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Research article

Differential geometry of parallel curves derived from pedal curves of a given curve in Euclidean 3-space

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Abstract: In this paper, we present a comprehensive study on the differential geometry of parallel curves derived from a tripartite family of pedal curves, specifically the tangent ($T-$pedal), principal normal ($N-$pedal), and binormal ($B-$pedal) tracks of a given regular space curve in Euclidean 3-space E^3. By utilizing the classical moving Frenet-Serret frame, we explicitly derive the parametric equations of the parallel curves constructed from each spatial pedal family. Furthermore, the complete Frenet-Serret apparatus, including the localized moving trihedrons, curvature κ, and torsion τ functions of these parallel curves obtained from pedal curves of a given curve by using Frenet vectors as position vectors, were systematically determined.

Keywords: space curve; $T-$pedal curve; $N-$pedal curve; $B-$pedal curve; parallel curves; Frenet frame; curvatures
Mathematics Subject Classification: 53A04, 53A05

1. Introduction

In differential geometry, generating parallel curves from pedal curves is a fascinating way to study how geometric properties like curvature, singularity formation, and envelopes transform under classical curve evolution techniques. The study of curve evolution, geometric transformations, and kinematic processes forms a cornerstone of classical differential geometry and modern computer-aided geometric design (CAGD). Among geometric operations, the pedal transformation and the parallel offset transformation have historically been treated as distinct processes with fundamental applications in mapping singularities, envelopes, and focal ridges. In classical literature, foundational textbooks by

Do Carmo [1], Gray et al. [2], and O'Neill [3] have thoroughly documented the individual behaviors and intrinsic properties of curves and surfaces undergoing these geometric transformations.

The classical pedal curve is traditionally defined as the locus of orthogonal projections from a fixed reference point (pole) to the tangent lines of a given base curve. Over the years, the concept of pedal trajectories has been widely extended and generalized in geometric spaces. For instance, Tuncer et al. explored the pedal and contrapedal curves of fronts in the Euclidean plane [4], while Li and Pei investigated the pedal curves of frontals [5] and Ceylan and Kara applied it to Bézier curves [6]. Extensions to non-Euclidean geometries include the study of pedal curves of fronts in the sphere [7] and non-lightlike curves in Minkowski 3-space [8]. Furthermore, singularity behaviors of pedal curves produced by dual curve germs have been successfully classified [9]. Beyond curves, the differential geometric aspects of pedal transformations have been adapted to surfaces, including pedal surfaces with constant support functions [10], characteristic properties of a-pedal surfaces [11], and the investigation of Gaussian and mean curvatures along specific pedal tracks [12,13]. Researchers have also generalized this concept to alternative frames and ruled surfaces, such as the spatial pedal curves generated by alternative frame vectors [14], their Smarandache curves [15], and $T-$pedal ruled surfaces structured via the Frenet frame [16]. Moreover, the boundaries of these transformations have been advanced into pseudo-Riemannian space forms, such as the comprehensive analysis of (contra)pedals and (anti) orthotomics of frontals in de Sitter 2-space [17].

In parallel, the parallel offset transformation, defined as the uniform displacement of a trajectory along its localized normal vector field, constitutes another essential tool in curve design and kinematics. Parallel offsets are crucial for understanding the motion of curves and surfaces in Euclidean 3-space [18], the construction of parallel curves based on normal vectors [19], and the properties of quasi-parallel curves [20]. In surface theory, parallel configurations are extensively used to analyze specific architectures, such as the parallel surfaces of Weingarten tubular surfaces [21].

While individual properties of pedals and parallel offsets are thoroughly documented in the literature, their sequential composition introduces a fascinating and structurally rich geometric problem. Specifically, the interactions between these two operations can be analyzed via two distinct pathways: Generating parallel curves from a tripartite family of spatial pedal curves ($T-$pedal, $N-$pedal, and $B-$pedal curves generated along each axis of the moving frame) versus deriving pedal curves from parallel tracks. The primary motivation for investigating parallel curves derived from spatial pedal tracks stems from the interplay between non-linear projection transformations and uniform normal displacements. While traditional studies focus on isolated curve transformations, analyzing parallel offsets of pedal families reveals how localized geometric features, such as inflection points, scalar speed distortions, and singular footprints induced by the pedal mapping, propagate outward.

Rather than merely offering mechanical derivations, this approach provides explicit characterizations of curvature κ and torsion τ variation along the offset fields. Consequently, these geometric formulations establish structural criteria for avoiding self-intersections and envelope formation in spatial curve modeling and CAGD applications.

The mathematical justification for studying parallel offsets of spatial pedal curves lies in understanding how non-linear geometric projection transformations interact with uniform displacement operations.

While constructing pedal curves from parallel tracks, analyzes how initial spatial clearance influences the resulting projection footprints, the reverse process presented in this work (i.e., generating parallel curves from pedal families) systematically reveals how intrinsic structural features (such as focal points, localized singular behaviors, and inflection points induced by the pedal operation) propagate outward under uniform normal offsets.

Specifically, this approach enables a direct evaluation of the scalar speed distortion, principal normal vector orientation shifts, and evolution of differential invariants (κ and τ) starting from a geometry that has undergone a projection transformation.

In this paper, we provide a rigorous synthesis of parallel curves obtained from pedal curves in Euclidean 3-space. Utilizing the standard Frenet-Serret frame, we explicitly derive the parametric equations, Frenet-Serret vectors, and fundamental invariants (curvature κ and torsion τ) for the parallel curves obtained from $T-$pedal, $N-$pedal, and $B-$pedal curves. Geometrically, the construction of the parallel curves of the $T-$pedal, $N-$pedal, and $B-$pedal curves provide geometric framework for investigating how the local geometry induced by the tangential, principal normal and binormal projection evolves under constant normal offset. This provides additional insight into the local geometric behaviors and characteristics of the induced curve family.

2. Preliminaries

Definition 2.1. Let $\alpha(s)=(x(s),y(s),z(s))$ be a smooth space curve parameterized by arc-length s. The velocity vector satisfies the unit speed condition $\|\alpha'(s)\|=1$, establishing the standard orthonormal Frenet-Serret frame $\{T(s),N(s),B(s)\}$, where $T(s)$ represents the unit tangent, $N(s)$ the principal normal, and $B(s)$ the binormal vector field. The structural variations of this moving trihedron are governed by the classical Frenet-Serret differential equations.

$$T=\frac{\alpha'}{\|\alpha'\|}\ ,\ N=B\wedge T=\frac{(\alpha'\wedge\alpha'')\wedge\alpha'}{\|\alpha'\wedge\alpha''\|\|\alpha'\|}\ ,\ B=\frac{\alpha'\wedge\alpha''}{\|\alpha'\wedge\alpha''\|},$$

$$\frac{dT}{ds}=\kappa(s)N(s)\,,\quad \frac{dN}{ds}=-\kappa(s)T(s)+\tau(s)B(s),\quad \frac{dB}{ds}=-\tau(s)N(s),$$

$$\kappa=\frac{\|\alpha'\wedge\alpha''\|}{\|\alpha'\|^3},\quad \tau=\frac{\det(\alpha',\alpha'',\alpha''')}{\|\alpha'\wedge\alpha''\|^2},$$

where $\kappa(s)$ and $\tau(s)$ denote the curvature and torsion functions of the main curve $\alpha(s)$.

Remark 2.1. The Frenet-Serret frame $\{T(s),N(s),B(s)\}$ provides a localized, orthonormal reference system along the space curve $\alpha(s)$ [1,3]. Understanding the evolution of these vector fields is fundamental when applying localized normal displacements to construct parallel offsets [2].

Definition 2.2. Let $\alpha(s)$ be a smooth space curve parameterized by its arc-length parameter s, and $N(s)$ represents the unit principal normal vector field of $\alpha(s)$ derived from its Frenet frame. The parallel curve of $\alpha(s)$ at a constant displacement distance d (where $d\in\Re/\{0\}$) is defined as the parameterized curve given by the position vector equation,

$$\beta_d(s)=\alpha(s)+d.N(s).$$

Remark 2.2. The parallel offset transformation shifts each point of the original trajectory by a uniform displacement distance d along its localized principal normal field [19]. This offset operation plays a crucial role in kinematics, motion planning, tool-path trajectory generation, and offset surface modeling [18,20].

Definition 2.3. Let us choose the coordinate origin $O(0,0,0)$ as the projection reference pole, we define a tripartite family of spatial pedal curves such as $T-$pedal curve $\alpha_T(s)$, $N-$pedal curve $\alpha_N(s)$, and $B-$pedal curve $\alpha_B(s)$, generated along each axis of the moving frame, respectively,

$$\alpha_T(s) = \alpha(s) + \langle \alpha(s), T(s) \rangle T(s),$$

$$\alpha_N(s) = \alpha(s) + \langle \alpha(s), N(s) \rangle N(s),$$

$$\alpha_B(s) = \alpha(s) + \langle \alpha(s), B(s) \rangle B(s),$$

where $\langle .,. \rangle$ denote the standard Euclidean inner product in $\Re^3$ [14,21].

