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Research article

A study of weak dominance for t-conorms and null-norms of ordinal sums

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Abstract: This study focuses on the weak dominance between t-conorms and nullnorms given by the following inequality:

$$S(N(x, y), z) \geq N(S(x, z), y), \quad x, y, z \in [0, 1],$$

where N is a nullnorm and S is a t-conorm. The paper mainly discusses the case where N has an ordinal sum structure, its underlying operators are Archimedean, and S is Archimedean. The established relations are proved through the concavity and convexity of functions constructed from the additive generators corresponding to the underlying operators.

Keywords: weak dominance; nullnorms; t-conorms; ordinal sum; convex; concave
Mathematics Subject Classification: 39B52, 39B55

1. Introduction

Aggregation functions, as mathematical tools for combining multiple input values into representative outputs, play a significant role in multi-attribute decision-making, image processing, data fusion, fuzzy logic, and artificial intelligence [1–3]. The concept of dominance was initially introduced in probability metric spaces [4] and is closely related to the construction of Cartesian products and the transitivity of sets in fuzzy relations [5–7].

However, the dominance relationship is too strict. Weak dominance, as an extension of the dominance relationship, has received increasing attention from the academic community in recent years. The concept of weak dominance was first introduced by Alsina et al. [1] As a generalization of the dominance relationship, it is closely linked to the modular equation. Due to its wider application scope, it has attracted increasing attention.

Most studies have focused on the weak dominance of t-norms [8–10]. In recent years, Li et al. extended the research to the weak dominance between t-norms and t-conorms [11, 12]. However, research on weak dominance, especially involving more complex operational structures, such as nullnorms, uninorms, and other mixed-type aggregation functions, related to classical operators, is still relatively scarce. For instance, the weak dominance between t-norms and semi-t-operators has been studied in fuzzy logic [7].

This paper is structured as follows: Section 2 sorts out and reviews basic preliminary concepts. Section 3 puts forward the core research results of this paper, with the focus on the weak dominance relation between ordinal sum structured nullnorms and t-conorms.

2. Preliminaries

Some basic concepts are recalled here, including nullnorms, ordinal sums, weak dominance, and convexity. For more details, please refer to [14, 15, 17].

Definition 2.1 ([13, 16]). Let $(T_i)_{i\in I}$ (resp. $(S_i)_{i\in I}$) be a family of t-norms (t-conorm) and let $\{(a_i, b_i)\}_{i\in I}$ be a pairwise disjoint open subintervals of $[0, 1]$ with $0 \le a_i < b_i \le 1$ and $\bigcup_{i\in I} [a_i, b_i] \subseteq [0, 1]$. The ordinal sum $T = \langle (a_i, b_i, T_i)\rangle_{i\in I}$ is the t-norm defined by

$$T(x, y) = \begin{cases} a_i + (b_i - a_i)\, T_i\left(\dfrac{x - a_i}{b_i - a_i}, \dfrac{y - a_i}{b_i - a_i}\right), & \text{if } x, y \in [a_i, b_i], \\ \min\{x, y\}, & \text{otherwise.} \end{cases}$$

Ordinal sum $S = \langle (a_i, b_i, S_i)\rangle_{i\in I}$ is the t-conorm given by

$$S(x, y) = \begin{cases} a_i + (b_i - a_i)\, S_i\left(\dfrac{x - a_i}{b_i - a_i}, \dfrac{y - a_i}{b_i - a_i}\right), & \text{if } x, y \in [a_i, b_i], \\ \max\{x, y\}, & \text{otherwise.} \end{cases}$$

The common t-norms (t-conorms) are as follows:

- Minimum t-norm T_M:

$$T_M(x, y) = \min\{x, y\}.$$

- Product t-norm T_P:

$$T_P(x, y) = x \cdot y.$$

- Lukasiewicz t-norm T_L:

$$T_L(x, y) = \max\{0,\ x + y - 1\}.$$

- Maximum t-conorm S_M:

$$S_M(x, y) = \max\{x, y\}.$$

- Probabilistic sum t-conorm S_P:

$$S_P(x, y) = x + y - xy.$$

- Łukasiewicz t-conorm S_L:

$$S_L(x, y) = \min\{1,\ x + y\}.$$

Theorem 2.1 ([13]). *A binary function $T(S)\colon [0,1]^2 \to [0,1]$: If there exists a continuous, strictly decreasing mapping $t(s)\colon [0,1] \to [0,\infty)$ with $t(1) = 0$ (resp., $s(0) = 0$) such that*

$$T(x,y) = t^{-1}(\min(t(x) + t(y),\ t(0)))$$

(resp., $S(x,y) = s^{-1}(\min(s(x) + s(y),\ s(1)))$) for all $x, y \in [0,1]$, then it is a continuous Archimedean t-norm (t-conorm). The functions t and s are called additive generators. The reverse also holds.

For a continuous Archimedean t-norm T (t-conorm S), it belongs either to the strict class or to the nilpotent class.

Definition 2.2 ([13]). A binary operation $N\colon [0,1]^2 \to [0,1]$ is called a nullnorm if it satisfies the following conditions for all $x, y, z \in [0,1]$:

- Commutativity: $N(x,y) = N(y,x)$;
- Associativity: $N(N(x,y),\ z) = N(x,\ N(y,z))$;
- Monotonicity: N is non-decreasing in each variable;
- There exists an absorbing element $k \in [0,1]$ such that $N(k,x) = k$ for all $x \in [0,1]$;
- Boundary conditions: $N(0,x) = x$ for all $x \leq k$, and $N(1,x) = x$ for all $x \geq k$.

Theorem 2.2 ([13]). *Let N be a nullnorm with $k \in [0,1]$. There exist a t-conorm S_N on $[0,k]$ and a t-norm T_N on $[k,1]$ such that*

$$N(x,y) = \begin{cases} S_N(x,y), & \textit{if } (x,y) \in [0,k]^2, \\ T_N(x,y), & \textit{if } (x,y) \in [k,1]^2, \\ k, & \textit{otherwise}. \end{cases}$$

We denote this nullnorm by $N \equiv \langle S_N, T_N, k\rangle$.

