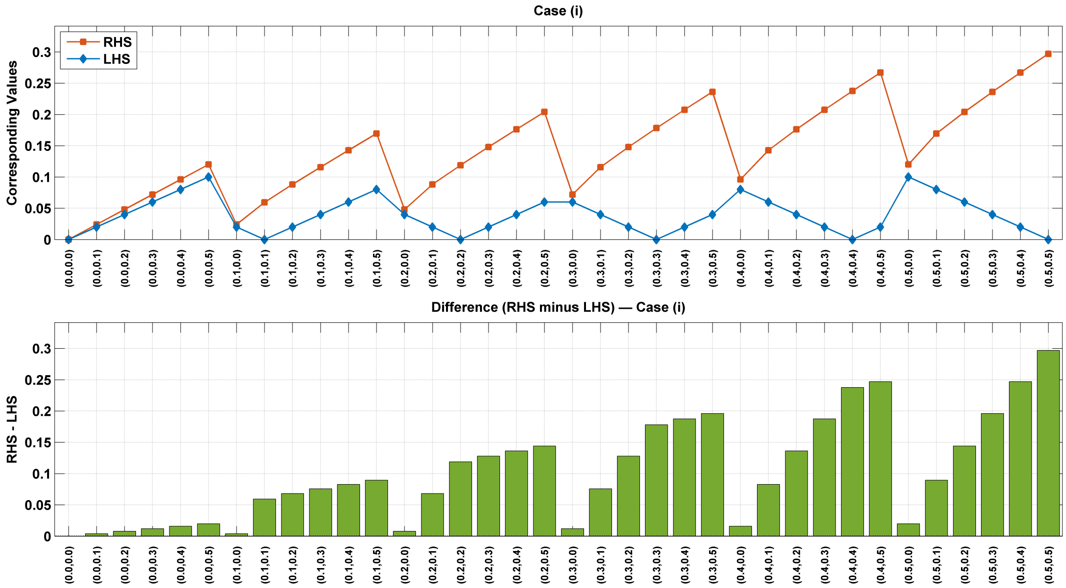

Figure 3. Comparison of the left-hand side and right-hand side of Case (i) of Example 2.

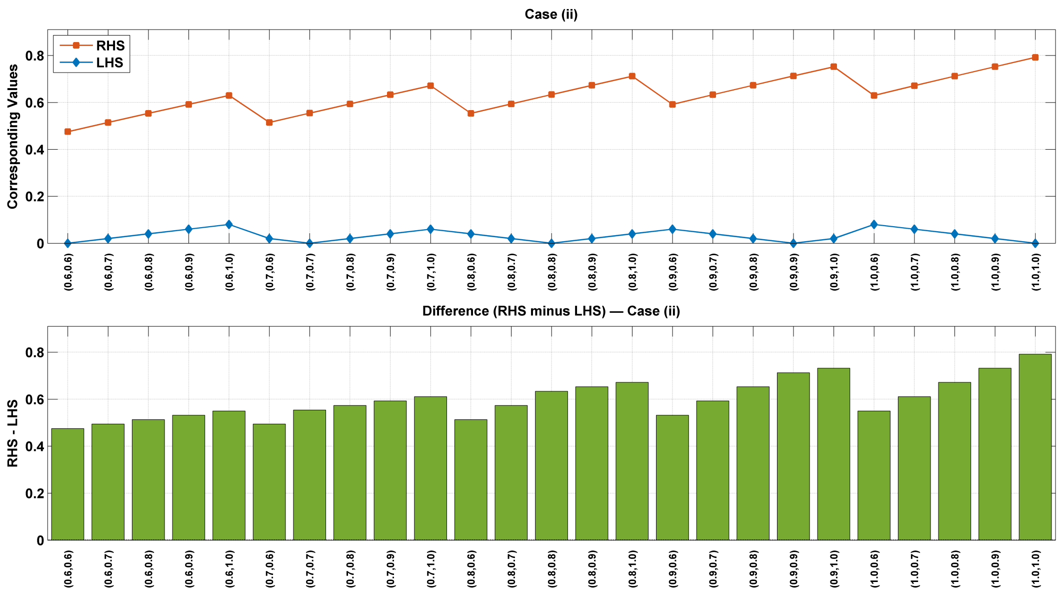

Figure 4. Comparison of the left-hand side and right-hand side of Case (ii) of Example 2.

***Case (iii)**: For $\dot{\varpi} \in [0, 1/2], \dot{\vartheta} \in (1/2, 1]$, then the hybrid interpolative Kannan* (3.1) *inequality becomes*

$$\left|\frac{\dot{\varpi}+1}{5} - \frac{\dot{\vartheta}}{5}\right| \le \lambda \left|\frac{3\dot{\varpi}}{5}\right|^{\alpha} \left(\frac{4\dot{\vartheta}}{5}\right)^{1-\alpha} + \gamma\left(\frac{3\dot{\varpi}}{5} + \frac{4\dot{\vartheta}}{5}\right),$$

which also holds for all $\dot{\varpi} \in [0, 1/2], \dot{\vartheta} \in (1/2, 1]$, as shown in Figure 5.

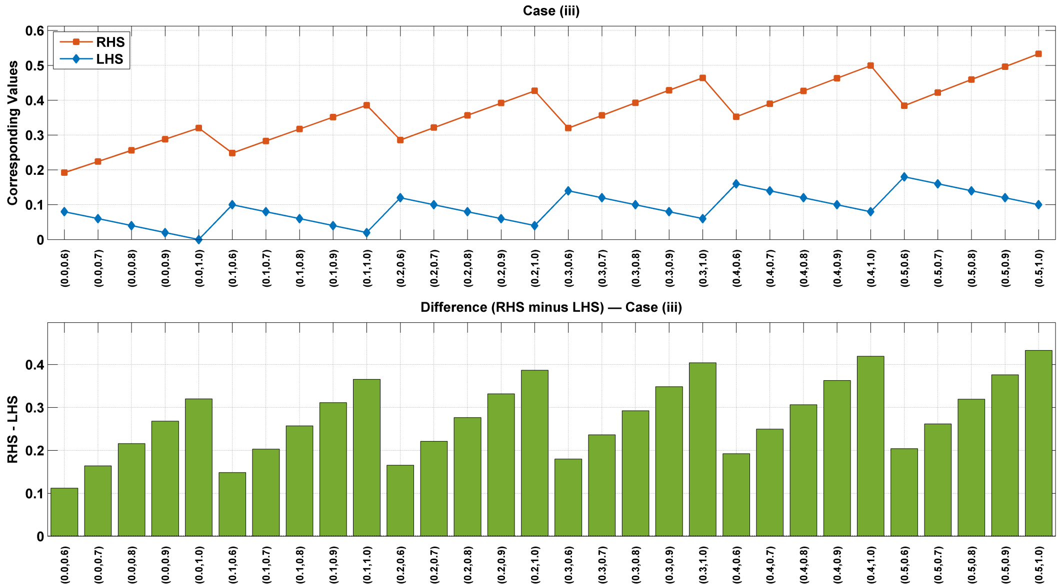

Figure 5. Comparison of the left-hand side and right-hand side of Case (iii) of Example 2.

For the interpolative Kannan condition mentioned in Definition 1*, if we take* $\dot{\varpi} = 0$ *and* $\dot{\vartheta} = 1$*, we arrive at*

$$\frac{1}{5} \leq \lambda[0]^{\alpha}\left[\frac{4}{5}\right]^{1-\alpha}.$$

From the inequality above, we can observe that no λ *and* α *exists such that interpolative Kannan contraction holds. Therefore, all conditions of Theorem* 1 *are satisfied,* $\dot{\varpi} = 1/4$ *is the unique fixed point of* $\mathcal{P}$*, and the example demonstrates that the hybrid interpolative Kannan contraction extends the classical definition by allowing the inequality to hold globally, even at the fixed point. Furthermore, if we take any initial guess from* $[0, 0.5]$*, it will converge towards the fixed point shown in Figure* 6*.*

To overcome the restriction that interpolative Reich-Rus-Čirić contractions are required to hold only for points $\dot{\varpi}, \dot{\vartheta} \in \mathscr{U} \setminus \mathrm{Fix}(\mathcal{P})$ *and hence fail to act on the entire domain, in Definition* 6*, we introduce a new class of mappings called hybrid interpolative Reich-Rus-Čirić contractions. This concept blends the fundamental features of the classical Reich contraction (No.* 3*) and the interpolative Reich-Rus-Čirić contraction (No.* 1*), while eliminating the restriction to* $\mathscr{U} \setminus \mathrm{Fix}(\mathcal{P})$ *by ensuring that the contractive condition holds for all* $\dot{\varpi}, \dot{\vartheta} \in \mathscr{U}$*. Within this generalized framework, Theorem* 2 *establishes a fixed point result guaranteeing both the existence and uniqueness of a fixed point for mappings satisfying the proposed hybrid interpolative Reich-Rus-Čirić condition.*