3. Differential geometry of parallel curves derived from $T-$pedal, $N-$pedal and $B-$pedal curves

3.1. The parallel curve of $T-$pedal curve

Theorem 3.1. The equation of the parallel curve of $T-$pedal curve of a given regular curve with the arc-length parameter s and choosing the coordinate origin $O(0,0,0)$ as the projection reference pole as follows:

$$\beta_{T,d}(s) = \left(\alpha(s) + \langle \alpha(s), T(s) \rangle T(s)\right) + d.\frac{T'_{\alpha_T}(s)}{\left\| T'_{\alpha_T}(s) \right\|}. \tag{1}$$

Proof. Let take the position vector $\alpha(s) = \left(x(s), y(s), z(s)\right)$ and the unit tangent vector $T(s) = \left(x_T(s), y_T(s), z_T(s)\right)$. The structural inner product component yields:

$$\langle \alpha(s), T(s) \rangle = x(s)x_T(s) + y(s)y_T(s) + z(s)z_T(s).$$

Substituting this scalar mapping into the operator equation results in the explicit Cartesian parameterization of $T-$pedal curve $\alpha_T(s) = \left(x_T^P(s), y_T^P(s), z_T^P(s)\right)$.

$$x_T^P(s) = x(s) + \left(x(s)x_T(s) + y(s)y_T(s) + z(s)z_T(s)\right)x_T(s),$$

$$y_T^P(s) = y(s) + \left(x(s)x_T(s) + y(s)y_T(s) + z(s)x_T(s)\right)y_T(s),$$

$$z_T^P(s) = z(s) + \left(x(s)x_T(s) + y(s)y_T(s) + z(s)z_T(s)\right)z_T(s).$$

To investigate the intrinsic properties of $\alpha_T(s)$, we compute its velocity vector field by differentiating with respect to the arc-length parameter *s*.

$$\alpha'_T(s) = \alpha'(s) + \left(\frac{d}{ds}\langle \alpha(s), T(s) \rangle\right)T(s) + \langle \alpha(s), T(s) \rangle T'(s).$$

Applying the identity,

$$\frac{d}{ds}\langle \alpha(s),T(s)\rangle = \langle T(s),T(s)\rangle + \langle \alpha(s),\kappa(s)N(s)\rangle = 1+\kappa(s)\langle \alpha(s),N(s)\rangle,$$

and substituting $T'(s)=\kappa(s)N(s)$ produces:

$$\alpha_T'(s) = T(s) + \big(1+\kappa(s)\langle \alpha(s),N(s)\rangle\big)T(s) + \kappa(s)\langle \alpha(s),T(s)\rangle N(s),$$

$$\alpha_T'(s) = \big(2+\kappa(s)\langle \alpha(s),N(s)\rangle\big)T(s) + \kappa(s)\langle \alpha(s),T(s)\rangle N(s).$$

Setting the coefficient functions as $\mathrm{f}(s)=2+\kappa(s)\langle \alpha(s),N(s)\rangle$ and $\mathrm{g}(s)=\kappa(s)\langle \alpha(s),\mathrm{T}(s)\rangle$. the speed metric (norm) of the pedal curve is given by:

$$\|\alpha_T'(\mathrm{s})\| = \sqrt{f(s)^2+g(s)^2}\ .$$

Additionally, the coefficient functions's first derivatives are as follows:

$$\begin{aligned}\mathrm{f}'(s) &= \kappa'(s)\langle \alpha(s),N(s)\rangle - \kappa^2(s)\langle \alpha(s),\mathrm{T}(s)\rangle + \kappa(s)\tau(\mathrm{s})\langle \alpha(s),\mathrm{B}(s)\rangle,\\ \mathrm{g}'(s) &= \kappa'(s)\langle \alpha(s),\mathrm{T}(s)\rangle + \kappa(s) + \kappa^2(s)\langle \alpha(s),\mathrm{N}(s)\rangle.\end{aligned}$$

The parallel envelope of the derived spatial $T-$pedal curve at a uniform offset distance $d\in\Re$ requires the determination of its own unit principal normal vector field, denoted by $N_{\alpha_T}(s)$.

The unit tangent vector field $T_{\alpha_T}(s)$ of $T-$pedal curve is established by normalizing its first derivative:

$$T_{\alpha_T}(s) = \frac{\alpha_T'(s)}{\|\alpha_T'(s)\|} = \frac{\big(2+\kappa(s)\langle \alpha(s),N(s)\rangle\big)T(s) + \kappa(s)\langle \alpha(s),T(s)\rangle N(s)}{\sqrt{\big(2+\kappa(s)\langle \alpha(s),N(s)\rangle\big)^2 + \kappa^2(s)\big(\langle \alpha(s),T(s)\rangle\big)^2}}\ .$$

The unique unit principal normal vector field $N_{\alpha_T}(s)$ of the $T-$pedal curve is isolated via differentiation:

$$N_{\alpha_T}(s) = \frac{T_{\alpha_T}'(s)}{\|T_{\alpha_T}'(s)\|}\ .$$

By shifting $T-$pedal trajectory along this localized normal vector field, the parametric equation for parallel curve of $T-$pedal curve $\beta_{T,d}(s)$ is expressed as:

$$\beta_{T,d}(s) = \alpha_T(s) + d.N_{\alpha_T}(s),$$

$$\beta_{T,d}(s) = \big(\alpha(s) + \langle \alpha(s),T(s)\rangle T(s)\big) + d.\frac{T_{\alpha_T}'(s)}{\|T_{\alpha_T}'(s)\|},$$

where the expression $T_{\alpha_T}'(s)$, $\|T_{\alpha_T}'(s)\|$, $\Omega(s)$ and $\big(f(s)g'(s)-g(s)f'(s)\big)$ are calculated as follows, respectively,

$$
\begin{aligned}
T'_{\alpha_T}(s) = & \left[\left(2+\kappa(s)\langle \alpha(s), N(s)\rangle\right)^2 + \left(\kappa(s)\langle \alpha(s), T(s)\rangle\right)^2\right] \\
& \cdot \begin{bmatrix} \left(\kappa'(s)\langle \alpha(s), N(s)\rangle - 2\kappa^2(s)\langle \alpha(s), T(s)\rangle + \kappa(s)\tau(s)\langle \alpha(s), B(s)\rangle\right)T(s) \\ +\left(3\kappa(s)+\kappa'(s)\langle \alpha(s), T(s)\rangle + 2\kappa^2(s)\langle \alpha(s), N(s)\rangle\right)N(s) \\ +\left(\kappa(s)\tau(s)\langle \alpha(s), T(s)\rangle\right)B(s) \end{bmatrix} \\
& - \begin{bmatrix} \left(2+\kappa(s)\langle \alpha(s), N(s)\rangle\right)\left(\kappa'(s)\langle \alpha(s), N(s)\rangle - \kappa^2(s)\langle \alpha(s), T(s)\rangle + \kappa(s)\tau(s)\langle \alpha(s), B(s)\rangle\right) \\ +\left(\kappa(s)\langle \alpha(s), T(s)\rangle\right)\left(\kappa'(s)\langle \alpha(s), T(s)\rangle - \kappa(s) + \kappa^2(s)\langle \alpha(s), N(s)\rangle\right) \end{bmatrix} \\
& \cdot \begin{bmatrix} \left(2+\kappa(s)\langle \alpha(s), N(s)\rangle\right)T(s) \\ +\left(\kappa(s)\langle \alpha(s), T(s)\rangle\right)N(s) \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\left\|T'_{\alpha_T}(s)\right\| = & \sqrt{\left(2+\kappa(s)\langle \alpha(s), N(s)\rangle\right)^2 + \kappa^2(s)\left(\langle \alpha(s), T(s)\rangle\right)^2} \\
& \cdot \sqrt{\Omega^2(s) + \kappa^2(s)\tau^2(s)\left(\langle \alpha(s), T(s)\rangle\right)^2 \left[\left(\left(2+\kappa(s)\langle \alpha(s), N(s)\rangle\right)^2 + \left(\kappa^2(s)\left(\langle \alpha, T\rangle\right)^2\right)\right)\right]},
\end{aligned}
$$

$$
\Omega(s) = \kappa(s)\left[\left(2+\kappa(s)\langle \alpha(s), N(s)\rangle\right)^2 + \left(\kappa(s)\langle \alpha(s), T(s)\rangle\right)^2\right] - \left(f(s)g'(s) - g(s)f'(s)\right),
$$

and

$$
\begin{aligned}
f(s)g'(s) - g(s)f'(s) = & \, 2\kappa(s) + 2\kappa'(s)\langle \alpha, T\rangle + 3\kappa^2\langle \alpha, N\rangle \\
& + \kappa^3\langle \alpha, N\rangle^2 + \kappa^3\langle \alpha, T\rangle^2 - \kappa^2\tau\langle \alpha, T\rangle\langle \alpha, B\rangle.
\end{aligned}
$$