In this representation, $S_N = N|_{[0,k]^2}$ is a t-conorm on $[0,k]$ with $e = 0$; $T_N = N|_{[k,1]^2}$ is a t-norm on $[k,1]$ with $e = 1$.

Definition 2.3. Let $\{(a_i,\ b_i,\ S_i)\}_{i\in I}$ be a collection where $\{(a_i,\ b_i)\}_{i\in I}$ are pairwise disjoint open subintervals of $[0,k]$, and each S_i is a t-conorm on $[a_i, b_i]$ (with neutral a_i and absorbing b_i). The ordinal sum t-conorm $S_N = \langle(a_i,\ b_i,\ S_i)\rangle_{i\in I}$ on $[0,k]$ is defined by

$$S_N(x,y) = \begin{cases} S_i\,(x,y), & \text{if } (x,y) \in [a_i,b_i]^2, \\ \max\,(x,y), & \text{otherwise}. \end{cases}$$

A completely analogous definition holds for the ordinal sum t-norm $T_N = \langle(c_j, d_j, T_j)\rangle_{j\in J}$ on $[k,1]$. S_i is a continuous Archimedean t-conorm on $[a_i, b_i]$, then there is an $s_i\colon [a_i,b_i] \to [0,\infty]$ and $s_i(a_i) = 0$, $s_i(b_i) = \infty$, then

$$S(x,y) = s_i^{-1}(s_i(x) + s_i(y))$$

for all $x, y \in [a_i, b_i]$. Meanwhile, T_j is a continuous Archimedean t-norm on $[c_j, d_j]$, and there is an $t_j\colon [c_j,d_j] \to [0,\infty]$ and $t_j(d_j) = 0$, $t_j(c_j) = \infty$, then

$$T(x,y) = t_j^{-1}(t_j(x) + t_j(y))$$

for all $x, y \in [c_j, d_j]$.

Definition 2.4 ([1]). Consider binary operations A and B defined on a partially ordered set $(S, \leq)$. If for all $x, y, z, w \in S$,

$$A\big(B(x,z),\ B(y,w)\big) \geq B\big(A(x,y),\ A(z,w)\big),$$

then A is said to dominate B, denoted $A \gg B$.

Definition 2.5 ([1]). Let A and B be binary operations on $(S, \leq)$. If for all $x, y, z \in S$,

$$A(B(x,y),z) \geq B(A(x,z),y),$$

then A is said to weakly dominate B, denoted $A \geq B$.

Definition 2.6 ([14]). A function f, if for any $x, y \in I$ and any $\theta \in [0,1]$,

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y),$$

is called convex.

Definition 2.7 ([14]). A function f, if for any $x, y \in I$ and any $\theta \in [0,1]$:

$$f(\theta x + (1-\theta)y) \geq \theta f(x) + (1-\theta) f(y),$$

is called concave.

Remark 2.1. If f is second-order differentiable, convexity is equivalent to $f''(x) \geq 0$, and concavity is equivalent to $f''(x) \leq 0$.

The interplay between convexity, Lipschitz conditions, and aggregation functions is further explored in [19].

3. Main results

In this section, we investigate the weak dominance between a t-conorm and nullnorm whose underlying operators have an ordinal sum. For the construction of such nullnorms, see [18], please.

Theorem 3.1. *Let S be a t-conorm. Then there exists no nullnorm N with $k \in (0,1)$ such that for all $x, y, z \in [0,1]$,*

$$N(S(x,y),\ z) \geq S\ (N(x,z),\ y)$$

holds.

Proof. Suppose such an N exists with absorbing element $k \in (0,1)$. Consider

$$x = \frac{k}{2}, \quad y = \frac{1+k}{2}, \quad z = \frac{k}{2},$$

then $x, z \in (0,k)$ and $y \in (k,1)$.

Left-hand side: Since $x < k < y$, we have $S(x,y) \geq \max\{x,y\} = y > k$. Because $S(x,y) > k$ and $z \leq k$, the definition of a nullnorm yields

$$N(S(x,y),z) = k.$$

Right-hand side: Both $x \in [0,k]$, $z \in [0,k]$, hence $N(x,z) \in [0,k]$. Since $y > k$ and S is non-decreasing,

$$S(N(x,z),\, y) \geq S(0,\, y) = y > k.$$

Thus the inequality $k \geq S(N(x,z),\, y)$ would require $k > k$, a contradiction. Therefore, no such nullnorm exists.

The relevant conclusions regarding the t-conorm S weakly dominating the nullnorm N are as follows:

Theorem 3.2. *Let S_e be a strict t-conorm with an additive generator $s\colon [0,1] \to [0,\infty]$ that is strictly increasing and satisfies $s(0) = 0$, $s(1) = \infty$. Let $N \equiv \langle S_N, T_N, k\rangle$ be a nullnorm such that:*
(i) $S_N = \langle (a_i,\, b_i,\, S_i)\rangle_{i\in I}$ is an ordinal sum t-conorm on $[0,k]$, where each S_i is a strict t-conorm on $[a_i, b_i]$ with additive generator $s_i\colon [a_i, b_i] \to [0,\infty]$ that is strictly increasing and satisfies

$$s_i(a_i) = 0,\ s_i(b_i) = \infty.$$

(ii) $T_N = \langle (c_j, d_j, T_j)\rangle_{j\in J}$ is an ordinal sum t-norm on $[k,1]$, where each T_j is a strict t-norm on $[c_j, d_j]$ with additive generator $t_j\colon [c_j, d_j] \to [0,\infty]$ that is strictly decreasing and satisfies

$$t_j(d_j) = 0,\ t_j(c_j) = \infty.$$

Define the functions:

$$\begin{aligned} f_i(u) &= s(s_i^{-1}(u)), \quad u \in [0,\infty),\ i \in I,\\ g_j(v) &= s(t_j^{-1}(v)), \quad v \in [0,\infty),\ j \in J. \end{aligned}$$

Then S weakly dominates N if and only if:
(i) *For each $i \in I$, the function f_i is concave;*
(ii) *For each $j \in J$, the function g_j is concave.*

Proof. Necessity: $S \gg N \Rightarrow f_i,\ g_j$ is concave.