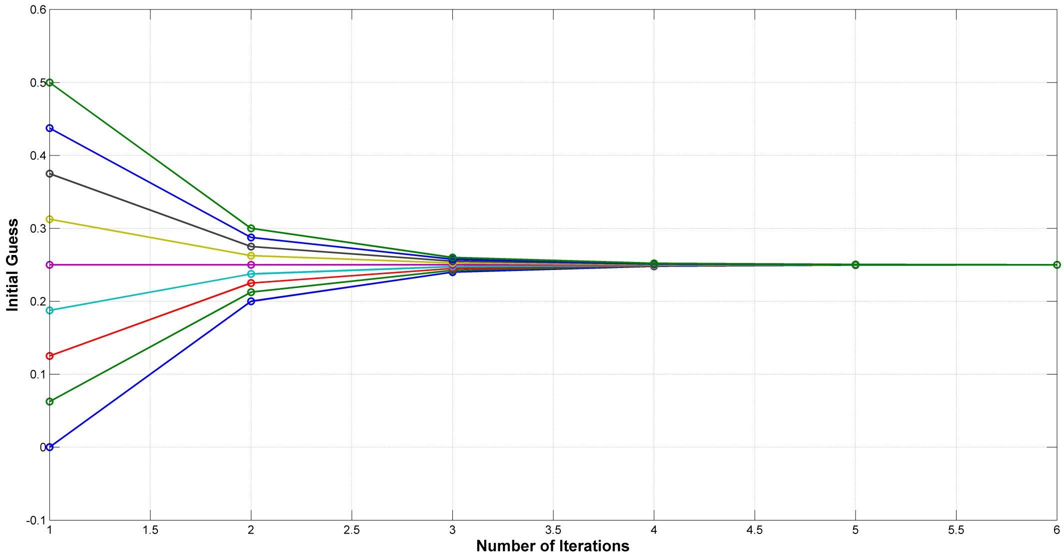

Figure 6. All initial guesses converge to the fixed point.

Definition 6. *Let* $(\mathscr{U}, \ddot{\mathbb{Q}})$ *be a metric space and let* $\mathcal{P} : \mathscr{U} \to \mathscr{U}$ *be a self-map. We say that* $\mathcal{P}$ *is a hybrid interpolative Reich-Rus-Čirić contraction if constants* $\lambda, \gamma \in [0, 1)$ *with* $\lambda + 3\gamma < 1$ *and* $\alpha, \beta \in (0, 1)$ *exists such that for all* $\dot{\varpi}, \dot{\vartheta} \in \mathscr{U}$*,*

$$\begin{aligned}\ddot{\mathbb{Q}}(\mathcal{P}\dot{\varpi}, \mathcal{P}\dot{\vartheta}) \leq \lambda\,[\ddot{\mathbb{Q}}(\dot{\varpi}, \mathcal{P}\dot{\varpi})]^{\alpha}\,[\ddot{\mathbb{Q}}(\dot{\varpi}, \dot{\vartheta})]^{\beta}\,[\ddot{\mathbb{Q}}(\dot{\vartheta}, \mathcal{P}\dot{\vartheta})]^{1-\alpha-\beta} \\ + \gamma(\ddot{\mathbb{Q}}(\dot{\varpi}, \dot{\vartheta}) + \ddot{\mathbb{Q}}(\dot{\varpi}, \mathcal{P}\dot{\varpi}) + \ddot{\mathbb{Q}}(\dot{\vartheta}, \mathcal{P}\dot{\vartheta})).\end{aligned} \tag{3.4}$$

Remark 2. *The hybrid interpolative Reich-Rus-Čirić contraction encompasses several existing contraction frameworks as special cases. Specifically, when* $\lambda = 0$*, Inequality* (3.5) *reduces to the classical Reich contraction [20]. Conversely, by choosing* $\gamma = 0$*, the hybrid condition simplifies to the interpolative Reich-Rus-Čirić contraction introduced in [2].*

Theorem 2. *Let* $\mathcal{P} : \mathscr{U} \to \mathscr{U}$ *act on a complete metric space* $(\mathscr{U}, \ddot{\mathbb{Q}})$ *and assume that it satisfies the hybrid interpolative Reich-Rus-Čirić contraction condition. Then a unique* $\dot{\varpi} \in \mathscr{U}$ *exists with* $\mathcal{P}(\dot{\varpi}) = \dot{\varpi}$*, and the sequence* $\dot{\varpi}_{n+1} = \mathcal{P}\dot{\varpi}_n$ *converges to* ϖ *for all initial choices* $\dot{\varpi}_0 \in \mathscr{U}$*.*

Proof. Let $\dot{\varpi}_0 \in \mathscr{U}$ and define the Picard iteration $\dot{\varpi}_{n+1} = \mathcal{P}\dot{\varpi}_n$, $n \geq 0$. For each $n \geq 1$, apply (3.4) with $\dot{\varpi} = \dot{\varpi}_{n-1}$ and $\dot{\vartheta} = \dot{\varpi}_n$, we have

$$\begin{aligned}\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}) = \ddot{\mathbb{Q}}(\mathcal{P}\dot{\varpi}_{n-1}, \mathcal{P}\dot{\varpi}_n) \leq \lambda\,[\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n)]^{\alpha}[\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n)]^{\beta}[\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1})]^{1-\alpha-\beta} \\ + \gamma\,(\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n) + \ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n) + \ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1})).\end{aligned} \tag{3.5}$$

If $\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}) = 0$ for some n, then $\dot{\varpi}_n = \dot{\varpi}_{n+1}$, and the sequence has reached a fixed point. Otherwise, assume $\ddot{\mathbb{Q}}(\dot{\varpi}_k, \dot{\varpi}_{k+1}) > 0$ for all k.

Case I: Suppose $\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n) \leq \ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1})$ in Eq (3.5); in this case, we arrive at

$$[\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n)]^{\alpha+\beta}[\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1})]^{1-\alpha-\beta} \leq \ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}),$$

and the inequality becomes

$$\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}) \leq \lambda\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}) + \gamma(\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}) + 2\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n)).$$

Using $\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n) \leq \ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1})$, we get

$$\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}) \leq (\lambda + 3\gamma)\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}).$$

Since $\lambda + 3\gamma < 1$, which leads to a contradiction.

Case II: Suppose $\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n) \geq \ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1})$ in Eq (3.5); in this case we conclude that

$$[\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n)]^{\alpha+\beta}[\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1})]^{1-\alpha-\beta} \leq \ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n),$$

and the inequality becomes

$$\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}) \leq \lambda\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n) + \gamma(\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}) + 2\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n)).$$

Rearranging, we get

$$(1-\gamma)\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}) \leq (\lambda + 2\gamma)\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n).$$

The condition $\lambda + 2\gamma < 1 - \gamma$ can be simplified as $\lambda + 3\gamma < 1$. Take $q = \frac{\lambda+2\gamma}{1-\gamma} \in (0, 1)$ and we arrive at

$$\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}) \leq q\, \ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n).$$

Iterating this inequality gives

$$\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}) \leq q^n\ddot{\mathbb{Q}}(\dot{\varpi}_0, \dot{\varpi}_1).$$

For $m > n$, by the triangle inequality, we have

$$\ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_m) \leq \sum_{k=n}^{m-1} q^k\ddot{\mathbb{Q}}(\dot{\varpi}_0, \dot{\varpi}_1) \leq \frac{q^n}{1-q}\ddot{\mathbb{Q}}(\dot{\varpi}_0, \dot{\varpi}_1) \to 0.$$

Hence $\{\dot{\varpi}_n\}$ is a Cauchy sequence, and by the completeness of $\mathscr{U}$, we get $\dot{\varpi}_n \to \varpi \in \mathscr{U}$. Using the triangle inequality, we have

$$\ddot{\mathbb{Q}}(\varpi, \mathcal{P}\varpi) \leq \ddot{\mathbb{Q}}(\varpi, \dot{\varpi}_{n+1}) + \ddot{\mathbb{Q}}(\dot{\varpi}_{n+1}, \mathcal{P}\varpi).$$

The first term tends to 0. For the second term, apply (3.4) with $\dot{\varpi} = \dot{\varpi}_n$, $\dot{\vartheta} = \varpi$ as follows

$$\begin{aligned}\ddot{\mathbb{Q}}(\dot{\varpi}_{n+1}, \mathcal{P}\varpi) \leq\ & \lambda[\ddot{\mathbb{Q}}(\dot{\varpi}_{n-1}, \dot{\varpi}_n)]^{\alpha}[\ddot{\mathbb{Q}}(\dot{\varpi}_n, \varpi)]^{\beta}[\ddot{\mathbb{Q}}(\varpi, \mathcal{P}\varpi)]^{1-\alpha-\beta} \\ & + \gamma(\ddot{\mathbb{Q}}(\dot{\varpi}_n, \varpi) + \ddot{\mathbb{Q}}(\dot{\varpi}_n, \dot{\varpi}_{n+1}) + \ddot{\mathbb{Q}}(\varpi, \mathcal{P}\varpi)).\end{aligned}$$

Letting $n \to \infty$ gives

$$\ddot{\mathbb{Q}}(\varpi, \mathcal{P}\varpi) \leq \gamma\ddot{\mathbb{Q}}(\varpi, \mathcal{P}\varpi) \implies \ddot{\mathbb{Q}}(\varpi, \mathcal{P}\varpi) = 0,$$

which shows that ϖ is the fixed point.