The Frenet-Sererjiret apparatus of parallel curve obtained from $T-$pedal curve

Theorem 3.2. The Frenet frame $\left\{T_{\beta_{T,d}}(s), N_{\beta_{T,d}}(s), B_{\beta_{T,d}}(s)\right\}$ of the parallel curve obtained from $T-$ pedal curve are structured as follows, respectively,

$$
T_{\beta_{T,d}}(s) = \frac{B_T(s)T(s) + B_N(s)N(s) + B_B(s)B(s)}{\sqrt{B_T^2(s) + B_N^2(s) + B_B^2(s)}}, \tag{2}
$$

$$
N_{\beta_{T,d}}(s) = \frac{D_T(s)T(s) + D_N(s)N(s) + D_B(s)B(s)}{\sqrt{R_T^2(s) + R_N^2(s) + R_B^2(s)}\sqrt{B_T^2(s) + B_N^2(s) + B_B^2(s)}}, \tag{3}
$$

$$
B_{\beta_{T,d}}(s) = \frac{\left(B_N C_B - B_B C_N\right)T(s) + \left(B_T C_B - B_B C_T\right)N(s) + \left(B_T C_N - B_N C_T\right)B(s)}{\sqrt{\left(B_N C_B - B_B C_N\right)^2 + \left(B_T C_B - B_B C_T\right)^2 + \left(B_T C_N - B_N C_T\right)^2}}. \tag{4}
$$

Proof. Since the parallel curve $\beta_{T,d}(s)$ is a non-unit speed curve field, we differentiate its position vector with respect to the arc-length parameter s. By implementing the chain rule along the Frenet projections, the first derivative vector field resolves into the following orthogonal decomposition:

$$\beta'_{T,d}(s) = B_T(s)T(s) + B_N(s)N(s) + B_B(s)B(s)\,.$$

To avoid algebraic complexity within the major formulations, the invariant scalar coefficients $B_i(s)$ are mapped explicitly in terms of $\kappa(s)$ and $\tau(s)$ as:

$$B_T(s) = 2 + \kappa(s)\langle\alpha(s), N(s)\rangle + d.\left[\frac{-\kappa(s)\langle\alpha(s), T(s)\rangle}{\nu_T(s)}\right]' - \frac{d.\kappa(s)\left(2 + \kappa(s)\langle\alpha(s), N(s)\rangle\right)}{\nu_T(s)},$$

$$B_N(s) = \kappa(s)\langle\alpha(s), T(s)\rangle + d.\left[\frac{2 + \kappa(s)\langle\alpha(s), N(s)\rangle}{\nu_T(s)}\right]' - \frac{d.\kappa^2(s)\langle\alpha(s), T(s)\rangle}{\nu_T(s)},$$

$$B_B(s) = \frac{d.\tau(s)\left(2 + \kappa(s)\langle\alpha(s), N(s)\rangle\right)}{\nu_T(s)},$$

where the internal speed metric $\nu_T(s) = \|\alpha'_T(s)\|$ satisfies:

$$\nu_T(s) = \sqrt{\left(2 + \kappa(s)\langle\alpha(s), N(s)\rangle\right)^2 + \left(\kappa(s)\langle\alpha(s), T(s)\rangle\right)^2}\,.$$

Consequently, the scalar speed field $\nu_B(s) = \|\beta'_{T,d}(s)\|$ of the parallel offset system is defined by:

$$\nu_B(s) = \|\beta'_{T,d}(s)\| = \sqrt{B_T^2(s) + B_N^2(s) + B_B^2(s)}\,.$$

By utilizing the metric definitions established above, the Frenet frame $\{T_{\beta_{T,d}}(s), N_{\beta_{T,d}}(s), B_{\beta_{T,d}}(s)\}$ of the parallel curve family is structured as follows.

The unit tangent vector field is obtained as follows:

$$T_{\beta_{T,d}}(s) = \frac{B_T(s)T(s) + B_N(s)N(s) + B_B(s)B(s)}{\sqrt{B_T^2(s) + B_N^2(s) + B_B^2(s)}}\,.$$

Differentiating the vector equation of $\beta'_{T,d}(s)$ with respect to s using the product rule yields:

$$\beta''_{T,d}(s) = \left(B'_T T + B_T T'\right) + \left(B'_N N + B_N N'\right) + \left(B'_B B + B_B B'\right).$$

By substituting the Frenet-Serret formulas into the equation of $\beta''_{T,d}(s)$, the following equality is obtained,

$$\beta''_{T,d}(s) = B'_T T + B_T(\kappa N) + B'_N N + B_N(-\kappa T + \tau B) + B'_B B + B_B(-\tau N)\,.$$

Grouping the independent directional terms along $T(s)$, $N(s)$, and $B(s)$ axes yields the final expressions tangential component $C_T(s)$, principal normal component $C_N(s)$, and binormal component $C_B(s)$:

$$C_T(s) = B'_T(s) - \kappa(s)B_N(s)\,,$$

$$C_N(s) = B'_N(s) - \kappa(s)B_T(s) - \tau(s)B_B(s)\,,$$

$$C_B(s) = B_B'(s) - \tau(s)B_N(s).$$

Derivatives of the tangential velocity B_T', normal velocity B_N', and binormal velocity B_B' are given by the following equalities.

$$B_T'(s) = \kappa'(s)\langle\alpha(s), N(s)\rangle + \kappa(s)\left(-\kappa(s)\langle\alpha(s), T(s)\rangle + \tau(s)\langle\alpha(s), B(s)\rangle\right)$$
$$+d.\left[\frac{-\kappa(s)\langle\alpha(s), T(s)\rangle}{v_T(s)}\right]'' - d.\left[\frac{\kappa(s)\left(2+\kappa(s)\langle\alpha(s), N(s)\rangle\right)}{v_T(s)}\right]',$$

$$B_T'(s) = \kappa'(s)\langle\alpha(s), T(s)\rangle + \kappa(s)\left(1+\kappa(s)\langle\alpha(s), N(s)\rangle + \tau(s)\langle\alpha(s), B(s)\rangle\right)$$
$$+d.\left[\frac{2+\kappa(s)\langle\alpha(s), N(s)\rangle}{v_T(s)}\right]'' - d.\left[\frac{\kappa^2(s)\langle\alpha(s), T(s)\rangle}{v_T(s)}\right]',$$

$$B_T'(s) = d.\left[\frac{\tau(s)\left(2+\kappa(s)\langle\alpha(s), N(s)\rangle\right)}{v_T(s)}\right]'.$$

To extract the secondary normal field, we track the structural acceleration field,

$$\beta_{T,d}''(s) = C_T(s)T(s) + C_N(s)N(s) + C_B(s)B.$$

The binormal vector field is obtained as follows:

$$B_{\beta_{T,d}}(s) = \frac{(B_N C_B - B_B C_N)T(s) + (B_T C_B - B_B C_T)N(s) + (B_T C_N - B_N C_T)B(s)}{\sqrt{(B_N C_B - B_B C_N)^2 + (B_T C_B - B_B C_T)^2 + (B_T C_N - B_N C_T)^2}}.$$

The principal normal vector field is given by following equality,

$$N_{\beta_{T,d}}(s) = \frac{D_T(s)T(s) + D_N(s)N(s) + D_B(s)B(s)}{\sqrt{R_T^2(s) + R_N^2(s) + R_B^2(s)}\sqrt{B_T^2(s) + B_N^2(s) + B_B^2(s)}},$$

where the invariants scalars are structured as:

$$D_T(s) = R_N(s)B_B(s) - R_B(s)B_N(s),$$

$$D_N(s) = R_B(s)B_T(s) - R_T(s)B_B(s),$$

$$D_B(s) = R_T(s)B_N(s) - R_N(s)B_T(s),$$

and the expressions $R_T(s)$, $R_N(s)$ and $R_B(s)$ are given the following equalities:

$$R_T(s) = B_N C_B - B_B C_N,$$

$$R_N(s) = -[B_T C_B - B_B C_T],$$

$$R_B(s) = B_T C_N - B_N C_T.$$

Theorem 3.3. The curvature $\kappa_{\beta_{T,d}}(s)$ and torsion $\tau_{\beta_{T,d}}(s)$ of the parallel curve obtained from $T-$

pedal curve are structured as follows, respectively,

$$\kappa_{\beta_{T,d}}(s)=\frac{\sqrt{\left(B_N C_B-B_B C_N\right)^2+\left(B_T C_B-B_B C_T\right)^2+\left(B_T C_N-B_N C_T\right)^2}}{\left(B_T^2(s)+B_N^2(s)+B_B^2(s)\right)^{\frac{3}{2}}}, \tag{5}$$

$$\tau_{\beta_{T,d}}(s)=\frac{R_T(s)K_T(s)+R_N(s)K_N(s)+R_B(s)K_B(s)}{R_T^2(s)+R_N^2(s)+R_B^2(s)}, \tag{6}$$

where the expressions $K_T(s)$, $K_N(s)$ and $K_N(s)$ are as follows:

$$K_T(s)=C_T'(s)-\kappa(s)C_N(s),$$

$$K_N(s)=C_N'(s)+\kappa(s)C_T(s)-\tau(s)C_B(s),$$

$$K_B(s)=C_B'(s)+\tau(s)C_N(s).$$

Proof. This is seen by utilizing the Frenet-Serret formulas from Definition 2.1. and proof of Theorem 3.1.