Assume S weakly dominates N, i.e., $S(N(x,y),\, z) \geq N(S(x,z),\, y)$ for all $x, y, z \in [0,1]$.

Case 1: Concavity of f_i.

Fix $i \in I$ and take any x, y lie in the same ordinal sum block of S_N, i.e., $x, y \in [a_i, b_i]$. Set

$$a = s_i(x),\ b = s_i(y),\ a, b \in [0,\infty).$$

Due to $S(x,0) = x$ and S being continuous, we can choose z sufficiently small such that $S(x,z) \leq k$ and $S(x,z) \in [a_i, b_i]$. For such z,

$$N(x,y) = S_i(x,y), \qquad N(S(x,z),\, y) = S_i(S(x,z),\, y).$$

The weak dominance inequality becomes

$$S(S_i(x,y), z) \geq S_i(S(x,z), y). \tag{1}$$

Apply the strictly increasing generator s to both sides:

$$s(S_i(x,y)) + s(z) \geq s(S_i(S(x,z), y)). \tag{2}$$

Due to s_i being an additive generator with S_i,

$$s(S_i(x,y)) = s(s_i^{-1}(a+b)) = f_i(a+b).$$

Moreover, set

$$c = s(z), \quad S(x,z) = s^{-1}\big(s(x)+s(z)\big) = s^{-1}\big(f_i(a)+c\big).$$

Because $S(x,z) \in [a_i, b_i]$,

$$s_i\left(S(x,z)\right) = s_i\left(s^{-1}\left(f_i(a)+c\right)\right) = f_i^{-1}\left(f_i(a)+c\right).$$

Hence

$$s_i\left(S_i\left(S(x,z),\ y\right)\right) = s_i(S(x,z)) + s_i(y) = f_i^{-1}\left(f_i(a)+c\right)+b,$$
$$s\left(S_i\left(S(x,z),\ y\right)\right) = f_i\left(f_i^{-1}\left(f_i(a)+c\right)+b\right).$$

Substituting into (2) gives

$$f_i(a+b)+c \geq f_i\left(f_i^{-1}\left(f_i(a)+c\right)+b\right). \tag{3}$$

Because f_i is increasing, now set $c = f_i(a+b) - f_i(a) \geq 0$. Then

$$f_i^{-1}\left(f_i(a)+c\right) = f_i^{-1}\left(f_i(a+b)\right) = a+b,$$

and (3) becomes

$$2f_i(a+b) - f_i(a) \geq f_i(a+2b). \tag{4}$$

Let $p = a$, $q = a+2b$. Then $a+b = \frac{p+q}{2}$ and (4) lead to

$$2f_i\left(\frac{p+q}{2}\right) \geq f_i(p)+f_i(q), \qquad \forall\, p,q \geq 0.$$

Since f_i is continuous, this midpoint inequality implies that f_i is concave on $[0,\infty)$.

Case 2: Concavity of g_j.

The proof is dual. Fix $j \in J$ and take $x, y \in [c_j, d_j]$. Set $a = t_j(x),\ b = t_j(y),\ a, b \in [0,\infty)$. Choose z sufficiently large such that $S(x,z) \geq k$ and $S(x,z) \in [c_j, d_j]$, then

$$N(x,y) = T_j(x,y), \qquad N(S(x,z),\ y) = T_j(S(x,z),\ y),$$

and the weak dominance gives

$$S(T_j(x,y), z) \geq T_j(S(x,z), y). \tag{5}$$

Apply s to (5) and set $w = s(z)$.

$$s\left(T_j(x,y)\right) + c \geq s\left(T_j(S(x,z),\ y)\right).$$

Due to t_j being an additive generator with T_j,

$$s\left(T_j(x,y)\right) = s\left(t_j^{-1}(a+b)\right) = g_j(a+b).$$

Using t_j,

$$t_j\left(T_j(x,y)\right) = a+b, \qquad s\left(T_j(x,y)\right) = g_j(a+b).$$

Also, $S(x,z) = s^{-1}\left(g_j(a) + w\right)$, and because $S(x,z) \in [c_j, d_j]$,

$$t_j(S(x,z)) = t_j\left(s^{-1}\left(g_j(a) + w\right)\right) = g_j^{-1}\left(g_j(a) + w\right).$$

Thus

$$t_j\left(T_j(S(x,z),\ y)\right) = g_j^{-1}\left(g_j(a) + w\right) + b,$$
$$s\left(T_j(S(x,z),\ y)\right) = g_j\left(g_j^{-1}\left(g_j(a) + w\right) + b\right).$$

Hence (5) becomes

$$g_j(a+b) + w \geq g_j\left(g_j^{-1}\left(g_j(a) + w\right) + b\right). \tag{6}$$

Since g_j is decreasing, we take $w = g_j(a) - g_j(a+b) \geq 0$. Then $g_j^{-1}\left(g_j(a) + w\right) = g_j^{-1}\left(g_j(a)\right) = a$.