Uniqueness: Suppose that ϖ and ϑ are two fixed points of $\mathcal{P}$; that is, $\mathcal{P}\varpi = \varpi$ and $\mathcal{P}\vartheta = \vartheta$. Applying the hybrid interpolative Reich-Rus-Čirić contraction (3.4) to ϖ and ϑ gives

$$\begin{aligned}\ddot{\mathbb{Q}}(\varpi, \vartheta) &= \ddot{\mathbb{Q}}(\mathcal{P}\varpi, \mathcal{P}\vartheta) \\ &\leq \lambda\, [\ddot{\mathbb{Q}}(\varpi, \mathcal{P}\varpi)]^{\alpha}\, [\ddot{\mathbb{Q}}(\varpi, \vartheta)]^{\beta}\, [\ddot{\mathbb{Q}}(\vartheta, \mathcal{P}\vartheta)]^{1-\alpha-\beta} \\ &\quad + \gamma(\ddot{\mathbb{Q}}(\varpi, \vartheta) + \ddot{\mathbb{Q}}(\varpi, \mathcal{P}\varpi) + \ddot{\mathbb{Q}}(\vartheta, \mathcal{P}\vartheta)).\end{aligned}$$

Since ϖ and ϑ are fixed points, we have $\mathcal{P}\varpi = \varpi$ and $\mathcal{P}\vartheta = \vartheta$. Therefore, $\ddot{\mathbb{Q}}(\varpi, \mathcal{P}\varpi) = \ddot{\mathbb{Q}}(\varpi, \varpi) = 0$ and $\ddot{\mathbb{Q}}(\vartheta, \mathcal{P}\vartheta) = \ddot{\mathbb{Q}}(\vartheta, \vartheta) = 0$. Substituting these into the inequality above yields

$$\ddot{\mathbb{Q}}(\varpi, \vartheta) \leq \lambda \cdot 0 \cdot [\ddot{\mathbb{Q}}(\varpi, \vartheta)]^{\beta} \cdot 0 + \gamma(\ddot{\mathbb{Q}}(\varpi, \vartheta) + 0 + 0) = \gamma\, \ddot{\mathbb{Q}}(\varpi, \vartheta).$$

Since $\gamma < 1$, the inequality above implies $\ddot{\mathbb{Q}}(\varpi, \vartheta) = 0$, and hence $\varpi = \vartheta$. Therefore, the fixed point is unique. □

Theorem 2 extends the classical interpolative Reich–Rus–Čirić contraction by allowing additional distance terms involving the images of points under the mapping. This modification enables the contractive inequality to hold globally, including at fixed points, where the classical interpolative condition may fail. The following example illustrates this improvement for a continuous self-mapping, showing that the hybrid condition is satisfied on the entire space while the classical interpolative Reich–Rus–Čirić inequality does not hold.

Example 3. *Let* $\mathscr{U} = [0, 1]$ *be equipped with the usual metric* $\ddot{\mathbb{Q}}(\dot{\varpi}, \dot{\vartheta}) = |\dot{\varpi} - \dot{\vartheta}|$ *for all* $\dot{\varpi}, \dot{\vartheta} \in [0, 1]$. *Define a mapping* $\mathcal{P} : \mathscr{U} \to \mathscr{U}$ *such that*

$$\mathcal{P}(\dot{\varpi}) = \frac{2\dot{\varpi} + 1}{5}, \quad \dot{\varpi} \in [0, 1].$$

Choose the following constants for the hybrid interpolative Reich-Rus-Čirić contraction:

$$\lambda = 0.24, \quad \gamma = 0.25, \quad \alpha = 0.35, \quad \beta = 0.4,$$

which satisfy $\lambda + 3\gamma < 1$, $\lambda, \gamma \in [0, 1)$, *and* $\alpha, \beta \in (0, 1)$. *For any* $\dot{\varpi}, \dot{\vartheta} \in [0, 1]$, *we have:*

$$\ddot{\mathbb{Q}}(\mathcal{P}\dot{\varpi}, \mathcal{P}\dot{\vartheta}) = \frac{2}{5}|\dot{\varpi} - \dot{\vartheta}|, \;\; \ddot{\mathbb{Q}}(\dot{\varpi}, \mathcal{P}\dot{\varpi}) = \left|\frac{3\dot{\varpi} - 1}{5}\right|, \;\; \ddot{\mathbb{Q}}(\dot{\vartheta}, \mathcal{P}\dot{\vartheta}) = \left|\frac{3\dot{\vartheta} - 1}{5}\right|.$$

*Now substituting all the values in the hybrid interpolative Reich-Rus-Čirić condition (*3.4*), we conclude that*

$$\frac{2}{5}|\dot{\varpi} - \dot{\vartheta}| \leq \lambda \left[\left|\frac{3\dot{\varpi} - 1}{5}\right|^{\alpha} |\dot{\varpi} - \dot{\vartheta}|^{\beta} \left|\frac{3\dot{\vartheta} - 1}{5}\right|^{1-\alpha-\beta} \right] + \gamma \left(|\dot{\varpi} - \dot{\vartheta}| + \left|\frac{3\dot{\varpi} - 1}{5}\right| + \left|\frac{3\dot{\vartheta} - 1}{5}\right| \right).$$

The inequality above holds globally for all $\dot{\varpi}, \dot{\vartheta} \in [0, 1]$, *as shown in Figure* 7.
For the classical interpolative Reich-Rus-Čirić condition, if we take $\dot{\varpi} = 0.5$ *(the fixed point) and* $\dot{\vartheta} = 1$, *we arrive at*

$$\ddot{\mathbb{Q}}(\mathcal{P}\dot{\varpi}, \mathcal{P}\dot{\vartheta}) = 0.2 > \lambda[\ddot{\mathbb{Q}}(\dot{\varpi}, \mathcal{P}\dot{\varpi})]^{\alpha}[\ddot{\mathbb{Q}}(\dot{\varpi}, \dot{\vartheta})]^{\beta}[\ddot{\mathbb{Q}}(\dot{\vartheta}, \mathcal{P}\dot{\vartheta})]^{1-\alpha-\beta} = 0,$$

showing that the classical condition fails. Therefore, all conditions of the hybrid theorem are satisfied, $\dot{\varpi} = 0.3333$ *is the unique fixed point of* $\mathcal{P}$, *and the example demonstrates that the hybrid interpolative Reich-Rus-Čirić contraction extends the classical definition by allowing the inequality to hold globally, including at the fixed point. Furthermore, if we take any initial guess from* $[0, 1]$, *it will converge towards the fixed point as shown in Figure* 8.

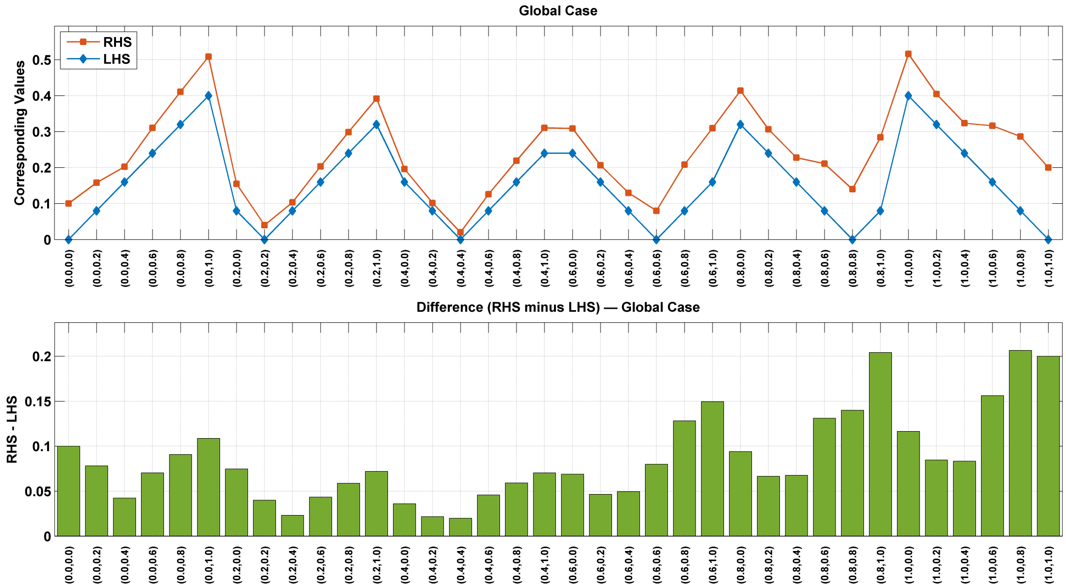

Figure 7. Validation of the contraction condition of Example 3.