Example 3.1. Let us consider the space curve parameterized as $\alpha(s)=\left(3\cos\left(\frac{s}{5}\right),3\sin\left(\frac{s}{5}\right),\frac{4s}{5}\right)$. $T-$pedal curve of the space curve $\alpha(s)$ and the parallel curve of $T-$pedal curve according to the origin are illustrated in Figure 1.

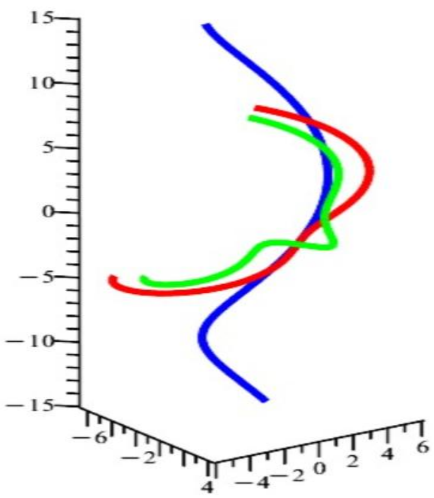

Figure 1. The original space curve (blue), the corresponding $T-$pedal curve (red) and the parallel curve of the $T-$pedal generated along its principal normal vector for d=1.5 (green).

$$T-pedal \Rightarrow \alpha_T(s)=\left(3\cos\left(\frac{s}{5}\right)+\frac{48s}{125}\sin\left(\frac{s}{5}\right),3\sin\left(\frac{s}{5}\right)-\frac{48s}{125}\cos\left(\frac{s}{5}\right),\frac{36s}{125}\right),$$

The parallel curve of $T-pedal \Rightarrow \beta_{T,d}(s)$ is obtained as follows:

$$\beta_{T,d}(s)=\begin{pmatrix} 3\cos\left(\frac{s}{5}\right)+\frac{48s}{125}\sin\left(\frac{s}{5}\right) \\ +\frac{d\left(-4096\sin\left(\frac{s}{5}\right)s^3-11520\cos\left(\frac{s}{5}\right)s^2-32400s\sin\left(\frac{s}{5}\right)+196875\cos\left(\frac{s}{5}\right)\right)}{\sqrt{256s^2+5625}\sqrt{65536s^4+115200s^2+6890625}}, \\ 3\sin\left(\frac{s}{5}\right)-\frac{48s}{125}\cos\left(\frac{s}{5}\right) \\ +\frac{d\left(4096\cos\left(\frac{s}{5}\right)s^3-11520\sin\left(\frac{s}{5}\right)s^2+32400s\cos\left(\frac{s}{5}\right)+196875\sin\left(\frac{s}{5}\right)\right)}{\sqrt{256s^2+5625}\sqrt{65536s^4+115200s^2+6890625}}, \\ \frac{36s}{125}-\frac{76800sd}{\sqrt{256s^2+5625}\sqrt{65536s^4+115200s^2+6890625}} \end{pmatrix}.$$

3.2. The parallel curve of $N-$*pedal curve*

Theorem 3.4. The equation of the parallel curve of the $N-$pedal curve of a give regular curve with the arc-length parameter s and choosing the coordinate origin $O(0,0,0)$ as the projection reference pole as follows:

$$\beta_{N,d}(s)=\left(\alpha(s)+\langle\alpha(s),N(s)\rangle N(s)\right)+d.\frac{T'_{\alpha_N}(s)}{\left\|T'_{\alpha_N}(s)\right\|}. \tag{7}$$

Proof. Expanding the position vector $\alpha(s)=\left(x(s),y(s),z(s)\right)$ and the unit principal normal vector $N(s)=\left(x_N(s),y_N(s),z_N(s)\right)$, the structural inner product component maps as:

$$\langle\alpha(s),N(s)\rangle=x(s)x_N(s)+y(s)y_N(s)+z(s)z_N(s).$$

Substituting this scalar mapping into the operator equation, $\alpha_N(s)=\alpha(s)+\langle\alpha(s),N(s)\rangle N(s)$ results in the explicit Cartesian parameterization of $N-$pedal curve $\alpha_N(s)=\left(x_N^P(s),y_N^P(s),z_N^P(s)\right)$:

$$x_N^P(s)=x(s)+\left(x(s)x_N(s)+y(s)y_N(s)+z(s)z_N(s)\right)x_N(s),$$

$$y_N^P(s)=y(s)+\left(x(s)x_N(s)+y(s)y_N(s)+z(s)z_N(s)\right)y_N(s),$$

$$z_N^P(s)=z(s)+\left(x(s)x_N(s)+y(s)y_N(s)+z(s)z_N(s)\right)z_N(s).$$

To investigate the intrinsic properties of $\alpha_N(s)$, we compute the first derivative with respect to arc-length parameter s. Applying the product rule and Frenet-Serret relations, we obtain

$$\alpha'_N(s)=\alpha'(s)+\left(\frac{d}{ds}\langle\alpha(s),N(s)\rangle\right)N(s)+\langle\alpha(s),N(s)\rangle N'(s).$$

Evaluating the derivative of the inner product component:

$$\frac{d}{ds}\langle \alpha(s), N(s)\rangle = \langle \alpha'(s), N(s)\rangle + \langle \alpha(s), N'(s)\rangle$$
$$= \langle T(s), N(s)\rangle + \langle \alpha(s), -\kappa(s)T(s) + \tau(s)B(s)\rangle,$$

$$\frac{d}{ds}\langle \alpha(s), N(s)\rangle = 0 - \kappa(s)\langle \alpha(s), T(s)\rangle + \tau(s)\langle \alpha(s), B(s)\rangle .$$

Substituting this back into the velocity equation along with $N'(s)$ gives

$$\alpha_N'(s) = T(s) + \left(\kappa(s)\langle \alpha(s), T(s)\rangle + \tau(s)\langle \alpha(s), B(s)\rangle\right) N(s)$$
$$+ \langle \alpha(s), N(s)\rangle\left(-\kappa(s)T(s) + \tau(s)B(s)\right).$$

Gathering coefficients for orthonormal basis vectors $\{T(s), N(s), B(s)\}$ yields the structural velocity field:

$$\begin{aligned}\alpha_N'(s) = {} & \left(1-\kappa(s)\langle \alpha(s), N(s)\rangle\right)T(s) + \left(-\kappa(s)\langle \alpha(s), T(s)\rangle + \tau(s)\langle \alpha(s), B(s)\rangle\right)N(s) \\ & +\tau(s)\langle \alpha(s), N(s)\rangle B(s)\end{aligned}.$$

The unit tangent vector field $T_{\alpha_N}(s)$ of $N-$pedal curve is established by dividing its velocity field by its scalar speed function $\|\alpha_N'(s)\|$:

$$\begin{aligned}T_{\alpha_N}(s) &= \frac{\alpha_N'(s)}{\|\alpha_N'(s)\|} \\ &= \frac{\left(1-\kappa(s)\langle \alpha\ (s), N(s)\rangle\right)T(s) + \left(-\kappa(s)\langle \alpha\ (s), T(s)\rangle + \tau(s)\langle \alpha\ (s), B(s)\rangle\right)N(s) + \tau(s)\langle \alpha\ (s), N(s)\rangle B(s)}{\sqrt{1-2\kappa(s)\langle \alpha\ (s), N(s)\rangle + \left(\kappa^2(s)+\tau^2(s)\right)\|\alpha\ (s)\|^2 - \left(\kappa(s)\langle \alpha\ (s), B(s)\rangle + \tau\langle \alpha\ (s), T(s)\rangle\right)^2}},\end{aligned}$$

where the normalization speed factor is explicitly mapped as:

$$\|\alpha_N'(s)\| = \sqrt{\left(1-\kappa(s)\langle \alpha(s), N(s)\rangle\right)^2 + \left(-\kappa(s)\langle \alpha(s), T(s)\rangle + \tau(s)\langle \alpha(s), B(s)\rangle\right)^2 + \tau^2(s)\left(\langle \alpha(s), B(s)\rangle\right)^2},$$

$$\|\alpha_N'(s)\| = \sqrt{1-2\kappa(s)\langle \alpha(s), N(s)\rangle + \left(\kappa^2(s)+\tau^2(s)\right)\ \|\alpha(s)\|^2 - \left(\kappa(s)\langle \alpha(s), B(s)\rangle + \tau\langle \alpha(s), T(s)\rangle\right)^2}.$$