To obtain concavity, we apply g_j^{-1} to both sides of (6) and reverse the inequality:

$$g_j^{-1}(g_j(a+b) + w) \leq g_j^{-1}(g_j(a) + w) + b. \tag{7}$$

Set $p = g_j(a)$, $q = g_j(a+b)$. Then $p \geq q$, $a = g_j^{-1}(p)$, $a + b = g_j^{-1}(q)$. Set $w = p - q$, then $g_j(a) + w = 2p - q$, and (7) becomes

$$g_j^{-1}(q) \leq g_j^{-1}(2p - q) + b.$$

Since $b = g_j^{-1}(q) - g_j^{-1}(p)$, we obtain

$$g_j^{-1}(q) \leq g_j^{-1}(2p - q) + g_j^{-1}(q) - g_j^{-1}(p) \Longrightarrow g_j^{-1}(p) \leq g_j^{-1}(2p - q). \tag{8}$$

Set $u = q$, $v = 2p - q$, then $p = \frac{u+v}{2}$ and (8) give

$$g_j^{-1}\left(\frac{u+v}{2}\right) \leq \frac{g_j^{-1}(u) + g_j^{-1}(v)}{2}.$$

Hence g_j^{-1} is convex. A continuous, strictly decreasing function whose inverse is convex is itself concave. Therefore, g_j is concave on $[0, \infty)$.

In conclusion, the weak dominance holds for all $x, y, z \in [0,1]$, and the sufficiency is proved.

Sufficiency: f_i, g_j **concave** $\Rightarrow S \gg N$**.**

Assume all f_i and g_j are concave. We need to prove that for any $x, y, z \in [0,1]$,

$$S(N(x,y),\ z) \geq N(S(x,z),\ y)$$

holds.

We distinguish cases based on the positions of x and y relative to k and the ordinal sum blocks.

Case 1: $x, y \in [0, k]$ **and belongs to the same block** $[a_i, b_i]$ **of** S_N**.**

Set $x = s_i^{-1}(a)$, $y = s_i^{-1}(b)$ with $a, b \geq 0$. For any $z \in [0,1]$, set $c = s(z)$. Concavity of f_i is equivalent to

$$f_i(a+b) + c \geq f_i\left(f_i^{-1}\left(f_i(a) + c\right) + b\right). \tag{9}$$

Because S_i is strict,

$$S_i(x,y) = s_i^{-1}(a+b), \quad s(S_i(x,y)) = f_i(a+b),$$
$$S(x,z) = s^{-1}(s(x)+c) = s^{-1}(f_i(a)+c), \quad s_i(S(x,z)) = f_i^{-1}(f_i(a)+c),$$
$$S_i(S(x,z),y) = s_i^{-1}\left(f_i^{-1}(f_i(a)+c)+b\right), \quad s(S_i(S(x,z),y)) = f_i\left(f_i^{-1}(f_i(a)+c)+b\right).$$

This is exactly $s(S_i(x,y)) + c \geq s(S_i(S(x,z),\ y))$. Applying s^{-1} gives

$$S(S_i(x,y),\ z) \geq S_i(S(x,z),\ y). \tag{10}$$

Now $N(x,y) = S_i(x,y)$.

If $S(x,z) \leq k$, and for z such that $S(x,z) \in [a_i, b_i]$, we have

$$N(S(x,z),y) = S_i(S(x,z),y).$$

If $S(x,z) > k$ or $S(x,z) \notin [a_i, b_i]$ leaves the block, then

$$N(S(x,z),y) = k.$$

In all such subcases, monotonicity of S and N together with the fact that $S_i(\cdot,\cdot) \leq k$ guarantees that (10) still holds. The inequality holds for all z by continuity.

Case 2: $x, y \in [k,1]$ **in same block** $[c_j, d_j]$**.**

Set $a = t_j(x)$, $b = t_j(y)$, $c = s(z)$. Concavity of g_j gives

$$g_j(a+b) + c \geq g_j\left(g_j^{-1}\left(g_j(a)+c\right)+b\right). \tag{11}$$

Using the generators of T_j and S,

$$T_j(x,y) = t_j^{-1}(a+b), \quad s(T_j(x,y)) = g_j(a+b),$$
$$S(x,z) = s^{-1}\left(g_j(a)+c\right), \quad t_j(S(x,z)) = g_j^{-1}\left(g_j(a)+c\right),$$
$$T_j(S(x,z),y) = t_j^{-1}\left(g_j^{-1}\left(g_j(a)+c\right)+b\right), \quad s\left(T_j(S(x,z),y)\right) = g_j\left(g_j^{-1}\left(g_j(a)+c\right)+b\right).$$

Thus (11) is equivalent to

$$s(T_j(x,y)) + s(z) \geq s(T_j(S(x,z),\ y)),$$

and after applying s^{-1},

$$S\left(T_j(x,y), z\right) \geq T_j\left(S(x,z), y\right). \tag{12}$$

Since $N(x,y) = T_j(x,y)$ and for z with $S(x,z) \geq k$ and $S(x,z) \in [c_j, d_j]$, we have

$$N(S(x,z),\ y) = T_j(S(x,z),\ y).$$

Inequality (12) yields the weak dominance. The remaining z (where $S(x,z) < k$ or $S(x,z)$ leaves the block) are handled by monotonicity as in Case 1. Therefore, the weak dominance inequality holds in the upper region.

Case 3: $x, y \in [0,k]$ **belongs to the interior of the different block, or one of them is not in any block.**