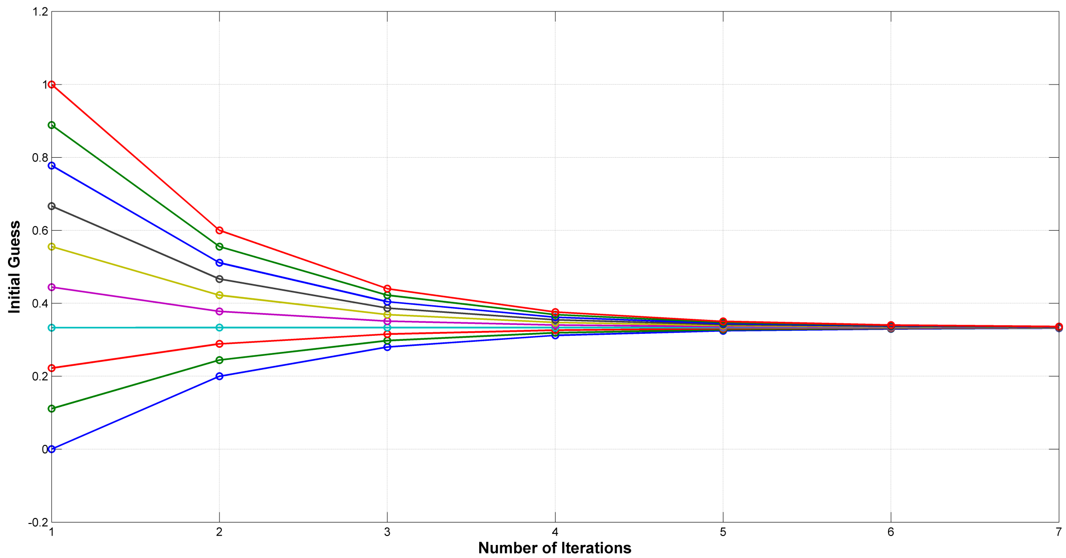

Figure 8. Convergence of different initial guesses to the fixed point.

The applicability of the hybrid interpolative Reich–Rus–Čirić contraction is not limited to continuous operators. In contrast to classical interpolative contractions, which generally break down in the presence of discontinuities, the hybrid framework preserves the contractive structure globally. The next example demonstrates this advantage by considering a discontinuous mapping for which the hybrid condition holds on the whole domain, whereas the classical interpolative Reich–Rus–Čirić condition fails.

Example 4. *Let $\mathscr{U} = [0,1]$ be equipped with the usual metric $\ddot{\mathbb{Q}}(\dot{\varpi},\dot{\vartheta}) = |\dot{\varpi} - \dot{\vartheta}|$, for all $\dot{\varpi},\dot{\vartheta} \in [0,1]$. Define a mapping $\mathcal{P} : \mathscr{U} \to \mathscr{U}$ such that*

$$\mathcal{P}(\dot{\varpi}) = \begin{cases} \dfrac{3-\dot{\varpi}}{6}, & \dot{\varpi} \in [0, 0.5], \\ \dfrac{1+\dot{\varpi}}{6}, & \dot{\varpi} \in (0.5, 1]. \end{cases}$$

Choose the following constants for the hybrid interpolative Reich-Rus-Čirić contraction:

$$\lambda = 0.24, \quad \gamma = 0.25, \quad \alpha = 0.85, \quad \beta = 0.75,$$

which satisfy $\lambda + 3\gamma < 1$, $\lambda,\gamma \in [0,1)$, and $\alpha,\beta \in (0,1)$. The expression above is not a Banach contraction, as it is discontinuous at $\dot{\varpi} = 0.5$.

Case (i): *For $\dot{\varpi},\dot{\vartheta} \in [0,0.5]$, the hybrid interpolative Reich-Rus-Čirić (3.4) inequality becomes*

$$\frac{|\dot{\vartheta} - \dot{\varpi}|}{4} \leq \lambda \left|\frac{5\dot{\varpi}-3}{4}\right|^{\alpha} |\dot{\varpi} - \dot{\vartheta}|^{\beta} \left|\frac{5\dot{\vartheta}-3}{4}\right|^{1-\alpha-\beta} + \gamma\left(|\dot{\varpi} - \dot{\vartheta}| + \left|\frac{5\dot{\varpi}-3}{4}\right| + \left|\frac{5\dot{\vartheta}-3}{4}\right|\right),$$

which holds for all $\dot{\varpi},\dot{\vartheta} \in [0,0.5]$, as shown in Figure 9.

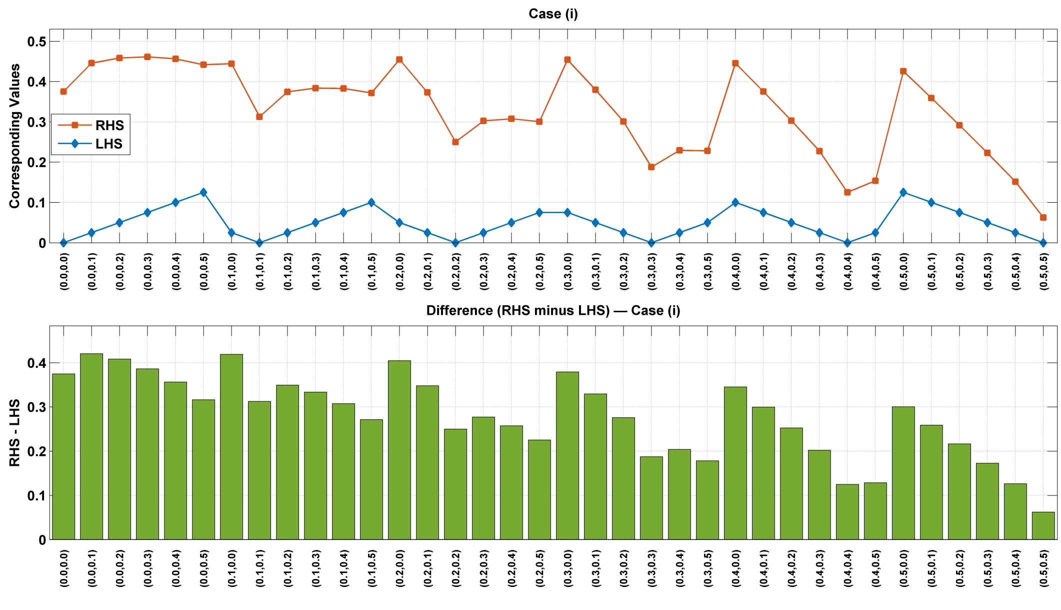

Figure 9. Comparison of the left-hand side and right-hand side of Case (i) of Example 4.

Case (ii): *For $\dot{\varpi},\dot{\vartheta} \in (0.5,1]$, the hybrid interpolative Reich-Rus-Čirić (3.4) inequality becomes*

$$\frac{|\dot{\varpi} - \dot{\vartheta}|}{5} \leq \lambda \left|\frac{1+\dot{\varpi}}{6} - \frac{1}{6}\right|^{\alpha} |\dot{\varpi} - \dot{\vartheta}|^{\beta} \left|\frac{1+\dot{\vartheta}}{6} - \frac{1}{6}\right|^{1-\alpha-\beta} + \gamma\left(|\dot{\varpi} - \dot{\vartheta}| + \frac{1+\dot{\varpi}}{6} - \frac{1}{6} + \frac{1+\dot{\vartheta}}{6} - \frac{1}{6}\right),$$

which holds for all $\dot{\varpi},\dot{\vartheta} \in (0.5,1]$, as shown in Figure 10.

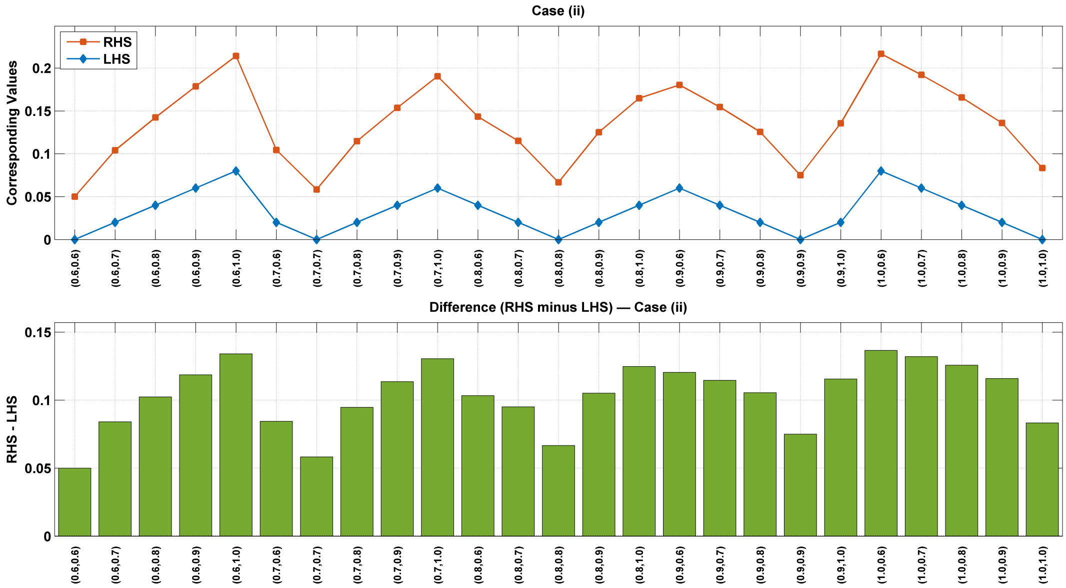

Figure 10. Comparison of the left-hand side and right-hand side of Case (ii) of Example 4.