Differentiating this unit tangent field gives its principal normal vector field

$$N_{\alpha_N}(s) = \frac{T_{\alpha_N}'(s)}{\|T_{\alpha_N}'(s)\|}.$$

The parametric equation for the parallel curve of $N-$pedal curve, denoted as $\beta_{N,d}(s)$, is expressed as:

$$\beta_{N,d}(s) = \alpha_N(s) + d.N_{\alpha_N}(s).$$

Finalized parallel curve $\beta_{N,d}(s)$ is

$$\beta_{N,d}(s) = \left(\alpha(s) + \langle \alpha(s), N(s)\rangle N(s)\right) + d.\frac{T_{\alpha_N}'(s)}{\|T_{\alpha_N}'(s)\|},$$

where the expressions $T'_{\alpha_N}(s)$ and $\left\|T'_{\alpha_N}(s)\right\|$ are

$$T'_{\alpha_N}(s)=\frac{1}{\left(f_N^2+g_N^2+h_N^2\right)^{\frac{3}{2}}}\left[\begin{array}{c}\left(f_N^2+g_N^2+h_N^2\right)\begin{pmatrix}\left(2\kappa^2\langle\alpha,T\rangle-\kappa'\langle\alpha,N\rangle-2\kappa\tau\langle\alpha,B\rangle\right)T(s)\\+\left(-\kappa'\langle\alpha,T\rangle-2\left(\kappa^2+\tau^2\right)\langle\alpha,N\rangle+\tau'\langle\alpha,B\rangle\right)N(s)\\\left(-2\kappa\tau\langle\alpha,T\rangle+\tau'\langle\alpha,N\rangle+2\kappa^2\langle\alpha,B\rangle\right)B(s)\end{pmatrix}\\-W_N(s)\left(\left(1-\kappa\langle\alpha,N\rangle\right)T(s)+\left(-\kappa\langle\alpha,T\rangle+\tau\langle\alpha,B\rangle\right)N(s)+\left(\tau\langle\alpha,N\rangle\right)B(s)\right)\end{array}\right],$$

$$\left\|T'_{\alpha_N}(s)\right\|=\frac{\sqrt{\left(f_N^2+g_N^2+h_N^2\right)\left(A_N^2(s)+B_N^2(s)+C_N^2(s)\right)-W_N^2(s)}}{f_N^2+g_N^2+h_N^2},$$

where the analytic multiplier $W_N(s)$, metric coefficients f_N, g_N, h_N, their derivatives f'_N, g'_N, and h'_N, and differential coefficients components $A_N(s)$, $B_N(s)$, $C_N(s)$ are given by following equalities:

$$W_N(s)=f_N(s)f'_N(s)+g_N(s)g'_N(s)+h_N(s)h'_N(s),$$

$$\begin{aligned}f_N(s)&=1-\kappa(s)\langle\alpha(s),N(s)\rangle,\\g_N(s)&=-\kappa(s)\langle\alpha(s),T(s)\rangle+\tau(s)\langle\alpha(s),B(s)\rangle,\\h_N(s)&=\tau(s)\langle\alpha(s),N(s)\rangle,\end{aligned}$$

$$\begin{aligned}f'_N(s)&=-\kappa(s)\langle\alpha(s),N(s)\rangle-\kappa^2(s)\langle\alpha(s),T(s)\rangle-\kappa(s)\tau(s)\langle\alpha(s),B(s)\rangle,\\g'_N(s)&=-\kappa(s)-\kappa'(s)\langle\alpha(s),T(s)\rangle-\left(\kappa^2(s)+\tau^2(s)\right)\langle\alpha(s),N(s)\rangle+\tau'(s)\langle\alpha(s),B(s)\rangle,\\h'_N(s)&=\tau'(s)\langle\alpha(s),N(s)\rangle-\kappa(s)\tau(s)\langle\alpha(s),T(s)\rangle+\tau^2(s)\langle\alpha(s),B(s)\rangle,\end{aligned}$$

$$\begin{aligned}A_N(s)&=2\kappa^2(s)\langle\alpha(s),T(s)\rangle+\kappa'(s)\langle\alpha(s),N(s)\rangle-2\kappa(s),\tau(s)\langle\alpha(s),B(s)\rangle,\\B_N(s)&=-\kappa'(s)\langle\alpha(s),T(s)\rangle-2\left(\kappa^2(s)+\tau^2(s)\right)\langle\alpha(s),N(s)\rangle+\tau'(s)\langle\alpha(s),B(s)\rangle,\\C_N(s)&=-2\kappa(s),\tau(s)\langle\alpha(s),T(s)\rangle+\tau'(s)\langle\alpha(s),N(s)\rangle+2\tau^2(s)\langle\alpha(s),B(s)\rangle.\end{aligned}$$

The Frenet-Serret apparatus of parallel curve obtained from $N-$*pedal curve*

Theorem 3.5. The Frenet frame $\{T_{\beta_{N,d}}(s),N_{\beta_{N,d}}(s),B_{\beta_{N,d}}(s)\}$ of the parallel curve obtained from $N-$ pedal curve are obtained as follows, respectively,

$$T_{\beta_{N,d}}(s)=\frac{J_T(s)T(s)+J_N(s)N(s)+J_B(s)B(s)}{\sqrt{J_T^2(s)+J_N^2(s)+J_B^2(s)}},\tag{8}$$

$$N_{\beta_{N,d}}(s)=\frac{\delta_T(s)T(s)+\delta_N(s)N(s)+\delta_B(s)B(s)}{\sqrt{S_T^2(s)+S_N^2(s)+S_B^2(s)}\sqrt{J_T^2(s)+J_N^2(s)+J_B^2(s)}},\tag{9}$$

$$B_{\beta_{N,d}}(s)=\frac{\left(J_N Z_B-J_B Z_N\right)\ T(s)+\left(J_T Z_B-J_B Z_T\right)\ N(s)+\left(J_T Z_N-J_N Z_T\right)\ B(s)}{\sqrt{\left(J_N Z_B-J_B Z_N\right)^2+\left(J_T Z_B-J_B Z_T\right)^2+\left(J_T Z_B-J_B Z_T\right)^2}}. \tag{10}$$

Proof. Since the parallel curve $\beta_{N,d}(s)$ is a non-unit speed curve field, we differentiate its position vector with respect to the arc-length parameter s. By implementing the chain rule along the Frenet projections, the first derivative vector field resolves into the following orthogonal decomposition:

$$\beta'_{N,d}(s)=J_T(s)T(s)+J_N(s)N(s)+J_B(s)B(s).$$

To avoid algebraic complexity within the major formulations, the invariant scalar coefficients $J_i(s)$ are mapped explicitly in terms of $\kappa(s)$ and $\tau(s)$ as:

$$J_T(s)=1-\kappa(s)\big(d-\langle\alpha(s),N(s)\rangle\big),$$

$$J_N(s)=\kappa(s)\langle\alpha(s),T(s)\rangle-\tau(s)\langle\alpha(s),B(s)\rangle,$$

$$J_B(s)=\tau(s)\big(d-\langle\alpha(s),N(s)\rangle\big).$$

Consequently, the scalar speed field $\left\|\beta'_{N,d}(s)\right\|$ of the parallel offset system is defined by

$$\left\|\beta'_{T,d}(s)\right\|=\sqrt{J_T^2(s)+J_N^2(s)+J_B^2(s)}.$$

By utilizing the the metric definitions established above, the Frenet frame $\left\{T_{\beta_{N,d}}(s),N_{\beta_{N,d}}(s),B_{\beta_{N,d}}(s)\right\}$ of the parallel curve family is structured as follows.

The unit tangent vector field is obtained as follows:

$$T_{\beta_{N,d}}(s)=\frac{J_T(s)T(s)+J_N(s)N(s)+J_B(s)B(s)}{\sqrt{J_T^2(s)+J_N^2(s)+J_B^2(s)}}.$$

Differentiating the vector equation of $\beta'_{N,d}(s)$ with respect to s using the product rule yields:

$$\beta''_{N,d}(s)=Z_T(s)T(s)+Z_N(s)N(s)+Z_B(s)B(s),$$

where the components Z_T, Z_N and Z_B are given by the following equalities:

$$Z_T=J'_T-\kappa\ J_N=-\kappa'\left(d-\langle\alpha,N\rangle\right),$$

$$Z_N=J'_N+\kappa J_T-\tau J_B=\left(\kappa\langle\alpha,T\rangle-\tau\langle\alpha,B\rangle\right)'+\kappa\left[1-\kappa\left(d-\langle\alpha,N\rangle\right)\right]-\tau^2\left(d-\langle\alpha,N\rangle\right),$$

$$Z_B=J'_B+\tau J_N=\tau'\left(\left(d-\langle\alpha,N\rangle\right)\right)+2\tau\left(\kappa\langle\alpha,T\rangle-\tau\langle\alpha,B\rangle\right).$$

The coefficients of the third derivative ($\beta'''_{T,d}(s)$), the following equality is obtained

$$\beta'''_{T,d}(s)=U_T(s)T(s)+U_N(s)N(s)+U_B(s)B(s),$$

where the components U_T, U_N and U_B are given by the following equalities:

$$U_T(s) = Z_T(s) - \kappa(s)Z_N(s),$$

$$U_N(s) = Z'_N(s) + \kappa(s)Z_T(s) - \tau(s)Z_B(s),$$

$$U_B(s) = Z'_B(s) + \tau(s)Z_N(s).$$

The binormal vector field is obtained as follows:

$$B_{\beta_{N,d}}(s) = \frac{\left(J_N Z_B - J_B Z_N\right)\, T(s) + \left(J_T Z_B - J_B Z_T\right)\, N(s) + \left(J_T Z_N - J_N Z_T\right)\, B(s)}{\sqrt{\left(J_N Z_B - J_B Z_N\right)^2 + \left(J_T Z_B - J_B Z_T\right)^2 + \left(J_T Z_B - J_B Z_T\right)^2}}.$$