In this case, $N(x,y) = \max(x,y)$. So, the weak dominance inequation becomes

$$S(\max(x,y),\ z) \geq N(S(x,z),\ y). \tag{13}$$

Subcase 3.1: $S(x,z) \leq k$. Since $S(x,z) \in [0,k]$ and $y \in [0,k]$, $N(S(x,z),y) \in [0,k]$. Moreover,

$$N(S(x,z),\ y) = \max(S(x,z),\ y).$$

On the other hand,

$$S(\max(x,y),z) \geq S(x,z), \quad S(\max(x,y),z) \geq S(y,z) \geq y.$$

Hence $S(\max(x,y),\ z) \geq \max(S(x,z),\ y)$, which is established.
Subcase 3.2: $S(x,z) > k$. Then $S(x,z) \in (k,1]$ while $y \leq k$. By the definition of a nullnorm,

$$N(S(x,z),\ y) = k.$$

Because $\max(x,y) \geq x$, the monotonicity of S yields

$$S(\max(x,y),\ z) \geq S(x,z) > k.$$

Consequently, $S(\max(x,y),\ z) \geq k = N(S(x,z),\ y)$, so (13) holds.
Case 4: $x, y \in [k,1]$ **but belongs to different blocks of** T_N**, or at least one lies outside all blocks.** Here, $T_N(x,y) = \min(x,y)$, so $N(x,y) = \min(x,y)$. The inequality becomes

$$S(\min(x,y),\ z) \geq N(S(x,z),\ y). \tag{14}$$

Again, we consider two subcases.
Subcase 4.1: $S(x,z) \geq k$. Then $S(x,z) \in [k,1]$ and $y \in [k,1]$. Hence

$$N(S(x,z),\ y) = \min(S(x,z),\ y).$$

- If $\min(x,y) = x$, then $N(S(x,z),y) = S(x,z)$ and $\min(S(x,z),y) \leq (S(x,z)$; therefore, (14) is true.
- If $\min(x,y) = y$, then $S(\min(x,y),z) = S(y,z) \geq y$ by monotonicity. Moreover, $\min(S(x,z),y) \leq y$, so (14) follows.

Subcase 4.2: $S(x,z) < k$. Here, $S(x,z) \in [0,k)$ while $y \geq k$; therefore,

$$N(S(x,z),\ y) = k.$$

Since $\min(x,y) \geq k$, we obtain $S(\min(x,y),\ z) \geq \min(x,y) \geq k$, which proves (14).
Case 5: $x \leq k \leq y$ **and** $y \leq k \leq x$**.**

Assume $x \leq k \leq y$ (the symmetric case is analogous). Then $N(x,y) = k$. The inequality to verify is

$$S(k,z) \geq N(S(x,z),\ y). \tag{15}$$

Subcase 5.1: $S(x,z) \leq k$**.** Then, because $y \geq k$, the nullnorm definition yields

$$N(S(x,z),\ y) = k.$$

Since $S(k,z) \geq S(k,0) = k$, (15) follows.

Subcase 5.2: $S(x,z) > k$. Now both $S(x,z) \in [k,1]$ and $y \in [k,1]$. Hence

$$N(S(x,z),\ y) = T_N(S(x,z),\ y).$$

For any t-norm T_N on $[k,1]$, we have $T_N(u,v) \le u$ for all $u, v \in [k,1]$. Consequently,

$$N(S(x,z),\ y) = T_N(S(x,z),\ y) \le S(x,z).$$

Monotonicity of S gives $S(x,z) \le S(k,z)$, because $x \le k$ and S is increasing, thus

$$N(S(x,z),\ y) \le S(x,z) \le S(k,z),$$

which establishes (15).

Subcase 5.3: Boundary cases ($S(x,z) = k$ or $y = k$) are directly covered by the continuity argument below.

Case 6: Ordinal sum boundary points.

Since S and N are continuous on $[0,1]^2$, for any boundary point (x_0, y_0, z_0), we can take a sequence of interior points (x_n, y_n, z_n) converging to it, such that the inequality holds at each interior point. Taking limits and using the order-preserving property of limits yields

$$S(N(x_0,y_0),\ z_0) \ge N(S(x_0,z_0),\ y_0).$$

Therefore, the weak dominance relation holds on the boundaries as well.

Combining the above six parts,

$$S(N(x,y),\ z) \ge N(S(x,z),\ y)$$

for every $x, y, z \in [0,1]$ holds. Therefore, the condition is sufficient.

Remark 3.1. The concavity arguments used in the proof are based on [14].

Remark 3.2. The continuity of ordinal sum operators follows from [13].

Remark 3.3. The only part where the concavity of f_i and g_j is needed is when x and y belong to the same ordinal sum block of the corresponding underlying operator (the lower region $[0,k]^2$ with a common block S_i or the upper region $[k,1]^2$ with a common block T_j). All remaining cases follow directly from the basic properties of t-conorms, t-norms, and nullnorms, together with the continuity of the operators.

Example 3.1. Take $k = \frac{1}{2}$. Define S as the probabilistic sum t-conorm

$$S(x,y) = x + y - xy, \qquad s(x) = -\ln(1-x), \quad x \in [0,1).$$

Consider a nullnorm N with absorbing element $k = \frac{1}{2}$ such that:

- S_N consists of a single block $[0, \frac{1}{2}]$ with a strict t-conorm S_1 given by

$$S_1(x,y) = x + y - 2xy, \qquad s_1(x) = -\ln(1-2x), \quad x \in [0, \frac{1}{2}].$$

- T_N consists of a single block $[\frac{1}{2}, 1]$ with a strict t-norm T_1 given by

$$T_1(x, y) = 2x + 2y - 2xy - 1, \qquad t_1(x) = -\ln(2(1 - x)), \quad x \in [\frac{1}{2}, 1].$$

Compute the composite functions:

$$f_1(u) = s(s_1^{-1}(u)), \qquad g_1(v) = s(t_1^{-1}(v)), \qquad u, v \in [0, \infty).$$

Since $s_1^{-1}(u) = \frac{1}{2}(1 - e^{-u})$, we have

$$f_1(u) = -\ln(1 - \frac{1}{2}(1 - e^{-u})) = -\ln(\frac{1}{2} + \frac{1}{2}e^{-u}).$$

Its second derivative is

$$f_1''(u) = -\frac{e^{-u}}{(1 + e^{-u})^2} < 0, \qquad \forall\, u \geq 0,$$

hence f_1 is strictly concave. For t_1, we obtain $t_1^{-1}(v) = 1 - \frac{1}{2}e^{-v}$, and

$$g_1(v) = -\ln(1 - (1 - \tfrac{1}{2}e^{-v})) = -\ln(\tfrac{1}{2}e^{-v}) = v + \ln 2.$$

Therefore, all conditions of the theorem are satisfied, and consequently S weakly dominates N.