Case (iii): *For* $\dot{\varpi} \in [0, 0.5], \dot{\vartheta} \in (0.5, 1]$*, the hybrid interpolative Reich-Rus-Čirić* (3.4) *inequality becomes*

$$\left|\frac{3-\dot{\varpi}}{6} - \frac{1+\dot{\vartheta}}{6}\right| \leq \lambda \left|\frac{5\dot{\varpi}-3}{4}\right|^{\alpha} |\dot{\varpi}-\dot{\vartheta}|^{\beta} \left|\frac{1+\dot{\vartheta}}{6} - \frac{1}{6}\right|^{1-\alpha-\beta} + \gamma\left(|\dot{\varpi}-\dot{\vartheta}| + \left|\frac{5\dot{\varpi}-3}{4}\right| + \frac{1+\dot{\vartheta}}{6} - \frac{1}{6}\right),$$

which also holds for all $\dot{\varpi} \in [0, 0.5], \dot{\vartheta} \in (0.5, 1]$*, as shown in Figure* 11.

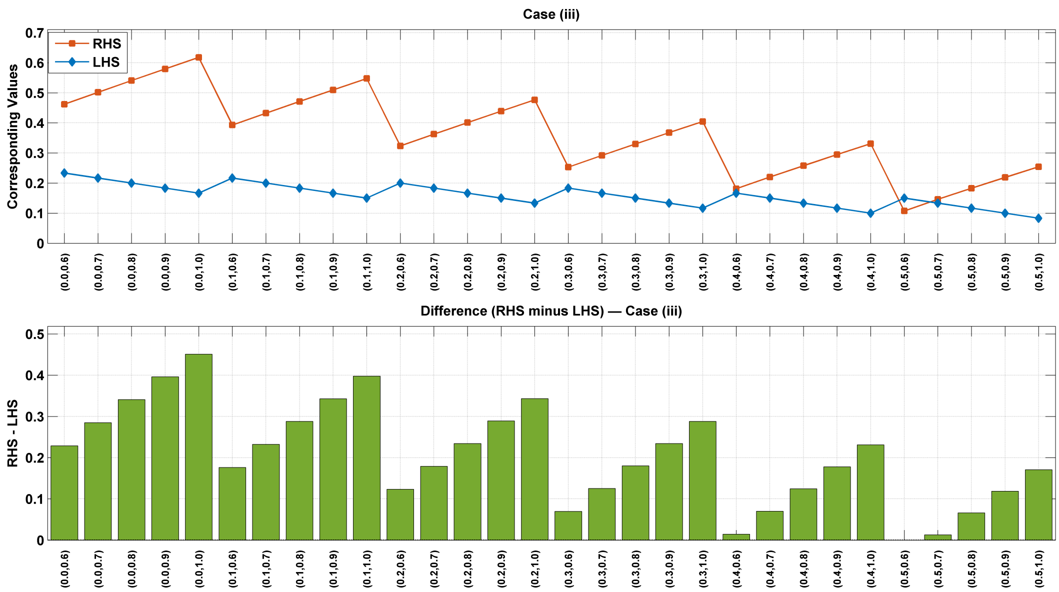

Figure 11. Comparison of the left-hand side and right-hand side of Case (iii) of Example 4.

For the classical interpolative Reich-Rus-Čirić condition, if we take $\dot{\varpi} = 0.5$ *(fixed point) and* $\dot{\vartheta} = 0.9$*, we arrive at*

$$\left|\frac{3-0.5}{6} - \frac{1+0.9}{6}\right| = 0.2333 \nleq \lambda[\ddot{\mathbb{Q}}(\dot{\varpi}, \mathcal{P}\dot{\vartheta})]^{\alpha}[\ddot{\mathbb{Q}}(\dot{\varpi}, \dot{\vartheta})]^{\beta}[\ddot{\mathbb{Q}}(\dot{\vartheta}, \mathcal{P}\dot{\vartheta})]^{1-\alpha-\beta} = 0,$$

showing that there is no such λ, α, β *for which the classical interpolative Reich-Rus-Čirić condition holds. Therefore, all conditions of the hybrid theorem are satisfied,* $\dot{\varpi} = \frac{3}{7} \approx 0.4286$ *is the unique fixed point of* $\mathcal{P}$*, and the example demonstrates that the hybrid interpolative Reich-Rus-Čirić contraction extends the classical definition by allowing the inequality to hold globally, including across the discontinuity. Furthermore, if we take any initial guess from* $[0, 0.5]$*, it will converge towards the fixed point, as shown in Figure* 12.

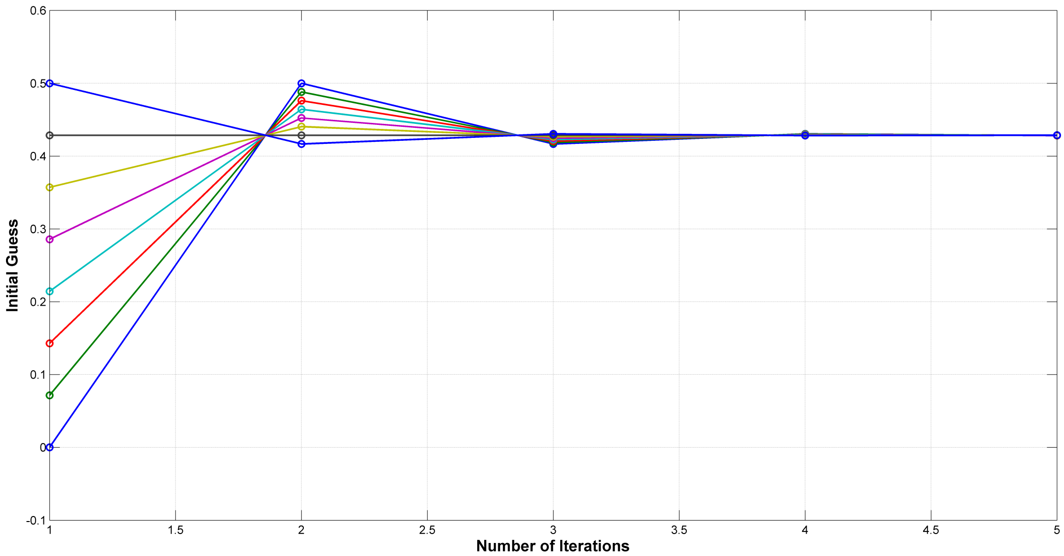

Figure 12. Convergence of different initial guesses to the fixed point.

Remark 3. *The control terms introduced in the hybrid contractions* (3.1) *and* (3.4) *admit a natural geometric interpretation. The interpolative component* $\lambda[\ddot{\mathbb{Q}}(\dot{\varpi}, \mathcal{P}\dot{\varpi})]^{\alpha}[\ddot{\mathbb{Q}}(\dot{\vartheta}, \mathcal{P}\dot{\vartheta})]^{1-\alpha}$ *(respectively, the three-factor analog in* (3.4)*) is a weighted geometric mean of the self-distances of* $\dot{\varpi}$ *and* $\dot{\vartheta}$ *under* $\mathcal{P}$*. A defining feature of the geometric mean is that it vanishes whenever any single factor vanishes, regardless of the magnitude of the remaining factor(s); this is exactly why the classical interpolative contractions fail whenever one of* $\dot{\varpi}, \dot{\vartheta}$ *is a fixed point of* $\mathcal{P}$*. The additional term* $\gamma(\ddot{\mathbb{Q}}(\dot{\varpi}, \mathcal{P}\dot{\varpi}) + \ddot{\mathbb{Q}}(\dot{\vartheta}, \mathcal{P}\dot{\vartheta}))$ *in* (3.1)*, and* $\gamma(\ddot{\mathbb{Q}}(\dot{\varpi}, \dot{\vartheta}) + \ddot{\mathbb{Q}}(\dot{\varpi}, \mathcal{P}\dot{\varpi}) + \ddot{\mathbb{Q}}(\dot{\vartheta}, \mathcal{P}\dot{\vartheta}))$ *in* (3.4) *is instead an arithmetic (linear) combination of the same quantities, and therefore does not degenerate when only one self-distance is zero. Geometrically, the hybrid condition may be viewed as a convex-type combination of a multiplicative (geometric-mean) estimate and an additive (Kannan/Reich-type) estimate, so that the contractive bound remains strictly positive unless both points are simultaneously fixed. This is what allows the hybrid contractions to hold on the entire domain* $\mathscr{U}$ *including at the fixed point, as illustrated in Examples 1–4 and exploited in the applications of Sections 4 and 5.*

4. Application to an RLC circuit

We consider a classical series RLC electrical circuit and use $u(\mathrm{t})$ to denote the voltage across the capacitor at time t. By applying Kirchhoff's voltage law, the dynamics of the circuit are governed by the second-order linear differential equation

$$L\,u''(\mathrm{t}) + R\,u'(\mathrm{t}) + \frac{1}{C}u(\mathrm{t}) = f(\mathrm{t}), \qquad \mathrm{t} \in [0,1], \tag{4.1}$$

as shown in Figure 13, where $L > 0$ represents the inductance, $R > 0$ is the resistance, $C > 0$ the capacitance, and $f(\mathrm{t})$ is an external voltage source. To analyze the system using Green's functions and fixed-point methods, we impose the homogeneous Dirichlet boundary conditions, $u(0) = 0$ and $u(1) = 0$. These boundary conditions correspond to a situation in which the capacitor voltage is zero at the initial and final times, ensuring bounded energy and allowing the problem to be formulated as a well-posed boundary value problem on the compact interval $[0, 1]$.