The principal normal vector field is given by the following equality:

$$N_{\beta_{N,d}}(s) = \frac{\delta_T(s)T(s) + \delta_N(s)N(s) + \delta_B(s)B(s)}{\sqrt{S_T^2(s) + S_N^2(s) + S_B^2(s)}\sqrt{J_T^2(s) + J_N^2(s) + J_B^2(s)}},$$

where the invariants scalars are structured as:

$$\delta_T(s) = \left(J_T Z_B - J_B Z_T\right)(s)J_B(s) - \left(J_T Z_B - J_B Z_T\right)(s)J_N(s),$$

$$\delta_N(s) = \left(J_T Z_N - J_N Z_T\right)(s)J_T(s) - \left(J_N Z_B - J_B Z_N\right)(s)J_B(s),$$

$$\delta_B(s) = \left(J_N Z_B - J_B Z_N\right)(s)J_N(s) - \left(J_T Z_B - J_B Z_T\right)(s)J_T(s),$$

and the expressions S_T, S_N and S_B are given the following equalities:

$$S_T = J_N Z_B - J_B Z_N,$$

$$S_N = J_T Z_B - J_B Z_T,$$

$$S_B = j_T Z_N - J_N Z_T.$$

Theorem 3.6. The curvature $\kappa_{\beta_{N,d}}(s)$ and torsion $\tau_{\beta_{N,d}}(s)$ of the parallel curve obtained from $N-$pedal curve are structured as follows, respectively,

$$\kappa_{\beta_{N,d}}(s) = \frac{\sqrt{\left(J_N Z_B - J_B Z_N\right)^2 + \left(J_T Z_B - J_B Z_T\right)^2 + \left(J_T Z_N - J_N Z_T\right)^2}}{\left[\left(J_T\right)^2 + \left(J_N\right)^2 + \left(J_B\right)^2\right]^{\frac{3}{2}}}, \tag{11}$$

$$\tau_{\beta_{N,d}}(s) = \frac{\left(J_N Z_B - J_B Z_N\right)(s)U_T(s) + \left(J_T Z_B - J_B Z_T\right)(s)U_N(s) + \left(J_T Z_N - J_N Z_T\right)(s)U_B(s)}{\left(J_T Z_N - J_N Z_T\right)^2(s) + \left(J_T Z_B - J_B Z_T\right)^2(s) + \left(J_T Z_B - J_B Z_T\right)^2(s)}. \tag{12}$$

Proof. This is clear by utilizing the Frenet-Serret formulas from the Definition 2.1. and proof of the Theorem 3.5.

By recalling Example 3.1, the $N-$pedal curve and the parallel curve of the $N-$pedal curve according to the origin are illustrated in Figures 2 and 3.

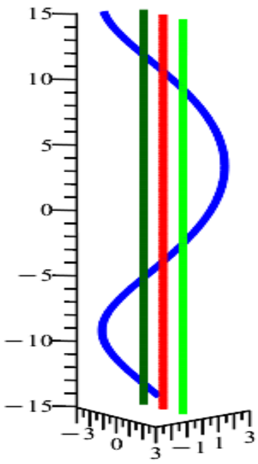

Figure 2. The original space curve (blue), and the corresponding $N-$pedal curve (red) with respect to the origin (red) and the parallel lines of the $N-$pedal translated along the positive and negative x-direction for the offset distances d=±1.5 (green and dark green).

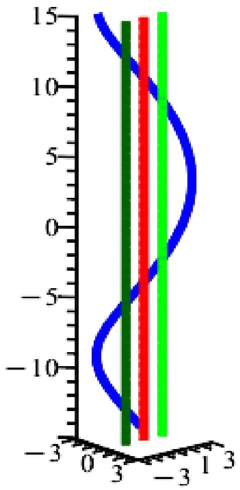

Figure 3. The original space curve (blue), and the corresponding $N-$pedal curve (red) with respect to the origin (red) and the parallel lines of the $N-$pedal translated along the positive and negative y-direction for the offset distances d=±1.5 (green and dark green).

3.3. The parallel curve of $B-$pedal curve

Theorem 3.7. The equation of the parallel curve of $B-$pedal curve of a given regular curve with the arc-length parameter s and choosing choosing the coordinate origin $O(0,0,0)$ as the projection reference pole gives:

$$\beta_{B,d}(s) = \left(\alpha(s) + \langle \alpha(s), B(s)\rangle B(s)\right) + d.\frac{T'_{\alpha_B}(s)}{\left\|T'_{\alpha_B}(s)\right\|}. \tag{13}$$

Proof. Expanding the position vector $\alpha(s) = (x(s), y(s), z(s))$ and binormal vector $B(s) = (x_B(s), y_B(s), z_B(s))$, the structural inner product component maps as:

$$\langle \alpha(s), B(s)\rangle = x(s)x_B(s) + y(s)y_B(s) + z(s)z_B(s).$$

Substituting this scalar mapping into the operator equation, $\alpha_B(s) = \alpha(s) + \langle \alpha(s), B(s)\rangle B(s)$ results in the explicit Cartesian parameterization of $B-$pedal curve $\alpha_B(s) = \left(x_B^P(s), y_B^P(s), z_B^P(s)\right)$:

$$x_B^P(s) = x(s) + \left(x(s)x_B(s) + y(s)y_B(s) + z(s)z_B(s)\right)x_B(s),$$

$$y_B^P(s) = y(s) + \left(x(s)x_B(s) + y(s)y_B(s) + z(s)z_B(s)\right)y_B(s),$$

$$z_B^P(s) = z(s) + \left(x(s)x_B(s) + y(s)y_B(s) + z(s)z_B(s)\right)z_B(s).$$

To investigate the intrinsic properties of $\alpha_B(s)$, we compute the first derivative with respect to arc-length parameter s:

$$\alpha'_B(s) = \alpha'(s) + \left(\frac{d}{ds}\langle \alpha(s), B(s)\rangle\right)B(s) + \langle \alpha(s), B(s)\rangle B'(s).$$

Applying the identity,

$$\frac{d}{ds}\langle \alpha(s), B(s)\rangle = \langle T(s), B(s)\rangle + \langle \alpha(s), -\tau(s)N(s)\rangle = -\tau(s)\langle \alpha(s), N(s)\rangle,$$

and substituting $B'(s) = -\tau(s)N(s)$ produces:

$$\alpha'_B(s) = T(s) + \left(-\tau(s)\langle \alpha(s), N(s)\rangle\right)B(s) + \langle \alpha(s), B(s)\rangle\left(-\tau(s)N(s)\right).$$

Gathering the coefficients systematically across the orthonormal moving frame yields the finalized kinematic velocity vector:

$$\alpha'_B(s) = T(s) - \tau(s)\langle \alpha(s), B(s)\rangle N(s) - \tau(s)\langle \alpha(s), N(s)\rangle B(s).$$

Taking the scalar norm of this velocity distribution across the orthonormal basis $\{T(s), N(s), B(s)\}$, the raw scalar speed of the $B-$pedal curve is given by:

$$\left\|\alpha'_B(s)\right\| = \sqrt{1 + \tau^2(s)\left(\langle \alpha(s), B(s)\rangle\right)^2 + \tau^2(s)\left(\langle \alpha(s), N(s)\rangle\right)^2}.$$

By factoring out the common torsion term $\tau^2(s)$ from the normal and binormal projections, we can apply the partial norm identity

$$\left(\langle \alpha(s), B(s)\rangle\right)^2 + \left(\langle \alpha(s), N(s)\rangle\right)^2 = \|\alpha(s)\|^2 - \left(\langle \alpha(s), T(s)\rangle\right)^2.$$

This yields the finalized, algebraically reduced scalar speed expression for the $B-$pedal curve.

$$\|\alpha_B'(s)\| = \sqrt{1+\tau^2(s)\left(\|\alpha(s)\|^2 - \left(\langle \alpha(s), T(s)\rangle\right)^2\right)}.$$

The parallel envelope of the derived spatial $B-$pedal curve at a uniform offset distance $d \in \Re$ requires the determination of its own unit principal normal vector field, denoted by $N_{\alpha_B}(s)$.