Remark 3.4. The result forms a dual complement to the conclusions on semi-t-operators in Theorems 5–7 [7].

Theorem 3.3. *Let S_e be a nilpotent t-conorm with additive generator $s\colon [0, 1] \to [0, 1]$ strictly increasing, satisfying $s(0) = 0$, $s(1) = 1$.*

Let N be a nullnorm with absorbing element $k \in (0, 1)$ such that:
(i) *Its underlying t-conorm on $[0, k]$ is an ordinal sum, i.e.,*

$$S_N = \langle (a_i,\ b_i,\ S_i) \rangle_{i \in I}\,,$$

where each S_i is a nilpotent t-conorm on $[a_i, b_i]$ with additive generator $s_i\colon [a_i, b_i] \to [0, 1]$ strictly increasing, satisfying $s_i(a_i) = 0$, $s_i(b_i) = 1$.
(ii) *Its underlying t-norm on $[k, 1]$ is an ordinal sum,i.e.,*

$$T_N = \big\langle (c_j,\ d_j,\ T_j) \big\rangle_{j \in J},$$

where each T_j is a nilpotent t-norm on $[c_j, d_j]$ with additive generator $t_j : [c_j, d_j] \to [0, 1]$ strictly decreasing, satisfying $t_j(d_j) = 0$, $t_j(c_j) = 1$.

Define the composite functions

$$f_i(u) = s(s_i^{-1}(u)),\ u \in [0, 1],\ i \in I,$$
$$g_j(v) = s(t_j^{-1}(v)),\ v \in [0, 1],\ j \in J.$$

Then S weakly dominates N (i.e., $S\big(N(x, y),\ z\big) \geq N\big(S(x, z),\ y\big)$ for all $x, y, z \in [0, 1]$) if and only if:
(a) *For each $i \in I$, the function f_i is concave;*
(b) *For each $j \in J$, the function g_j is concave.*

Proof. Necessity: $S \gg N \;\Rightarrow\; f_i, g_j$ is concave.

Assume S weakly dominates N, i.e., for all $x, y, z \in [0, 1]$,

$$S(N(x, y),\ z) \geq N(S(x, z),\ y).$$

Case 1: Proving f_i is concave.

Fix $i \in I$ and take any $x, y \in [a_i, b_i]$. Set

$$a = s_i(x), \qquad b = s_i(y), \qquad a, b \in [0, 1].$$

Because S_i is nilpotent, the interior of S_i is strict. We consider the case $a + b \leq 1$.

Choose z sufficiently small by continuity such that

$$S(x, z) \leq k, \quad S(x, z) \in [a_i, b_i], \quad s_i(S(x, z)) + b \leq 1, \quad s(x) + s(z) \leq 1,$$

$$N(x, y) = S_i(x, y), \quad N(S(x, z), y) = S_i(S(x, z), y).$$

Weak dominance gives

$$S(S_i(x, y),\ z) \geq S_i(S(x, z),\ y). \tag{16}$$

Apply strictly increasing s and denote $c = s(z) \in [0, 1]$:

$$s(S_i(x, y)) + c \geq s(S_i(S(x, z),\ y)). \tag{17}$$

Since $a + b \leq 1$,

$$S_i(x, y) = s_i^{-1}(a + b), \qquad s(S_i(x, y)) = s\big(s_i^{-1}(a + b)\big) = f_i(a + b).$$

Also, $S(x, z) = s^{-1}(s(x) + c) = s^{-1}(f_i(a) + c)$, where $f_i(a) + c \leq 1$ by z. Because $S(x, z) \in [a_i, b_i]$,

$$s_i(S(x, z)) = s_i\big(s^{-1}(f_i(a) + c)\big) = f_i^{-1}(f_i(a) + c).$$

Then

$$S_i(S(x, z), y) = s_i^{-1}\big(s_i(S(x, z)) + b\big) = s_i^{-1}\big(f_i^{-1}(f_i(a) + c) + b\big),$$

and therefore

$$s(S_i(S(x, z), y)) = f_i\big(f_i^{-1}(f_i(a) + c) + b\big).$$

Substituting the above into (17) and applying the monotonicity of s yields

$$f_i(a + b) + c \geq f_i\big(f_i^{-1}(f_i(a) + c) + b\big). \tag{18}$$

Since f_i is strictly increasing, take $c = f_i(a + b) - f_i(a) \geq 0$. Then

$$f_i^{-1}(f_i(a) + c) = f_i^{-1}(f_i(a + b)) = a + b.$$

Substituting into (18) gives

$$2f_i(a + b) - f_i(a) \geq f_i(a + 2b). \tag{19}$$

Set $p = a$, $q = a + 2b$, then $a + b = \frac{p+q}{2}$, and we have

$$2f_i\left(\frac{p+q}{2}\right) \geq f_i(p) + f_i(q), \qquad \forall\, p, q \geq 0,$$

therefore f_i is concave.

Case 2: Proving is g_j concave.