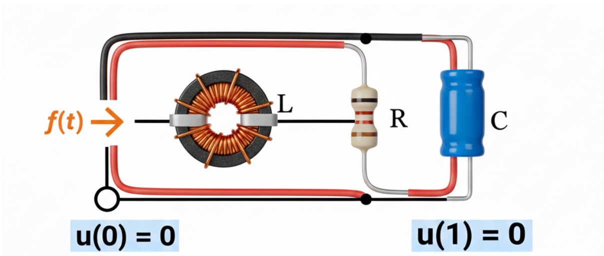

Figure 13. RLC system of model 4.1.

In order to simplify the mathematical analysis while retaining physical relevance, we select the normalized parameter values

$$L = 1, \qquad R = 2, \qquad C = 1.$$

With this choice, the RLC model reduces to the following normalized form:

$$u''(\mathrm{t}) + 2u'(\mathrm{t}) + u(\mathrm{t}) = f(\mathrm{t}), \qquad \mathrm{t} \in [0,1].$$

Such normalization is commonly adopted in existence and uniqueness studies because it eliminates unnecessary scaling constants and leads to explicit expressions for the associated Green's function. Setting L to 1 and C to 1 normalizes both the inductance and capacitance, so that the energy in the magnetic and electric fields of the circuit become standardized. Selecting $R = 2$ has a significant impact on the system's overall behavior. Now, consider the characteristic equation for the homogeneous part of the equation: $\mathrm{r}^2 + 2\mathrm{r} + 1 = 0$. This equation has a root that is repeated at $\mathrm{r} = -1$, which tells us that the circuit is critically damped. Physically, critical damping implies that the circuit returns to equilibrium as rapidly as possible without exhibiting oscillatory behavior. All transient responses decay exponentially, and no overshoot occurs. From a mathematical perspective, this damping regime guarantees strong stability, the positivity of the corresponding Green's function, and favorable boundedness properties. These details are necessary to be able to construct a contractive integral operator, which is useful, and to apply fixed-point theorems to work out whether a solution to the equation is unique and exists.

We now construct the Green's function corresponding to the boundary value problem (4.1) subject to the homogeneous Dirichlet boundary conditions. The Green's function $\mathcal{G}(\mathrm{t}, \mathfrak{s})$ is defined as the solution of

$$\frac{\partial^2 \mathcal{G}}{\partial \mathrm{t}^2}(\mathrm{t}, \mathfrak{s}) + 2\frac{\partial \mathcal{G}}{\partial \mathrm{t}}(\mathrm{t}, \mathfrak{s}) + \mathcal{G}(\mathrm{t}, \mathfrak{s}) = \delta(\mathrm{t} - \mathfrak{s}),$$

with $\mathcal{G}(0, \mathfrak{s}) = \mathcal{G}(1, \mathfrak{s}) = 0$, where $\delta(\cdot)$ denotes the Dirac delta function. For the critically damped case with the characteristic root $\mathfrak{r} = -1$, the Green's function can be expressed in the standard piecewise form as

$$\mathcal{G}(\mathrm{t}, \mathfrak{s}) = \begin{cases} \mathrm{t}(\mathfrak{s} - 1)e^{-(\mathrm{t}-\mathfrak{s})}, & 0 \leq \mathrm{t} \leq \mathfrak{s} \leq 1, \\ \mathfrak{s}(\mathrm{t} - 1)e^{-(\mathrm{t}-\mathfrak{s})}, & 0 \leq \mathfrak{s} \leq \mathrm{t} \leq 1. \end{cases}$$

This representation clearly distinguishes the two cases: When the observation time t is less than the impulse time $\mathfrak{s}$, and when t is greater than $\mathfrak{s}$. The function $\mathcal{G}(\mathrm{t}, \mathfrak{s})$ is continuous, non-negative on the domain $[0, 1] \times [0, 1]$, and has the role of representing the impulse response of the critically damped RLC circuit. Being bounded and positive is what lets the boundary value problem be reworked into an equivalent integral equation, which, in turn, is what we can use to apply fixed-point theorems to prove the existence and uniqueness of the solutions.

Let $\mathscr{U} = C[0, 1]$ with the distance function $\ddot{\mathbb{Q}}(u(\mathrm{t}), v(\mathrm{t})) = \sup_{\mathrm{t}\in[0,1]} |u(\mathrm{t}) - v(\mathrm{t})|$. Clearly one can see that $(\mathscr{U}, \ddot{\mathbb{Q}})$ is a complete metric space, and we define the operator $\mathcal{P} : \mathscr{U} \to \mathscr{U}$ as follows

$$\mathcal{P}u(\mathrm{t}) = \int_0^1 \mathcal{G}(\mathrm{t}, \mathfrak{s}) f(\mathfrak{s}, u(\mathfrak{s}))\, d\mathfrak{s}$$

where $\mathcal{G}(\mathrm{t}, \mathfrak{s})$ is the Green's function for the linear operator $Lu = -u''$ with the boundary conditions $u(0) = u(1) = 0$.

Next, we investigate the existence and uniqueness of solutions to the RLC electrical circuit with the boundary conditions specified in (4.1), given the framework of the hybrid interpolative Kannan contraction presented in Theorem 1.

Theorem 3. *Consider the RLC electrical circuit with the boundary conditions given in (*4.1*). Define the operator $\mathcal{P} : \mathscr{U} \to \mathscr{U}$ as previously described, and suppose that the following condition holds for all $u, v \in \mathscr{U}$ and for all $\mathrm{t} \in [0, 1]$:*

$$|f(\mathrm{t}, u(\mathrm{t})) - f(\mathrm{t}, v(\mathrm{t}))|^2 \leq |u(\mathrm{t}) - \mathcal{P}u(\mathrm{t})|\, |v(\mathrm{t}) - \mathcal{P}v(\mathrm{t})|. \tag{4.2}$$

*Then the RLC electrical circuit with the boundary conditions (*4.1*) admits a unique solution $u \in C[0, 1]$.*

Proof. We prove that the operator $\mathcal{P}$ is a hybrid interpolative Kannan contraction on $\mathscr{U}$. For any $u, v \in \mathscr{U}$, we have:

$$\begin{aligned} |\mathcal{P}u(\mathrm{t}) - \mathcal{P}v(\mathrm{t})|^2 &= \left| \int_0^1 \mathcal{G}(\mathrm{t}, \mathfrak{s}) \left[f(\mathfrak{s}, u(\mathfrak{s})) - f(\mathfrak{s}, v(\mathfrak{s})) \right] d\mathfrak{s} \right|^2 \\ &\leq \left(\int_0^1 |\mathcal{G}(\mathrm{t}, \mathfrak{s})| d\mathfrak{s} \right)^2 |f(\mathfrak{s}, u(\mathfrak{s})) - f(\mathfrak{s}, v(\mathfrak{s}))|^2 \\ &\leq \left(\int_0^1 |\mathcal{G}(\mathrm{t}, \mathfrak{s})| d\mathfrak{s} \right)^2 [|u(\mathrm{t}) - \mathcal{P}u(\mathrm{t})| \times |v(\mathrm{t}) - \mathcal{P}v(\mathrm{t})|] \end{aligned}$$

Taking the square root on both sides, we obtain

$$|\mathcal{P}u(\mathrm{t}) - \mathcal{P}v(\mathrm{t})| \leq \left(\int_0^1 |\mathcal{G}(\mathrm{t},\mathfrak{s})| d\mathfrak{s}\right)\left[|(\mathcal{P}u)(\mathrm{t}) - \mathcal{P}u(\mathrm{t})|^{\frac{1}{2}} \times |(\mathcal{P}v)(\mathrm{t}) - \mathcal{P}v(\mathrm{t})|^{\frac{1}{2}}\right].$$

Because $\mathcal{G}(\mathrm{t},\mathfrak{s})$ is defined piecewise, we split the integral at $\mathfrak{s} = \mathrm{t}$ as follows

$$\int_0^1 \mathcal{G}(\mathrm{t},\mathfrak{s})\, d\mathfrak{s} = \int_0^{\mathrm{t}} \mathfrak{s}(1-\mathrm{t})e^{-(\mathrm{t}-\mathfrak{s})}\, d\mathfrak{s} + \int_{\mathrm{t}}^1 \mathrm{t}(1-\mathfrak{s})e^{-(\mathrm{t}-\mathfrak{s})}\, d\mathfrak{s}.$$

The first integral represents the contribution from impulses applied before time t, while the second integral accounts for contributions from impulses applied after time t. Each integral can be evaluated using standard techniques for exponential functions. Specifically, for the first integral, we have

$$\int_0^{\mathrm{t}} \mathfrak{s}(\mathrm{t}-1)e^{-(\mathrm{t}-\mathfrak{s})}\, d\mathfrak{s} = (\mathrm{t}-1)e^{-\mathrm{t}} \int_0^{\mathrm{t}} \mathfrak{s}e^{\mathfrak{s}}\, d\mathfrak{s} = (\mathrm{t}-1)e^{-\mathrm{t}}[(\mathrm{t}-1)e^{\mathrm{t}} + 1].$$