The unit tangent vector field $T_{\alpha_B}(s)$ of the $B-$pedal curve is established by normalizing its first derivative:

$$T_{\alpha_B}(s) = \frac{\alpha_B'(s)}{\|\alpha_B'(s)\|} = \frac{T(s) - \tau(s)\langle \alpha(s), B(s)\rangle N(s) - \tau(s)\langle \alpha(s), N(s)\rangle B(s)}{\sqrt{1+\tau^2(s)\left(\|\alpha(s)\|^2 - \left(\langle \alpha(s), T(s)\rangle\right)^2\right)}}.$$

The unique unit principal normal vector field $N_{\alpha_B}(s)$ of the $B-$pedal curve is isolated via differentiation:

$$N_{\alpha_B}(s) = \frac{T_{\alpha_B}'(s)}{\|T_{\alpha_b}'(s)\|}.$$

By displacing $B-$pedal trajectory along this localized normal vector field, the parametric equation for parallel curve of $B-$pedal curve, denoted as $\beta_{B,d}(s)$, is expressed as:

$$\beta_{B,d}(s) = \alpha_B(s) + d.N_{\alpha_B}(s).$$

Substituting geometric operators yields the finalized

$$\beta_{B,d}(s) = \left(\alpha(s) + \langle \alpha(s), B(s)\rangle B(s)\right) + d.\frac{T_{\alpha_B}'(s)}{\|T_{\alpha_B}'(s)\|},$$

where the expressions $T_{\alpha_B}'(s)$ and $\|T_{\alpha_B}'(s)\|$ are obtained by the following equalities:

$$T_{\alpha_B}'(s) = \frac{1}{\left(1+f_B^2+g_B^2\right)^{\frac{3}{2}}}\left[\left(1+f_B^2+g_B^2\right)\begin{pmatrix} \left(\kappa\tau\langle \alpha, B\rangle\right)T(s) \\ +\left(\kappa+2\tau^2\langle \alpha, N\rangle - \tau'\langle \alpha, B\rangle\right)N(s) \\ \left(\kappa\tau\langle \alpha, T\rangle + \tau'\langle \alpha, N\rangle - 2\tau^2\langle \alpha, B\rangle\right)B(s)\end{pmatrix} - W_B(s)\left(T(s) + \left(-\tau\langle \alpha, B\rangle\right)N(s) + \left(\tau\langle \alpha, N\rangle\right)B(s)\right)\right],$$

where the coefficient $W_B(s)$, scalar multipliers $f_B(s), g_B(s)$ and scalar multipliers' derivatives $f_B'(s)$ and $g_B'(s)$ are given by following equalities, respectively,

$$W_N(s) = f_B(s)f_B'(s) + g_B(s)g_B'(s),$$

$$g_B(s) = -\tau(s)\langle \alpha(s), B(s)\rangle,$$

$$h_B(s) = -\tau(s)\langle \alpha(s), N(s)\rangle,$$

and

$$f_B'(s) = \tau^2(s)\langle \alpha(s), N(s)\rangle - \tau'(s)\langle \alpha(s), B(s)\rangle,$$
$$g_B'(s) = \kappa(s)\tau(s)\langle \alpha(s), T(s)\rangle - \tau'(s)\langle \alpha(s), N(s)\rangle - \tau^2(s)\langle \alpha(s), B(s)\rangle.$$

The Frenet-Serret apparatus of parallel curve obtained from $B-$*pedal curve*

Theorem 3.8. The Frenet frame $\{T_{\beta_{B,d}}(s), N_{\beta_{B,d}}(s), B_{\beta_{B,d}}(s)\}$ of the parallel curve obtained from $B-$ pedal curve is structured as follows, respectively,

$$T_{\beta_{T,d}}(s) = \frac{(1-d.\kappa(s))T_{\alpha_B}(s) + d.\tau(s)B_{\alpha_B}(s)}{\sqrt{(1-d.\kappa(s))^2 + d^2.\tau^2(s)}}, \tag{14}$$

$$N_{\beta_{B,d}}(s) = N_{\alpha_B}(s), \tag{15}$$

$$B_{\beta_{B,d}}(s) = \frac{-d.\tau(s)T_{\alpha_B}(s) + (1-d.\kappa(s)B_{\alpha_B}(s)}{\sqrt{(1-d.\kappa(s))^2 + (d.\tau(s))^2}}. \tag{16}$$

Proof. Let assume $\alpha_B(s)$ is a pedal curve parameterized by its arc-length s and $\{T_{\beta_{B,d}}(s), N_{\beta_{T,dB}}(s), B_{\beta_{T,d}}(s)\}$ represents its Frenet frame. The given parallel curve is:

$$\beta_{B,d}(s) = \alpha_B(s) + d.N_{\alpha_B}(s).$$

Since the parallel curve is a non-unit speed curve field, we differentiate its position vector with respect to the arc-length parameter s. By applying the Frenet-Serret equations, the first derivative $\beta_{B,d}'(s)$ is expanded as:

$$\beta_{B,d}'(s) = T_{\alpha_B}(s) + d.(-\kappa(s)T_{\alpha_B}(s) + \tau(s)B_{\alpha_B}(s)).$$

Now, group the terms by their Frenet vectors:

$$\beta_{B,d}'(s) = (1-d.\kappa(s))T_{\alpha_B}(s) + d.\tau(s)B_{\alpha_B}(s).$$

Therefore, the unit tangent vector field of the parallel curve $\beta_{B,d}(s)$ obtained from the $B-$ pedal curve is calculated as the following equality:

$$T_{\beta_{T,d}}(s) = \frac{\beta_{B,d}'(s)}{\|\beta_{B,d}'(s)\|},$$

$$T_{\beta_{T,d}}(s) = \frac{(1-d.\kappa(s))T_{\alpha_B}(s) + d.\tau(s)B_{\alpha_B}(s)}{\sqrt{(1-d.\kappa(s))^2 + d^2.\tau^2(s)}}.$$

Differentiating $\beta_{B,d}'(s)$ with respect to s, we get the following equality:

$$\beta_{B,d}''(s) = (-d.\kappa'(s))T_{\alpha_B}(s) + (1-d.\kappa(s))T_{\alpha_B}'(s) + d.\tau'(s)B_{\alpha_B}(s) + d.\tau(s)B_{\alpha_B}'(s).$$

Applying the Frenet-Serret formulas of the $B-$ pedal curve and grouping coefficients yields:

$$\beta''_{B,d}(s)=\left(-d.\kappa'(s)\right)T_{\alpha_B}(s)+\left[\kappa(s)\left(1-d.\kappa(s)\right)-d.\tau^2(s)\right]N_{\alpha_B}(s)+d.\tau'(s)B_{\alpha_B}(s).$$

Let us compute the vector cross product $\beta'_{B,d}(s)\times\beta''_{B,d}(s)$,

$$\begin{aligned}\beta'_{B,d}\times\beta''_{B,d}&=\left[-d.\tau\left(\kappa\left(1-d.\kappa\right)-d.\tau^2\right)\right]T_{\alpha_B}-\left[-d.\tau'\left(\left(1-d.\kappa\right)-d^2.\tau\kappa'\right)\right]N_{\alpha_B}\\&+\left[\left(1-d.\kappa\right)\left(\kappa\left(1-d.\kappa\right)-d.\tau^2\right)\right]B_{\alpha_B}.\end{aligned}$$

The binormal vector field of parallel curve $\beta_{B,d}(s)$ is given by the following equality.

$$\begin{aligned}&B_{\beta_{B,d}}(s)\\&=\frac{1}{\Lambda}\left(\left[-d.\tau\left(\kappa\left(1-d.\kappa\right)-d.\tau^2\right)\right]T_{\alpha_B}-\left[-d.\tau'\left(\left(1-d.\kappa\right)-d^2.\tau\kappa'\right)\right]N_{\alpha_B}+\left[\left(1-d.\kappa\right)\left(\kappa\left(1-d.\kappa\right)-d.\tau^2\right)\right]B_{\alpha_B}\right),\end{aligned}$$

where the normalization scale factor $\Lambda=\left\|\beta'_{B,d}\times\beta''_{B,d}\right\|$ is

$$\Lambda=\sqrt{\left[d.\tau\left(\kappa\left(1-d.\kappa\right)-d.\tau^2\right)\right]^2+d^2.\left[\tau'\left(\left(1-d.\kappa\right)-d^2.\tau\kappa'\right)\right]^2+\left[\left(1-d.\kappa\right)\left(\kappa\left(1-d.\kappa\right)-d.\tau^2\right)\right]^2}\ .$$

If the binormal vector equation is arranged in terms of T_{α_B} and B_{α_B}, the following equality is obtained.

$$B_{\beta_{B,d}}(s)=\frac{-d.\tau(s)T_{\alpha_B}(s)+(1-d.\kappa(s)B_{\alpha_B}(s)}{\sqrt{\left(1-d.\kappa(s)\right)^2+\left(d.\tau(s)\right)^2}}.$$

By computing the following equality, the principal normal vector field of the parallel curve is given by following equality:

$$N_{\beta_{B,d}}(s)=B_{\beta_{B,d}}(s)\times T_{\beta_{B,d}}(s)=\left[\frac{\left(1-d.\kappa(s)\right)^2+d^2\tau^2(s)}{\sigma^2}\right]N_{\alpha_B}(s),$$

where the shared denominator σ is denoted as follows:

$$\sigma=\sqrt{\left(1-d.\kappa(s)\right)^2+\left(d.\tau(s)\right)^2}\ .$$

Therefore, the following equality is obtained

$$N_{\beta_{B,d}}(s)=N_{\alpha_B}(s).$$

Corollary 3.1. Let $\alpha_B(s)$ be a pedal curve of a given curve $\alpha(s)$ parameterized by its arc-length s with the Frenet-Serret frame $\{T_{\alpha_B}(s),N_{\alpha_B}(s),B_{\alpha_B}(s)\}$. If $\beta_{B,d}(s)$ is a parallel curve constructed a constant distance d along the normal vector $N_{\alpha_B}(s)$ of $B-$pedal curve $\alpha_B(s)$ with Frenet-Serret frame $\{T_{\beta_{B,d}}(s),N_{\beta_{B,d}}(s),B_{\beta_{B,d}}(s)\}$, defined by $\beta_{B,d}(s)=\alpha_B(s)+d.N_{\alpha_B}(s)$, then the principal normal vector field $N_{\beta_{B,d}}(s)$ of the parallel curve $\beta_{B,d}(s)$ is identically equal to the unit normal vector field $N_{\alpha_B}(s)$ of the $B-$pedal curve $\alpha_B(s)$. This means

$$N_{\beta_{B,d}}(s)=N_{\alpha_B}(s).$$

Proof. This is seen from Definition 2.1 and proof of Theorem 3.2.