The proof is dual. Fix $j \in J$ and $x, y \in [c_j, d_j]$. Set

$$a = t_j(x), \qquad b = t_j(y), \qquad a, b \in [0, 1].$$

Choose z sufficiently large so that

$$S(x,z) \geq k, \qquad S(x,z) \in [c_j, d_j], \qquad t_j(S(x,z)) + b \leq 1, \qquad s(x) + s(z) \leq 1.$$

Then $N(x,y) = T_j(x,y)$, $N(S(x,z), y) = T_j(S(x,z), y)$. Weak dominance gives

$$S(T_j(x,y),\ z) \geq T_j(S(x,z),\ y). \tag{20}$$

Apply s and set $c = s(z)$:

$$s(T_j(x,y)) + c \geq s(T_j(S(x,z),\ y)). \tag{21}$$

Since $a + b \leq 1$,

$$T_j(x,y) = t_j^{-1}(a+b), \qquad s(T_j(x,y)) = s(t_j^{-1}(a+b)) = g_j(a+b).$$

Also, $S(x,z) = s^{-1}(g_j(a) + c)$, and because $S(x,z) \in [c_j, d_j]$,

$$t_j(S(x,z)) = t_j(s^{-1}(g_j(a) + c)) = g_j^{-1}(g_j(a) + c).$$

Hence

$$T_j(S(x,z), y) = t_j^{-1}(t_j(S(x,z)) + b) = t_j^{-1}(g_j^{-1}(g_j(a) + c) + b),$$
$$s(T_j(S(x,z), y)) = g_j(g_j^{-1}(g_j(a) + c) + b).$$

Thus (21) becomes

$$g_j(a+b) + c \geq g_j(g_j^{-1}(g_j(a) + c) + b). \tag{22}$$

Take

$$c = g_j(a) - g_j(a+b) \geq 0,$$

then

$$g_j^{-1}(g_j(a) + c) = a.$$

Apply g_j^{-1} to (22) and reverse the inequality:

$$g_j^{-1}(g_j(a+b) + c) \leq g_j^{-1}(g_j(a) + c) + b. \tag{23}$$

Set $p = g_j(a)$, $q = g_j(a+b)$. Then $p \geq q$, $a = g_j^{-1}(p)$, $a + b = g_j^{-1}(q)$, and $c = p - q$ into (23) yields

$$g_j^{-1}(q) \leq g_j^{-1}(2p - q) + b, \qquad \text{where } b = g_j^{-1}(q) - g_j^{-1}(p).$$

Therefore,

$$g_j^{-1}(q) \leq g_j^{-1}(2p-q) + g_j^{-1}(q) - g_j^{-1}(p) \Longrightarrow g_j^{-1}(p) \leq g_j^{-1}(2p-q). \tag{24}$$

Set $u = q$, $v = 2p - q$. Then $p = (u+v)/2$, and (24) is

$$g_j^{-1}\left(\frac{u+v}{2}\right) \leq \frac{g_j^{-1}(u) + g_j^{-1}(v)}{2},$$

for all $u, v \geq 0$.

Thus g_j^{-1} is convex. Hence g_j is concave on $[0,1]$. This completes the necessity.

This completes the proof of sufficiency.

Sufficiency (f_i, g_j **is concave** $\Rightarrow S \gg N$**).**

Assume all f_i and g_j are concave. We verify

$$S(N(x,y),\, z) \geq N(S(x,z),\, y), \qquad \forall\, x,y,z \in [0,1].$$

Case 1: $x, y \in [0,k]$ **belongs to the interior of the same block** S_i**.**

Set $a = s_i(x)$, $b = s_i(y)$, $c = s(z)$. Concavity of f_i implies

$$\begin{aligned} & f_i(a+b) + c \geq f_i(f_i^{-1}(f_i(a)+c)+b), \\ & \text{provided } a+b \leq 1, \\ & \qquad f_i(a) + c \leq 1, \\ & \qquad f_i^{-1}(f_i(a)+c) + b \leq 1. \end{aligned}$$

Define the function

$$H(c) = f_i(f_i^{-1}(f_i(a)+c)+v) - f_i(a+b) - c.$$

Set $p = f_i^{-1}(f_i(a)+c) > a$. By concavity,

$$\frac{f_i(p+b) - f_i(a+b)}{p-a} \leq \frac{f_i(p) - f_i(a)}{p-a} = \frac{c}{p-a},$$

which implies $f_i(p+b) - f_i(a+b) \leq c$, i.e., $H(c) \leq 0$. Moreover, $H(0) = 0$. Hence $H(c) \leq 0$ for all $c \geq 0$.

If $f_i(a)+c \geq 1$ or $s_i(S(x,z)/k)+b = 1$, then either $S(x,z) = 1$ or $N(S(x,z),y) = k$, and the inequality can be verified directly by monotonicity or by taking limits. Thus the inequality holds for all interior points of this region.

Case 2: $x, y \in [k,1]$ **belongs to the interior of the same block** $[c_j, d_j]$ **of** T_N**.**

Set

$$a = t_j(x), \qquad b = t_j(y), \qquad c = s(z) \in [0,1].$$

Concavity of g_j implies

$$g_j(a+b) + c \geq g_j(g_j^{-1}(g_j(a)+c)+b).$$

This is equivalent to

$$S(T_j(x,y),\, z) \geq T_j(S(x,z),\, y). \tag{25}$$

Because $N(x,y) = T_j(x,y)$ and $N(S(x,z),y) = T_j(S(x,z),y)$, when $S(x,z) \geq k$ and is inside the block, (25) gives the weak dominance for interior z. Continuity extends it to all z.

Cases 3–6: We now consider the other cases, including x, y in different blocks, mixed region and boundary points, which do not involve the internal sums of the generators; they rely only on the monotonicity of S, the definition of nullnorms, and continuity. The arguments are identical to those in Theorem 3.2 (Case 3 to Case 6).

Since all possible cases satisfy the required inequality, we conclude $S \gg N$. Thus, the theorem is proved.

Remark 3.5. The proof is analogous to that of Theorem 3.2, with details adapted from [15].

Example 3.2. Take $k = \frac{1}{2}$. Define S as the Lukasiewicz t-conorm

$$S(x, y) = \min(1, x + y), \qquad s(x) = x.$$

Consider a nullnorm N with absorbing element $k = \frac{1}{2}$ such that:
- S_N consists of a single block $[0, \frac{1}{2}]$ with the t-conorm

$$S_1(x, y) = \min(\frac{1}{2}, x + y),$$

  whose normalized additive generator is $s_1(x) = 2x$, mapping $[0, \frac{1}{2}]$ to $[0, 1]$.
- T_N consists of a single block $[\frac{1}{2}, 1]$ with the t-norm

$$T_1(x, y) = \max(\frac{1}{2}, x + y - 1),$$

  whose normalized additive generator is $t_1(x) = 2(1 - x)$, mapping $[\frac{1}{2}, 1]$ to $[0, 1]$.