For the second integral, we have:

$$\int_{\mathrm{t}}^1 \mathrm{t}(\mathfrak{s}-1)e^{-(\mathrm{t}-\mathfrak{s})}\, d\mathfrak{s} = \mathrm{t}e^{-\mathrm{t}} \int_{\mathrm{t}}^1 (\mathfrak{s}-1)e^{\mathfrak{s}}\, d\mathfrak{s} = \mathrm{t}e^{-\mathrm{t}}[(2-\mathrm{t})e^{\mathrm{t}} - e] = \mathrm{t}(2-\mathrm{t}) - \mathrm{t}e^{1-\mathrm{t}}.$$

Combining both parts, the integral of the Green's function over $\mathfrak{s} \in [0,1]$ is

$$\int_0^1 \mathcal{G}(\mathrm{t},\mathfrak{s})\, d\mathfrak{s} = (\mathrm{t}-1)[(\mathrm{t}-1) + e^{-\mathrm{t}}] + \mathrm{t}(2-\mathrm{t}) - \mathrm{t}e^{1-\mathrm{t}}.$$

Simplifying further, we can write

$$\int_0^1 \mathcal{G}(\mathrm{t},\mathfrak{s})\, d\mathfrak{s} = 1 + (\mathrm{t}-1)e^{-\mathrm{t}} - \mathrm{t}e^{1-\mathrm{t}}.$$

This expression is continuous on $[0,1]$ and satisfies $0 \leq \int_0^1 |\mathcal{G}(\mathrm{t},\mathfrak{s})|\, d\mathfrak{s} < 1$, which can be replaced by $\lambda \in [0,1)$. Now taking the supremum over $\mathrm{t} \in [0,1]$, we get

$$\ddot{\mathbb{Q}}(\mathcal{P}u,\mathcal{P}v) \leq \lambda\, [\ddot{\mathbb{Q}}(u,\mathcal{P}u)]^{\frac{1}{2}} [\ddot{\mathbb{Q}}(v,\mathcal{P}v)]^{\frac{1}{2}}$$

As we know that $\lambda \in [0,1)$ and $\alpha \in (0,1)$, the inequality above can be rewritten as

$$\ddot{\mathbb{Q}}(\mathcal{P}u,\mathcal{P}v) \leq \lambda\, [\ddot{\mathbb{Q}}(u,\mathcal{P}u)]^{\alpha} [\ddot{\mathbb{Q}}(v,\mathcal{P}v)]^{1-\alpha}$$

For any $\gamma \geq 0$, we can write the inequality as

$$\ddot{\mathbb{Q}}(\mathcal{P}u,\mathcal{P}v) \leq \lambda\, [\ddot{\mathbb{Q}}(u,\mathcal{P}u)]^{\alpha} [\ddot{\mathbb{Q}}(v,\mathcal{P}v)]^{1-\alpha} + \gamma(\ddot{\mathbb{Q}}(u,\mathcal{P}u) + \ddot{\mathbb{Q}}(v,\mathcal{P}v)).$$

Therefore, all the conditions of Theorem 1 are fulfilled and $\mathcal{P}$ has a unique fixed point in $\mathscr{U} = C[0,1]$. This fixed point is the unique solution of the RLC electrical circuit (4.1). □

In the context of Theorem 3, the quantity $\ddot{\mathbb{Q}}(u, \mathcal{P}u) = \sup_{\mathrm{t}\in[0,1]} |u(\mathrm{t}) - \mathcal{P}u(\mathrm{t})|$ measures the residual between a trial voltage profile u and the voltage profile reconstructed from the critically damped Green's function $\mathcal{G}(\mathrm{t}, \mathfrak{s})$. Physically, this residual quantifies the extent to which a candidate voltage fails to satisfy the governing equation $u'' + 2u' + u = f$ exactly, i.e., an energy mismatch of the transient response. Since $R = 2$ places the circuit exactly at critical damping, the homogeneous solutions decay as $e^{-\mathrm{t}}$ and $\mathrm{t}e^{-\mathrm{t}}$, and the associated Green's function is bounded with $\lambda = \sup_{\mathrm{t}\in[0,1]} \int_0^1 \mathcal{G}(\mathrm{t}, \mathfrak{s})\, d\mathfrak{s} < 1$. The control term $\gamma(\ddot{\mathbb{Q}}(u, \mathcal{P}u) + \ddot{\mathbb{Q}}(v, \mathcal{P}v))$ in the hybrid condition then acts as a *damping margin*: Even if one candidate voltage profile exactly satisfies the equilibrium condition (zero residual), the term retains the residual of the other profile, consistent with the physical fact that any nonzero perturbation in a critically damped circuit is still dissipated and controlled by the resistive term R, rather than relying solely on the multiplicative gain of Green's function.

5. Nonlinear biharmonic problem with hinged boundary conditions

We consider the following nonlinear biharmonic boundary value problem as shown in Figure 14:

$$\begin{cases} u''''(\mathrm{t}) = f(\mathrm{t}, u(\mathrm{t})), & \mathrm{t} \in [0, 1], \\ u(0) = u(1) = 0, \qquad u''(0) = u''(1) = 0, & \end{cases} \tag{5.1}$$

Problem (5.1) models the transverse deflection of a slender elastic beam whose ends are hinged. The fourth–order differential operator arises from the classical Euler–Bernoulli beam theory, where the bending stiffness of the beam balances the applied loading. The nonlinear term $f(\mathrm{t}, u(\mathrm{t}))$ lets one incorporate things like nonlinear elasticity, big deflections, or external forces that depend on how much displacement is happening. The boundary conditions u(0) = u(1) = 0 tell us that the beam is fixed at both ends. Then $u''(0) = u''(1) = 0$ means there are no forces putting a bend in it, which is exactly what occurs with hinged or simply supported supports. This is a realistic model for things like bridge decks, railway tracks, and most common structural beams, as it allows the ends to rotate freely but keep the beam from moving up or down. In practical terms, this is a common situation, and it is mathematically useful because with these kinds of boundary conditions, one gets a positive Green's function, which is very important when you are working with fixed points and integral equations.

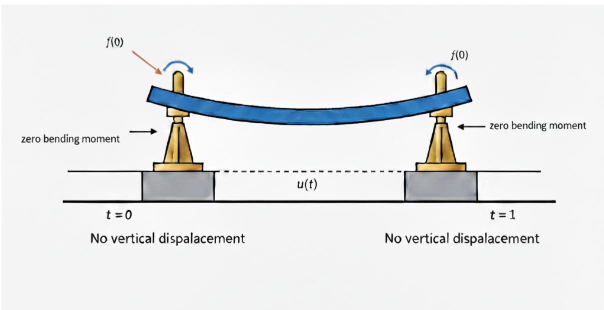

Figure 14. Nonlinear biharmonic problem (5.1).

For the associated linear biharmonic problem corresponding to (5.1), the Green's function $\mathcal{G}(\mathrm{t}, \mathfrak{s})$ is explicitly given by

$$\mathcal{G}(\mathrm{t},\mathfrak{s}) = \begin{cases} \dfrac{\mathrm{t}(1-\mathfrak{s})}{6}(2\mathfrak{s}-\mathfrak{s}^2-\mathrm{t}^2), & 0 \le \mathrm{t} \le \mathfrak{s} \le 1, \\ \dfrac{\mathfrak{s}(1-\mathrm{t})}{6}(2\mathrm{t}-\mathrm{t}^2-\mathfrak{s}^2), & 0 \le \mathfrak{s} \le \mathrm{t} \le 1. \end{cases}$$

The kernel $\mathcal{G}(\mathrm{t},\mathfrak{s})$ is continuous, symmetric, and non-negative on $[0,1]\times[0,1]$. Moreover, it satisfies the bounds

$$\max_{\mathrm{t},\mathfrak{s}\in[0,1]} \mathcal{G}(\mathrm{t},\mathfrak{s}) = \frac{1}{48}, \qquad 0 < \int_0^1 \mathcal{G}(\mathrm{t},\mathfrak{s})\, d\mathfrak{s} \le \frac{1}{24} < 1.$$

Consequently, Problem (5.1) can be equivalently rewritten in the integral form

$$u(\mathrm{t}) = \int_0^1 \mathcal{G}(\mathrm{t},\mathfrak{s})\, f(\mathfrak{s}, u(\mathfrak{s}))\, d\mathfrak{s}.$$

This integral equation provides a natural and effective framework for applying the hybrid interpolative Reich-Rus-Čirić contractive fixed-point techniques to Problem (5.1).