Theorem 3.9. The curvature $\kappa_{\beta_{B,d}}(s)$ and torsion $\tau_{\beta_{B,d}}(s)$ of the parallel curve obtained from the $B-$pedal curve are structured as follows, respectively,

$$\kappa_{\beta_{B,d}}(s)=\frac{\sqrt{\left[\kappa(1-d.\kappa)-d.\tau^2\right]^2\sigma^2+d^2.\left[\tau'(1-d.\kappa)+d.\tau\kappa'\right]^2}}{\sigma^3}, \tag{17}$$

$$\tau_{\beta_{B,d}}(s)=\frac{\begin{array}{l}\left[\kappa(1-d.\kappa)-d.\tau^2\right]^2.\tau+d.\left[\kappa(1-d.\kappa)-d.\tau^2\right](\kappa'\tau-\kappa\tau')\\ +d.\left[\tau'(1-d.\kappa)+d.\tau\kappa'\right]\left[\kappa'(1-d.\kappa)-\kappa(1-d.\kappa)'+\left(d\tau^2\right)'\right]\\ -\left[\kappa(1-d.\kappa)-d.\tau^2\right]\left[\tau''(1-d.\kappa)+d.\kappa'\tau'\right]\end{array}}{\left[\kappa(1-d.\kappa)-d.\tau^2\right]^2\sigma^2+d^2.\left[\tau'(1-d.\kappa)+d.\tau\kappa'\right]}, \tag{18}$$

where the expression

$$\sigma=\left\|\beta'_{B,d}(s)\right\|=\sqrt{(1-d.\kappa(s))^2+(d.\tau(s))^2}.$$

Proof. This is seen from Definition 2.1 and proof of Theorem 3.2.

By recalling Example 3.1, $B-$pedal curve and the parallel curve of $B-$pedal curve according to the origin are illustrated in Figure 4.

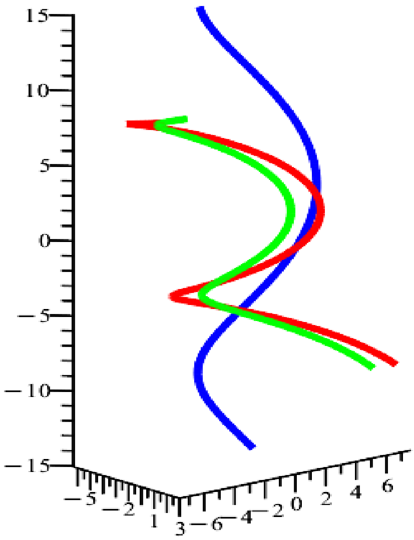

Figure 4. The original space curve (blue), and the $B-$pedal curve with respect to the origin (red) and the parallel curve generated from the $B-$pedal curve for the offset distance d=1.5 (green).

4. Conclusions

In this paper, we present a comprehensive study on the differential geometry of parallel curves derived from a tripartite family of pedal curves specifically, the tangent ($T-$pedal), principal normal ($N-$pedal), and binormal ($B-$pedal) tracks of a given regular space curve in Euclidean 3-space E^3. First, the equation of the parallel curve obtained from the pedal curves ($T-$pedal, $N-$pedal, $B-$pedal curves) of a space curve are determined by taking the Frenet vectors as position vectors. Second, Frenet vectors, curvature, and torsion for each parallel curve obtained from the pedal curves were calculated. Thus, new types of curves were added to the curve family.

Furthermore, our comparative analysis demonstrates that deriving parallel curves from spatial pedal tracks offers distinct advantages over the inverse scheme. Specifically, it provides an explicit formulation for the propagation of curvature and torsion singularities under uniform displacement, yielding valuable tools for kinematic trajectory design and envelope theory in Euclidean 3-space.

Use of Generative-AI tools declaration

The author declares that no Artificial Intelligence (AI) tools have been used in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

References

1. M. P. Do Carmo, *Differential geometry of curves and surfaces*, Prentice Hall, Inc., Englewood Cliffs, 1976.
2. A. Gray, *Modern differential geometry of curves and surfaces with mathematica*, CRC Press, 1996.
3. B. O'Neill, *Elementary differential geometry, revised second edition*, Elsevier, 2006.
4. O. O. Tuncer, H. Ceyhan, İ. Gök, F. N. Ekmekci, Notes on pedal and contrapedal curves of fronts in the Euclidean plane, *Math. Method. Appl. Sci.*, **41** (2018), 5096–5111. https://doi.org/10.1002/mma.5056
5. Y. Li, D. Pei, Pedal curves of frontals in the Euclidean plane, *Math. Methods Appl. Sci.*, **41** (2018), 1988–1997. https://doi.org/10.1002/mma.4724
6. A. Y. Ceylan, M. Kara, On pedal and contrapedal curves of Bezier curves, *Konuralp J. Math.*, **9** (2021), 217–221.
7. Y. Li, D. Pei, Pedal curves of fronts in the sphere, *J. Nonlinear Sci. Appl.*, **9** (2016), 836–844. https://doi.org/10.22436/jnsa.009.03.12
8. M. Li, K. Yao, P. Li, D. Pei, Pedal curves of non-lightlike curves in Minkowski 3-space, *Symmetry*, **14** (2022), 59. https://doi.org/10.3390/sym14010059
9. T. Nishimura, Normal forms for singularities of pedal curves produced by non-singular dual curve germs in S^n, *Geom. Dedicata*, **133** (2008), 59–66. https://doi.org/10.1007/s10711-008-9233-5
10. E. As, A. Sarıoğlugil, On the pedal surfaces of 2-D surfaces with the constant support function in E^4, *Pure Math. Sci.*, **4** (2015), 105–120. https://doi.org/10.12988/pms.2015.545
11. N. Kuruoğlu, A. Sarıoğlugil, On the characteristic properties of the a-pedal surfaces in the Euclidean space E_3, *Commun. Fac. Univ. Ank. Ser. A1 Math. Stat.*, **42** (1993), 19–25. https://doi.org/10.1501/Commua1_0000000483
12. F. E. Kaya, Differential geometric aspects of pedal curves on surfaces, *J. Math.*, **2024** (2024), 6237091. https://doi.org/10.1155/2024/6237091

13. F. E. Kaya, Gaussian and mean curvatures of the surfaces obtained along T-pedal curve of a given curve, *Turk. J. Sci.*, **10** (2025), 105–120.
14. S. Şenyurt, F. E. Kaya, D. Canlı, Pedal curves obtained from Frenet vector of a space curve and Smarandache curves belonging to these curves, *AIMS Math.*, **9** (2024), 20136–20162. https://doi.org/10.3934/math.2024981
15. D. Canlı, S. Şenyurt, F. E. Kaya, L. Grilli, The pedal curves generated by alternative frame vectors and their Smarandache curves, *Symmetry*, **16** (2024), 1012. https://doi.org/10.3390/sym16081012
16. A. Elsharkawy, H. K. Elsayied, M. E. Desouky, C. Cesarano, T-pedal ruled surface with the Frenet frame of the original curve in E^3, *AIMS Math.*, **10** (2025), 25606–25623. https://doi.org/10.3934/math.20251134
17. Y. Li, O. O. Tuncer, On (contra)pedals and (anti)orthotomics of frontals in de Sitter 2-space, *Math. Methods Appl. Sci.*, **46** (2023), 11157–11171. https://doi.org/10.1002/mma.9173
18. M. T. Aldossary, M. A. Gazwani, Motion of parallel curves and surfaces in Euclidean 3-space $\mathbb{R}^3$, *Global J. Adv. Res. Classical Mod. Geom.*, **9** (2020), 43–56.
19. Y. Sağıroğlu, G. Köse, Parallel curves based on normal vector, *Sinop Uni. J. Nat. Sci.*, **9** (2024), 1–13. https://doi.org/10.33484/sinopfbd.1315640
20. F. Güler, The quasi parallel curve of a space curve, *CBUJOS*, **18** (2022), 203–206. https://doi.org/10.18466/cbayarfbe.955974
21. S. Kiziltug, Ö. Tarakci, On parallel surfaces of Weingarten tubular surface in Euclidean 3-space, *Global J. Adv. Res. Classical Mod. Geom.*, **2** (2013), 30–35.