Then compute the composite functions:

$$f_1(u) = s(s_1^{-1}(u)) = s(\frac{u}{2}) = \frac{u}{2}, \quad u \in [0, 1],$$
$$g_1(v) = s(t_1^{-1}(v)) = s(1 - \frac{v}{2}) = 1 - \frac{v}{2}, \quad v \in [0, 1].$$

Hence f_1 and g_1 are concave. By Theorem 3.3, S weakly dominates N.

Take $x = 0.2$, $y = 0.3$, $z = 0.6$:

$$N(x, y) = S_1(0.2, 0.3) = \min(0.5, 0.5) = 0.5,$$
$$S(0.5, 0.6) = \min(1, 1.1) = 1,$$
$$S(x, z) = \min(1, 0.8) = 0.8,$$
$$N(0.8, 0.3) = 0.5.$$

Therefore, the conditions of the theorem are satisfied, and S weakly dominates N.

Theorem 3.4. *Let S_e be a strict t-conorm with additive generator $s\colon [0, 1] \to [0, \infty]$ strictly increasing, satisfying $s(0) = 0$, $s(1) = \infty$. Let N be a nullnorm with an absorbing element $k \in (0, 1)$, and its underlying operators satisfy:*
(i) *$S_N = \langle (a_i, b_i, S_i) \rangle_{i \in I}$, where each S_i is a nilpotent t-conorm on $[a_i, b_i]$ with the additive generator strictly increasing, satisfying $s_i\colon [a_i, b_i] \to [0, 1]$ ($s_i(a_i) = 0$, $s_i(b_i) = 1$).*

(ii) *$T_N = \langle (c_j, d_j, T_j)\rangle_{j\in J}$, where each T_j is a strict t-norm on $[c_j, d_j]$ with the additive generator strictly decreasing, satisfying $t_j\colon [c_j, d_j] \to [0, \infty]$, ($t_j(d_j) = 0$, $t_j(c_j) = \infty$).*

Define

$$f_i(u) = s(s_i^{-1}(u)), \qquad u \in [0,1],\ i \in I,$$
$$g_j(v) = s(t_j^{-1}(v)), \qquad v \in [0,\infty),\ j \in J.$$

Then S weakly dominates N (i.e., $S(N(x,y),z) \geq N(S(x,z),y)$ for all $x, y, z \in [0,1]$) if and only if:
(i) *For each $i \in I$, the function f_i is concave;*
(ii) *For each $j \in J$, the function g_j is concave.*

Remark 3.6. This theorem extends the results in [15] to the case of mixed strict and nilpotent structures.

Example 3.3. Take $k = \frac{1}{2}$. Define S as the probabilistic sum t-conorm: $S(x,y) = x + y - xy$, with additive generator $s(x) = -\ln(1-x)$.
Lower block (nilpotent t-conorm): S_N consists of a single block $[0, \frac{1}{2}]$ with

$$S_1(x,y) = \min(\frac{1}{2}, x+y), \qquad s_1(x) = 2x, \qquad x \in [0, \frac{1}{2}].$$

Then
$$f_1(u) = s(s_1^{-1}(u)) = s(\frac{u}{2}) = -\ln(1 - \frac{u}{2}).$$

Compute
$$f_1''(u) = -\frac{1}{4(1-\frac{u}{2})^2} < 0$$

on $[0,1]$, so f_1 is concave.
Upper block (strict t-norm): T_N consists of a single block $[\frac{1}{2}, 1]$ with

$$T_1(x,y) = 1 - \frac{(1-x)(1-y)}{1/2}.$$

Use the product t-norm shifted:

$$T_1(x,y) = \frac{1}{2} + 2(x - \frac{1}{2})(y - \frac{1}{2}), \qquad t_1(x) = -\ln(2(x - \frac{1}{2})).$$

Then $t_1^{-1}(v) = \frac{1}{2} + \frac{1}{2}e^{-v}$, and

$$g_1(v) = s(t_1^{-1}(v)) = -\ln(1 - (\frac{1}{2} + \frac{1}{2}e^{-v})) = -\ln(\frac{1}{2} - \frac{1}{2}e^{-v}) = \ln 2 - \ln(1 - e^{-v}).$$

And
$$g_1''(v) = -\frac{e^{-v}}{(1-e^{-v})^2} < 0,$$

so g_1 is concave.

Therefore, the conditions of the theorem are satisfied, and S weakly dominates N.

4. Conclusions

The relevant results clarify the necessary and sufficient conditions for a t-conorm to weakly dominate a nullnorm, with the underlying operators with ordinal sums whether the functions constructed from the generators possess concavity. Complete concavity characterizations are established via additive generators defined directly on the original intervals—thereby avoiding linear transformations. Future research will focus on the weak dominance relationships between uninorms and related operators and explore their applications in fuzzy logic, fuzzy control, and study the new properties and applications brought about by weak dominance relations when the additive generators are nonlinear functions.

Author contributions

Xiaodong Hou: Conceptualization, data curation; Jiajia Fan: Writing and original draft; Nan Zhang: Formal analysis, writing and review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

During the preparation of this work, we used DeepSeek for language polishing, grammar refinement, and preliminary structural organization of the manuscript. After using this tool, we carefully reviewed and edited the content as necessary and take full responsibility for the final content of the published work. We confirm that all generative AI outputs were verified and corrected by the authors, and no AI-generated content is directly reproduced without thorough validation.

Conflict of interest

We declare that the material presented in this manuscript has not been published or simultaneously submitted to publication elsewhere; this paper does not infringe on intellectual property rights of any third parties.

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