Let $\mathscr{U} = C[0,1]$ be endowed with the metric

$$\ddot{\mathbb{Q}}(u,v) = \sup_{\mathrm{t}\in[0,1]} |u(\mathrm{t}) - v(\mathrm{t})|.$$

Then $(\mathscr{U}, \ddot{\mathbb{Q}})$ is a complete metric space. Define the operator $\mathcal{P} : \mathscr{U} \to \mathscr{U}$ by

$$\mathcal{P}u(\mathrm{t}) = \int_0^1 \mathcal{G}(\mathrm{t},\mathfrak{s})\, f(\mathfrak{s}, u(\mathfrak{s}))\, d\mathfrak{s}.$$

We now establish the existence and uniqueness of solutions of Problem (5.1) using the hybrid interpolative Reich-Rus-Čirić contraction framework stated in Theorem 2.

Theorem 4. *Let $\mathcal{P} : \mathscr{U} \to \mathscr{U}$ be defined as above. Suppose that for all $u, v \in \mathscr{U}$ and all $\mathrm{t} \in [0,1]$, we have*

$$|f(\mathrm{t}, u(\mathrm{t})) - f(\mathrm{t}, v(\mathrm{t}))|^2 \le |u(\mathrm{t}) - v(\mathrm{t})|\, |u(\mathrm{t}) - \mathcal{P}u(\mathrm{t})|\, |v(\mathrm{t}) - \mathcal{P}v(\mathrm{t})|.$$

Then Problem (5.1) *admits a unique solution $u \in C[0,1]$.*

Proof. We prove that the operator $\mathcal{P}$ is a hybrid interpolative Reich-Rus-Čirić contraction on $\mathscr{U}$. For any $u, v \in \mathscr{U}$ and $\mathrm{t} \in [0,1]$, we have

$$\begin{aligned} |(\mathcal{P}u)(\mathrm{t}) - (\mathcal{P}v)(\mathrm{t})|^2 &= \left| \int_0^1 \mathcal{G}(\mathrm{t},\mathfrak{s})[f(\mathfrak{s}, u(\mathfrak{s})) - f(\mathfrak{s}, v(\mathfrak{s}))]\, d\mathfrak{s} \right|^2 \\ &\le \left(\int_0^1 \mathcal{G}(\mathrm{t},\mathfrak{s})\, d\mathfrak{s} \right)^2 |f(\mathfrak{s}, u(\mathfrak{s})) - f(\mathfrak{s}, v(\mathfrak{s}))|^2 \\ &\le \left(\int_0^1 \mathcal{G}(\mathrm{t},\mathfrak{s})\, d\mathfrak{s} \right)^2 |u(\mathrm{t}) - v(\mathrm{t})|\, |u(\mathrm{t}) - \mathcal{P}u(\mathrm{t})|\, |v(\mathrm{t}) - \mathcal{P}v(\mathrm{t})|. \end{aligned}$$

Taking the square root on both sides, we arrive at

$$|(\mathcal{P}u)(\mathrm{t}) - (\mathcal{P}v)(\mathrm{t})| \le \left(\int_0^1 \mathcal{G}(\mathrm{t},\mathfrak{s})\, d\mathfrak{s} \right) [|u(\mathrm{t}) - v(\mathrm{t})|\, |u(\mathrm{t}) - \mathcal{P}u(\mathrm{t})|\, |v(\mathrm{t}) - \mathcal{P}v(\mathrm{t})|]^{\frac{1}{2}}$$

Using the assumed condition and the bound, we have

$$\int_0^1 \mathcal{G}(\mathrm{t}, \mathfrak{s})\, d\mathfrak{s} \leq \frac{1}{24} =: \lambda < 1.$$

After substituting, we obtain

$$|(\mathcal{P}u)(\mathrm{t}) - (\mathcal{P}v)(\mathrm{t})| \leq \lambda \left[|u(\mathrm{t}) - v(\mathrm{t})|\, |u(\mathrm{t}) - \mathcal{P}u(\mathrm{t})|\, |v(\mathrm{t}) - \mathcal{P}v(\mathrm{t})|\right]^{\frac{1}{2}}.$$

Taking the supremum over $\mathrm{t} \in [0, 1]$, it follows that

$$\ddot{\mathbb{Q}}(\mathcal{P}u, \mathcal{P}v) \leq \lambda\, [\ddot{\mathbb{Q}}(u, v)]^{\frac{1}{2}} [\ddot{\mathbb{Q}}(u, \mathcal{P}u)]^{\frac{1}{2}} [\ddot{\mathbb{Q}}(v, \mathcal{P}v)]^{\frac{1}{2}}.$$

Equivalently, for any α, $\beta \in (0, 1)$, we can express the inequality above as

$$\ddot{\mathbb{Q}}(\mathcal{P}u, \mathcal{P}v) \leq \lambda\, [\ddot{\mathbb{Q}}(u, \mathcal{P}u)]^{\alpha}\, [\ddot{\mathbb{Q}}(u, v)]^{\beta}\, [\ddot{\mathbb{Q}}(v, \mathcal{P}v)]^{1-\alpha-\beta}.$$

For any $\gamma \geq 0$, the inequality above can be expressed as

$$\ddot{\mathbb{Q}}(\mathcal{P}u, \mathcal{P}v) \leq \lambda\, [\ddot{\mathbb{Q}}(u, \mathcal{P}u)]^{\alpha}\, [\ddot{\mathbb{Q}}(u, v)]^{\beta}\, [\ddot{\mathbb{Q}}(v, \mathcal{P}v)]^{1-\alpha-\beta} + \gamma(\ddot{\mathbb{Q}}(u, v) + \ddot{\mathbb{Q}}(u, \mathcal{P}u) + \ddot{\mathbb{Q}}(v, \mathcal{P}v)).$$

Thus, all the hypotheses of Theorem 2 are satisfied. Consequently, the operator $\mathcal{P}$ has a unique fixed point in $\mathscr{U} = C[0, 1]$, which is precisely the unique solution of Problem (5.1). □

In the context of Theorem 4, the residual $\ddot{\mathbb{Q}}(u, \mathcal{P}u) = \sup_{\mathrm{t}\in[0,1]} |u(\mathrm{t}) - \mathcal{P}u(\mathrm{t})|$ measures the mismatch between a trial deflection profile u and the deflection reconstructed from the biharmonic Green's function $\mathcal{G}(\mathrm{t}, \mathfrak{s})$ of the hinged beam. Because the hinged boundary conditions enforce a zero bending moment at both supports ($u''(0) = u''(1) = 0$), the classical interpolative estimate is most susceptible to failure precisely near configurations where the beam's deflection is at, or close to, its equilibrium profile at a support, where the local self-distance vanishes. The control term $\gamma(\ddot{\mathbb{Q}}(u, v) + \ddot{\mathbb{Q}}(u, \mathcal{P}u) + \ddot{\mathbb{Q}}(v, \mathcal{P}v))$ then provides a structural compliance margin: It retains the direct difference between two deflection profiles together with whichever residual is nonzero, consistent with the fact that the positivity and boundedness of the Green's function ($\max_{\mathrm{t},\mathfrak{s}\in[0,1]} \mathcal{G}(\mathrm{t}, \mathfrak{s}) = 1/48$, $\int_0^1 \mathcal{G}(\mathrm{t}, \mathfrak{s})\, ds \leq 1/24$) guarantee the finite compliance of the beam everywhere on $[0, 1]$, including at the hinge points where the classical interpolative bound degenerates.

6. Conclusions

In this work, we overcome a significant weakness of classical interpolative contraction mappings, whose conditions of contraction break down at fixed points, by introducing the notions of hybrid interpolative Kannan and hybrid interpolative Reich–Rus–Čirić contractions. These new hybrid contractions break the limitation of classical mappings in offering a more general and flexible structure which applies to all the points of the domain. The suggested hybrid framework with the supplementary control terms will assure the validity of the contractive conditions throughout the domain, including on the fixed points. We strictly proved the existence and uniqueness of fixed points of these mappings in complete metric spaces.

The effectiveness and novelty of the methodology were proved by continuous and discontinuous examples that do not satisfy the classical interpolative contractions but meet the conditions of the hybrid, which reveal the true extension and intensification of the current theory. Moreover, the results were extended to the practical uses of the hybrid interpolative contractions, namely an RLC circuit system and a nonlinear biharmonic with hinged boundary conditions, and confirmed the generality of these contractions to both physical and mathematical frameworks.
In general, the suggested hybrid interpolative contractions are not only able to address the natural limitations of classical interpolative mappings but also offer a flexible and powerful tool to analyze a large variety of fixed-point problems and their uses in science and engineering.

Open problems

1) Can Theorems 1 and 2 be extended to multivalued mappings?
2) Can the hybrid interpolative contractions (Definitions 5 and 6) be utilized to establish best proximity and coincidence point results?
3) Can Theorems 1 and 2 be applied to establish the existence and uniqueness of solutions for the fractional-order models studied in [11, 12]?

Author contributions

Wardah: Investigation, software, methodology, writing-original draft preparation, writing-review and editing; Mujahid Abbas: Validation, formal analysis, project administration, writing-review and editing; Haroon Ahmad: Conceptualization, methodology, writing-original draft preparation; Mudasir Younis: Validation, formal analysis, project administration, writing-review and editing; Lateef Ahmad Wani: Writing-review and editing. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The author declares that no Artificial Intelligence (AI) tools were used in the creation of this article.

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Conflict of interest

All authors declare that they have no conflict of interest in this paper.

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